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Projectively flat Finsler manifolds with infinite dimensional holonomy

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Abstract. Recently, we developed a method for the study of holonomy properties of non-Riemannian Finsler manifolds and obtained that the holonomy group cannot be a compact Lie group if the Finsler manifold of dimension > 2 has non-zero constant flag curvature. The purpose of this paper is to move further, exploring the holonomy properties of projectively flat Finsler manifolds of non-zero constant flag curvature. We prove in particular that projectively flat Randers and Bryant–Shen manifolds of non-zero constant flag curvature have infinite dimensional holonomy group.

Keywords. Finsler geometry, holonomy, infinite dimensional Lie groups.

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1 Introduction

In the recent papers [9, 10] we have developed a method for the study of holonomy properties of non-Riemannian Finsler manifolds. Particularly, we obtained in [10], that the holonomy group cannot be a compact Lie group if the Finsler manifold of dimension > 2 has non-zero constant flag curvature. We described the first example of a Finsler manifold with infinite dimensional holonomy group, namely a left invariant Berwald–Moór metric on the 3-dimensional Heisenberg group.

The purpose of the present paper is to investigate families of Finsler manifolds with interesting geometric structure which have infinite dimensional holonomy group. From the viewpoint of non-Euclidean geometry the most important Riemann–Finsler manifolds are the projectively flat spaces of constant flag curvature. We will turn our attention to non-Riemannian projectively flat Finsler manifolds of non-zero constant flag curvature.

There are many examples of non-Riemannian projectively flat Finsler manifolds of non-zero constant flag curvature. Their classification is related to the smooth version of Hilbert's Fourth Problem: characterize (not necessarily reversible) distance functions on an open subset in $\mathbb{R}^n$ such that straight lines are the short-

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est paths. The famous Beltrami's theorem states that the locally projectively flat Riemannian manifolds are exactly the manifolds of constant curvature. But for Finsler manifolds Beltrami's theorem is not true. In fact, any projectively flat Finsler manifold has scalar flag curvature, but there are Finsler manifolds of constant flag curvature, which are not projectively flat. The constancy of the flag curvature is a very restrictive property for complete projectively flat Finsler manifolds, cf. [1–3,5–7], but there are many non-complete examples defined on open domains in $\mathbb{R}^n$, constructed and studied by Z. Shen, cf. [4, 12–15].

We will consider the following classes of locally projectively flat non-Riemannian Finsler manifolds of non-zero constant flag curvature:

(i) Randers manifolds (cf. [15]),

(ii) manifolds having a 2-dimensional subspace in the tangent space at some point, on which the Finsler norm is an Euclidean norm (cf. [14, Theorem 1.2, p. 1715]),

(iii) manifolds having a 2-dimensional subspace in the tangent space at some point, on which the Finsler norm and the projective factor are linearly dependent (cf. [14, Example 7.1, pp. 1725–1727]).

The first class consists of positively complete Finsler manifolds of negative curvature, the second class contains a large family of (not necessarily complete) Finsler manifolds of negative curvature, and the third class contains a large family of not necessarily complete Finsler manifolds of positive curvature. The metrics belonging to these classes can be considered as (local) generalizations of a one-parameter family of complete Finsler manifolds of positive curvature defined on S^2 by R. Bryant in [1, 2] and on S^n by Z. Shen in [14, Example 7.1]. We prove that the holonomy group of Finsler manifolds belonging to these classes and satisfying some additional technical assumption is infinite dimensional.

2 Preliminaries

Throughout this article, M denotes a C^∞-manifold, $\mathfrak{X}^\infty(M)$ denotes the vector space of smooth vector fields on M and $\mathsf{Diff}^\infty(M)$ denotes the group of all C^∞-diffeomorphism of M with the C^∞-topology.

Spray manifold, horizontal distribution, canonical connection, parallelism

A *spray* on a manifold M is a smooth vector field $\mathcal{S}$ on $\hat{T}M := TM \setminus \{0\}$ expressed in a standard coordinate system (x^i, y^i) on TM as

$$\mathcal{S} = y^i \frac{\partial}{\partial x^i} - 2\Gamma^i(x, y) \frac{\partial}{\partial y^i}, \tag{2.1}$$

where $\Gamma^i(x,y)$ are local functions on TM satisfying

$$\Gamma^i(x,\lambda y) = \lambda^2 \Gamma^i(x,y), \quad \lambda > 0.$$

A manifold M with a spray $\mathcal{S}$ is called a *spray manifold* $(M,\mathcal{S})$.

A curve $c(t)$ is called *geodesic* if its coordinate functions $c^i(t)$ satisfy the differential equations

$$\ddot{c}^i(t) + \Gamma^i(c(t),\dot{c}(t)) = 0, \tag{2.2}$$

where the functions $\Gamma^i(x,y)$ are called the *geodesic coefficients* of the spray manifold $(M,\mathcal{S})$. The associated *homogeneous (nonlinear) parallel translation*

$$\tau_c : T_{c(0)}M \to T_{c(1)}M$$

along a curve $c(t)$ is defined by parallel vector fields $X(t) = X^i(t)\frac{\partial}{\partial x^i}$ along $c(t)$ satisfying

$$D_{\dot{c}}X(t) := \left(\frac{dX^i(t)}{dt} + \Gamma^i_j(c(t),X(t))\dot{c}^j(t)\right)\frac{\partial}{\partial x^i} = 0, \quad \Gamma^i_j := \frac{\partial \Gamma^i}{\partial y^j}. \tag{2.3}$$

Let (TM,π,M) and (TTM,τ,TM) denote the first and the second tangent bundle of the manifold M, respectively. The *horizontal distribution* $\mathcal{H}TM \subset TTM$ associated to the spray manifold $(M,\mathcal{S})$ is the image of the horizontal lift which is a vector space isomorphism $l_y : T_xM \to \mathcal{H}_yTM$ for every $x \in M$ and $y \in T_xM$ defined by

$$l_y\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} - \Gamma^k_i(x,y)\frac{\partial}{\partial y^k}. \tag{2.4}$$

Then a vector field $X(t)$ along a curve $c(t)$ is parallel if and only if it is a solution of the differential equation $\frac{d}{dt}X(t) = l_{X(t)}(\dot{c}(t))$.

If $\mathcal{V}TM \subset TTM$ is the vertical distribution on TM defined by

$$\mathcal{V}_yTM := \operatorname{Ker}\pi_{*,y},$$

then we have the decomposition $T_yTM = \mathcal{H}_yTM \oplus \mathcal{V}_yTM$.

Let $(\hat{\mathcal{V}}TM,\tau,\hat{T}M)$ be the vertical bundle over $\hat{T}M := TM \setminus \{0\}$. We denote by $\hat{\mathfrak{X}}^\infty(TM)$ the vector space of smooth sections of the bundle $(\hat{\mathcal{V}}TM,\tau,\hat{T}M)$. The *horizontal covariant derivative* of a section $\xi \in \hat{\mathfrak{X}}^\infty(TM)$ by a vector field $X \in \mathfrak{X}^\infty(M)$ is given by $\nabla_X\xi := [h(X),\xi]$. We can express the horizontal covariant derivative of the section

$$\xi(x,y) = \xi^i(x,y)\frac{\partial}{\partial y^i}$$

by the vector field

$$X(x) = X^i(x)\frac{\partial}{\partial x^i}$$

as

$$\nabla_X \xi = \left(\frac{\partial \xi^i(x,y)}{\partial x^j} - \Gamma^k_j(x,y) \frac{\partial \xi^i(x,y)}{\partial y^k} + \Gamma^i_{jk}(x,y)\xi^k(x,y) \right) X^j \frac{\partial}{\partial y^i}, \quad (2.5)$$

where

$$\Gamma^i_{jk}(x,y) := \frac{\partial \Gamma^i_j(x,y)}{\partial y^k}.$$

Let $(\pi^* TM, \bar{\pi}, \hat{T}M)$ be the pull-back bundle of $(\hat{T}M, \pi, M)$ by the map

$$\pi : TM \to M.$$

Clearly, the mapping

$$\left(x, y, \xi^i \frac{\partial}{\partial y^i} \right) \mapsto \left(x, y, \xi^i \frac{\partial}{\partial x^i} \right) : \hat{\mathcal{V}}TM \to \pi^* TM \quad (2.6)$$

is a canonical bundle isomorphism. In the following we will use the isomorphism (2.6) for the identification of these bundles.

The *curvature tensor* field $K_{(x,y)} = K^i_{jk}(x,y) dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^i}$ on the pull-back bundle $(\pi^* TM, \bar{\pi}, \hat{T}M)$ of the spray manifold $(M, \mathcal{S})$ in a local coordinate system is given by

$$K^i_{jk}(x,y) = \frac{\partial \Gamma^i_j(x,y)}{\partial x^k} - \frac{\partial \Gamma^i_k(x,y)}{\partial x^j} + \Gamma^m_j(x,y)\Gamma^i_{km}(x,y) - \Gamma^m_k(x,y)\Gamma^i_{jm}(x,y). \quad (2.7)$$

Finsler manifold, canonical connection, Berwald connection

A *Finsler manifold* is a pair $(M, \mathcal{F})$, where M is an n-dimensional smooth manifold and $\mathcal{F} : TM \to [0, \infty)$ is a continuous function, smooth on $\hat{T}M := TM \setminus \{0\}$, such that its restriction $\mathcal{F}_x = \mathcal{F}|_{T_x M}$ for any $x \in M$ is a positively 1-homogeneous function and the symmetric bilinear form

$$g_{x,y} : (u,v) \mapsto g_{ij}(x,y) u^i v^j = \frac{1}{2} \frac{\partial^2 \mathcal{F}_x^2(y + su + tv)}{\partial s\, \partial t} \bigg|_{t=s=0}$$

is positive definite at every $y \in \hat{T}_x M$.

We call $\mathcal{F}$ the Finsler function, $g_{x,y}$ the metric tensor of the Finsler manifold $(M, \mathcal{F})$. The Finsler function is called *absolutely homogeneous* at $x \in M$ if $\mathcal{F}_x(\lambda y) = |\lambda| \mathcal{F}_x(y)$. If $\mathcal{F}$ is absolutely homogeneous at every $x \in M$, then the Finsler manifold $(M, \mathcal{F})$ is *reversible*.

The associated spray is locally given by $\mathcal{S} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, where the functions

$$G^i(x,y) := \frac{1}{4} g^{il}(x,y)\left(2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\right) y^j y^k \tag{2.8}$$

are the *geodesic coefficients* of the Finsler manifold $(M, \mathcal{F})$. The corresponding homogeneous (nonlinear) parallel translation $\tau_c : T_{c(0)}M \to T_{c(1)}M$ along a curve $c(t)$ is called the *canonical homogeneous (nonlinear) parallel translation* of the Finsler manifold $(M, \mathcal{F})$. The horizontal covariant derivative with respect to the spray associated to the Finsler manifold $(M, \mathcal{F})$ is called the *horizontal Berwald covariant derivative*. If we define

$$\nabla_X \phi = \left(\frac{\partial \phi}{\partial x^j} - G^k_j(x,y)\frac{\partial \phi(x,y)}{\partial y^k}\right) X^j$$

for a smooth function $\phi : \hat{T}M \to \mathbb{R}$, the horizontal Berwald covariant derivation (2.5) can be extended to the tensor bundle over $(\pi^*TM, \bar{\pi}, \hat{T}M)$.

The *Riemannian curvature tensor* field $R = R^i_{jk}(x,y)dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^i}$ on the pull-back bundle $(\pi^*TM, \bar{\pi}, \hat{T}M)$ is

$$R^i_{jk}(x,y) = \frac{\partial G^i_j(x,y)}{\partial x^k} - \frac{\partial G^i_k(x,y)}{\partial x^j} + G^m_j(x,y)G^i_{km}(x,y) - G^m_k(x,y)G^i_{jm}(x,y).$$

The tensor field R characterizes the integrability of the horizontal distribution associated with the (nonlinear) parallel translation. Namely, $R = 0$ if and only if the horizontal distribution is integrable and hence the parallel translation is path-independent on simply connected manifolds.

The manifold is called of constant flag curvature $\lambda \in \mathbb{R}$ if for any $x \in M$ the local expression of the Riemannian curvature is

$$R^i_{jk}(x,y) = \lambda\big(\delta^i_k g_{jm}(x,y)y^m - \delta^i_j g_{km}(x,y)y^m\big). \tag{2.9}$$

In this case the flag curvature of the Finsler manifold does not depend on the point, nor on the 2-flag (cf. [4, Section 2.1, pp. 43–46]).

A Finsler function $\mathcal{F}$ on an open subset $D \subset \mathbb{R}^n$ is said to be *projectively flat* if all geodesic curves are straight lines in D. A Finsler manifold is said to be *locally projectively flat* if at any point there is a local coordinate system (x^i) in which $\mathcal{F}$ is projectively flat.

Let $(x^1, \dots, x^n)$ be a local coordinate system on M corresponding to the canonical coordinates of the Euclidean space which is projectively related to $(M, \mathcal{F})$.

Then the geodesic coefficients (2.8) are of the form

$$\begin{aligned} G^i(x,y) &= \mathcal{P}(x,y)y^i, \\ G^i_k &= \frac{\partial \mathcal{P}}{\partial y^k}y^i + \mathcal{P}\delta^i_k, \\ G^i_{kl} &= \frac{\partial^2 \mathcal{P}}{\partial y^k \partial y^l}y^i + \frac{\partial \mathcal{P}}{\partial y^k}\delta^i_l + \frac{\partial \mathcal{P}}{\partial y^l}\delta^i_k. \end{aligned} \tag{2.10}$$

where $\mathcal{P}$ is a 1-homogeneous function in y, called the projective factor of $(M, \mathcal{F})$. Clearly, any 2-plane in this coordinate system $(x^1, \dots, x^n)$ is a totally geodesic submanifold of $(M, \mathcal{F})$.

Remark 2.1. The canonical homogeneous parallel translation

$$\tau_c : T_{c(0)}M \to T_{c(1)}M$$

in a locally projectively flat Finsler manifold $(M, \mathcal{F})$ along curves $c(t)$ contained in the domain of the coordinate system $(x^1, \dots, x^n)$ are linear maps if and only if the projective factor $\mathcal{P}(x,y)$ is a linear function in y. Hence the non-linearity in y of the projective factor implies that the locally projectively flat Finsler manifold is non-Riemannian.

Projectively flat Randers manifolds with constant flag curvature were classified by Z. Shen in [15]. He proved that any projectively flat Randers manifold $(M, \mathcal{F})$ with non-zero constant flag curvature has negative curvature. These metrics can be normalized by a constant factor so that the curvature is $-\frac{1}{4}$. In this case $(M, \mathcal{F})$ is isometric to the Finsler manifold defined by the metric function

$$\mathcal{F}(x,y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)}}{1-|x|^2} \pm \left(\frac{\langle x,y\rangle}{1-|x|^2} + \frac{\langle a,y\rangle}{1+\langle a,x\rangle}\right) \tag{2.11}$$

on the unit ball $\mathbb{D}^n \subset \mathbb{R}^n$, where $a \in \mathbb{R}^n$ is any constant vector with $|a| < 1$. According to [4, Lemma 8.2.1, p. 155] the projective factor $\mathcal{P}(x,y)$ can be computed by the formula

$$\mathcal{P}(x,y) = \frac{1}{2\mathcal{F}}\frac{\partial \mathcal{F}}{\partial x^i}y^i.$$

An easy calculation yields

$$\pm\frac{\partial \mathcal{F}}{\partial x^i}y^i = \left(\frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)} \pm \langle x,y\rangle}{1-|x|^2}\right)^2 - \left(\frac{\langle a,y\rangle}{1+\langle a,x\rangle}\right)^2,$$

hence

$$\mathcal{P}(x,y) = \frac{1}{2}\left(\frac{\pm\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x,y\rangle^2)} + \langle x,y\rangle}{1-|x|^2} - \frac{\langle a,y\rangle}{1+\langle a,x\rangle}\right). \tag{2.12}$$

Holonomy group

Let $(M, \mathcal{F})$ be an n-dimensional Finsler manifold. We denote by $(\mathcal{I}M, \pi, M)$ the *indicatrix bundle* of $(M, \mathcal{F})$, the *indicatrix* $\mathcal{I}_x M$ at $x \in M$ is the compact hypersurface

$$\mathcal{I}_x M := \{y \in T_x M : \mathcal{F}(y) = 1\},$$

of $T_x M$ diffeomorphic to the standard $(n-1)$-sphere.

The homogeneous (nonlinear) parallel translation $\tau_c : T_{c(0)}M \to T_{c(1)}M$ along a curve $c : [0,1] \to M$ preserves the value of the Finsler function, hence it induces a map $\tau_c : \mathcal{I}_{c(0)}M \longrightarrow \mathcal{I}_{c(1)}M$ between the indicatrices.

The group of diffeomorphisms $\mathsf{Diff}^\infty(\mathcal{I}_x M)$ of the indicatrix $\mathcal{I}_x M$ is a regular infinite dimensional Lie group modeled on the vector space $\mathfrak{X}^\infty(\mathcal{I}_x M)$. In this category of groups one can define the exponential mapping and the group structure is locally determined by the Lie algebra. The Lie algebra of $\mathsf{Diff}^\infty(\mathcal{I}_x M)$ is $\mathfrak{X}^\infty(\mathcal{I}_x M)$ equipped with the negative of the usual Lie bracket.

The *holonomy group* $\mathsf{Hol}_x(M)$ of a Finsler space $(M, \mathcal{F})$ at a point $x \in M$ is the subgroup of the group of diffeomorphisms $\mathsf{Diff}^\infty(\mathcal{I}_x M)$ of the indicatrix $\mathcal{I}_x M$ generated by (nonlinear) parallel translations of $\mathcal{I}_x M$ along piece-wise differentiable closed curves initiated at the point $x \in M$. The holonomy groups at different points of M are isomorphic.

If the Riemannian curvature R of (M, F) vanishes identically, then the (nonlinear) parallel translation is path-independent and hence the holonomy group is trivial.

Infinitesimal holonomy algebra

A vector field $\xi \in \mathfrak{X}^\infty(\mathcal{I}M)$ on the indicatrix bundle $\mathcal{I}M$ is a *curvature vector field* of the Finsler manifold $(M, \mathcal{F})$ if there exist vector fields $X, Y \in \mathfrak{X}^\infty(M)$ on M such that $\xi = R(X, Y)$.

If $x \in M$ is fixed and $X, Y \in T_x M$, then the vector field $y \to R(X,Y)(x,y)$ on $\mathcal{I}_x M$ is a *curvature vector field at* x (see [10]).

The Lie algebra $\mathfrak{R}(M)$ of vector fields generated by the curvature vector fields of $(M, \mathcal{F})$ is called the *curvature algebra* of the Finsler manifold $(M, \mathcal{F})$. For a fixed $x \in M$ the Lie algebra $\mathfrak{R}_x(M)$ of vector fields generated by the curvature vector fields at x is called the *curvature algebra at* x.

The *infinitesimal holonomy algebra* of the Finsler manifold $(M, \mathcal{F})$ is the smallest Lie algebra $\mathfrak{hol}^*(M)$ of vector fields on the indicatrix bundle $\mathcal{I}M$ satisfying the following properties:

(i) any curvature vector field ξ belongs to $\mathfrak{hol}^*(M)$,

(ii) if $\xi, \eta \in \mathfrak{hol}^*(M)$, then $[\xi, \eta] \in \mathfrak{hol}^*(M)$,

(iii) if $\xi \in \mathfrak{hol}^*(M)$ and $X \in \mathfrak{X}^\infty(M)$, then the horizontal Berwald covariant derivative $\nabla_X \xi$ also belongs to $\mathfrak{hol}^*(M)$.

The *infinitesimal holonomy algebra at a point* $x \in M$ is the Lie algebra

$$\mathfrak{hol}^*_x(M) := \{\xi|_{\mathcal{I}_x M} : \xi \in \mathfrak{hol}^*(M)\} \subset \mathfrak{X}^\infty(\mathcal{I}_x M)$$

of vector fields on the indicatrix $\mathcal{I}_x M$.

One has $\mathfrak{R}(M) \subset \mathfrak{hol}^*(M)$ and $\mathfrak{R}_x(M) \subset \mathfrak{hol}^*_x(M)$ for any $x \in M$ (see [9]).

Let H be a subgroup of the diffeomorphism group $\mathsf{Diff}^\infty(M)$ of a differentiable manifold M. A vector field $X \in \mathfrak{X}^\infty(M)$ is called *tangent to* $H \subset \mathsf{Diff}^\infty(M)$ if there exists a $\mathcal{C}^1$-differentiable 1-parameter family $\{\Phi(t) \in H\}_{t\in\mathbb{R}}$ of diffeomorphisms of M such that $\Phi(0) = \mathsf{Id}$ and $\frac{\partial \Phi(t)}{\partial t}|_{t=0} = X$. A Lie subalgebra $\mathfrak{g}$ of $\mathfrak{X}^\infty(M)$ is called *tangent to* H if all elements of $\mathfrak{g}$ are tangent vector fields to H.

A subgroup H of the diffeomorphism group $\mathsf{Diff}^\infty(M)$ of a manifold M will be called *infinite dimensional* if H has an infinite dimensional tangent Lie algebra of vector fields.

The following assertion will be an important tool in the next discussion:

- *The infinitesimal holonomy algebra* $\mathfrak{hol}^*(x)$ *at any point* $x \in M$ *is tangent to the holonomy group* $\mathsf{Hol}(x)$ ([9, Theorem 6.3].)

Hence we have:

- *If the infinitesimal holonomy algebra* $\mathfrak{hol}^*(x)$ *at a point* $x \in M$ *is infinite dimensional, then the holonomy group* $\mathsf{Hol}(x)$ *is infinite dimensional.*

3 Holonomy of projective Finsler surfaces of constant curvature

A Finsler manifold $(M, \mathcal{F})$ of dimension 2 is called *Finsler surface*. In this case the indicatrix is 1-dimensional at any point $x \in M$, hence the curvature vector fields at $x \in M$ are proportional to any given non-vanishing curvature vector field. It follows that the curvature algebra $\mathfrak{R}_x(M)$ has a simple structure: it is at most 1-dimensional and commutative. Even in this case, the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ can be higher dimensional, or potentially infinite dimensional. For the investigation of such examples we use a classical result of S. Lie on the classification of Lie group actions on 1-manifolds (cf. [8] and [11, Theorem 2.70]):

- *If a finite dimensional connected Lie group acts locally effectively on a* 1*-dimensional manifold without fixed points, then its dimension is less than* 4.

Proposition 3.1. *If the infinitesimal holonomy algebra* $\mathfrak{hol}^*_x(M)$ *contains four simultaneously non-vanishing* $\mathbb{R}$*-linearly independent vector fields, then the holonomy group* $\mathsf{Hol}_x(M)$ *is an infinite dimensional subgroup of* $\mathsf{Diff}^\infty(\mathcal{I}_x M)$.

Proof. If the infinitesimal holonomy algebra was finite dimensional, then the dimension of the corresponding Lie group acting locally effectively on the 1-dimensional indicatrix would be at least 4, which is a contradiction. □

Let $(M, \mathcal{F})$ be a locally projectively flat Finsler surface of non-zero constant curvature, let (x^1, x^2) be a local coordinate system centered at $x \in M$, corresponding to the canonical coordinates of the Euclidean space which is projectively related to $(M, \mathcal{F})$ and let (y^1, y^2) be the induced coordinate system in the tangent plane $T_x M$.

In the sequel we identify the tangent plane $T_x M$ with $\mathbb{R}^2$ with help of the coordinate system (y^1, y^2). We will use the Euclidean norm

$$\|(y^1, y^2)\| = \sqrt{(y^1)^2 + (y^2)^2}$$

of $\mathbb{R}^2$ and the corresponding polar coordinate system (e^r, t), too.

Let $\varphi(y^1, y^2)$ be a positively 1-homogeneous function on $\mathbb{R}^2$ and let $r(t)$ be the 2π-periodic smooth function $r : \mathbb{R} \to \mathbb{R}$ determined by

$$\varphi(e^{r(t)} \cos t, e^{r(t)} \sin t) = 1 \quad \text{or} \quad \varphi(y^1, y^2) = e^{-r(t)} \sqrt{(y^1)^2 + (y^2)^2}, \tag{3.1}$$

where

$$\cos t = \frac{y^1}{\sqrt{(y^1)^2 + (y^2)^2}}, \quad \sin t = \frac{y^2}{\sqrt{(y^1)^2 + (y^2)^2}} \quad \text{and} \quad \tan t = \frac{y^2}{y^1},$$

i.e. the level set $\{\varphi(y^1, y^2) \equiv 1\}$ of the 1-homogeneous function φ in $\mathbb{R}^2$ is given by the parametrized curve $t \to (e^{r(t)} \cos t, e^{r(t)} \sin t)$.

Since the curvature κ of a smooth curve $t \to (e^{r(t)} \cos t, e^{r(t)} \sin t)$ in $\mathbb{R}^2$ is

$$\kappa = -\frac{e^r}{\sqrt{\dot{r}^2 + 1}} (\ddot{r} - \dot{r}^2 - 1), \tag{3.2}$$

the vanishing of the expression $\ddot{r} - \dot{r}^2 - 1$ means the infinitesimal linearity of the corresponding positively homogeneous function in $\mathbb{R}^2$.

Definition 3.2. Let $\varphi(y^1, y^2)$ be a positively 1-homogeneous function on $\mathbb{R}^2$ and let $\kappa(t)$ be the curvature of the curve $t \to (e^{r(t)} \cos t, e^{r(t)} \sin t)$ defined by the equations (3.1). We say that $\varphi(y^1, y^2)$ is *strongly convex* if $\kappa(t) \neq 0$ for all $t \in \mathbb{R}$.

Conditions (A), (B), (C) in the following theorem imply that the projective factor $\mathcal{P}$ at $x_0 \in M$ is a non-linear function, and hence, according to Remark 2.1, $(M, \mathcal{F})$ is a non-Riemannian Finsler manifold.

Theorem 3.3. *Let $(M, \mathcal{F})$ be a projectively flat Finsler surface of non-zero constant curvature. Assume that there exists a point $x_0 \in M$ such that one of the following conditions holds:*

(A) *$\mathcal{F}$ induces a scalar product on $T_{x_0}M$ and the projective factor $\mathcal{P}$ at x_0 is a strongly convex positively* 1*-homogeneous function,*

(B) *$\mathcal{F}(x_0, y)$ is a strongly convex absolutely* 1*-homogeneous function on $T_{x_0}M$, and the projective factor $\mathcal{P}(x_0, y)$ on $T_{x_0}M$ satisfies $\mathcal{P}(x_0, y) = c \cdot \mathcal{F}(x_0, y)$ with $0 \neq c \in \mathbb{R}$,*

(C) *there is a projectively related Euclidean coordinate system of $(M, \mathcal{F})$ centered at x_0 and one has*

$$\mathcal{F}(0, y) = |y| \pm \langle a, y \rangle \quad \textit{and} \quad \mathcal{P}(0, y) = \frac{1}{2}\big(\pm|y| - \langle a, y \rangle\big). \tag{3.3}$$

Then the holonomy group $\mathsf{Hol}_{x_0}(M)$ is infinite dimensional.

Proof. Let $(M, \mathcal{F})$ be a Finsler surface of constant flag curvature covered by a coordinate system (x^1, x^2). Assume that the vector fields

$$U = U^i \frac{\partial}{\partial x^i}, V = V^i \frac{\partial}{\partial x^i} \in \mathfrak{X}^\infty(M)$$

have constant coordinate functions. Let $\xi = R(U, V)$ be the corresponding curvature vector field.

Since $(M, \mathcal{F})$ is of constant flag curvature, we can write

$$R^i_{jk}(x, y) = \lambda\big(\delta^i_j g_{km}(x, y) y^m - \delta^i_k g_{jm}(x, y) y^m\big) \quad \text{with } \lambda = \text{const.}$$

It is well known that the horizontal Berwald covariant derivative $\nabla_W R$ of the tensor field $R = R^i_{jk}(x, y) dx^j \wedge dx^k \frac{\partial}{\partial x^i}$ vanishes. Indeed, [12, Lemma 6.2.2, p. 85] yields

$$\nabla_W g_{(x,y)}(u, v) = -2L(u, v, w) \quad \text{for any } u, v, w \in T_x M.$$

Moreover we have $\nabla_W y = 0$, $\nabla_W \mathsf{Id}_{TM} = 0$ for any vector field $W \in \mathfrak{X}^\infty(M)$, and $L_{(x,y)}(y, v, w) = 0$ (cf. [12, equation (6.28), p. 85]). Hence we obtain that $\nabla_W R = 0$.

Since the curvature tensor field is skew-symmetric, $R_{(x,y)}$ acts on the 1-dimensional wedge product $T_x M \wedge T_x M$. The covariant derivative $\nabla_W \xi$ of the curvature vector field

$$\xi = R(U, V) = \frac{1}{2} R(U \otimes V - V \otimes U) = R(U \wedge V)$$

can be written in the form

$$\nabla_W \xi = R(\nabla_W (U \wedge V)) = R(\nabla_W U \wedge V + U \wedge \nabla_W V).$$

We have $U \wedge V = \frac{1}{2}(U^1V^2 - U^2V^1)\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}$ and hence

$$\nabla_W \xi = (U^1V^2 - V^1U^2)W^k R\left(\nabla_k\left(\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}\right)\right), \tag{3.4}$$

where $\nabla_k \xi := \nabla_{\frac{\partial}{\partial x^k}} \xi$. Since

$$\begin{aligned}\nabla_k\left(\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}\right) &= \left(\nabla_k \frac{\partial}{\partial x^1}\right) \wedge \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^1} \wedge \left(\nabla_k \frac{\partial}{\partial x^2}\right) \\ &= G^l_{k1}\frac{\partial}{\partial x^l} \wedge \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^1} \wedge G^m_{k2}\frac{\partial}{\partial x^m} \\ &= \left(G^1_{k1} + G^2_{k2}\right)\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}\end{aligned}$$

we obtain

$$\nabla_W \xi = \left(G^1_{k1} + G^2_{k2}\right)W^k R(U,V) = \left(G^1_{k1} + G^2_{k2}\right)W^k \xi.$$

Since the geodesic coefficients are given by (2.10), we have

$$\nabla_W \xi = G^m_{km} W^k \xi = 3\frac{\partial \mathcal{P}}{\partial y^k} W^k \xi. \tag{3.5}$$

Hence

$$\begin{aligned}\nabla_Z(\nabla_W \xi) &= 3\nabla_Z\left(\frac{\partial \mathcal{P}}{\partial y^k} W^k \xi\right) \\ &= 3\left\{\nabla_Z\left(\frac{\partial \mathcal{P}}{\partial y^k} W^k\right)\xi + \left(\frac{\partial \mathcal{P}}{\partial y^k} W^k\right)\left(\frac{\partial \mathcal{P}}{\partial y^l} Z^l\right)\right\}\xi.\end{aligned}$$

Let W be a vector field with constant coordinate functions. Then, using (2.10) we get

$$\begin{aligned}\nabla_Z\left(\frac{\partial \mathcal{P}}{\partial y^k} W^k\right) &= \left(\frac{\partial^2 \mathcal{P}}{\partial x^j \partial y^k} - G^m_j \frac{\partial^2 \mathcal{P}}{\partial y^m \partial y^k}\right)W^k Z^j \\ &= \left(\frac{\partial^2 \mathcal{P}}{\partial x^j \partial y^k} - \mathcal{P}\frac{\partial^2 \mathcal{P}}{\partial y^k \partial y^j}\right)W^k Z^j,\end{aligned}$$

and hence

$$\nabla_Z(\nabla_W \xi) = 3\left\{\frac{\partial^2 \mathcal{P}}{\partial x^j \partial y^k} - \mathcal{P}\frac{\partial^2 \mathcal{P}}{\partial y^k \partial y^j} + \frac{\partial \mathcal{P}}{\partial y^k}\frac{\partial \mathcal{P}}{\partial y^j}\right\}W^k Z^j \xi. \tag{3.6}$$

Let $x_0 \in M$ be the point with coordinates $(0,0)$ in the local coordinate system of $(M, \mathcal{F})$ corresponding to the canonical coordinates of the projectively related

Euclidean plane. According to [4, Lemma 8.2.1, p. 155] we have

$$\frac{\partial^2\mathcal{P}}{\partial x^1\partial y^2}-\mathcal{P}\frac{\partial^2\mathcal{P}}{\partial y^1\partial y^2}+\frac{\partial\mathcal{P}}{\partial y^1}\frac{\partial\mathcal{P}}{\partial y^2}=2\frac{\partial\mathcal{P}}{\partial y^1}\frac{\partial\mathcal{P}}{\partial y^2}-\frac{1}{2}\frac{\partial^2\mathcal{F}^2}{\partial y^1\partial y^2}=2\frac{\partial\mathcal{P}}{\partial y^1}\frac{\partial\mathcal{P}}{\partial y^2}-\lambda g_{12}.$$

Hence the vector fields $\xi\big|_{x_0}$, $\nabla_1\xi|_{x_0}$, $\nabla_2\xi|_{x_0}$ and $\nabla_1(\nabla_2\xi)|_{x_0}$ are linearly independent if and only if the functions

$$1,\quad \frac{\partial\mathcal{P}}{\partial y^1}\bigg|_{x_0},\quad \frac{\partial\mathcal{P}}{\partial y^2}\bigg|_{x_0},\quad \left(2\frac{\partial\mathcal{P}}{\partial y^1}\frac{\partial\mathcal{P}}{\partial y^2}-\lambda\, g_{12}\right)\bigg|_{x_0} \tag{3.7}$$

are linearly independent, where $g_{12}=g_y(\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2})$ is the component of the metric tensor of $(M,\mathcal{F})$.

Lemma 3.4. *The functions*

$$\frac{\partial\mathcal{P}(0,y)}{\partial y^1},\qquad \frac{\partial\mathcal{P}(0,y)}{\partial y^2}\qquad \textit{and}\qquad \mathcal{P}(0,y)\frac{\partial^2\mathcal{P}(0,y)}{\partial y^1\partial y^2}$$

can be expressed in the polar coordinate system (e^r,t) by

$$\begin{aligned}\frac{\partial\mathcal{P}(0,y)}{\partial y^1}&=(\cos t+\dot r\sin t)e^{-r},\\ \frac{\partial\mathcal{P}(0,y)}{\partial y^2}&=(\sin t-\dot r\cos t)e^{-r},\\ \mathcal{P}(0,y)\frac{\partial^2\mathcal{P}(0,y)}{\partial y^1\partial y^2}&=(\dot r^2+1-\ddot r)e^{-2r}\sin t\cos t,\end{aligned}$$

where the dot refers to differentiation with respect to the variable t.

Proof. We obtain from $\frac{\partial e^{-r}}{\partial y^1}=-e^{-r}\dot r\frac{\partial t}{\partial y^1}$ and from

$$-\frac{y^2}{(y^1)^2}=\frac{\partial}{\partial y^1}\left(\frac{y^2}{y^1}\right)\frac{d\tan t}{dt}\frac{\partial t}{\partial y^1}=\frac{1}{\cos^2 t}\frac{\partial t}{\partial y^1}$$

that

$$\frac{\partial e^{-r}}{\partial y^1}=e^{-r}\dot r\cos^2 t\frac{y^2}{(y^1)^2}=e^{-r}\dot r\frac{y^2}{(y^1)^2+(y^2)^2}.$$

Hence

$$\begin{aligned}\frac{\partial\mathcal{P}(0,y)}{\partial y^1}&=\frac{\partial\left(e^{-r}\sqrt{(y^1)^2+(y^2)^2}\right)}{\partial y^1}\\ &=e^{-r}\left(\dot r\frac{y^2}{\sqrt{(y^1)^2+(y^2)^2}}+\frac{y^1}{\sqrt{(y^1)^2+(y^2)^2}}\right).\end{aligned}$$

Similarly, we have $\frac{\partial e^{-r}}{\partial y^2} = -e^{-r}\dot{r}\cos^2 t\,\frac{1}{y^1} = -e^{-r}\dot{r}\frac{y^1}{(y^1)^2+(y^2)^2}$. Hence

$$\begin{aligned}\frac{\partial \mathcal{P}(0,y)}{\partial y^2} &= \frac{\partial\big(e^{-r}\sqrt{(y^1)^2+(y^2)^2}\big)}{\partial y^2}\\ &= e^{-r}\Big(-\dot{r}\frac{y^1}{\sqrt{(y^1)^2+(y^2)^2}} + \frac{y^2}{\sqrt{(y^1)^2+(y^2)^2}}\Big).\end{aligned}$$

Finally we have

$$\frac{\partial^2 \mathcal{P}(0,y)}{\partial y^1 \partial y^2} = \frac{\partial(\sin t - \dot{r}\cos t)e^{-r}}{\partial y^1} = (\ddot{r}-\dot{r}^2-1)e^{-r}\sin t\cos t\frac{1}{\sqrt{(y^1)^2+(y^2)^2}}.$$

Replacing now φ by the function $\mathcal{P}(0,y)$ in the expression (3.1) we get the assertion. □

Lemma 3.5. *Let $r:\mathbb{R}\to\mathbb{R}$ be a 2π-periodic smooth function such that the inequality $\ddot{r}(t)-\dot{r}^2(t)-1\neq 0$ holds on a dense subset of $\mathbb{R}$. Then the functions*

$$1,\ (\cos t+\dot{r}\sin t)e^{-r},\ (\sin t-\dot{r}\cos t)e^{-r},\ (\cos t+\dot{r}\sin t)(\sin t-\dot{r}\cos t)e^{-2r} \tag{3.8}$$

are linearly independent.

Proof. The derivative of

$$(\cos t+\dot{r}\sin t)e^{-r} \quad \text{and} \quad (\sin t-\dot{r}\cos t)e^{-r}$$

are

$$(\ddot{r}-\dot{r}^2-1)e^{-r}\sin t \quad \text{and} \quad (\ddot{r}-\dot{r}^2-1)e^{-r}\cos t,$$

respectively, hence the functions (3.8) do not vanish identically. Let us consider a linear combination

$$\begin{aligned}A + B(\cos t+\dot{r}\sin t)e^{-r} &+ C(\sin t-\dot{r}\cos t)e^{-r}\\ &+ D(\cos t+\dot{r}\sin t)(\sin t-\dot{r}\cos t)e^{-2r} = 0\end{aligned}$$

with constant coefficients A, B, C, D. Differentiate and divide by $e^{-t}(\ddot{r}-\dot{r}^2-1)$ and we have

$$B\sin t - C\cos t - D(\cos 2t + \dot{r}\sin 2t)e^{-r} = 0.$$

Putting $t=0$ and $t=\pi$, we get

$$C = -De^{-r(0)} = De^{-r(\pi)}.$$

Since $e^{-r(0)}, e^{-r(\pi)} > 0$, we get

$$C = D = 0$$

and hence $A=B=C=D=0$. □

Now, assume that condition (A) of Theorem 3.3 is fulfilled. According to Proposition 3.1 if the functions (3.7) are linearly independent, then the holonomy group $\mathsf{Hol}_{x_0}(M)$ is an infinite dimensional subgroup of $\mathsf{Diff}^\infty(\mathcal{I}_{x_0}M)$. The function $\mathcal{F}(x_0, y)$ induces a scalar product on $T_{x_0}M$, consequently the component g_{12} of the metric tensor is constant on $T_{x_0}M$. Hence $\mathsf{Hol}_{x_0}(M)$ is infinite dimensional if the functions

$$1, \quad \frac{\partial \mathcal{P}}{\partial y^1}\bigg|_{x_0}, \quad \frac{\partial \mathcal{P}}{\partial y^2}\bigg|_{x_0}, \quad \frac{\partial \mathcal{P}}{\partial y^1}\frac{\partial \mathcal{P}}{\partial y^2}\bigg|_{x_0} \tag{3.9}$$

are linearly independent. This follows from Lemma 3.5 and hence the assertion of the theorem is true.

Assume that condition (B) is satisfied. We denote $\varphi(y) = \mathcal{F}(x_0, y)$. Using the expressions (3.7) we obtain that the vector fields $\xi|_{x_0}$, $\nabla_1\xi|_{x_0}$, $\nabla_2\xi|_{x_0}$ and $\nabla_1(\nabla_2\xi)|_{x_0}$ are linearly independent if and only if the functions

$$1, \quad \frac{\partial \mathcal{P}}{\partial y^1}\bigg|_{x_0} = c\frac{\partial \varphi}{\partial y^1}, \quad \frac{\partial \mathcal{P}}{\partial y^2}\bigg|_{x_0} = c\frac{\partial \varphi}{\partial y^2}$$

$$\left(2\frac{\partial \mathcal{P}}{\partial y^1}\frac{\partial \mathcal{P}}{\partial y^2} - \lambda g_{12}\right)\bigg|_{x_0} = (2c^2 - \lambda)\frac{\partial \varphi}{\partial y^1}\frac{\partial \varphi}{\partial y^2} - \lambda\varphi\frac{\partial^2 \varphi}{\partial y^1 \partial y^2}$$

are linearly independent. According to Lemma 3.4 this is equivalent to the linear independence of the functions

$$1, \quad (\cos t + \dot r \sin t)\, e^{-r}, \quad (\sin t - \dot r \cos t)\, e^{-r},$$

$$(2c^2 - \lambda)(\cos t + \dot r \sin t)(\sin t - \dot r \cos t)\, e^{-2r} - \lambda(\ddot r - \dot r^2 - 1)\, e^{-2r} \sin t \cos t.$$

If $r = \text{const}$, then these functions are $1, \cos t\, e^{-r}, \sin t\, e^{-r}, 2c^2 \cos t \sin t\, e^{-2r}$, hence the assertion follows from Lemma 3.5. In the following we can assume that $r(t) \neq \text{const}$. Let $t_0 \in \mathbb{R}$ such that $\dot r(t_0) = 0$ and $\kappa(t_0) \neq 0$. We rotate the coordinate system at the angle $-t_0$ with respect to the euclidean norm $\sqrt{(y^1)^2 + (y^2)^2}$, then we get in the new polar coordinate system that $\dot r(0) = 0$ and $\kappa(0) \neq 0$. Consider the linear combination

$$\begin{aligned} A + B(\cos t + \dot r \sin t)e^{-r} + C(\sin t - \dot r \cos t)\, e^{-r} \\ + D\big((2c^2 - \lambda)(\cos t + \dot r \sin t)(\sin t - \dot r \cos t)\, e^{-2r} \\ - \lambda(\ddot r - \dot r^2 - 1)\, e^{-2r} \sin t \cos t\big) = 0 \end{aligned} \tag{3.10}$$

with some constants A, B, C, D. Since the function φ is absolutely homogeneous, the function $r(t)$ is π-periodic. Putting $t + \pi$ into t, the value of

$$A + D(2c^2 - \lambda)(\cos t + \dot r \sin t)(\sin t - \dot r \cos t)\, e^{-2r} - \lambda(\ddot r - \dot r^2 - 1)\, e^{-2r} \sin t \cos t$$

does not change, but the value of

$$B(\cos t + \dot r \sin t)\, e^{-r} + C(\sin t - \dot r \cos t)\, e^{-r}$$

changes its sign. Since Lemma 3.5 implies that the functions $(\cos t + \dot r \sin t)\, e^{-r}$ and $(\sin t - \dot r \cos t)\, e^{-r}$ are linearly independent, we have $B = C = 0$ and (3.10) becomes

$$A\, e^{2r} + D\left((2c^2-\lambda)\left[-\dot r \cos 2t + \frac{1}{2}(1-\dot r^2)\sin 2t\right] - \frac{\lambda}{2}(\ddot r - \dot r^2 - 1)\sin 2t\right) = 0. \tag{3.11}$$

Since $\dot r(0) = 0$ at $t = 0$, we have $A = 0$. If $D \neq 0$, then (3.11) gives

$$(2c^2-\lambda)\left[-\dot r \cos 2t + \frac{1}{2}(1-\dot r^2)\sin 2t\right] - \frac{\lambda}{2}(\ddot r - \dot r^2 - 1)\sin 2t = 0.$$

By derivation and putting $t = 0$ we obtain

$$(2c^2-\lambda)[-\ddot r(0) + 1] - \lambda(\ddot r(0) - 1) = 2c^2(1 - \ddot r(0)) = 0.$$

Using the relation (3.2) condition (B) gives $\kappa(0) = e^{r(0)}(1 - \ddot r(0)) \neq 0$, which is a contradiction. Hence $D = 0$ and the vector fields $\xi\big|_{x_0}$, $\nabla_1\xi|_{x_0}$, $\nabla_2\xi|_{x_0}$ and $\nabla_1(\nabla_2\xi)|_{x_0}$ are linearly independent. Using Proposition 3.1 we obtain the assertion.

Suppose now that condition (C) holds. Hence we have

$$\frac{\partial \mathcal{F}}{\partial y^1}(0,y) = \frac{y^1}{|y|} \pm a^1, \quad \frac{\partial \mathcal{F}}{\partial y^2}(0,y) = \frac{y^2}{|y|} \pm a^2, \quad \frac{\partial^2 \mathcal{F}}{\partial y^1 \partial y^2}(0,y) = -\frac{y^1 y^2}{|y|^3},$$

and

$$g_{12} = \left(\frac{y^1}{|y|} \pm a^1\right)\left(\frac{y^2}{|y|} \pm a^2\right) - \left(1 \pm \left\langle a, \frac{y}{|y|}\right\rangle\right)\frac{y^1 y^2}{|y|^2}. \tag{3.12}$$

Similarly, we obtain from condition (C) that

$$\frac{\partial \mathcal{P}}{\partial y^1}(0,y) = \pm\frac{y^1}{|y|} - a^1, \quad \frac{\partial \mathcal{P}}{\partial y^2}(0,y) = \pm\frac{y^2}{|y|} - a^2.$$

Using the expressions (3.7) we get that the vector fields $\xi\big|_{x_0}$, $\nabla_1\xi|_{x_0}$, $\nabla_2\xi|_{x_0}$, $\nabla_1(\nabla_2\xi)|_{x_0}$ are linearly independent if and only if the functions

$$1, \quad \frac{\partial \mathcal{P}}{\partial y^1}\bigg|_{(0,y)} = \pm\frac{y^1}{|y|} - a^1, \quad \frac{\partial \mathcal{P}}{\partial y^2}\bigg|_{(0,y)} = \pm\frac{y^2}{|y|} - a^2$$

and

$$2\frac{\partial \mathcal{P}}{\partial y^1}\frac{\partial \mathcal{P}}{\partial y^2} - \lambda\, g_{12}\bigg|_{(0,y)} = \mp\left\langle a, \frac{y}{|y|}\right\rangle\frac{y^1 y^2}{|y|^2} + (1-\lambda)\frac{y^1 y^2}{|y|^2}$$
$$\mp (2+\lambda)\left(a_2\frac{y^1}{|y|} + a_1\frac{y^2}{|y|}\right) + (2-\lambda)a_1 a_2$$

are linearly independent. Putting

$$\cos t = \frac{y^1}{|y|}, \quad \sin t = \frac{y^2}{|y|}$$

we obtain that this condition is true, since the trigonometric polynomials

$$1, \quad \cos t, \quad \sin t, \quad (1-\lambda)\cos t \sin t \mp (a_1 \cos t + a_2 \sin t)\cos t \sin t$$

are linearly independent. Hence $\mathsf{Hol}_{x_0}(M)$ is infinite dimensional. □

4 Holonomy of projective Finsler manifolds of constant curvature

Now we will prove that the infinitesimal holonomy algebra of a totally geodesic submanifold of a Finsler manifold can be embedded into the infinitesimal holonomy algebra of the entire manifold. This result yields a lower estimate for the dimension of the holonomy group.

Totally geodesic and auto-parallel submanifolds

Let $(M, \mathcal{S})$ be a spray manifold. A submanifold $\bar{M}$ is called *totally geodesic* if any geodesic of $(M, \mathcal{S})$ which is tangent to $\bar{M}$ at some point is contained in $\bar{M}$.

A totally geodesic submanifold $\bar{M}$ of the spray manifold $(M, \mathcal{S})$ is called *auto-parallel* if the homogeneous (nonlinear) parallel translations

$$\tau_c : T_{c(0)}M \to T_{c(1)}M$$

along curves in the submanifold $\bar{M}$ leave invariant the tangent bundle $T\bar{M}$ and for every $\xi \in \hat{\mathfrak{X}}^\infty(T\bar{M})$ the horizontal covariant derivative $\nabla_X \xi$ belongs to $\hat{\mathfrak{X}}^\infty(T\bar{M})$.

Let $X, Y \in T_x M$ be tangent vectors at $x \in M$ and let K denote the curvature tensor of $(M, \mathcal{S})$ (cf. equation (2.7)). The mapping

$$y \to K(X,Y)(x,y) : T_x M \mapsto T_x M$$

is called *curvature vector field at x* of the spray manifold $(M, \mathcal{S})$.

Lemma 4.1. *Let $\bar{M}$ be a totally geodesic submanifold in a spray manifold $(M, \mathcal{S})$. The following assertions hold:*

(a) *the spray $\mathcal{S}$ induces a spray $\bar{\mathcal{S}}$ on the submanifold $\bar{M}$,*

(b) *$\bar{M}$ is an auto-parallel submanifold,*

(c) *the curvature vector fields at any point of $\bar{M}$ can be extended to a curvature vector field of M.*

Proof. Assume that the manifolds $\bar{M}$ and M are k-, respectively $n = k+p$-dimensional. Let $(x^1, \dots, x^k, x^{k+1}, \dots, x^n)$ be an adapted coordinate system, i.e. the submanifold $\bar{M}$ is locally given by the equations $x^{k+1} = \dots = x^n = 0$. We denote the indices running on the values $\{1, \dots, k\}$ or $\{k+1, \dots, n\}$ by α, β, γ or σ, τ, respectively. The differential equation (2.2) of geodesics yields that the geodesic coefficients $\Gamma^\sigma(x, y)$ satisfy

$$\Gamma^\sigma(x^1, \dots, x^k, 0, \dots, 0; y^1, \dots, y^k, 0, \dots, 0) = 0$$

identically, hence their derivatives with respect to $y^1, \dots, y^k$ are also vanishing. It follows that

$$\Gamma^\sigma_\alpha = 0 \quad \text{and} \quad \Gamma^\sigma_{\alpha\beta} = 0$$

at any $(x^1, \dots, x^k, 0, \dots, 0; y^1, \dots, y^k, 0, \dots, 0)$. Hence the induced spray $\bar{\mathcal{S}}$ on $\bar{M}$ is defined by the geodesic coefficients

$$\bar{\Gamma}^\beta(x^1, \dots, x^k; y^1, \dots, y^k) = \Gamma^\beta(x^1, \dots, x^k, 0, \dots, 0; y^1, \dots, y^k, 0, \dots, 0).$$

The homogeneous (nonlinear) parallel translation $\tau_c : T_{c(0)}M \to T_{c(1)}M$ along curves in the submanifold $\bar{M}$ and the horizontal covariant derivative on $\bar{M}$ with respect to the spray $\mathcal{S}$ coincide with the translation and the horizontal covariant derivative on $\bar{M}$ with respect to the spray $\bar{\mathcal{S}}$. Hence the first two assertions are true.

If $y, X, Y \in T_x\bar{M}$ are tangent vectors at $x \in \bar{M}$, then $K(X,Y)(x,y)$ can be expressed by

$$\Big(\frac{\partial \Gamma^i_\alpha(x,y)}{\partial x^\beta} - \frac{\partial \Gamma^i_\beta(x,y)}{\partial x^\alpha} + \Gamma^m_\alpha(x,y)\Gamma^i_{\beta m}(x,y) - \Gamma^m_\beta(x,y)\Gamma^i_{\alpha m}(x,y)\Big) X^\alpha Y^\beta \frac{\partial}{\partial x^i}.$$

Since $\Gamma^\sigma_\alpha = 0$ and $\Gamma^\sigma_{\alpha\beta} = 0$ at any $(x^1, \dots, x^k, 0, \dots, 0; y^1, \dots, y^k, 0, \dots, 0)$, we have

$$\frac{\partial \Gamma^\sigma_\alpha}{\partial x^\beta} - \frac{\partial \Gamma^\sigma_\beta}{\partial x^\alpha} + \Gamma^\tau_\alpha \Gamma^\sigma_{\beta\tau} - \Gamma^\tau_\beta \Gamma^\sigma_{\alpha\tau} + \Gamma^\gamma_\alpha \Gamma^\sigma_{\beta\gamma} - \Gamma^\gamma_\beta \Gamma^\sigma_{\alpha\gamma} = 0$$

at $(x^1, \dots, x^k, 0, \dots, 0; y^1, \dots, y^k, 0, \dots, 0)$. Hence the curvature tensors $\bar{K}$ corresponding to the spray $\bar{\mathcal{S}}$ and K corresponding to the spray $\mathcal{S}$ satisfy

$$\bar{K}(X,Y)(x,y) = K(X,Y)(x,y)$$

if $x \in \bar{M}$ and $y, X, Y \in T_x\bar{M}$. It follows that for any given $X, Y \in T_x\bar{M}$ the curvature vector field $\bar{\xi}(y) = \bar{K}(X,Y)(x,y)$ at $x \in \bar{M}$ defined on $T_x\bar{M}$ can be extended to the curvature vector field $\xi(y) = K(X,Y)(x,y)$ at $x \in \bar{M}$ defined on T_xM. □

Theorem 4.2. *Let $\bar{M}$ be a totally geodesic 2-dimensional submanifold of a Finsler manifold $(M, \mathcal{F})$ such that the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(\bar{M})$ of $\bar{M}$ is infinite dimensional. Then the holonomy group $\mathsf{Hol}_x(M)$ is infinite dimensional.*

Proof. According to Lemma 4.1 any curvature vector field of $\bar{M}$ at $x \in \bar{M} \subset M$ defined on $\mathcal{I}_x\bar{M}$ can be extended to a curvature vector field on the indicatrix $\mathcal{I}_x M$. Hence the curvature algebra $\mathfrak{R}_x(\bar{M})$ of the submanifold $\bar{M}$ can be embedded into the curvature algebra $\mathfrak{R}_x(M)$ of the manifold $(M, \mathcal{F})$. Assume that $\bar{\xi}$ is a vector field belonging to the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(\bar{M})$ which can be extended to the vector field ξ belonging to the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$. Any given vector field $\bar{X} \in \mathfrak{X}^\infty(\bar{M})$ can be extended to a vector field $X \in \mathfrak{X}^\infty(M)$, hence the Berwald horizontal covariant derivative along $\bar{X} \in \mathfrak{X}^\infty(\bar{M})$ of $\bar{\xi}$ can be extended to the Berwald horizontal covariant derivative along $X \in \mathfrak{X}^\infty(M)$ of the vector field ξ. It follows that the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(\bar{M})$ of the submanifold $\bar{M}$ can be embedded into the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ of the Finsler manifold $(M, \mathcal{F})$. Consequently, the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ is infinite dimensional and hence the holonomy group $\mathsf{Hol}_x(M)$ is an infinite dimensional subgroup of $\mathsf{Diff}^\infty(\mathcal{I}_x M)$. □

This result can be applied to locally projectively flat Finsler manifolds, as they have for each tangent 2-plane a totally geodesic submanifold which is tangent to this 2-plane.

Corollary 4.3. *If a locally projectively flat Finsler manifold has a 2-dimensional totally geodesic submanifold satisfying one of the conditions of Theorem 3.3, then its holonomy group is infinite dimensional.*

According to equations (2.11) and (2.12) the projectively flat Randers manifolds of non-zero constant curvature satisfy condition (C) of Theorem 3.3. We can apply Corollary 4.3 to these manifolds and we get the following

Theorem 4.4. *The holonomy group of any projectively flat Randers manifolds of non-zero constant flag curvature is infinite dimensional.*

R. Bryant in [1, 2] introduced and studied complete Finsler metrics of positive curvature on S^2. He proved that there exists exactly a 2-parameter family of Finsler metrics on S^2 with curvature $= 1$ with great circles as geodesics. Z. Shen generalized a 1-parameter family of complete Bryant metrics to S^n satisfying

$$\mathcal{F}(0, y) = |y| \cos\alpha, \quad \mathcal{P}(0, y) = |y| \sin\alpha \tag{4.1}$$

with $|\alpha| < \frac{\pi}{2}$ in a coordinate neighbourhood centered at $0 \in \mathbb{R}^n$ (cf. [14, Example 7.1] and [4, Example 8.2.9]).

We investigate the holonomy groups of two families of metrics, containing the 1-parameter family of complete Bryant–Shen metrics (4.1). The first family in the following theorem is defined by condition (A), which is motivated by [4, Theorem 8.2.3]. There is given the following construction:

If $\psi = \psi(y)$ is an arbitrary Minkowski norm on $\mathbb{R}^n$ and $\varphi = \varphi(y)$ is an arbitrary positively 1-homogeneous function on $\mathbb{R}^n$, then there exists a projectively flat Finsler metric $\mathcal{F}$ of constant flag curvature -1, defined on a neighbourhood of the origin, such that $\mathcal{F}$ and its projective factor $\mathcal{P}$ satisfy $\mathcal{F}(0, y) = \psi(y)$ and $\mathcal{P}(0, y) = \varphi(y)$.

Condition (B) in the next theorem is confirmed by [14, Example 7, p. 1726], where it is proved that for an arbitrary given Minkowski norm φ and $|\vartheta| < \frac{\pi}{2}$ there exists a projectively flat Finsler function $\mathcal{F}$ of constant curvature $= 1$ defined on a neighbourhood of $0 \in \mathbb{R}^n$, such that

$$\mathcal{F}(0, y) = \varphi(y) \cos \vartheta \quad \text{and} \quad \mathcal{P}(0, y) = \varphi(y) \sin \vartheta.$$

Conditions (A) and (B) in Theorem 3.3 together with Corollary 4.3 yield the following

Theorem 4.5. *Let $(M, \mathcal{F})$ be a projectively flat Finsler manifold of non-zero constant curvature. Assume that there exists a point $x_0 \in M$ and a 2-dimensional totally geodesic submanifold $\bar{M}$ through x_0 such that one of the following conditions holds:*

(A) *$\mathcal{F}$ induces a scalar product on $T_{x_0}\bar{M}$, and the projective factor $\mathcal{P}$ on $T_{x_0}\bar{M}$ is a strongly convex positively 1-homogeneous function,*

(B) *$\mathcal{F}(x_0, y)$ on $T_{x_0}\bar{M}$ is a strongly convex absolutely 1-homogeneous function on $T_{x_0}M$, and the projective factor $\mathcal{P}(x_0, y)$ on $T_{x_0}\bar{M}$ satisfies*

$$\mathcal{P}(x_0, y) = c \cdot \mathcal{F}(x_0, y)$$

with $0 \neq c \in \mathbb{R}$.

Then the holonomy group $\mathsf{Hol}_{x_0}(M)$ is infinite dimensional.

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