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Geometrical transformations and the concept of cyclic ordering

In this paper we describe a research on the connection between geometrical transformations and orientation. We discuss the particulars of the thinking process and typical difficulties connected to the field of geometrical transformations. We pay special attention to the problem of cyclic order. We investigate pupils' competence in primary school, especially in Grade 2 (age 7–8).

Introduction

This study is part of a research on spatial orientation competence in primary school.

Spatial orientation describes the visualization of a spatial arrangement in which the observer is part of the situation (Maier, 1999).

On the basis of mathematical and historical analysis we may divide the relevant mathematics curriculum regarding the topic of spatial orientation into 6 subtopics (Kónya, 2006).

1. Using words to describe spatial relations
2. Describing routes (using simple maps)
3. Ordering cyclically
4. The coordinate system
5. Geometrical transformations
6. The front-, side-, and top-view of an object

We will discuss the particulars of the thinking process and typical difficulties connected to the 3rd and 5th subtopic.

Theoretical background

We use Guilford's interpretation of spatial ability, especially spatial orientation. Spatial ability has two main components: visualization and spatial orientation. Spatial orientation has five components: factor S3 of Thurstone, spatial relations, spatial perception, mental rotation and kinesthetic imagery (Maier, 1999).

Mental rotation and kinesthetic imagery are important preconditions of the development of spatial ability. (Aman and Roberts, 1993)

We studied the results of mathematical, historical and didactical theories connected to the spatial orientation and particularly to the geometrical transformations. Our analysis is based mainly on the work of Hilbert (1956), Kerékjártó (1937) and Freudenthal (1983).

Hilbert introduced the circulation sense of a triangle with the help of the concept of the left side of an oriented line. The circulation sense is the basis of the orientation on a plane, furthermore of the well-known property of transformations: preserving or inverting of orientation.

Kerékjártó introduced the concept of orientation in another way. The starting concept in his work was the cyclic order. If a, b, c are three half-lines with a common start point O on the plane, he says that the cyclic orientation of a, b, c is a function, which orders to them one of their permutations (Kerékjártó, 1937, p. 116). It is easy to see that permutations *(abc)*, *(bca)*, *(cab)* and permutations *(cba)*, *(bac)*, *(acb)* determine the same cyclic orientations. One of the two cyclic orientations corresponds to one of the two directions of rotation around point *O*. Kerékjártó highlighted the link between cyclic permutation and orientation.

Freudenthal compared the cyclic order with the linear one on the level of mental objects (Freudenthal, 1983, p. 414). He established that cyclic orientation is not deducible from linear orientation directly, so it is worth teaching it separated. He pointed out that cyclic orders are probably early mental objects and arranging cyclically is an early mental activity in the individual learning process then linear order and arranging linearly. He referred to such kind of activities as sitting around a table, standing or dancing in a circle, counting out, etc. Freudenthal called one's attention to two phenomenologically important sources of orientation: to the reflection and to the angle (Freudenthal, 1983, p. 424–425).

Research questions and methodology

Our research questions are the following:

1. Do pupils in grade 2 have the competence to construct the reflected image or the rotated image of an arrangement?
2. Can they identify the transformed image of a certain arrangement?
3. Which are the activities we can enlarge pupils' knowledge with or correct their recognized faults of thinking?

We assumed that in this age it is worth dealing with these questions through specific activities.

Our investigation consists of the following phases:

We planned a pilot study (in spring 2005) with pupils of grades 1–4. Our goal was to gauge the problems of elementary school-pupils in different ages in order to adjust the actual knowledge level for the full experiment. We chose three elementary schools in Debrecen, in Hungary. The first was the practicing school of the teacher training college. The pupils had very good abilities; they had been accepted to the school after a selection. We can say that average pupils attend the second school, and in the third school there are pupils whose abilities are average or below average, and whose social backgrounds are not optimal. We chose, in all, three classes from each grade. The classes were without any specification, their learning based on the normal curriculum of their school. With the selection of the classes participating in our experiment we tried to represent the real situation in the grades 1–4 in Hungary.

Grades	Grade 1	Grade 2	Grade 3	Grade 4	All
Number of participants	63	78	73	62	276

Table 1. The number of participants in the pilot study

We prepared the following paper-pencil tasks: (The first was used only for Grades 1–3, the second only for Grades 1–2.)

Problem of rotation

You can see the same disk in different situations. Colour the white squares!

(■: green: □: yellow, ■: red, ■: blue, Figure 1–3.)

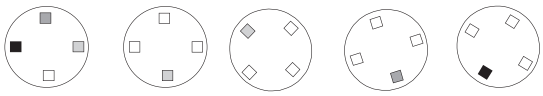


Figure 1. Rotation task for Grade 3

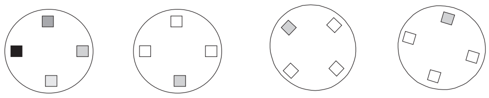
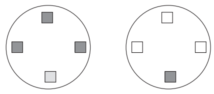


Figure 2. Rotation task for Grade 2

Figure 3. Rotation task for Grade 1

Problem of reflection



Colour the reflected images of the disks if the black lines means the position of the mirror! (Figure 4.)

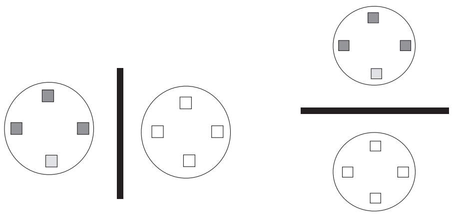


Figure 4. Reflection task

After this pilot study, in the next school year (in autumn 2005) we carried out a classroom experiment with pupils of Grade 2 (7–8 years old children). We chose the same class (27 pupils) from the practicing school of the teacher training college which participated as Grade 1 in our pilot study. Our aim was to try our ideas to develop pupil's ability in the field of spatial orientation, particularly of geometrical transformations. Grade 2 seemed a good choice because pupils are already familiar with school life, reading and writing. The results of the pilot study in grades 1, 3 and 4 were useful because of identification problems which remained and knowledge that was getting in every day life in this age. We prepared 3 problems on the topic of geometrical transformations in 3 different lessons. One problem required 10–15 minutes from the lesson. We planned the lessons together with the classroom-teacher, and discussed the problems after the lessons, but we did not teach.

We finished the classroom experiment with a post-test (in January 2006) and prepared a delayed-test for "our" second graders two months after. The post-test was solved not by the experimental class, but by other Grade 2 class from the same school (control class) too. We were interested in the development of "our" pupils comparing their results with other pupils' results. We wanted to know also about the spontaneous development of pupils who didn't pay special attention to the topic of orientation.

Results

Pilot study

Problem of rotation:

To solve the problem pupils have to imagine the process of rotation, have to do a mental rotation. Diagram 1 shows an overview of solutions.

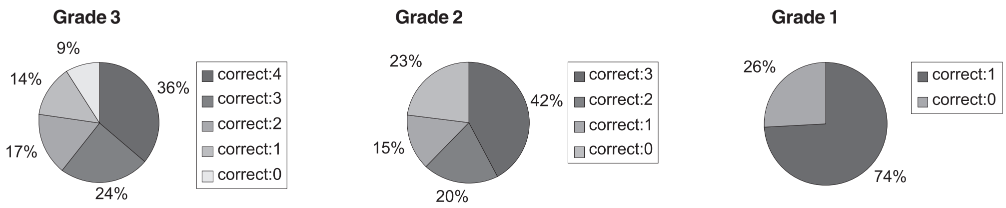


Diagram 1. Solutions of the problem of rotation in the pilot study

The direction of the rotation is indifferent; the cyclic order of colours will be the same in both directions. We wanted to know which graders are able to construct correct cyclic order. We can assume that pupils who colour all disks correctly, or made only one mistake have the competence to construct the rotated image of a discrete arrangement.

The number of these pupils is relatively low in any grade. In Grade 340% of pupils couldn't solve the problem. They didn't understand the task or couldn't construct the cyclic order.

Problem of reflection:

We allowed pupils to use mirror to colour the disks, but this tool didn't give support for everybody. Some of them weren't able to use it. Diagram 2 shows the solving strategies of first and second graders.

76% of pupils in both grades use the same strategies by colouring of disks independently from the position of the mirror.

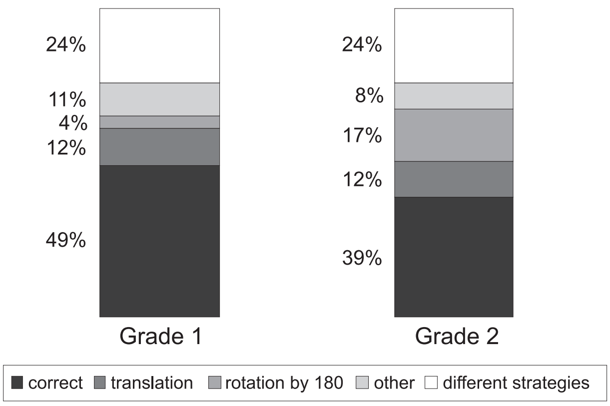


Diagram 2. Solving strategies in problem of reflection in the pilot study

"Translation" means that order of colours neither vertically nor horizontally changes (Y: yellow, R: red, B: blue, G: green, Figure 5).

"Rotation by 180°" means that order of colours changes both vertically and horizontally (Figure 6).

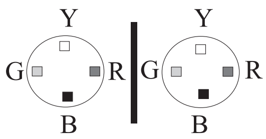


Figure 5. The “translation” strategy

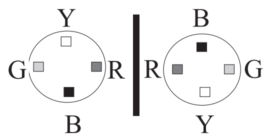


Figure 6. The “rotation by 180°” strategy

We can compare the results of the problem of rotation and reflection in Grade 1 and 2. (Diagram 3)

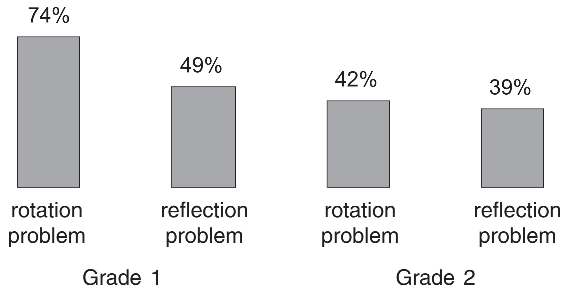


Diagram 3. Correct solutions percentages in the two problems of pilot study

Constructing of a reflected image is more difficult than constructing of a rotated image notwithstanding the use of the mirror.

Teaching experiment

Lesson 1:

We gave pupils a coloured hexagon from cardboard (Figure 7) and a mirror. We asked them to colour 3 rotated (Figure 8) and 3 reflected hexagons (Figure 9) on the paper adequately.

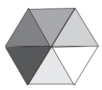


Figure 7

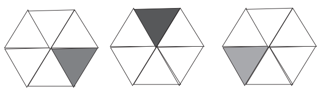


Figure 8

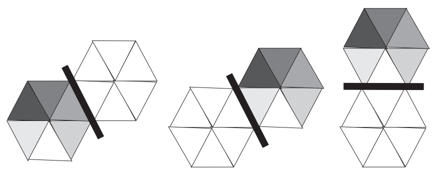


Figure 9

The work was very successful, almost all pupils coloured the hexagons correctly. (25 pupils, 5 hexagonal per person, only 5 hexagons from 125 was wrong)

Lesson 2:

Pupils worked with the same hexagon, but now we drew six coloured hexagons on a paper and they had to mark which the rotated image of the original one is.

From 27 solutions 14 were correct in all the six case.

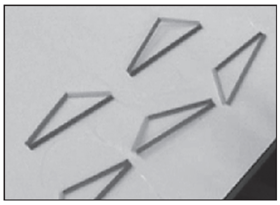
Lesson 3:

Pupils had to create different ordering of 3 coloured straws (Picture 1) then form triangles from them. (Picture 2). This activity helped children identify triangles by rotation and understand the differences between an image and a reflected image.

We had an interesting observation: Pupils sorted triangles in two groups, but they weren’t able to conceive the connection between the two groups, e.g. they are reflections of each other (Picture 3).

Teacher: What did you recognise? These two are ……. of each other.
Gábor: Öööööö…
Teacher: Each other…..
Gábor and Fanni: ……….(listen)
Teacher: If I would put here something…..What should I put here? What should I put here to make it the same?
Gábor and Fanni: ………(listen)
Teacher: What?
Fanni: Red?
Teacher: Oh, no! I didn't think on a colour! Look at the triangles! You can see that they aren't the same. What are these two of each other?
Gábor: Pairs!
Teacher: What happen if I put a mirror here in the middle? What can we see in it?
Gábor: The same!
Teacher: Then they are ……
Gábor and Fanni: …….(They listen and don't know the word waiting the teacher.)

Picture 1



Picture 2

Picture 3

Pupils are able to distinguish the image and the reflected image on the level of manipulation. *Gábor* said that they are "pairs", so they are close to each other visually, but he isn't able to express this situation verbally.

The post- and delayed-test

Problem of rotation:

The problem in the post- and the delayed-test was the same as the problem of rotation for Grade 3 in the pilot study. Diagram 4 shows the results of the post- and delayed-test.

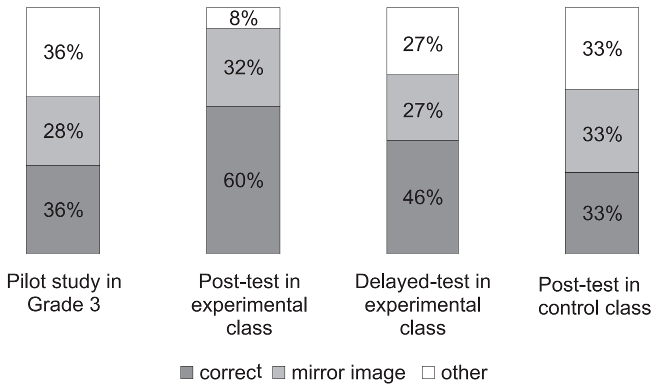
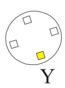


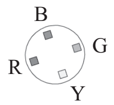
Diagram 4. The solutions of problem of rotation

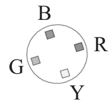
The "correct" answer means that all the 4 squares are well coloured.

The "mirror image" means that the order of colours is correct, but the direction of mental rotation is not.

For example in the case of disk 

:

If the correct answer: 

then the „mirror image”: 

Third graders in the pilot study and the control class achieved near the same result. The experimental class in the post-test was quite successful but the delayed-test shows that although the activities doing through the teaching experiment were useful, the stabile knowledge needs more experience on mental rotation.

Problem of reflection:

The problem of the delayed-test was the same as the problem of reflection in the pilot study for Grade 1–2.

Comparing the correct answers on rotation and reflection problems we see a slight incrase in „our” class in both cases (Diagram 5).

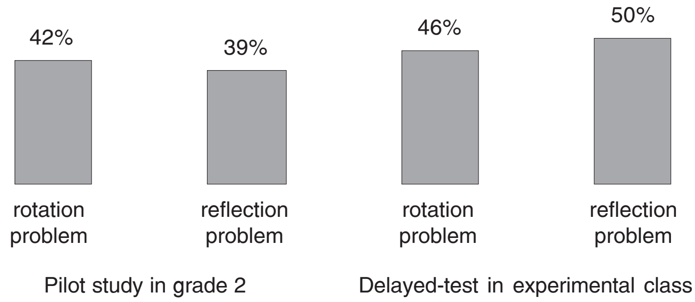


Diagram 5. Correct solutions percentages in the two problems of pilot study and delayed-test

Conclusion

- Whereas in everyday life we use the cyclic order and cyclic orientation several times, the problem situations linking to them are almost unknown for pupils.
- Cyclic orientation assumes a dynamic situation, a rotation. Mental rotation especially in discrete case is quite difficult, while rotation with some concrete instrument is not.
- Construction of a rotated image is a simpler task than deciding whether an image is the rotated image of the others.
- The concept of cyclic order with different instruments and activities is developable effectively, but the development is a long-term period.
- The second graders are not familiar with construction of a reflected image of an arrangement. If they are experienced in using mirror, it can help drawing the image.
- The construction of the reflected image of an arrangement is more difficult then of the rotated image.
- Lots of different activities are preconditions of successful mental rotation and reflection.

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Building the concept of line symmetry

In his paper I discuss the key processes characteristic to the levels of reasoning in geometry. I have discussed particular of the thinking processes and their functioning by 12–16 year old students in relation to the concept of line symmetry on the plane.

Introduction

The nature of the concept of line symmetry is very composed. At primary school the students start to learn some aspects of this concept by using different tools: mirror (they observe mirror reflections), ink – stains and paper cut – outs. This kind of activities help the students to build the correct understanding of a concept. Van Hiele (1986) expressed important implications of his theory: students cannot show adequate performances at a certain level without having experiences which enable them to reason intuitively at each proceeding level. Hoffer (1986) claims that hands – on activities usually help students to perform at level 1 (Visualisation) and to move towards level 2 (Analysis).

A question arises whether or not using of these different tools (like mirror, paper cut – outs or ink – stains) is sufficient to build a correct properties of the concept of line symmetry? What kind of relationships do students observe while using these tools?

Van Hiele's levels of reasoning integrated by several key thinking processes which are characteristic of the levels may be useful for answering these questions. In order to evaluate a student's thinking level we have to evaluate the way in which the student uses the key thinking process.

Theoretical background

Gutierrez and Jamie have described different processes of reasoning as characteristic of several van Hiele levels (Gutierrez, Jaime 1994, 1998):

1. **Recognition** of types and families of geometric figures, identification of components and properties of the figures.
2. **Definition** of a geometrical concept. This process can be viewed in two ways: as the students formulate definition of the concept they are learning, and as the students use a given definition read in a textbook, or heard from the teacher or another student.
3. **Classification** of geometrical figures or concept into different families or classes.
4. **Proof** of properties or statements, that is to explain in some convincing way why such property or statement is true.

Table 1 summarises the key processes characteristics of each van Hiele level.

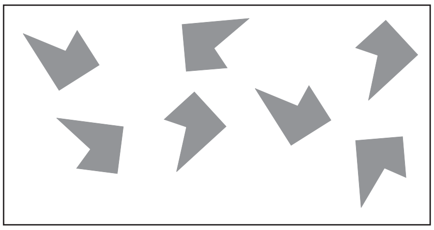
	recognition	Definition	classification	proof
Level 1	+	State	+	
Level 2	+	Read & state	+	+
Level 3		Read & state	+	+
Level 4		Read & state		+

Table 1. The key – processes characteristics of the Van Hiele levels

Each process is a component of two or more levels of reasoning. At each level the students show them in a different way. Based on the Gutierrez and Jaime proposition of a test for evaluating the level of student's thinking (Gutierrez, Jaime 1994, 1998) I prepared a test based on open – ended items that are not pre – assigned to a specific level, but to a range of the levels in which answers can be given (see *Appendix 1)*. Each of the key processes have been verified at least at two items. A test was solved by 15 students 12–13 years old from the 5^{th} and 6^{th} class of primary school and 15 students 16 years old from the 3^{rd} class of junior high school. I thought, that the students participating in my research project could be in the 1st or 2nd van Hiele level of reasoning according to the concept of line symmetry. Therefore I have restricted to analysis of the key – processes characteristic to 1–3 levels. In this paper I would like to consider the key – processes of reasoning characteristic to van Hiele levels but I would not like to establish in which the van Hiele's level of reasoning the students are.

Results – analysis of students' answers

The test began from the task connected with observation and manipulation. From among the congruent figures, children were to choose pairs of figures, which were their own mirror reflection (*picture 1*). Figures should have been cut-out and pasted to the test. For each pair of figures there was a need to draw a line of symmetry. Students were informed that they could paste figures in any way, but not coloured side to the paper (it was not possible to flip any figures). From the mathematical point of view the goal of this task was to focus students' attention on a very important mathematical property of congruent figures: it is not possible to transform a figure on the plane into a figure symmetric to it, only by the movement on the plane (shift or rotation). The results were the following:

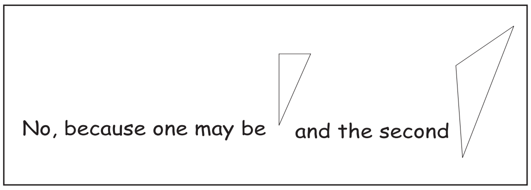


Picture 1

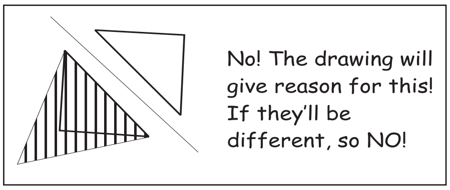
- 70% placements with a vertical line of symmetry with figures having a side parallel or perpendicular to the line of symmetry,
- 13% placements with a slanting line of symmetry with figures having a side parallel or perpendicular to the line of symmetry,
- 10% placements with a horizontal line of symmetry with figures placed oblique to the line of symmetry
- 7% wrong arrangement of figures (often point symmetry).

The second task: "*Is it possible for any two squares in the plane to be mirror reflections of one another?*" and the task no. 5: "*Is it possible for any two triangles on the plane to be mirror reflection of one another?*" concerned the classification of figures among one family of shapes (squares, triangles). In order to two triangles might fulfil the relation of mirror reflection they must be the same shape and size and proper orientation. All the students stated that the relation of mirror reflection on the plane fulfil only the triangles with the same shape and size. Students supported their thesis in the following way: Triangles can be different, because there are different kinds of triangles. Such triangles cannot be their own mirror reflection. Younger students, as opposed to older ones, often made a drawing (*Example 1, 2*).

Example 1

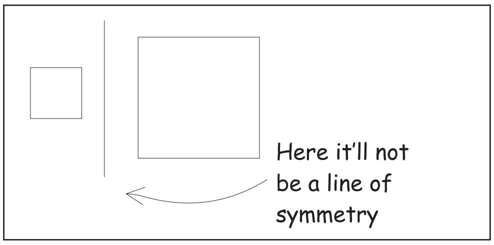


Example 2

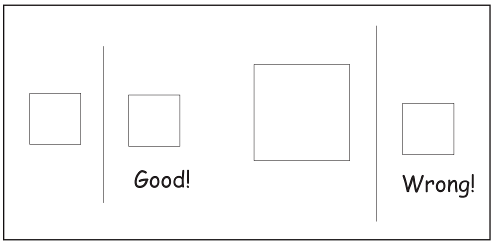


The question concerning squares had a different meaning for students. 2/3 of all students answered that the squares must be of the same sizes in order to fulfil the relation of mirror reflection. Among squares there are not figures with different shape. From that reason students did not mention about the same shape of figures. In this task younger students did the drawing as well. All the drawings were similar but they had different remarks (*Example 3, 4)*.

Example 3

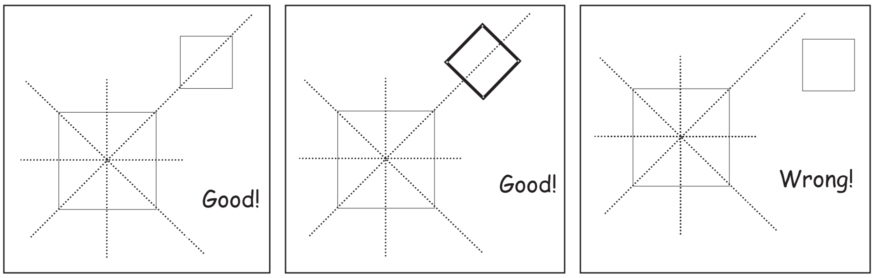


Example 4



In group of 12–13-year old kids, most of students (80%) answered that not each of squares on a plane is the mirror reflection of the other one. Among all remarks, the one stood out: "*I think that not, because if there is one and we shift the second one a bit irregularly that it will not be symmetry*". It was a different argumentation from the others such as: "Yes, because…", "No, because …". The lack of references to the same size of squares focuses our attention. Here "is one" square, so it will be whichever and freely placed. One added the other one, whichever size too. Adding the second square we can damage the symmetry, if "we shift the second one a bit irregularly". The question arises, how the child understand the essence of the task and what he/she expressed by this answer. In this case (maybe it has connection with school's experiences) the child thought about drawing of two squares on the plane and about the situations, in which that drawing has a line of symmetry. He/she focused attention on a specific placement of figures in mirror symmetry. A square has four lines of symmetry. Adding the second one we have to place it on the one of existing lines of symmetry in order to the whole drawing still has a line of symmetry.

Example 5



It is possible that the child have understood the question as follow: "does always the drawing of two squares present mirror reflection?" and gave the exact answer. At school she/he was checking different pictures – more or less complicated – if they have a line of symmetry. She/he had a lot of mental images of symmetric figures and concerned her/his attention of a placement's relationships.

In group of 16-years old a half of students answered that not each square could be place so they would be mirror reflection of each other, because "*squares would be of different sizes*". The second half of students answered that each square could be place so they would be mirror reflection of each other because "*all squares are the same*".

The arrangement of students' answers was very surprising. All of them focused their attention on a shape and size of figures. As a result of conversations with students took place after the test it turned out that in different ways they understood the question. It shows the dialog placed in appendix 2. At first understanding of the statement „does each square" meant for students participating in the conversation "the square and its reflection about freely placed line of symmetry. In this case there existed two squares. Having one square we always can get the second one as the image of the first one in the line symmetry. With regard for the special shape of square (it is the same from each side) it is possible to draw the line of symmetry in any place and in any direction. The figure after reflection always looks the same like the first one (has the same shape). Students clearly claimed that "each two" means the first square and that, which we received as its image in line symmetry. It is not important which of the squares is "the first". Always one of them is "its own image". Statement "is its own image" is used with reference to figures having line of symmetry. Here this statement was understanding in another way. This second figure is "its own reflection" in the same way as my reflection in the mirror is my reflection (not the reflection of other person).

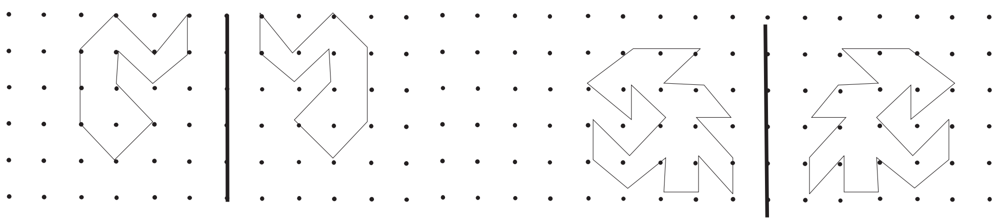
As a result of the conversation, drawing different squares – small and big – the meaning of "each two squares" was questionable. At first it meant two same figures (congruent). Bit by bit it extended to similar figures. One of students quickly understood the meaning of "each". He claimed that only one answer is correct and it is the answer "not each square". His college agreed that in the case of congruent squares we can answer "Yes" but in the case of different squares it must be the answer "No". She did not accept that these two particular answers give the general statement "no".

Summarize, the main reason of discrepancy between answers for the question "Is it possible to place each square on the plane so they would be mirror reflection of each other" was the language of the task. For 16-years old students the statement "each two squares" had different meaning either. It might concern any squares chosen from the whole family of squares on the plane. It might concern "any" squares that is "as I like to have".

Van Hiele emphasizes that each level of reasoning has its own language. Moving from one level to the other one manifests in a language. In that task the language referred to the level 3 or

4 and was different understanding by students because it might be a language from another level, which was inaccessible for students at that moment.

In the task no 3 children drew on dot paper any figure and its image in mirror reflection (an axis was not given). All the drawings were correct, even very complicated.



Among figures dominated squares (30%), figures with very complicated shapes (27%), triangles (20%). Therefore the shape of drawing figure does not indicate the level of concept reasoning. At a visual level a child is able to draw very complicated figures and their images if she/he make task on dot paper. She/he knows that she/he has to draw the same figure (keep the same shape and size) but "in the other side" (left – right figures). After drawing symmetric figures children explained why those figures are symmetric:

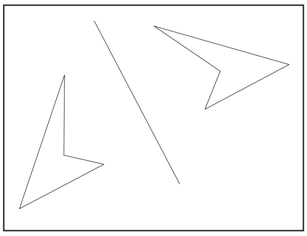
– Because if we put a mirror we will see that they are their own mirror reflection (30% of the whole answers)
– "they are the same" (40% of the whole answers), sometimes with remark „and between them there is an axis of symmetry" (but child did not draw an axis),
– "after folding a sheet of paper about the axis the figures overlaps (10% of the whole answers),
– "because every point in the figure has its own reflection on the other side" or "if we draw a line from vertex A it will touch A', similarly from B – B' etc." (10% of the whole answers), where student appeals intuitively to transforming vertexes.

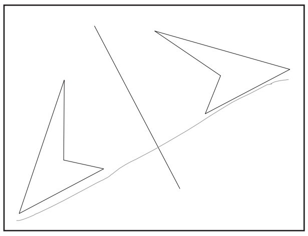
In one more task children were asked about justification how to check if two drawing figures were symmetric about an axis. Most students answered "with a mirror" – 2/3 of whole students. For younger children (12–13 years old) a mirror is a tool using during mathematics lessons. That is why those students know that this tool help to check a drawing, especially if the drawing is complicated. For these students line symmetry is closely associated with the mirror. They do not know the mathematical definition of this transformation. The older got to know this definition. In their justification they did not feel the need of referring to property from definition.

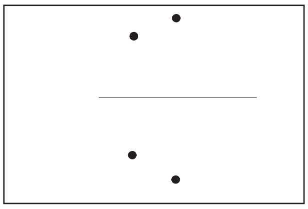
For both younger and older students the mirror still remains important tool. Among 15-years old 40% of them opined that it is need to use a mirror either, 20% that it is need to fold the sheet. The rest of them opined that it is need to measuring something: either the distance between figures or the distance between corresponding vertexes. It points at the students transformed the figure but not points (vertexes). It did not appear references to the point and its image like in the definition. Viener and Hershkowitz (1980) claim that in thinking, people do not use definitions of concepts, but rather concepts images, combinations of all the mental pictures and properties that have been associated with the concept.

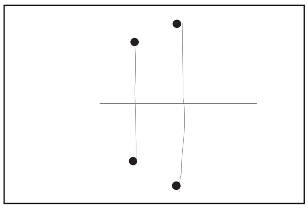
Various types of argumentation show that children function on different levels. However, contrary to expectation the older children (15-years old) did not function on the higher level.

Children recognized figures symmetric about a line very well. Occasional mistakes appeared when the axis was oblique, and figures did not have any side parallel or perpendicular to the axis. Justification why figures are symmetric about the drawing line children mentioned more property than shape and size, for example: “if one put a mirror, the figures would overlap”, “one can fold the sheet”. In a bit complicated situation for children (*picture 2, 3*) the deeper analysis and another argumentation appeared: “figures are placed equally from the line” or “figures begin on the same level (supported by a drawing of a segment connected to corresponding sides or vertexes. – *pic. 2a*). Children were not able to express by mathematical language relationship between a point and its image, although they intuitively felt it. They knew that figures must to have the same distance from the axis (what means the same distance between a point and its image from the axis) as well as figures begin on the same level (what means that point and its image lay on the same line perpendicular to the axis).


Picture 2


Picture 2a


Picture 3


Picture 3a

Similarly, for pic.3 children gave argumentation: “dots are in the same place from axis” or “connected points are in the same distance”.

I observed among all students justifications why figures on the drawing are / are not symmetric two types of argumentation:

I. “the line of symmetry is arranged correctly”

II. “figures are arranged correctly with respect to the axis”

According to the students opinion a part of drawings do not present symmetric figures about given axis for two reasons:

– “because figures were placed wrongly” – but not one figure placed wrongly (wrong image of a given figures about given axis)
– or “the axis of symmetry was placed wrongly”.

Explaining why figures on the drawing were not symmetric about given line children staked out: “the placement of the axis should be changed”, “the axis is placed wrongly”. Children justifying that the drawing is correct: “the axis is good placed”, “the line is perpendicular to the dots”. It was necessary to change the line of symmetry to correct the drawing and to change an image of a figure.

From the research of E. Swoboda (2006) results that in situation when children constructed axis-symmetrical mosaic the axis of symmetry existed in their mind though it was not drawn. Imaginary axis organised surface of a sheet. Children act in a different way when arranging figures on the plane and when analysing a placement of two figures on the plane. If they select figures self-dependently and compose them symmetrically on the sheet of paper, the axis exists in child’s mind and determines the placement of figures. On the other hand when they have two figures on a picture and they have to determine if figures are/ are not symmetric about drawing line a configuration of figures moves forward, to the first position, and dominates. It is possible only to change a position of an axis of symmetry. Mathematical definition is different: first a line is given and then we define the transformation. Farther activities we apply to existing axis of symmetry.

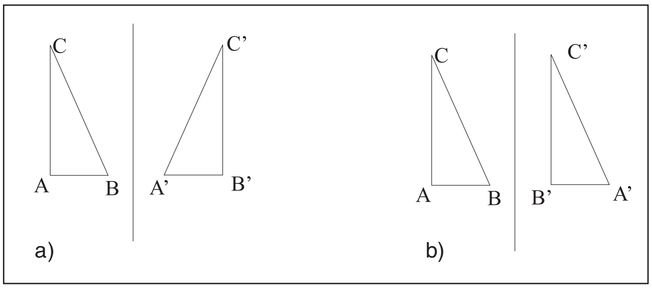
The key – processes characteristic to the van Hiele levels with the reference to the concept of line symmetry

Comparing my research to the Gutierrez and Jaime description of the key – processes characteristic to van Hiele levels I have made some remarks.

Recognition by the students at level 1 is limited to physical, global attributes of figures. They sometimes use geometric vocabulary, but such terms have a visual meaning more than a mathematical one. However students at level 2 or higher, are able to use and recognize mathematical properties of geometric concepts. For that reason the ability of recognition does not discriminate among students in the van Hiele levels 2, 3 or 4.

All the students recognized correctly symmetric figures. Both – the younger (12–13 years old) and the oldest (15 years old) used the geometrical language in a low degree. They used descriptions: "tip", "dot" instead of mathematical terms like vertex, point. They based on the visual assessment of properties but these properties had mathematical meaning for them. The same shape and size they referred to congruent figures.

When the students had two important visual information: one about a shape and the second about the vertexes, they have had difficulties in recognition which one of this information is more important (*picture 4*). Sometimes the information about the vertexes was stronger and they indicate the situation *4b* as correct.

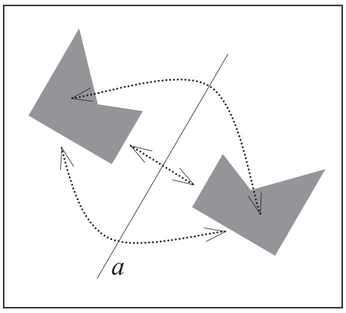


Picture 4

According to the next key – process, students at level 1 are not able to use a given mathematical **definitions**. The only definitions they can formulate consist of descriptions of physical attributes of the figure they are looking at and perhaps some basic mathematical property. The students mentioned generally the mathematical properties: the same / different shape and size.

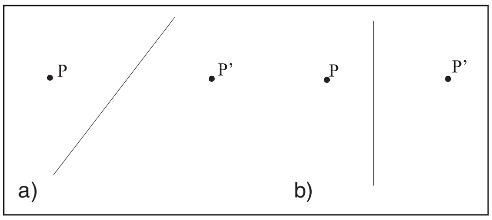
Gutierrez and Jaime discriminate to processes: "**read**" and "**state**" **definition**. Students participated in my research can both, "read" and "state" definition but they experienced difficulties with giving all important properties or only the necessary conditions.

While "state" definition, they referred to the basic mathematical properties: shape, size and distance. Their justification why the figures are/ are not symmetric about the given line referred to the shape and size of figures. It was very difficult to them to name any other property they have observed. They pointed at only one property to one given situation as: "the shape was not changed", "the size is the same", "the same distance between figure and a line", "it is mirror reflection", "if you put the mirror, the figures wholly overlaps". They did not write a complete list of properties. They mentioned sometimes that figures were "in different direction" (that means for them left-right figure). When drawing the image of the figure in mirror reflection they always put the ruler so as it was perpendicular to the line of symmetry and they did not verify if it is really a right angle. When they are asked why the figures are symmetric, they did not mention that property explicitly but they give the answer connected with this property. They stated that the figures "begin at the same place" or "at the same level" and they pointed at the line between appropriated sides or vertexes (the language of gesture was important and made their description easer).



Picture 5. First the student pointed at corresponding sides parallel to the line *a*, second – at sides perpendicular to the line *a* and then at the whole figures

When students at level 2 know every property contained in the definition, they can use it, but they may experience difficulties with understanding of the logical structure of definitions. When they are asked for a definition that has not been learned by rote, their answer may not include some necessary property that the students use implicitly. I observed that in the case of the known but not reminded definition the students also mentioned only one property: the distance from the line. They did not mention the other properties the definition includes: perpendicularity, point and its image, a distance between point and line of symmetry. When "read" definition the students focused very seldom on the necessity of lying of a point and its image on the line perpendicular to the given axis of symmetry. They felt this property intuitively drawing a line connected two corresponding vertexes. More frequently they pointed at corresponding sides (*pictures* 5).

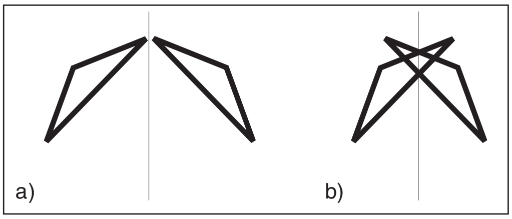


Picture 6

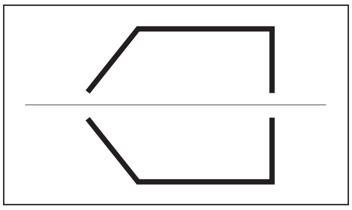
Easier for them was justification why the figures were not symmetric about drawing line then why they were symmetric. Students did not have difficulty in situation shown on pic.6b, were figures are not symmetric because they have different distance from the line. However in situation on pic.6a it was difficult for them to give a proper explanation (the figures do not lay on the line perpendicular to the axis of symmetry).

In "reading" definition students' attention focused on property concerning the distance of a point and its image from the axis. They were able to use it in the process of "state" definition. The second property – laying on the line perpendicular to the axis – functioned in the intuitive range.

Student at level 1 can understand only exclusive **classifications**, since they do not accept nor recognize any kind of logical relationships between classes nor, many times, among two elements of the same class having quite different physical appearance. When they are asked if the figures are symmetric about the given line they have accepted the situation on picture 7a, but they have had difficulties with estimate of the situation on picture 7b. In the case of *picture 8* they need to "close" a figure to the common shape.



Picture 7



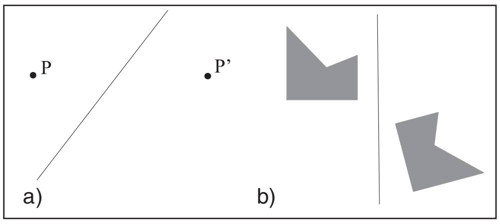
Picture 8

According to the concept of line symmetry or mirror reflection students in level 1 may accept only some of very typical situations: with horizontal or vertical line of symmetry or with figures having some of sides parallel or perpendicular to the line of symmetry. Otherwise they would "change" the position of line of symmetry in order to "fit" the position of figures on the plane.

A more accurate discrimination between students in level 2 or 3 is based on the ability to accept and identify non – equivalent definitions of the same concept and to change one's mind about the kind of classification, exclusive or inclusive, when the definitions are changed but in this research I did not discriminate among students in level 2 or 3.

Conclusion

The important conclusion from this research is the statement about the role of the line of symmetry which I have observed. For some children the figures were not symmetric about the given line because "the line was placed wrongly". One had to change the position of the line of symmetry in order to "repair" the given drawing. The students did not think about the figure and its image in this transformation. They consider pare of figures and a line between them. First the figures were placed and then the line of symmetry was added. Probably it may be the influence of the tools used at school during mathematics lessons.



Picture 9

Very common and useful for pupils was a mirror. It seems to me that the use of a mirror may get them accustomed to a pair of figures. There is a necessity to use at school different tools which emphasise the role of a line of symmetry and the relationship between the placement of an image of a given figure and the position of a line of symmetry (Jagoda 2005, 2007).

The research showed that regardless of the age difference between students they hardly used a mathematical terminology and they understood this language in a different manner. Also, regardless of the age difference they referred to basic mathematical properties (shape, size). The most difficult key thinking process for them was to "read and state" the definition.

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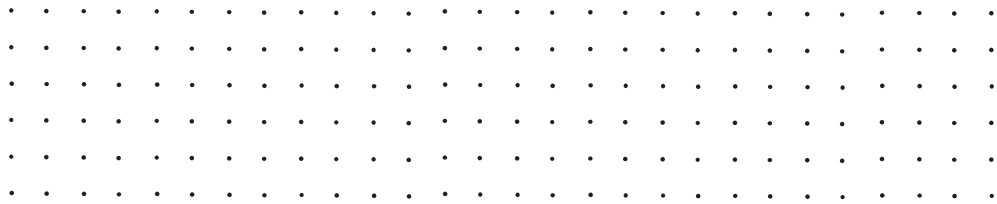
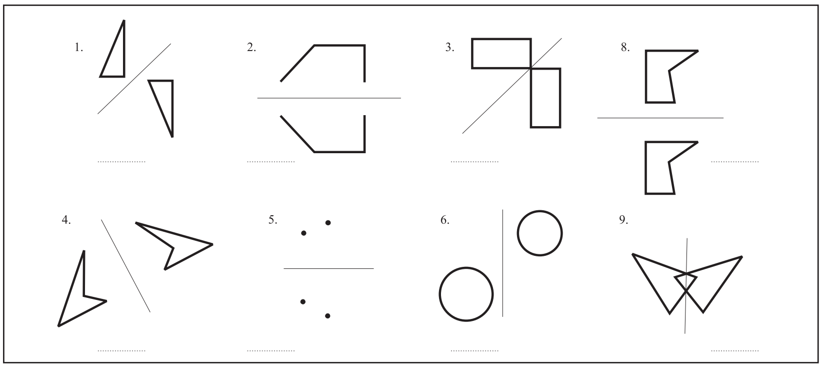
Appendix 1

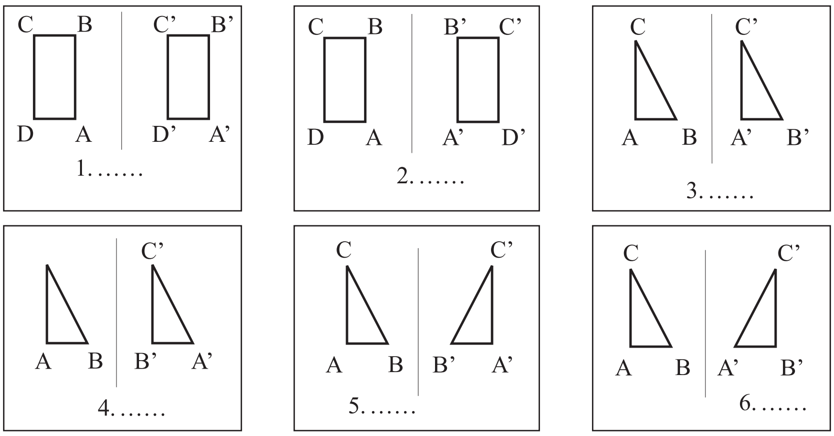
2. Is it possible to place any two squares on the plane so they would be mirror reflection of one another?
3. Draw any figure and a figure symmetric to the first one.



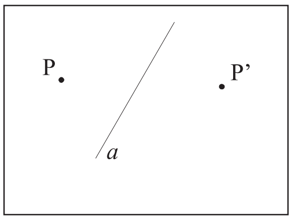
4. Explain why the figures you drew are symmetric?
5. Is it possible to place any two triangles on the plane so that they are mirror reflection of one another?
6. How can we verify if two figures are symmetric?
7. Write a proper letter under the drawing:
   S – if the figures are symmetric about the given line
   N – if the figures are not symmetric about the given line

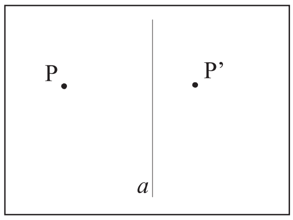


8. We adopt a convention: the image of a point X in the line of symmetry about any axis will be denoted by X'. Write a letter **S** if the figures in a picture are symmetric about a given line or a letter **N** if the figures are not symmetric about the given line.



9. Explain, why the points in the drawings below **are not symmetric** about the given line.





Appendix 2

T7: Task 2: *Is it possible for any two squares in the plane to be mirror reflections of one another?*

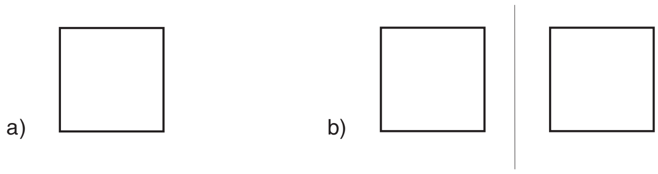
J8: Yes, because a square has equal sides, and all the angles have 90°.

K9: A square looks the same from each side. That is if we reflect it in any way, it would look the same. If it was a concave figure, then it would be a different matter. However a square is the same from each side.

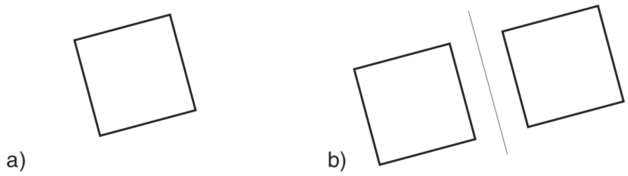
T10: What does it mean for you "any two"?

K11: The first and the reflected one.

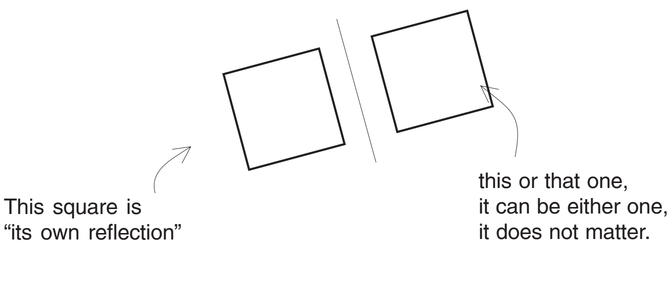
T12: So, if the first would be like this
(The teacher draws a square on a paper. Kasia draws an axis of symmetry and then the second square on the left side of the axis)

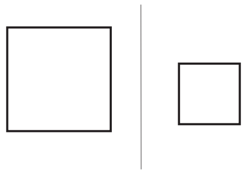


T13: What if you have this:
The teacher draws a square. Kasia draws an axis of symmetry and then the second square on the left side of the axis.



J14: It should be "any two squares". And then the answer should be "Yes", because a square has equal sides. *Kuba indetifies this in the picture:*





T15: What does it mean for you "any two"?
J16: The same two figures.
K17: But there would be small and big squares...
K18: It is not possible .., if one is larger than the other, then they won't reflect in the same way.
T19: What if the first is like this: *(The teacher draws a smaller square and Kasia draws a line of symmetry and the second square)*
J20: It is "reflection to scale". Does there exist something like that?
J21: The answer should be "Yes", provided both squares have the same size.
T22: Thus, "any two squares"...?
K23: Yes, but only if they have the same size. "No", if they have different sizes.
J24: Nevertheless "every" – but every means we take all squares into account, that is small ones and large ones.
J25: I think the answer should be "No". We have small and big squares, that is different squares. So, not all of them can be paired together.
K26: What about you? I think the answer should be "Yes" if the squares are the same and "No" if they are of different sizes. Here we do not have a specific set.
J27: We have – small and big squares. We talk about the whole family of squares – small and big ones. The answer will be "No". Definitely it should be "No".
…
T33: What about triangles?
K34: Not every triangle because there are different triangles.
T35: If the triangles are the same?
K36: So "Yes".
J37: No, because one has to be flipped. If it's not, it won't work.
K38: What?
T39: If they were cut–out triangles to be glued on the paper, like in the first task?
J39: If we paste them on the paper, one will be grey and the second white.
Kuba draws.

Example 10

a)
If there are triangles like these two, it fails.

b)
If you paste them on the paper, one will be gray and the second white.Well, it won't work.