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PROJECTIVE AND FINSLER METRIZABILITY: PARAMETERIZATION-RIGIDITY OF THE GEODESICS

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In this work we show that for the geodesic spray S of a Finsler function F, the most natural projective deformation $\widetilde{S} = S - 2\lambda F\mathbb{C}$ leads to a non-Finsler metrizable spray, for almost every value of $\lambda \in \mathbb{R}$. This result shows how rigid is the metrizablility property with respect to certain reparameterizations of the geodesics. As a consequence, we obtain that the projective class of an arbitrary spray contains infinitely many sprays that are not Finsler metrizable.

Keywords: Sprays; geodesics; projective metrizability; Finsler metrizability; sectional curvature.

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1. Introduction

A system of second order homogeneous ordinary differential equations (SODE), whose coefficients functions do not depend explicitly on time, can be identified with a special vector field, called spray.

The Finsler metrizability problem for a spray S seeks for a Finsler function whose geodesics coincide with the geodesics of S, [9, 16, 27]. In [22] a set of necessary and sufficient conditions for the Finsler metrizability problem were formulated in terms of the holonomy distribution of a spray. In this work, we will use these conditions to decide whether or not a spray is Finsler metrizable.

For the projective metrizability problem, one seeks for a Finsler function whose geodesics coincide with the geodesics of S, up to an orientation preserving reparameterization. The projective metrizability problem is known as the Finslerian version of Hilbert's fourth problem [1, 10]. In the general case it was Rapcsák [23] who obtained, in local coordinates, necessary and sufficient conditions for the projective metrizability problem of a spray.

The two problems can be viewed as particular cases of the inverse problem of the calculus of variation. We refer to the review articles [2, 17, 21, 24] for various approaches of the inverse problem of the calculus of variations. One of this approaches seeks for the existence of a multiplier matrix that satisfies four Helmholtz conditions [24]. In [7], these four Helmholtz conditions where reformulated in terms of a semi-basic 1-form. For the particular case of the Finsler metrizability problem, only three of the Helmholtz conditions are independent [7, 16], while for the projective metrizability problem, only two Helmholtz conditions are independent, [7]. The formal integrability of these two Helmholtz conditions was studied in [8] and it lead to some classes of sprays that are projectively metrizable: isotropic sprays and arbitrary sprays on 2-dimensional manifold. Within these classes, we searched for sprays that are not Finsler metrizable. Of great help for us, at the time, was given by Yang's example, which was just published online, [28]. Yang shows that for a flat spray of constant flag curvature its projective class contains sprays that are not projectively flat and hence cannot be Finsler metrizable. In this work, using different techniques, we extend Yang's example, and we show that for an arbitrary spray its projective class contains sprays that are not Finsler metrizable.

The structure of the paper is as follows. In Sec. 2 we give a brief introduction of the Frölicher–Nijenhuis theory and the canonical structures one can define on the tangent bundle of a manifold. In Sec. 3 we use the Frölicher–Nijenhuis theory to introduce the main structures one need to discuss the geometry of a spray: connection, Jacobi endomorphism, curvature, and covariant derivative. We pay a special attention to projectively related sprays and the role of parameterization for the corresponding metrizability problem. In Sec. 4 we discuss the Finsler metrizability problem and projective metrizability problem for a spray. For projectively related sprays we provide in Propositions 4.4 and 4.5 the relations between the corresponding geometric structures. In Sec. 5, in Theorem 5.1, we prove that for an arbitrary spray S, there are infinitely many values of a scalar λ such that the projectively related spray $\widetilde{S} = S - 2\lambda F\mathbb{C}$ is not Finsler metrizable, where F is a Finsler function and $\mathbb{C}$ the Liouville vector field. For these values of λ, we show how to reparameterize the geodesics of a Finsler function to transform them into parameterized curves that cannot be the geodesics of any Finsler function.

2. Preliminaries

In this work M is a real and smooth manifold of dimension $n > 1$. We denote by $C^\infty(M)$, the ring of smooth functions on M, and by $\mathfrak{X}(M)$, the $C^\infty(M)$-module of vector fields on M. Consider $\Lambda(M) = \bigoplus_{k\in\mathbb{N}} \Lambda^k(M)$ the graded algebra of differential forms on M. We also write $\Psi(M) = \bigoplus_{k\in\mathbb{N}} \Psi^k(M)$ for the graded algebra of vector-valued differential forms on M.

In this work, we will discuss some relations between Finsler and projective metrizability problems for a homogeneous system of second order ordinary differential equations using the Frölicher–Nijenhuis formalism associated to the system. For systematic treatments of the Frölicher–Nijenhuis theory, we refer to [12, 14, 15]. For a vector valued l-form A on M consider the inner product i_A, which is a derivation of degree $l-1$ and the Lie derivation d_A, which is a derivation of degree l, related by

$$d_A = i_A \circ d + (-1)^l d \circ i_A.$$

When $l = 0$, which means that A is a vector field, we have that $d_A = \mathcal{L}_A$, the usual Lie derivative. For two vector valued forms $A \in \Psi^l(M)$ and $B \in \Psi^s(M)$, the Frölicher–Nijenhuis bracket of A and B is the unique vector valued $(l+s)$-form $[A, B]$ on M such that

$$d_{[A,B]} = d_A \circ d_B - (-1)^{ls} d_B \circ d_A. \tag{2.1}$$

Consider (TM, π, M) the tangent bundle of M and $(\hat{T}M := TM\backslash\{0\}, \pi, M)$ the tangent bundle with the zero section removed. There are some canonical structures one can associate to the tangent bundle, such as the vertical distribution, Liouville vector field, and tangent structure. We will use the Frölicher–Nijenhuis theory associated to these structures to formulate a geometric setting for a system of SODE, viewed as a vector field on the tangent bundle, [6, 11, 20, 26].

The *vertical distribution* is defined as $V : u \in TM \mapsto V_u = \{\xi \in T_uTM, d_u\pi(\xi) = 0\}$. This distribution is n-dimensional and integrable, since it is tangent to the leaves of the regular foliation induced by the submersion π, whose leaves are tangent spaces to M, $\pi^{-1}(p) = T_pM$, for $p \in M$. We denote by (x^i) local coordinates on the base manifold M and by (x^i, y^i) the induced coordinates on TM. It follows that (y^i) are coordinates in the leaves of the foliation, while (x^i) are transverse coordinates for the foliation. An important vertical vector field is the *Liouville vector field*, which locally is given by $\mathbb{C} = y^i\partial/\partial y^i$. The Liouville vector field will be used to characterize homogeneous objects on $\hat{T}M$. For an integer s, we say that a vector valued form $A \in \Psi^l(\hat{T}M)$ is s-homogeneous if $\mathcal{L}_{\mathbb{C}}A = (s-1)A$. A form $\omega \in \Lambda^l(\hat{T}M)$ is s-homogeneous if $\mathcal{L}_{\mathbb{C}}\omega = s\omega$.

The *tangent structure* is the $(1,1)$-type tensor field J on TM, which locally is given by $J = \partial/\partial y^i \otimes dx^i$. Tensor J satisfies $J^2 = 0$ and $\operatorname{Ker} J = \operatorname{Im} J = V$. The tangent structure J is integrable, which means that the Frölicher–Nijenhuis bracket $[J, J]$ vanishes. Using formula (2.1) it follows that $2d_J^2 = d_{[J,J]} = 0$. Since $[\mathbb{C}, J] = -J$ it follows that the vector valued 1-form J is 0-homogeneous.

An important class of (vector valued) forms on $\hat{T}M$ that are compatible with the structures presented above is given by semi-basic forms. A form ω on $\hat{T}M$ is called *semi-basic* if it vanishes whenever one of its arguments is a vertical vector field. Locally, a semi-basic k-form ω on $\hat{T}M$ can be written as

$$\omega = \frac{1}{k!}\omega_{i_1\cdots i_k}(x, y)dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

A 1-form ω on $\hat{T}M$ is semi-basic if and only if $i_J\omega = \omega \circ J = 0$.

A vector valued form L on $\hat{T}M$ is called semi-basic if it takes vertical values and vanishes whenever one of its arguments is a vertical vector field. In local coordinates, a vector valued semi-basic l-form L on $\hat{T}M$ can be written as

$$L = \frac{1}{l!} L^j_{i_1 \cdots i_l}(x, y) \frac{\partial}{\partial y^j} \otimes dx^{i_1} \wedge \cdots \wedge dx^{i_l}.$$

A vector valued 1-form L on $\hat{T}M$ is semi-basic if and only if $J \circ L = 0$ and $i_J L = L \circ J = 0$. The tangent structure J is a vector valued semi-basic 1-form.

3. Sprays and Related Geometric Objects

A system of homogeneous second order ordinary differential equations, whose coefficients functions do not depend explicitly on time, can be identified with a special vector field on $\hat{T}M$ that is called a spray. In this section, we use the Frölicher–Nijenhuis theory to associate a geometric setting to a spray, [13]. Within this geometric setting we will discuss in the next sections the Finsler and projective metrizability problems and some relations between these two problems.

A vector field $S \in \mathfrak{X}(\hat{T}M)$ is called a *spray* if $JS = \mathbb{C}$ and $[\mathbb{C}, S] = S$. Locally, a spray can be expressed as follows

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i},$$

for some functions G^i defined on domains of induced coordinates on $\hat{T}M$. The homogeneity condition, $[\mathbb{C}, S] = S$, for a spray is equivalent with the fact that functions $G^i(x, y)$ are 2-homogeneous in the fiber coordinates. In this work, we will consider positive homogeneity only and hence will assume that $G^i(x, \lambda y) = \lambda^2 G^i(x, y)$ for all $\lambda > 0$.

A curve $c : I \to M$ is called *regular* if its tangent lift takes values in the slashed tangent bundle, $c' : I \to \hat{T}M$. A regular curve is called a *geodesic* of spray S if $S \circ c' = c''$. Locally, $c(t) = (x^i(t))$ is a geodesic of spray S if

$$\frac{d^2 x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0. \tag{3.1}$$

An orientation preserving reparameterization $t \to \tilde{t}(t)$ of the system (3.1) leads to a new spray $\widetilde{S} = S - 2P\mathbb{C}$, [3, 25]. The scalar function $P \in C^\infty(\hat{T}M)$ is 1-homogeneous and it is related to the new parameter by

$$\frac{d^2 \tilde{t}}{dt^2} = 2P\left(x^i(t), \frac{dx^i}{dt}\right) \frac{d\tilde{t}}{dt}, \quad \frac{d\tilde{t}}{dt} > 0. \tag{3.2}$$

Definition 3.1. Two sprays S and $\widetilde{S}$ are *projectively related* if their geodesics coincide up to an orientation preserving reparameterization.

We will refer to the map $S \to \widetilde{S} = S - 2P\mathbb{C}$, for $P \in C^\infty(\hat{T}M)$ a 1-homogeneous function, as to the *projective deformation* of spray S. The aim of this work is to show that projective deformations, or orientation preserving reparameterizations, are rigid with respect to an important problem associated to a spray, the Finsler metrizability problem.

A *nonlinear connection* is defined by an n-dimensional distribution $H : u \in \hat{T}M \to H_u \subset T_u(\hat{T}M)$ that is supplementary to the vertical distribution, which means that for all $u \in \hat{T}M$, we have $T_u(\hat{T}M) = H_u \oplus V_u$.

Every spray S induces a canonical nonlinear connection through the corresponding horizontal and vertical projectors, [13]

$$h = \frac{1}{2}(\mathrm{Id} - [S, J]), \quad v = \frac{1}{2}(\mathrm{Id} + [S, J]). \tag{3.3}$$

Equivalently, the canonical nonlinear connection induced by a spray can be expressed in terms of an *almost product structure* $\Gamma = -[S, J] = h - v$. With respect to the induced nonlinear connection, a spray S is horizontal, which means that $S = hS$. Locally, the two projectors h and v can be expressed as follows

$$h = \frac{\delta}{\delta x^i} \otimes dx^i, \quad v = \frac{\partial}{\partial y^i} \otimes \delta y^i, \quad \text{where}$$

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x, y)\frac{\partial}{\partial y^j}, \quad \delta y^i = dy^i + N_j^i(x, y)dx^j, \quad N_j^i(x, y) = \frac{\partial G^i}{\partial y^j}(x, y).$$

For a spray S consider the vector valued semi-basic 1-form

$$\Phi = v \circ [S, h] = R_j^i(x, y)\frac{\partial}{\partial y^i} \otimes dx^j, \quad R_j^i = 2\frac{\delta G^i}{\delta x^j} - S(N_j^i) + N_k^i N_j^k, \tag{3.4}$$

which will be called the *Jacobi endomorphism*.

Another important geometric structure induced by a spray S is the *curvature tensor* R. It is the vector valued semi-basic 2-form

$$R = \frac{1}{2}[h, h] = \frac{1}{2}R_{jk}^i \frac{\partial}{\partial y^i} \otimes dx^j \wedge dx^k, \quad R_{jk} = \frac{\delta N_j^i}{\delta x^k} - \frac{\delta N_k^i}{\delta x^j}. \tag{3.5}$$

All geometric objects induced by a spray S inherit the homogeneity condition. Therefore $[\mathbb{C}, h] = 0$, which means that the nonlinear connection is 1-homogeneous. Also $[\mathbb{C}, R] = 0$, $[\mathbb{C}, \Phi] = \Phi$ and hence the curvature tensor R is 1-homogeneous, while the Jacobi endomorphism Φ is 2-homogeneous.

The two vector valued semi-basic 1 and 2-forms Φ and R are related as follows:

$$\Phi = i_S R, \quad [J, \Phi] = 3R. \tag{3.6}$$

Locally, the two formulae (3.6) can be expressed as follows:

$$R_j^i = R_{kj}^i y^k, \quad R_{jk}^i = \frac{1}{3}\left(\frac{\partial R_k^i}{\partial y^j} - \frac{\partial R_j^i}{\partial y^k}\right). \tag{3.7}$$

For the Jacobi endomorphism Φ we say that a continuous function $\kappa \in C^0(\hat{T}M)$ is an *eigen function* if there exists a non-zero horizontal vector field $X \in \mathfrak{X}(\hat{T}M)$ such that $\Phi(X) = \kappa JX$. The horizontal vector field X is called an *eigen vector field*.

Since the Jacobi endomorphism is 2-homogeneous, it follows that its non-zero eigen functions are 2-homogeneous functions on $\hat{T}M$. See [14, p. 58] for more details on the eigen functions and eigen vector fields for vector valued semi-basic 1-forms on TM. From first formula (3.6) we obtain that $\Phi(S) = 0$ and hence $\kappa = 0$ is always an eigen function for the Jacobi endomorphism, the corresponding eigen vector field is the spray S. Therefore, $\operatorname{rank} \Phi \leq n - 1$.

For a spray S, consider $\mathcal{H}ol_S \subset T(\hat{T}M)$ the *homolnomy distribution* generated by horizontal vector fields and their successive Lie brackets, [22]. If the curvature R is non-zero, then $\mathcal{H}ol_S$ contains also vertical vector fields. From formula (3.5) it follows that $\operatorname{Im} R \subset \mathcal{H}ol_S$, while from first formula (3.6) it follows that $\operatorname{Im} \Phi \subset \operatorname{Im} R$.

The nonlinear connection induced by a spray S can be characterized also using an *almost complex structure*. It is the $(1,1)$-type tensor field on $\hat{T}M$ given by

$$\mathbb{F} = h \circ [S, h] - J = \frac{\delta}{\delta x^i} \otimes \delta y^i - \frac{\partial}{\partial y^i} \otimes dx^i.$$

For a spray S, consider the map $\nabla : \mathfrak{X}(\hat{T}M) \to \mathfrak{X}(\hat{T}M)$, given by

$$\nabla = h \circ \mathcal{L}_S \circ h + v \circ \mathcal{L}_S \circ v = \mathcal{L}_S + h \circ \mathcal{L}_S h + v \circ \mathcal{L}_S v \tag{3.8}$$

that will be called the *dynamical covariant derivative*. By setting $\nabla f = S(f)$, for $f \in C^\infty(\hat{T}M)$, using the Leibniz rule, and the requirement that ∇ commutes with tensor contraction, we extend the action of ∇ to arbitrary tensor fields and forms on $\hat{T}M$, see [7, Sec. 3.2]. The action of ∇ on semi-basic forms coincide with the semi-basic derivation introduced in [14, Def. 4.2]. From first formula in (3.8) it follows that $\nabla h = 0$ and $\nabla v = 0$, which means that ∇ preserves both the horizontal and the vertical distributions. Moreover, we have $\nabla J = 0$, which implies that ∇ has the same action on horizontal and vertical vector fields. Locally, we can see this from the following formulae:

$$\nabla \frac{\delta}{\delta x^i} = N_i^j \frac{\delta}{\delta x^j}, \quad \nabla \frac{\partial}{\partial y^i} = N_i^j \frac{\partial}{\partial y^j}.$$

Using the homogeneity condition $[\mathbb{C}, S] = S$ and formula (3.8) it follows that $\nabla S = 0$ and $\nabla \mathbb{C} = 0$.

Another geometric structure, induced by a spray, and very important for its geometry, is the *Berwald connection*. It is a linear connection on $\hat{T}M$, and it can be defined as follows $\mathcal{D} : \mathfrak{X}(\hat{T}M) \times \mathfrak{X}(\hat{T}M) \to \mathfrak{X}(\hat{T}M)$,

$$\mathcal{D}_X Y = v[hX, vY] + h[vX, hY] + J[vX, (\mathbb{F} + J)Y] + (\mathbb{F} + J)[hX, JY]. \tag{3.9}$$

Using formula (3.9), it follows that $\mathcal{D}h = 0$ and $\mathcal{D}v = 0$, which means that the Berwald connection preserves both the horizontal and vertical distribution. Moreover, we have $\mathcal{D}J = 0$, which implies that the Berwald connection has the same action on horizontal and vertical vector fields. Locally, we can see this from the following formulae:

$$\mathcal{D}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} = \frac{\partial N_i^k}{\partial y^j}\frac{\delta}{\delta x^k}, \quad \mathcal{D}_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j} = \frac{\partial N_i^k}{\partial y^j}\frac{\partial}{\partial y^k},$$

$$\mathcal{D}_{\frac{\partial}{\partial y^i}}\frac{\delta}{\delta x^j} = 0, \quad \mathcal{D}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = 0.$$

Using the fact that the spray S is horizontal, formulae (3.8) and (3.9) it follows that $\nabla = \mathcal{D}_S$. Therefore, $\mathcal{D}_S S = 0$ which means that integral curves of the spray S are geodesics of the Berwald connection.

4. Projectively Related Sprays

In this section, we discuss the two inverse problems of the calculus of variations that one can associate to a spray: The Finsler metrizability problem and the projective metrizability problem. Our aim, in the next section, will be to search for sprays that are not Finsler metrizable, within the projective class of a given spray. For this we will need some formulae that relate the geometric structures of two projectively related sprays: connections, Jacobi endomorphisms, and curvatures.

Definition 4.1. By a *Finsler function* we mean a continuous function $F : TM \to \mathbb{R}$ satisfying the following conditions:

(i) F is smooth on $\hat{T}M$;
(ii) F is positive on $\hat{T}M$ and $F(x, 0) = 0$;
(iii) F is positively homogeneous of order 1, which means that $F(x, \lambda y) = \lambda F(x, y)$, for all $\lambda > 0$ and $(x, y) \in TM$;
(iv) The *metric tensor* with components

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} \tag{4.1}$$

has rank n.

Conditions (ii) and (iv) of Definition 4.1 imply that the metric tensor g_{ij} of a Finsler function is positive definite, [18]. The regularity condition (iv) of Definition 4.1 is equivalent to the fact that the Poincaré–Cartan 2-form of F^2, $\omega_{F^2} = dd_J F^2$, is non-degenerate and hence it is a symplectic structure. Therefore, the equation

$$i_S dd_J F^2 = -dF^2 \tag{4.2}$$

uniquely determine a vector field S on $\hat{T}M$ that is called the *geodesic spray* of the Finsler function.

Definition 4.2. A spray S is called *Finsler metrizable* if there exists a Finsler function F that satisfies the Eq. (4.2).

Necessary and sufficient criteria for the Finsler metrizability problem for a spray S where formulated in [22] using the holonomy distribution $\mathcal{H}ol_S$. We will use such criteria, in the next section, to construct classes of sprays that are not Finsler metrizable.

One can reformulate condition (iv) of Definition 4.1 in terms of the Hessian of the Finsler function F as follows. Consider

$$h_{ij}(x, y) = F\frac{\partial^2 F}{\partial y^i \partial y^j} \tag{4.3}$$

the *angular metric* of the Finsler function. Using the homogeneity of the Finsler function F, the metric tensor g_{ij} and the angular tensor h_{ij} are related by

$$g_{ij} = h_{ij} + \frac{\partial F}{\partial y^i}\frac{\partial F}{\partial y^j} = h_{ij} + \frac{1}{F^2}y_i y_j. \tag{4.4}$$

In the above formula (4.4), we did use the following calculation, which follows from the homogeneity of the Finsler function F,

$$y_i := g_{ik}y^k = F\frac{\partial F}{\partial y^i} = \frac{1}{2}\frac{\partial F^2}{\partial y^i}. \tag{4.5}$$

Throughout this work, we will raise and lower indices using the metric tensor g_{ij}. For the covector field with components y_i, we show now that its horizontal covariant derivative, with respect to the Berwald connection, vanishes:

$$y_{i|j} := \frac{\delta y_i}{\delta x^j} - \frac{\partial N^k_i}{\partial y^j}y_k = 0. \tag{4.6}$$

Indeed, we have

$$\left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right] = \frac{\partial N^k_i}{\partial y^j}\frac{\partial}{\partial y^k}.$$

If we apply both sides of this formula to F^2, use formula (4.5), and the fact that $d_h F^2 = 0$ we obtain formula (4.6).

We consider, the components of the $(1,1)$-type tensor field

$$h^i_j := g^{ik}h_{kj} = \delta^i_j - \frac{1}{F^2}y^i y_j. \tag{4.7}$$

Metric tensor g_{ij} has rank n if and only if angular tensor h_{ij} has rank $(n-1)$, see [19]. Therefore, the regularity condition of the Finsler function F is equivalent with the fact that the Poincaré-Cartan 2-form $\omega_F = dd_J F$ has rank $2n-2$.

Definition 4.3. A spray S is *projectively metrizable* if it is projectively related to the geodesic spray of a Finsler function.

Equivalently, a spray S is projectively metrizable if its geodesics coincide with the geodesics of a Finsler function, up to an orientation preserving reparameterization.

Next proposition, provides the relation between the geometric structures of two projectively related sprays. We will specialize these relations in Proposition 4.5, when one of the spray is the geodesic spray of a Finsler function.

Proposition 4.4. *Consider S and $\widetilde{S}$ two projectively related sprays, and let $P \in C^\infty(\hat{T}M)$ be the 1-homogeneous function such that $\widetilde{S} = S - 2P\mathbb{C}$. The corresponding connections, Jacobi endomorphisms, and curvature tensors of the two sprays are related by the following formulae:*

$$\begin{aligned}
\widetilde{\Gamma} &= \Gamma - 2(PJ + d_J P \otimes \mathbb{C}),\\
\widetilde{h} &= h - PJ - d_J P \otimes \mathbb{C},\\
\widetilde{v} &= v + PJ + d_J P \otimes \mathbb{C},\\
\widetilde{\Phi} &= \Phi + (P^2 - S(P))J + (2d_h P - Pd_J P - \nabla d_J P) \otimes \mathbb{C},\\
\widetilde{R} &= R + d_J d_h P \otimes \mathbb{C} + (Pd_J P - d_h P) \wedge J.
\end{aligned} \tag{4.8}$$

Proof. Since $\widetilde{S} = S - 2P\mathbb{C}$ it follows $\widetilde{\Gamma} = -[\widetilde{S}, J] = -[S - 2P\mathbb{C}, J] = -[S, J] + 2[P\mathbb{C}, J]$. First formula in (4.8) follows, using the homogeneity condition $[\mathbb{C}, J] = -J$. Next two formulae are direct consequences of the first one, using the fact that $\widetilde{\Gamma} = \widetilde{h} - \widetilde{v}$ and $\Gamma = h - v$.

For the fourth formula, we have to relate $\widetilde{\Phi} = \widetilde{v} \circ [\widetilde{S}, \widetilde{h}]$ and $\Phi = v \circ [S, h]$. Using the homogeneity conditions of the involved geometric objects: $\mathbb{C}(P) = P$, $[\mathbb{C}, S] = S$, $[\mathbb{C}, h] = 0$, and $[\mathbb{C}, J] = -J$ it follows that

$$\begin{aligned}
[\widetilde{S}, \widetilde{h}] = [S, h] - PJ + P\Gamma - \mathcal{L}_S d_J P \otimes \mathbb{C} + d_J P \otimes \mathbb{C}\\
+ 2d_h P \otimes \mathbb{C} - 4Pd_J P \otimes \mathbb{C}.
\end{aligned}$$

Now, if we compose to the left both terms in the above formula by $\widetilde{v}$ and use third formula in (4.8) we obtain fourth formula in (4.8). In this formula we used the fact that the action of the dynamical covariant derivative ∇ on the semi-basic 1-form $d_J P$ is given by

$$\nabla d_J P = \mathcal{L}_S d_J P - d_v P.$$

For the last formula in (4.8), using the relation between $\widetilde{\Phi}$ and Φ, we obtain

$$\begin{aligned}
3\widetilde{R} &= [J, \widetilde{\Phi}]\\
&= 3R + d_J(P^2 - S(P)) \wedge J + [J, (2d_h P - Pd_J P - \nabla d_J P) \otimes \mathbb{C}].
\end{aligned} \tag{4.9}$$

Using the fact that $\omega = 2d_h P - Pd_J P - \nabla d_J P$ is a semi-basic form, it follows that

$$[J, \omega \otimes \mathbb{C}] = d_J\omega \otimes \mathbb{C} - \omega \wedge J.$$

In view of these, formula (4.9) becomes

$$3\widetilde{R} = 3R + d_J\omega \otimes \mathbb{C} + (-\omega + d_J(P^2 - S(P))) \wedge J. \tag{4.10}$$

We compute now $d_J\omega$. We have

$$d_J\omega = 2d_J d_h P - d_J(Pd_J P) - d_J\nabla d_J P.$$

Using the commutation formula

$$\nabla d_J - d_J\nabla = -d_h + 4i_R \tag{4.11}$$

and the fact that $d_J(Pd_JP) = 0$ it follows that $d_J\omega = 3d_Jd_hP$. Finally, we have

$$\begin{aligned} -\omega + d_JP^2 - d_JS(P) &= -2d_hP + 3Pd_JP + \nabla d_JP - d_J\nabla P \\ &= -3d_hP + 3Pd_JP. \end{aligned}$$

If we replace these formulae in (4.10) we obtain that last formula in (4.8) is true. □

Locally, fourth formula in (4.8) reads as follows

$$\widetilde{R}^i_j = R^i_j + (P^2 - S(P))\delta^i_j + \left(2\frac{\delta P}{\delta x^j} - P\frac{\partial P}{\partial y^j} - \nabla\left(\frac{\partial P}{\partial y^j}\right)\right)y^i, \tag{4.12}$$

which is formula (12.17) in [25, p. 176].

In our work, we will be interested in the particular case when the projective factor is of the form $P = \lambda F$, where F is a Finsler function and λ is a nonzero real number.

Proposition 4.5. *Consider F a Finsler function and let S be its geodesic spray. For a nonzero constant λ, consider the projectively related spray $\widetilde{S} = S - 2\lambda F\mathbb{C}$. The corresponding connections, Jacobi endomorphisms, and curvature tensors of the two sprays are related by the following formulae:*

$$\begin{aligned} \widetilde{\Gamma} &= \Gamma - 2\lambda(FJ + d_JF \otimes \mathbb{C}), \\ \widetilde{h} &= h - \lambda(FJ + d_JF \otimes \mathbb{C}), \\ \widetilde{v} &= v + \lambda(FJ + d_JF \otimes \mathbb{C}), \\ \widetilde{\Phi} &= \Phi + \lambda^2(F^2J - Fd_JF \otimes \mathbb{C}), \\ \widetilde{R} &= R + \lambda^2 Fd_JF \wedge J. \end{aligned} \tag{4.13}$$

Proof. First three formulae in (4.13) follow from first three formulae in (4.8) by replacing $P = \lambda F$.

Since S is the geodesic spray for the Finsler function F, using Eq. (4.2) it follows that $S(F) = 0$ and hence $S(P) = 0$. Moreover, we have $d_hF = 0$ and using the commutation formula (4.11) it follows that $\nabla d_JF = 0$. Therefore, for $P = \lambda F$, last two formulae in (4.8) imply the last two formulae in (4.13). □

Locally, fourth formula in (4.13), can be expressed, using formula (4.7), as follows:

$$\widetilde{R}^i_j = R^i_j + \lambda^2F^2\left(\delta^i_j - \frac{1}{F^2}y^iy_j\right) = R^i_j + \lambda^2F^2h^i_j. \tag{4.14}$$

Formula (4.14) corresponds to formula (4.12) for the particular case when the projective factor is $P = \lambda F$.

5. Parameterization-Rigidity of the Geodesics of a Finsler Space

In this section, we consider S the geodesic spray of a Finsler function F. We show that the most natural projective deformation $S \to \widetilde{S} = S - 2\lambda F\mathbb{C}$, $\lambda \in \mathbb{R}$ leads to non-Finsler metrizable sprays, for almost all values of λ. Consequently, we obtain that the projective class of an arbitrary spray contains infinitely many sprays that are not Finsler metrizable. This result shows how rigid is the Finsler metrizablility property with respect to certain reparameterization of the geodesics. We provide the corresponding reparameterization of the geodesics of a Finsler function F that transforms them into parameterized curves that cannot be the geodesics of any Finsler function.

Theorem 5.1. *Let S be the geodesic spray associated to the Finsler function F. Then the projective deformation $\widetilde{S} = S - 2\lambda F\mathbb{C}$ of S is not Finsler metrizable for almost every value of $\lambda \in \mathbb{R}$.*

Proof. Since S is the geodesic spray of the Finsler function F, it satisfies the Eq. (4.2). It follows that all Helmholtz conditions are satisfied for the spray S, [7], and hence the Jacobi endomorphism satisfies $d_\Phi d_J F^2 = 0$. In coordinates, this Helmholtz condition reads as follows: $g_{ik}R^k_j = g_{jk}R^k_i$. This symmetry condition implies that the Jacobi endomorphism is diagonalizable. We denote by $r = \operatorname{rank}\Phi$, where $r \in \{0, \dots, n-1\}$ and $\kappa_\alpha \in C^0(\hat{T}M)$, $\alpha \in \{1, \dots, n-1\}$, the eigen functions of Φ such that the first r eigen functions are not zero.

We fix a point $(x_0, y_0) \in \hat{T}M$ and choose $\lambda \in \mathbb{R}^*$

$$\lambda^2 F^2 + \kappa_\alpha \neq 0, \quad \forall\, \alpha \in \{1, \dots, n-1\}, \tag{5.1}$$

at the point (x_0, y_0). We remark that almost every $\lambda \in \mathbb{R}^*$ can be chosen, since only a finite number is not allowed. Due to the continuity of the Finsler function F and the eigen functions κ_α, it follows that there is an open subset $\mathcal{U} \subset \hat{T}M$, $(x_0, y_0) \in \mathcal{U}$, such that condition (5.1) is satisfied everywhere on $\mathcal{U}$. For the remaining of the proof, all geometric objects will be considered restricted to $\mathcal{U}$.

For the chosen value of λ, we consider the projectively related spray $\widetilde{S} = S - 2\lambda F\mathbb{C}$, with the corresponding projectors $\widetilde{h}$, $\widetilde{v}$, and Jacobi endomorphism $\widetilde{\Phi}$. They are related to the geometric structures induced by spray S through formulae (4.13). We will prove first that $\operatorname{rank}\widetilde{\Phi} = n-1$ on $\mathcal{U}$. Consider the vector fields

$$h_i = h^j_i \frac{\delta}{\delta x^j} = \frac{\delta}{\delta x^i} - \frac{1}{F^2} y_i S, \quad v_i = Jh_i = h^j_i \frac{\partial}{\partial y^j} = \frac{\partial}{\partial y^i} - \frac{1}{F^2} y_i \mathbb{C}. \tag{5.2}$$

Since $\operatorname{rank} h^i_j = n-1$ and $h^i_j y^j = 0$ it follows that $\mathcal{H}_{n-1} := \operatorname{Span}\{h_1, \dots, h_n\}$ is an $(n-1)$-dimensional horizontal sub-distribution, orthogonal to S. Similarly, it follows that $\mathcal{V}_{n-1} := \operatorname{Span}\{v_1, \dots, v_n\}$ is an $(n-1)$-dimensional vertical sub-distribution, orthogonal to $\mathbb{C}$. The above mentioned orthogonality is considered with respect to the Sasaki-type metric tensor on $\mathcal{U}$:

$$G = g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j. \tag{5.3}$$

Therefore, the tangent space to $\mathcal{U}$ can be decomposed into four subspaces, orthogonal to each other:

$$T\mathcal{U} = \mathcal{H}_{n-1} \oplus \mathrm{Span}\{S\} \oplus \mathcal{V}_{n-1} \oplus \mathrm{Span}\{\mathbb{C}\}. \tag{5.4}$$

From formula (5.4) it follows that $J\mathcal{H}_{n-1} = \mathcal{V}_{n-1}$. Above decomposition, with the corresponding distributions and some induced foliations, was studied by Bejancu and Farran in [5].

An important property of the vertical sub-distribution $\mathcal{V}_{n-1}$ is the following: For any $Y \in \mathcal{V}_{n-1}$ it follows $Y(F) = 0$. Indeed, using formulae (4.5) and (5.2) it follows that for any $v_i \in \mathcal{V}_{n-1}$ we have $v_i(F) = 0$.

We prove now that the vertical sub-distribution $\mathcal{V}_{n-1}$ is integrable. Since $v_i, v_j \in \mathcal{V}_{n-1} \subset V$ and the vertical distribution V is integrable, it follows that $[v_i, v_j] \in V = \mathcal{V}_{n-1} \oplus \mathrm{Span}\{\mathbb{C}\}$ and hence

$$[v_i, v_j] = A^l_{ij} v_l + B_{ij}\mathbb{C}, \tag{5.5}$$

for some locally defined functions A^l_{ij} and B_{ij} on $\mathcal{U}$. If we apply the vector fields in both sides of formula (5.5) to the Finsler function F, and use the fact that $v_l(F) = 0$ and $\mathbb{C}(F) = F$, we obtain $0 = B_{ij}F$. This implies that $B_{ij} = 0$ and from formula (5.5) it follows that vertical sub-distribution $\mathcal{V}_{n-1}$ is integrable.

For the Jacobi endomorphism Φ, consider $X_\alpha \in \mathcal{H}_{n-1}$, the eigen vector fields corresponding to the eigen functions κ_α, $\alpha \in \{1, \ldots, n-1\}$, which means that $\Phi(X_\alpha) = \kappa_\alpha J X_\alpha$. Since $X_\alpha \in \mathcal{H}_{n-1}$ it follows $JX_\alpha \in \mathcal{V}_{n-1}$ and therefore $d_J F(X_\alpha) = (JX_\alpha)F = 0$. Now, using fourth formula (4.13) it follows that $\widetilde{\Phi}(X_\alpha) = (\lambda^2 F^2 + \kappa_\alpha) J X_\alpha$, for all $\alpha \in \{1, \ldots, n-1\}$. With the choice (5.1) we made for λ it follows that $\mathrm{Im}\,\widetilde{\Phi} = \mathcal{V}_{n-1}$ and hence $\mathrm{rank}\,\widetilde{\Phi} = n-1$ on $\mathcal{U}$.

We will prove now that $\widetilde{S}$ is not Finsler metrizable on $\mathcal{U}$ by showing that its holonomy distribution $\mathcal{H}ol_{\widetilde{S}}$, contains the Liouville vector field $\mathbb{C}$. We have that $\widetilde{H} = \mathrm{Im}\,\widetilde{h} \subset \mathcal{H}ol_{\widetilde{S}}$ and $\mathcal{V}_{n-1} = \mathrm{Im}\,\widetilde{\Phi} \subset \mathcal{H}ol_{\widetilde{S}}$, which implies $\widetilde{h}_i = \widetilde{h}(h_i) \in \mathcal{H}ol_{\widetilde{S}}$ and $v_i \in \mathrm{Im}\,\widetilde{\Phi} \subset \mathcal{H}ol_{\widetilde{S}}$. Therefore we have $[\widetilde{h}_i, v_j] \in \mathcal{H}ol_{\widetilde{S}}$. We will show that one can choose a pair of indices (i, j) such that the vector field $[\widetilde{h}_i, v_j]$ has a component along the Liouville vector field.

Since $\widetilde{h}[\widetilde{h}_i, v_j] \in \mathcal{H}ol_{\widetilde{S}}$ it follows that $\widetilde{v}[\widetilde{h}_i, v_j] \in \mathcal{H}ol_{\widetilde{S}}$. From second formula (4.13) it follows that $\widetilde{h}_i = h_i - \lambda F v_i$ and hence we have

$$\widetilde{v}[\widetilde{h}_i, v_j] = \widetilde{v}[h_i - \lambda F v_i, v_j] = \widetilde{v}[h_i, v_j] - \lambda F [v_i, v_j]. \tag{5.6}$$

For the last equality in the above formula we did use that $v_j(F) = 0$ and hence $\widetilde{v}[\lambda F v_i, v_j] = \lambda F \widetilde{v}[v_i, v_j]$. From third formula (4.13) it follows that the restrictions of v and $\widetilde{v}$ to the vertical distribution V coincide. Since $v_i, v_j \in \mathcal{V}_{n-1}$ and $\mathcal{V}_{n-1}$ is integrable it follows that $[v_i, v_j] \in \mathcal{V}_{n-1}$. Therefore, $\widetilde{v}[v_i, v_j] = v[v_i, v_j] = [v_i, v_j] \in \mathcal{V}_{n-1} \subset \mathcal{H}ol_{\widetilde{S}}$, and using formula (5.6) it follows that $\widetilde{v}[h_i, v_j] \in \mathcal{H}ol_{\widetilde{S}}$.

Using third formula (4.13), we obtain

$$\widetilde{v}[h_i, v_j] = v[h_i, v_j] + \lambda F J[h_i, v_j] + \lambda(J[h_i, v_j])(F)\mathbb{C} \in \mathcal{H}ol_{\widetilde{S}}. \tag{5.7}$$

Using formula (3.9) it follows that the first two vector fields in the right-hand side of the above formula can be expressed in terms of the Berwald connection as follows

$$v[h_i, v_j] = \mathcal{D}_{h_i} v_j = h_i^l h_{j|l}^k \frac{\partial}{\partial y^k}, \quad J[h_i, v_j] = -\mathcal{D}_{v_j} v_i = v_i(h_j^k) \frac{\partial}{\partial y^k}. \tag{5.8}$$

In formula (5.8), $h_{j|l}^k$ represents the horizontal covariant derivative of the $(1,1)$-type tensor field h_j^k with respect to the Berwald connection:

$$h_{j|l}^k = \frac{\delta h_j^k}{\delta x^l} + h_j^i \frac{\partial N_i^k}{\partial y^l} - h_i^k \frac{\partial N_j^i}{\partial y^l}.$$

We show that $\mathcal{D}_{h_i} v_j \in \mathcal{V}_{n-1} \subset \mathcal{H}ol_{\widetilde{S}}$. Since the Berwald connection preserves the vertical distribution it follows that $\mathcal{D}_{h_i} v_j \in V$ and in view of the orthogonal decomposition $V = \mathcal{V}_{n-1} \oplus \mathrm{Span}\{\mathbb{C}\}$, it remains to show that $G(\mathcal{D}_{h_i} v_j, \mathbb{C}) = 0$. From first formula (5.8) and formula (5.3) we have

$$G(\mathcal{D}_{h_i} v_j, \mathbb{C}) = G\left(h_i^l h_{j|l}^k \frac{\partial}{\partial y^k}, y^s \frac{\partial}{\partial y^s} \right) = h_i^l h_{j|l}^k g_{ks} y^s = h_i^l h_{j|l}^k y_k = 0. \tag{5.9}$$

For the last equality in formula (5.9) we did use that $h_j^k y_k = 0$ and hence its horizontal covariant derivative with respect to the Berwald connection is zero as well: $0 = (h_j^k y_k)_{|l} = h_{j|l}^k y_k + h_j^k y_{k|l} = h_{j|l}^k y_k$, since due to formula (4.6) we have $y_{k|l} = 0$.

Let us evaluate now, the vector field $J[h_i, v_j]$ using the second formula (5.8). Using formula (4.7), we have

$$J[h_i, v_j] = -\mathcal{D}_{v_j} v_i = \frac{1}{F^2} y_i v_j + \frac{1}{F^2} h_{ij} \mathbb{C}. \tag{5.10}$$

Using formula (5.10) and the fact that $v[h_i, v_j] \in \mathcal{H}ol_{\widetilde{S}}$, it follows that the last two terms in formula (5.6) can be written as follows

$$\mathcal{H}ol_{\widetilde{S}} \ni \lambda F J[h_i, v_j] + \lambda(J[h_i, v_j])(F)\mathbb{C} = \frac{\lambda}{F} y_i v_j + \frac{2\lambda}{F} h_{ij} \mathbb{C}, \tag{5.11}$$

which implies that $h_{ij}\mathbb{C} \in \mathcal{H}ol_{\widetilde{S}}$ for all pairs of indices (i, j). Since $\mathrm{rank}(h_{ij}) = n - 1$ and $n > 1$ it follows that there is at least one pair (i, j) such that $h_{ij} \neq 0$. Therefore $\mathbb{C} \in \mathcal{H}ol_{\widetilde{S}}$ and according to Theorem 2 of [22] this proves that the restriction of $\widetilde{S}$ to $\mathcal{U}$ is not Finsler metrizable and hence the spray $\widetilde{S}$ is not Finsler metrizable. □

A direct consequence of the Theorem 5.1 is given by the following corollary.

Corollary 5.2. *For any spray its projective class contains infinitely many sprays that are not Finsler metrizable.*

In the case when the geodesic spray S of a Finsler function F has constant flag curvature κ [4], it follows that all eigen functions of the Jacobi endomorphism are $\kappa_\alpha = \kappa F^2$, $\alpha \in \{1, \ldots, n-1\}$. In this case, the condition (5.1) becomes

$$\kappa + \lambda^2 \neq 0. \tag{5.12}$$

It follows that one can choose $\lambda \in \mathbb{R}^*$ such that condition (5.12) and hence condition (5.1) is satisfied everywhere on $\hat{T}M$.

The particular case when the geodesic spray S of a Finsler function F is flat and has constant flag curvature κ was studied by Yang in [28] using different techniques but the same condition (5.12). Yang's example has been used also in [8] to provide examples of sprays that are projectively metrizable and not Finsler metrizable.

We will show now how to reparameterize the geodesics of a Finsler function F such that the new parameterized curves are not the geodesics for any Finsler function. Consider F a Finsler function with geodesic equations given by the system (3.1), where t is the arc length of the Finsler function F. Consider $\lambda \in \mathbb{R}^*$ satisfying the condition (5.12). According to Theorem 5.1 we have to search for a new parameterization $\tilde{t}$ that satisfies Eq. (3.2). It follows that the reparameterization $\tilde{t} = c_1 t + c_2 e^{2\lambda t}$ of the system (3.1) leads to a system of second order differential equations that is not Finsler metrizable, for c_1, c_2 real constants such that $c_1 > 0$ and $\lambda c_2 > 0$.

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