VOLUME
20/2012
No. 1
ISSN 1804-1388

COMMUNICATIONS IN MATHEMATICS

Guest Editor: Marcella Palese

Available online at
http://cm.osu.cz

UNIVERSITAS OSTRAVIENSIS

Communications in Mathematics 20 (2012) 33–40

Witt algebra and the curvature of the Heisenberg group

Zoltán Muzsnay, Péter T. Nagy

Abstract. The aim of this paper is to determine explicitly the algebraic structure of the curvature algebra of the 3-dimensional Heisenberg group with left invariant cubic metric. We show, that this curvature algebra is an infinite dimensional graded Lie subalgebra of the generalized Witt algebra of homogeneous vector fields generated by three elements.

1 Introduction

The notion of curvature algebra of a Finsler manifold is introduced in a previous paper [4] of the authors and it is proved that this algebra is tangent to the holonomy group. This property used for the proof that the holonomy group of Finsler manifolds of constant non-zero curvature cannot be a compact Lie group, if the dimension of the manifold is greater than 2. The 3-dimensional Heisenberg group with left invariant cubic metric was given as an example of Finsler manifolds having infinite dimensional curvature algebra and holonomy group. The aim of this paper is to describe explicitly the algebraic structure of this curvature algebra. We show, that it is a filtered subalgebra of the generalized Witt algebra of Laurent polynomial vector fields defined on a 3-dimensional vector space, which is generated by three elements. We determine the generators of the curvature algebra in this Witt algebra.

2 Preliminaries

Generalized Witt algebras

Let A be an abelian group, $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) = 0$ and T a vector space over $\mathbb{F}$. The group algebra $\mathbb{F}A$ of A over $\mathbb{F}$ generated by the basis elements t^J, $J \in A$, and the multiplication of $\mathbb{F}A$ is defined by $t^J t^K = t^{J+K}$. The unit of $\mathbb{F}A$ is the element $1 = t^0$.

2010 MSC: 53B40, 53C60, 17B65, 22E65
Key words: Finsler geometry, holonomy, infinite dimensional Lie algebra, Witt algebra

Let us consider the tensor product

$$W = \mathbb{F}A \otimes_{\mathbb{F}} T = \mathrm{Span}_{\mathbb{F}} \left\{ t^J \otimes \partial \,\middle|\, J \in A, \partial \in T \right\}.$$

The element of W is also denoted as $t^J\partial := t^J \otimes \partial$. Now, if a given map $(\partial, J) \mapsto \partial(J) : T \otimes A \to \mathbb{F}$ is $\mathbb{F}$-linear in the first variable and additive in the second variable, then the bracket

$$[t^J\partial_1, t^K\partial_2] := t^{J+K}(\partial_1(K)\partial_2 - \partial_2(J)\partial_1), \qquad J, K \in A, \quad \partial_1, \partial_2 \in T, \tag{1}$$

defines an infinite dimensional Lie algebra on the tensor product W. The Lie algebra W with the Lie multiplication (1) is called a *generalized Witt algebra over the vector space T graded by the abelian group A.*

Witt algebra $W_n(\mathbb{F})$ over the vector space $\mathbb{F}^n$

If A is the additive group of $\mathbb{Z}^n$ with $n > 0$, then the group algebra $\mathbb{F}A$ is isomorphic to the Laurent polynomial algebra $\mathbb{F}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ over $\mathbb{F}$. For an n-tuple $J = (j_1, \dots, j_n) \in \mathbb{Z}^n$ we write $t^J = t_1^{j_1} \cdots t_n^{j_n}$. Let T be the linear span $T = \oplus_{i=1}^n \mathbb{F}\partial_i$ of the operators $\partial_i = t_i \frac{\partial}{\partial t_i}$. If the map $(\partial, J) \mapsto \partial(J) : T \otimes A \to \mathbb{F}$ satisfies $\partial_i(J) = j_i$ then the corresponding generalized Witt algebra $W =: W_n(\mathbb{F})$ can be identified with the Lie algebra $\mathrm{Der}_{\mathbb{F}}(\mathbb{F}[t_1^{\pm 1}, \dots, t_n^{\pm 1}])$ of derivations of the Laurent polynomial algebra $\mathbb{F}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ over $\mathbb{F}$, consisting of the Laurent polynomial vector fields

$$w(J; i) = w(j_1, \dots, j_n; i) = t_1^{j_1} \cdots t_n^{j_n} t_i \frac{\partial}{\partial t_i},$$

where $(t_1, \dots, t_n) \in \mathbb{F}^n$ are the canonical coordinates in $\mathbb{F}^n$ (c. f. [2], [1]). A Lie algebra isomorphic to the Lie algebra $W_n(\mathbb{F})$ of Laurent polynomial vector fields is called *Witt algebra over the vector space* $\mathbb{F}^n$.

Lie subalgebras of $W_n(\mathbb{F})$

Let $\omega \colon \mathbb{Z}^{n-1} \to \mathbb{Z}$ be an additive map. We consider the linear subspace $\overline{W}_\omega$ of the Witt algebra $W_n(\mathbb{F})$ generated by the basis consisting of the elements

$$\bar{w}(\kappa; i) := w\big(\kappa, \omega(\kappa); i\big)$$

with $\kappa = (k_1, \dots, k_{n-1}) \in \mathbb{Z}^n$ and $i \in \{1, \dots, n\}$. The Lie multiplication of $W_n(\mathbb{F})$ induces a Lie multiplication

$$\big[\bar{w}(\kappa; i), \bar{w}(\lambda; j)\big] = \big[w\big(\kappa, \omega(\kappa); i\big), w\big(\lambda, \omega(\lambda); j\big)\big]$$

on $\overline{W}_\omega$ which makes it a Lie subalgebra of $W_n(\mathbb{F})$.

Definition 1. *If $\mathbb{F} = \mathbb{R}$ and $\omega(\kappa) = -(k_1 + \cdots + k_{n-1})$, then we denote the corresponding Lie algebra by $W_n^0(\mathbb{R})$, and $W_n^0(\mathbb{R}) \subseteq W_n(\mathbb{R})$ will be called the* Lie algebra of homogeneous vector fields on the vector space $\mathbb{R}^n$.

3 Curvature algebra of Finsler manifolds

A *Finsler manifold* $(M, \mathcal{F})$ is a pair of an n-dimensional manifold M and of a continuous function $\mathcal{F}\colon TM \to \mathbb{R}$ is (called *Finsler functional*) defined on the tangent bundle of M, smooth on $\hat{T}M := TM \setminus \{0\}$ and for any $x \in M$ the restriction $\mathcal{F}_x = \mathcal{F}|_{T_xM}$ of $\mathcal{F}$ to the tangent space T_xM is a 1-homogeneous continuous function such that for all $y \in \hat{T}_xM = TM_x \setminus \{0\}$ the symmetric bilinear form $g_y\colon T_xM \times T_xM \to \mathbb{R}$ defined by

$$g_y\colon (u,v) \;\mapsto\; g_{ij}(y)u^iv^j = \frac{1}{2}\frac{\partial^2\mathcal{F}^2(y+su+tv)}{\partial s\,\partial t}\Big|_{t=s=0} \tag{2}$$

is non-degenerate. $(M, \mathcal{F})$ is called a *singular Finsler manifold* if the condition (2) is assumed to be satisfied on an open dense cone in T_xM. In the following we will use the name *Finsler manifold* also for *singular Finsler manifolds*.

Geodesics of Finsler manifolds are determined by a system of 2nd order ordinary differential equation $\ddot{x}^i + 2G^i(x, \dot{x}) = 0$, $i = 1, \dots, n$ in a local coordinate system. The functions $G^i(x, y)$ are called the *spray coefficients* belonging to the coordinate system, which are given by

$$G^i(x,y) := \frac{1}{4}g^{ij}(x,y)\Big(2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\Big)y^jy^k.$$

A vector field $X(t)$ is *parallel along a curve* $c(t)$ if and only if it is a solution of the differential equation

$$\nabla_{\dot{c}}X(t) := \Big(\frac{dX^i(t)}{dt} + \Gamma^i_j(c(t), X(t))\dot{c}^j(t)\Big)\frac{\partial}{\partial x^i} = 0, \tag{3}$$

where $\Gamma^i_j = \frac{\partial G^i}{\partial y^j}$ are the parameters of the *associated non-linear connection*. The *curvature tensor* field

$$R_{(x,y)} = \left(\frac{\partial \Gamma^k_i}{\partial x^j} - \frac{\partial \Gamma^k_j}{\partial x^i} + \Gamma^m_i\frac{\partial \Gamma^k_j}{\partial y^m} - \Gamma^m_j\frac{\partial \Gamma^k_i}{\partial y^m}\right)dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^k}.$$

characterizes the integrability of the horizontal distribution $\mathcal{H}TM \subset TTM$, which is locally generated by the vector fields $\frac{\partial}{\partial x^i} + \Gamma^k_i(x,y)\frac{\partial}{\partial y^k}$, $i = 1, \dots, n$. The *indicatrix* $\mathfrak{I}_xM$ $(M, \mathcal{F})$ at $x \in M$ is defined by the hypersurface of T_xM:

$$\mathfrak{I}_xM := \{y \in T_xM;\ \mathcal{F}(y) = \pm 1\}.$$

Since the *parallel translation* $\tau_c\colon T_{c(0)}M \to T_{c(1)}M$ is a differentiable map between $\hat{T}_{c(0)}M$ and $\hat{T}_{c(1)}M$ preserving the value of the Finsler functional, it induces a map

$$\tau_c^{\mathfrak{I}}\colon \mathfrak{I}_{c(0)}M \longrightarrow \mathfrak{I}_{c(1)}M \tag{4}$$

between the indicatrices. The *holonomy group* $\mathsf{Hol}_x(M)$ of $(M, \mathcal{F})$ at $x \in M$ is the subgroup of the group of diffeomorphisms $\mathsf{Diff}(\mathfrak{I}_xM)$ of the indicatrix $\mathfrak{I}_xM$ determined by parallel translation of $\mathfrak{I}_xM$ along piece-wise differentiable closed curves initiated at the point $x \in M$.

Definition 2. *A vector field $\xi \in \mathfrak{X}(\mathfrak{I}_x M)$ on the indicatrix $\mathfrak{I}_x M$ is called a* curvature vector field *of the Finsler manifold $(M, \mathcal{F})$ at $x \in M$, if there exists $X, Y \in T_x M$ such that $\xi = r_x(X, Y)$, where*

$$r_x(X,Y)(y) := R_{(x,y)}(X,Y). \tag{5}$$

The Lie subalgebra $\mathfrak{R}_x := \langle r_x(X,Y);\ X, Y \in T_x M\rangle$ of $\mathfrak{X}(\mathfrak{I}_x M)$ generated by the curvature vector fields is called the curvature algebra *of the Finsler manifold $(M, \mathcal{F})$ at the point $x \in M$.*

The following assertion is proved in [4]:
Theorem *The curvature algebra $\mathfrak{R}_x$ of a Finsler manifold $(M, \mathcal{F})$ is tangent to the holonomy group $\mathsf{Hol}_x(M)$ for any $x \in M$.*

4 Heisenberg group with left invariant cubic metric

The Finsler functional $\mathcal{F}$ of a Finsler manifold $(M, \mathcal{F})$ is called *cubic metric* if it has the form $\mathcal{F}(x,y)^3 = a_{pqr}(x) y^p y^q y^r$, where $a_{pqr}(x)$ are components of a symmetric covariant tensor field. The tensor field $a_{ij}(x,y)$ defined by $\mathcal{F}(x,y) a_{ij}(x,y) = a_{ijr}(x) y^r$ is called the *basic tensor* of $(M, \mathcal{F})$. Let us denote

$$\{ijk, r\} = \frac{1}{4}\left(\frac{\partial a_{ijr}}{\partial x^k} + \frac{\partial a_{jkr}}{\partial x^i} + \frac{\partial a_{kir}}{\partial x^j} - \frac{\partial a_{ijk}}{\partial x^r}\right).$$

According to equation (1.6.2.6) in [3], p. 595, the spray coefficients $G^i(x,y)$ satisfy the system of linear equations

$$3\mathcal{F}(x,y) a_{ir}(x,y) G^r(x,y) = \{jkl, i\} y^j y^k y^l. \tag{6}$$

Let us consider the Heisenberg group H_3 consisting of 3×3-matrices

$$x = \begin{pmatrix} 1 & x^1 & x^3 \\ 0 & 1 & x^2 \\ 0 & 0 & 1 \end{pmatrix}, \quad (x^1, x^2, x^3) \in \mathbb{R}^3.$$

The vector $(x^1, x^2, x^3) \in \mathbb{R}^3$ is the coordinate representation of the element $x \in H_3$. The unit element of H_3 in this coordinate representation is $0 = (0,0,0) \in \mathbb{R}^3$ and the group multiplication has the form

$$(x^1, x^2, x^3) \cdot (x'^1, x'^2, x'^3) = (x^1 + x'^1,\ x^2 + x'^2,\ x^3 + x'^3 + x^1 x'^2,).$$

The Lie algebra $\mathfrak{h}_3 = T_0 H_3$ of H_3 has the matrix representation

$$y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} + y^3 \frac{\partial}{\partial x^3} \quad \longrightarrow \quad \begin{pmatrix} 0 & y^1 & y^3 \\ 0 & 0 & y^2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The left-invariant *Berwald-Moór* cubic Finsler functional $\mathcal{F}$ (c. f. [5], Example 1.1.5, p. 8) on the Heisenberg group H_3 is determined by the function $\mathcal{F}_0 \colon \mathfrak{h}_3 \to \mathbb{R}$ satisfying $\mathcal{F}_0(y)^3 := y^1 y^2 y^3$. If $y = (y^1, y^2, y^3)$ is a tangent vector at $x \in H_3$, then $\mathcal{F}(x,y) := \mathcal{F}_0(x^{-1} y)$, and hence its coordinate expression is of the form

$$\mathcal{F}(x,y)^3 = y^1 y^2 \left(y^3 - x^1 y^2\right).$$

Since $\mathcal{F}$ is left-invariant, the associated geometric structures (connection, geodesics, curvature) are also left-invariant and the curvature algebras at different points are isomorphic. The coefficients $a_{pqr}(x)$ are the following:

$$a_{122} = a_{212} = a_{221} = -\frac{x_1}{3}, \quad a_{123} = a_{231} = a_{312} = a_{321} = a_{213} = a_{132} = \frac{1}{6},$$
$$a_{ijj} = a_{jij} = a_{jji} = 0 \quad \text{with} \quad i,j \in \{1,2,3\} \quad \text{and} \quad (i,j) \neq (1,2).$$

Hence the right hand side of (6) gives $\{jkl,1\}y^j y^k y^l = \{jkl,3\}y^j y^k y^l = 0$ and

$$\{jkl,2\}y^j y^k y^l = \frac{3}{4}\frac{\partial a_{jk2}(x)}{\partial x^1} y^j y^k y^1 = \frac{3}{2}\frac{\partial a_{122}(x)}{\partial x^1} {y^1}^2 y^2 = -\frac{1}{2}{y^1}^2 y^2.$$

The matrix of the basic tensor field $a_{ij}(x,y)$ is

$$(a_{ij}(x,y)) = \frac{1}{\mathcal{F}(x,y)} \begin{pmatrix} 0 & -\frac{x^1}{3}y^2 + \frac{1}{6}y^3 & \frac{1}{6}y^2 \\ -\frac{x^1}{3}y^2 + \frac{1}{6}y^3 & -\frac{x^1}{3}y^1 & \frac{1}{6}y^1 \\ \frac{1}{6}y^2 & \frac{1}{6}y^1 & 0 \end{pmatrix}.$$

This matrix is non-singular on the open dense cone determined by $y^1 y^2 \left(y^3 - x^1 y^2\right) \neq 0$ in T_xM. In the following we will investigate on this domain. We obtain from equations (6)

$$\frac{1}{6}y^2 G^1(x,y) + \frac{1}{6}y^1 G^2(x,y) = 0.$$
$$\left(-\frac{x^1}{3}y^2 + \frac{1}{6}y^3\right) G^2(x,y) + \frac{1}{6}y^2 G^3(x,y) = 0,$$
$$\left(-\frac{x^1}{3}y^2 + \frac{1}{6}y^3\right) G^1(x,y) - \frac{x^1}{3}y^1 G^2(x,y) + \frac{1}{6}y^1 G^3(x,y) = -\frac{1}{6}{y^1}^2 y^2,$$

The solution yields

$$G^1(x,y) = -\frac{{y^1}^2 y^2}{2(y^3 - x^1 y^2)}, \; G^2(x,y) = \frac{y^1 {y^2}^2}{2(y^3 - x^1 y^2)}, \; G^3(x,y) = \frac{y^1 y^2 y^3}{2(y^3 - x^1 y^2)} - y^1 y^2.$$

The matrix of the parameters Γ^i_j of the associated non-linear connection is

$$(\Gamma^i_j(x,y)) = \begin{pmatrix} -\frac{y^1 y^2}{y^3 - x^1 y^2} & -\frac{{y^1}^2 y^3}{2(y^3 - x^1 y^2)^2} & \frac{{y^1}^2 y^2}{2(y^3 - x^1 y^2)^2} \\ \frac{{y^2}^2}{2(y^3 - x^1 y^2)} & \frac{2y^1 y^2 y^3 - y^1 {y^2}^2 x^1}{2(y^3 - x^1 y^2)^2} & -\frac{y^1 {y^2}^2}{2(y^3 - x^1 y^2)^2} \\ \frac{y^2 y^3}{2(y^3 - x^1 y^2)} - y^2 & \frac{y^1 {y^3}^2}{2(y^3 - x^1 y^2)^2} - y^1 & -\frac{x^1 y^1 {y^2}^2}{2(y^3 - x^1 y^2)^2} \end{pmatrix}.$$

The derivatives of Γ^i_j by x^2 and x^3 vanish, we compute their derivatives by x^1:

$$\left(\frac{\partial \Gamma^i_j}{\partial x^1}(x,y)\right) = \begin{pmatrix} -\frac{y^1 {y^2}^2}{(y^3 - x^1 y^2)^2} & -\frac{{y^1}^2 y^2 y^3}{(y^3 - x^1 y^2)^3} & \frac{{y^1}^2 {y^2}^2}{(y^3 - x^1 y^2)^3} \\ \frac{{y^2}^3}{2(y^3 - x^1 y^2)^2} & \frac{y^1 {y^2}^2 (3y^3 - y^2 x^1)}{2(y^3 - x^1 y^2)^3} & -\frac{y^1 {y^2}^3}{(y^3 - x^1 y^2)^3} \\ \frac{{y^2}^2 y^3}{2(y^3 - x^1 y^2)^2} & \frac{y^1 y^2 {y^3}^2}{(y^3 - x^1 y^2)^3} & -\frac{y^1 {y^2}^2 (y^3 + x^1 y^2)}{2(y^3 - x^1 y^2)^3} \end{pmatrix}.$$

In the following we put $x=0$ and we get

$$(\Gamma^i_j)=\begin{pmatrix} -\frac{y^1y^2}{y^3} & -\frac{{y^1}^2}{2y^3} & \frac{{y^1}^2y^2}{2{y^3}^2}\\ \frac{{y^2}^2}{2y^3} & \frac{y^1y^2}{y^3} & -\frac{y^1{y^2}^2}{2{y^3}^2}\\ -\frac{y^2}{2} & -\frac{y^1}{2} & 0\end{pmatrix},\quad \left(\frac{\partial\Gamma^i_j}{\partial x^1}\right)=\begin{pmatrix} -\frac{y^1{y^2}^2}{{y^3}^2} & -\frac{{y^1}^2y^2}{{y^3}^2} & \frac{{y^1}^2{y^2}^2}{{y^3}^3}\\ \frac{{y^2}^3}{2{y^3}^2} & \frac{3}{2}\frac{y^1{y^2}^2}{{y^3}^2} & -\frac{y^1{y^2}^3}{{y^3}^3}\\ \frac{{y^2}^2}{2y^3} & \frac{y^1y^2}{y^3} & -\frac{y^1{y^2}^2y^3}{2{y^3}^3}\end{pmatrix}.$$

Moreover

$$\left(\frac{\partial\Gamma^i_1}{\partial y^j}\right)=\begin{pmatrix} -\frac{y^2}{y^3} & -\frac{y^1}{y^3} & \frac{y^1y^2}{{y^3}^2}\\ 0 & \frac{y^2}{y^3} & -\frac{{y^2}^2}{2{y^3}^2}\\ 0 & -\frac{1}{2} & 0\end{pmatrix},\quad \left(\frac{\partial\Gamma^i_2}{\partial y^j}\right)=\begin{pmatrix} -\frac{y^1}{y^3} & 0 & \frac{{y^1}^2}{2{y^3}^2}\\ \frac{y^2}{y^3} & \frac{y^1}{y^3} & -\frac{y^1y^2}{{y^3}^2}\\ -\frac{1}{2} & 0 & 0\end{pmatrix}$$

and

$$\left(\frac{\partial\Gamma^i_3}{\partial y^j}\right)=\begin{pmatrix} \frac{y^1y^2}{{y^3}^2} & \frac{{y^1}^2}{2{y^3}^2} & -\frac{{y^1}^2y^2}{{y^3}^3}\\ -\frac{{y^2}^2}{2{y^3}^2} & -\frac{y^1y^2}{{y^3}^2} & \frac{y^1{y^2}^2}{{y^3}^3}\\ 0 & 0 & 0\end{pmatrix}.$$

We obtain for $\frac{\partial\Gamma^i_a}{\partial x^b}-\frac{\partial\Gamma^i_a}{\partial y^m}\Gamma^m_b$, $(a,b)=(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)$ the following expressions:

$$\left(\frac{\partial\Gamma^i_1}{\partial x^2}-\frac{\partial\Gamma^i_1}{\partial y^m}\Gamma^m_2\right)=\begin{pmatrix}\frac{{y^1}^2y^2}{{y^3}^2}\\ -\frac{5}{4}\frac{y^1{y^2}^2}{{y^3}^2}\\ \frac{1}{2}\frac{y^1y^2}{y^3}\end{pmatrix},\quad \left(\frac{\partial\Gamma^i_2}{\partial x^1}-\frac{\partial\Gamma^i_2}{\partial y^m}\Gamma^m_1\right)=\begin{pmatrix}-\frac{7}{4}\frac{{y^1}^2y^2}{{y^3}^2}\\ \frac{3}{2}\frac{y^1{y^2}^2}{{y^3}^2}\\ \frac{1}{2}\frac{y^1y^2}{y^3}\end{pmatrix},$$

$$\left(\frac{\partial\Gamma^i_1}{\partial x^3}-\frac{\partial\Gamma^i_1}{\partial y^m}\Gamma^m_3\right)=\begin{pmatrix}0\\ \frac{1}{2}\frac{y^1{y^2}^3}{{y^3}^3}\\ -\frac{1}{4}\frac{y^1{y^2}^2}{{y^3}^2}\end{pmatrix},\quad \left(\frac{\partial\Gamma^i_3}{\partial x^1}-\frac{\partial\Gamma^i_3}{\partial y^m}\Gamma^m_1\right)=\begin{pmatrix}\frac{5}{4}\frac{{y^1}^2{y^2}^2}{{y^3}^3}\\ -\frac{1}{2}\frac{y^1{y^2}^3}{{y^3}^3}\\ -\frac{1}{2}\frac{y^1{y^2}^2}{{y^3}^2}\end{pmatrix},$$

$$\left(\frac{\partial\Gamma^i_2}{\partial x^3}-\frac{\partial\Gamma^i_2}{\partial y^m}\Gamma^m_3\right)=\begin{pmatrix}\frac{1}{2}\frac{{y^1}^3y^2}{{y^3}^3}\\ 0\\ \frac{1}{4}\frac{{y^1}^2y^2}{{y^3}^2}\end{pmatrix},\quad \left(\frac{\partial\Gamma^i_3}{\partial x^2}-\frac{\partial\Gamma^i_3}{\partial y^m}\Gamma^m_2\right)=\begin{pmatrix}-\frac{1}{2}\frac{{y^1}^3y^2}{{y^3}^3}\\ \frac{5}{4}\frac{{y^1}^2{y^2}^2}{{y^3}^3}\\ 0\end{pmatrix}.$$

Hence we can obtain the following curvature vector fields on the indicatrix $\mathfrak{I}_0(M)$ at $x=0$:

$$r_0(1,2)=\frac{11}{4}\begin{pmatrix}\frac{{y^1}^2y^2}{{y^3}^2}\\ -\frac{y^1{y^2}^2}{{y^3}^2}\\ 0\end{pmatrix},\quad r_0(1,3)=\begin{pmatrix}-\frac{5}{4}\frac{{y^1}^2{y^2}^2}{{y^3}^3}\\ \frac{y^1{y^2}^3}{{y^3}^3}\\ \frac{1}{4}\frac{y^1{y^2}^2}{{y^3}^2}\end{pmatrix},\quad r_0(2,3)=\begin{pmatrix}\frac{{y^1}^3y^2}{{y^3}^3}\\ -\frac{5}{4}\frac{{y^1}^2{y^2}^2}{{y^3}^3}\\ \frac{1}{4}\frac{{y^1}^2y^2}{{y^3}^2}\end{pmatrix}.$$

where we use the notation $r_0(i,j)=r_0\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)$. These vector fields $r_0(i,j)$, $i<j$, $i,j=1,2,3$ generate the curvature algebra $\mathfrak{r}_0$ at $x=0$.

Let us consider the vector fields

$$A^{k,m}(a_1,a_2,a_3) := a_1Y^{k+1,m}E_1 + a_2Y^{k,m+1}E_2 + a_3Y^{k,m}E_3,$$

defined on $\mathfrak{h}_3 = T_0H_3$, where

$$(a_1,a_2,a_3)\in\mathbb{R}^3, \qquad E_i = \frac{\partial}{\partial y^i}\Big|_0, \ i=1,2,3,$$

and

$$Y^{k,m} := \frac{{y^1}^k {y^2}^m}{{y^3}^{k+m-1}}, \quad k,m\in\mathbb{N}.$$

Then the curvature vector fields $r_0(i,j),\ i=1,2,3$ can be written in the form

$$r_0(1,2)=\frac{11}{4}A^{1,1}(1,-1,0), \quad r_0(1,3)=\frac{1}{4}A^{1,2}(-5,4,1), \quad r_0(2,3)=\frac{1}{4}A^{2,1}(4,-5,1).$$

We have that

$$\big[A^{k,l}(a_1,a_2,a_3), A^{p,q}(b_1,b_2,b_3)\big] = A^{k+p,l+q}(c_1,c_2,c_3),$$

where

$$\begin{aligned}
c_1 &= b_1\big((p+1)a_1+qa_2-(p+q)a_3\big) - a_1\big((k+1)b_1+lb_2-(k+l)b_3\big),\\
c_2 &= b_2\big(pa_1+(q+1)a_2-(p+q)a_3\big) - a_2\big(kb_1+(l+1)b_2-(k+l)b_3\big),\\
c_3 &= b_3\big(pa_1+qa_2-(p+q-1)a_3\big) - a_3\big(kb_1+lb_2-(k+l-1)b_3\big).
\end{aligned}$$

With these preparations we are able to completely describe the structure of the curvature algebra of the Heisenberg group as a Lie subalgebra of the Witt algebra $W_3(\mathbb{R})$. We have the following

Theorem 1. *The curvature algebra $\mathfrak{r}_0$ of the Berwald-Moór left-invariant cubic metric $\mathcal{F}$ on the Heisenberg group H_3 is isomorphic to the Lie subalgebra $W_{\langle h_1,h_2,h_3\rangle}$ of $W_3^0(\mathbb{R})$ generated by the elements*

$$\begin{aligned}
h_1 &= \quad \bar{w}(1,1;1) - \bar{w}(1,1;2),\\
h_2 &= -5\bar{w}(1,2;1) + 4\bar{w}(1,2;2) + \bar{w}(1,2;3),\\
h_3 &= \quad 4\bar{w}(2,1;1) - 5\bar{w}(2,1;2) + \bar{w}(2,1;3).
\end{aligned}$$

In particular, $\mathfrak{r}_0$ is infinite dimensional, and we have the following sequence of Lie algebras:

$$\mathfrak{r}_0 \ \cong\ W_{\langle h_1,h_2,h_3\rangle} \ \subset\ W_3^0(\mathbb{R}) \ \subset\ W_3(\mathbb{R}).$$

where $W_3^0(\mathbb{R})$ is the Lie algebra of homogeneous vector fields and $W_3(\mathbb{R})$ is the Witt algebra of Laurent polynomial vector fields on the vector space $\mathfrak{h}_3 \cong \mathbb{R}^3$.

Acknowledgement

This research was supported by the Hungarian Scientific Research Fund (OTKA) Grant K 67617.

References

[1] D. Z. Dokovic, K. Zhao: Derivations, isomorphisms, and second cohomology of generalized Witt algebras. *T. Am. Math. Soc.* 350 (1998) 643–664.

[2] N. Kawamoto: Generalizations of Witt algebras over a field of characteristic zero. *Hiroshima Math.* 16 (1986) 417–426.

[3] M. Matsumoto: *Finsler Geometry in the 20th-Century, Handbook of Finsler Geometry*. Kluwer Academic Publishers (2003). 565–966.

[4] Z. Muzsnay, P. T. Nagy: Finsler manifolds with non-Riemannian holonomy. *Houston J. Math.* 38 (2012) 77–92.

[5] Z. Shen: *Differential Geometry of Spray and Finsler Spaces*. Kluwer Academic Publishers, Dordrecht (2001).

Authors' addresses:
Zoltán Muzsnay: Institute of Mathematics, University of Debrecen, 4010 Debrecen, Egyetem tér 1, Hungary

E-mail: muzsnay@science.unideb.hu

Péter T. Nagy: Institute of Applied Mathematics, Óbuda University 1034 Budapest, Bécsi út 96/b, Hungary

E-mail: nagy.peter@nik.uni-obuda.hu

Received: 15 September, 2012