THE ROLE OF TILING, CUTTING AND REARRANGING IN THE FORMATION OF THE CONCEPT OF AREA

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At the beginning of the teaching of the concept of area we focused on various activities. The tiling is emphasized particularly with congruent patterns. We make a distinction between tiling in plane and tiling polygons of a given area. Special attention has been paid to the development of the skill of estimation during the covering of the tiles, namely what kind of patterns are suitable for tiling the polygons on the one hand, and how many tiles from the selected patterns are needed to cover the polygons on the other. Another activity is cutting and rearranging polygons or tiles. These activities were carried out in classes 4, 5 and 6.

INTRODUCTION

In this work we want to show an experiment in grades 4, 5, 6, where we used the tiling to introduce the concept of area.

Research in this field of mathematical education often reveals poor understanding of the processes used for area measurement of plane figures. However, it is not only students who have difficulties in understanding the concepts of area and measurement; it is also student teachers (Simon, 1995; Baturo & Nason, 1996; Zacharos, 2006; Murphy, 2009). Some of our students have this kind of difficulties. Probably these difficulties are mainly attributed to the emphasis placed on the use of formulas, starting from the very first steps of introducing students to this subject. Though it is generally accepted that mathematics should be taught through understanding but in the topic of area it would seem that children often rely on the use of formulae with little understanding of the mathematical concepts involved (Dickson, Boyd & Davis, 1990). In our opinion the topic of measurement is very useful to develop problem solving, spatial sense, estimation, and concept of numbers.

THEORETICAL BACKGROUND

The area of a shape or object can be defined in everyday words as the ″amount of stuff″ needed to cover the shape. Common uses of the concept of area are finding the amount of tile needed to cover a floor, the amount of wallpaper needed to cover a wall etc. Measuring area is based on the notion of ′space filling′, i.e. tessellation. Tessellations can be very useful in education and in teaching mathematics. They can be used from kindergarten to high school. Since tessellations have patterns made from small sets of tiles they could be used for different counting activities. Two or more tiles usually make some other shape. Tiles can be used to teach students that area is a measure of covering. Tessellation patterns can produce many lines of symmetry. With young children you could have them learn about tessellations with pattern block pieces of geometric figures such the square, triangle, rectangle etc.

We applied these patterns to measure areas.

The starting point for the creation and development of geometry is the process of measurement, which presupposes comparison between quantities. Comparing different figures: Sometimes areas of different objects can be compared directly, without measurement. We can compare areas of two different objects that one of them divided into parts which, if appropriately recomposed, would form the other figure. We can compare areas of two different objects with tessellation, i.e. both of them are covered with congruent geometric figures (tiles) without gaps or overlaps. Activities related to each of three comparisons and evaluation of areas is carried out throughout the teaching process. At this point, either the logic of analysis and reconstruction or that of overlapping is introduced. In the case of overlapping, different shapes such as rectangles, triangles, squares, hexagon and trapeziums as well as irregular geometric shapes are used as measurement units. The teaching process provides an introduction to the concept of area and its measurement.

Measuring length can be realized by using a ruler, measurements of areas is more complex, since length is directly measured by a ruler, while area is indirectly measured through the lengths appearing in the formula for calculating it (Zacharos 2006; Murphy, 2009; Vighi & Marchetti 2011), but it can be made using other artefact, e.g. tiles which can be different geometrical figures. According to Santi & Sbaragli (2007) the use of ruler brings ′unavoidable misconceptions′, i.e. a misconception that ″does not depend directly on the teacher′s didactic transposition″. In early grades we assist to the blind application of formulae and according to our observations, the early use of formulas in area measurement has been criticized on the grounds that it generates misconceptions about area measurement.

In Hungarian schools comparison between areas is generally reduced to evaluating areas and to comparing numbers. Teachers tend to determine equivalence of the magnitude of area of two figures by means of measurement. But ″transferring the comparison to the numerical field, we are in fact working with numerical order which doesn’t consider the criterion of quantity of magnitude″ (Chamorro, 2001).

Estimation is very important in measuring area, too. Children can estimate which patterns are suitable for tiling the geometrical forms and how many tiles are need to cover the figure. Estimation is very important in real life for checking measurement and spatial ability.

To establish the concept of area tiling is a fairly important activity, but not the only one. When a polygon cannot be covered totally with a given pattern, the need to cut and rearrange the pattern into pieces obviously emerges. The aim of cutting and rearranging is to achieve the most appropriate covering of the polygon. This also happens in real life when the floor is tiled for instance. The two basic types of cutting and rearranging are the cutting of the pattern of and the cutting of the polygon. The first activity is closely related to the actual tiling, it is a more sophisticated version, whereas the latter one serves as a comparison of the areas of the polygons thus contributing to the preparation of the measurement of the area. In this way both activities contribute to the establishment of the concept of area.

RESEARCH QUESTIONS

1. What sort of activities based on tiling, cutting and rearranging contribute to the establishment of the stable concept of area?

2. To what extent are activities required in classes 5-6?

METHODOLOGY

A teaching experiment was carried out in classes 4, 5 and 6 of the demo primary school of the teacher training college. The teaching material and the teaching aids were compiled by the research team and the sessions were conducted by the class teacher in accordance with our guidelines. The tasks were done in groups of four, whereas setting the tasks and the discussion of the experience took place in the whole class. The sessions were recorded and photos were taken of the works produced and also the kids carrying out the tasks. Two 45 minute sessions were designed for all three years based mainly on the tiling activity, whereas another 45 minute session was planned for classes 5 and 6 relying on the activities of cutting and rearranging. Classes 5 and 6 had the same teacher and class 4 another one. Prior to the sessions none of the classes were involved with measurement of area. Learners in class 4 were not familiar with the concept of area at all, whereas learners in class 5 were introduced to the concepts of square and rectangle at the end of the previous term during some lessons. Learners in class 6 did tasks related to the area of rectangle and square two months before. They made use of the formulas of area and the SI measurement units of area were also introduced.

Session 1. Tiling the plane with various patterns. Tiling the rectangle with various patterns.

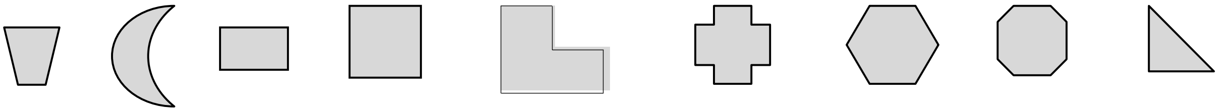
Session 2. Tiling various polygons with patterns selected appropriately and estimating the number of patterns required to cover the polygons.

Session 3. Cutting and rearranging patterns in order to cover a given polygon. Comparing the area of the two polygons by means of cutting and rearranging the polygons into each other.

DESCRIPTION OF THE SESSIONS AND RESULTS

Session 1

There were nine groups of four children. All the groups were given one kind of tiles from the set below.



Every group got several tiles of the same kind so that the rectangle could be covered.

Teacher: Try and cover as economically as you can the rectangle. Make use of the most of them but without overlapping.

The works were put on the board and the experience gained was discussed. We were wondering in what ways learners were able to put the coverings into groups.

Observations

Figure 1: The works on the board (Class 6)

In all three grades the sessions took place very much in the same way. Children were required the same amount of time to do the tasks and raising the problems and the interest in the topic was roughly similar.

Perceiving the difference in covering rectangle and plane: In class 4 learners did not perceive the concept of plane. They could not make a difference between coverings with gaps (such as octagon) and coverings without gaps and coverings as well as coverings deficient on the margins (for instance hexagon). In classes 5 and 6 the plane was illustrated by the tabletop and the geographical notion of the steppe (e.g. Hortobágy in Hungary). Children put the nine coverings into three groups according to the arrangement of the tiles: The plane cannot be covered with them without gaps (octagon, crescent) (set 1). The plane could be covered but not the rectangle (hexagon, trapezium, cross, L-shape) (set 2). Both the plane and the rectangle can be covered (rectangle, right angle triangle, square) (set 3).

In all three classes the demand for cutting emerged. During the discussions the idea came up in that case when the paper rectangles could not be covered:

S1: If one of the them could have been cut then it would have been put …(the trapezium on the L-shape) (Class 4)

S2: In set 2 the margins should be cut. (by margin they meant the uncovered parts) (Class 6)

Children have made an effort to create regular patterns of tiles.

S3: We created squares from triangles, at first we did it at random, but it was not really appropriate. (Class 4)

There were some hints at the types and the size of the tiles, as well as the connection between size and the possibility of covering:

S4: It was possible to make square from rectangles and triangles. (Class 4)

S5: It is impossible to cover, because the elements are too big. (hexagon, trapezium, octagon) (Class 4, 5)

Teacher: How many tiles do you need for set 3?

S6: The larger the plane figure, the fewer are needed. (the plane figure was meant to be a tile by the child) (Class 6)

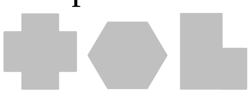
Teacher: What kind of tile would you like to plan?

S7: The trapezium would cover it, but it is ugly.

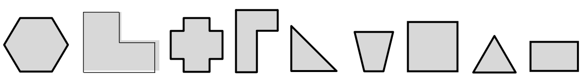
S8: The rectangle is rather thin that would reach as far as it is. (Class 6)

Session 2

We put three large polygons cut from large sheets of papers on the board:



At the bottom of the board the following patterns of tiles were seen:



Teacher: Which plane figures do you think could cover the shapes cut out of sheets of paper without gaps?

Children answer some plane figures, and then the teacher asks who agrees with the answer. The votes are counted.

Then children were put into nine groups of four, and they were given 3-3-3 large polygons together with a bag of tiles of the same pattern. Children were told to tile the large polygon with the tiles they were given. In case the large sheet could not be covered without gaps, then fewer tiles should be used so that they would not be overlapping. The idea was to make use of as many tiles as possible so that the large sheet could be covered totally. The nine tiling patterns children produced were put on the board.

Observations

It is shown in the Table 1 which tiles were selected by most of the children to tile some of the polygons.

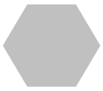
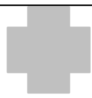
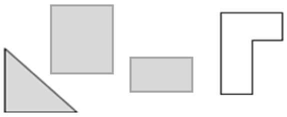
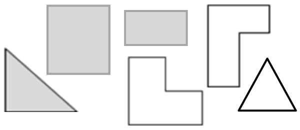
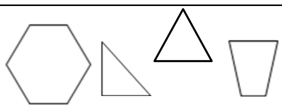
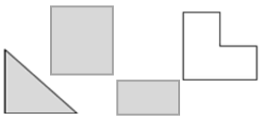
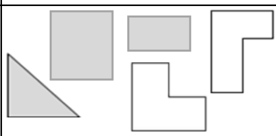
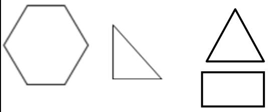
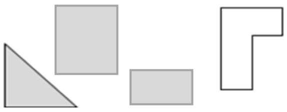
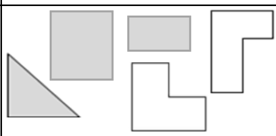
			
Class 4			
Class 5			
Class 6			

Table 1: Selected tiles

Right angle triangle was represented in every case.

Right angle triangle, square and rectangle can be found together except for one case; presumably they have recognised the relationship between them. A square can be covered by two rectangles or two right angle triangles.

S9: If the rectangle is OK, then so is the square, because two rectangles cover exactly as much as a square and we could have counted by two. (Class 4)

S10: I insist on the triangle, because if it can be covered with the square, then with this one too. With 24, because we said 12 squares” (Class 6)

S11: It seems that similar polygons can be covered with similar ones, hexagons with hexagons, L-shape with L-shape, irrespective of its shape. (Class 6)

To select the right pattern of tile for the given polygon is not that easy (Figure 2).

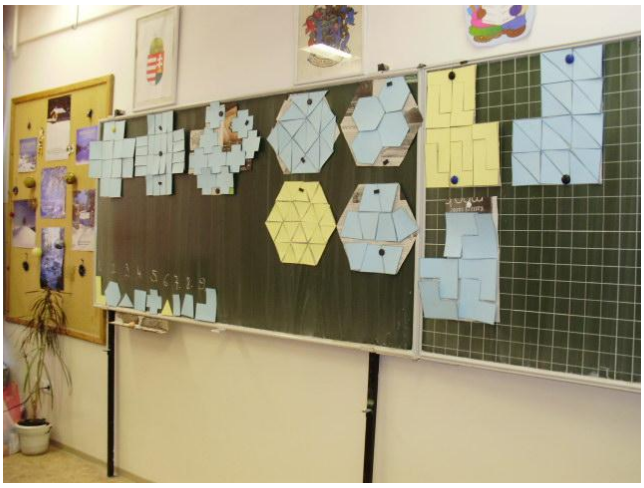


Figure 2: The works on the board (Class 6)

Covering similar shapes with similar ones does not always work.

S12: I was disappointed with the hexagon, because when I figured out my idea, I did not realize that it will be wider in the middle. (class 6)

S13: L-shape can be covered with the yellow L-shape, but not with the blue one. (He has another try with the blue one.) It can be done with another type of covering, no, it does not work...” (Class 6)

The difference in the size of the angles can be seen for them when they fit the tile to some of the vertexes of the polygon.

S14: I thought hexagon can be covered with right angle triangle, it can be done. But no, it cannot be done, because of the angles... (Class 6)

S15: I meant the trapezium in a slanting direction. The trouble is with the angle here as well.(Class 5)

Out of the two triangles and two L-shape polygons only one of them is appropriate for covering hexagon and L-shape respectively. Thus the fact that the names are identical, it is still not enough.

S16: We tried to turn around the right angle triangle to the hexagon at random, but it did not work. It cannot be done with this triangle, but it can be done with the other one, because its shape is different. (Class 5)

S17: The yellow L-shape could have been OK, one of its branches is not thick. (Class 5)

When estimating the tiles for covering children are able to make use of the relationships they have discovered between areas most of the time, but the position of the tile also matters.

The estimation of the number of tiles required for covering is useful and children are keen on it.

Session 3

To recall the previous lesson by making use of the photos of the coverings. For instance it can be seen what kind of tiles were used when they tried to cover the hexagon, and how many of them were needed.

> Teacher: How many hexagons or right angle triangles will be needed for covering if the tiles are cut into pieces?

During this session children were put into groups of four.

After the first task, children were asked to compare the area of two polygons of large sheets of paper by means of cutting and rearranging. This time it was the tiles but the polygons that had to be cut and rearranged.

Observations

Children were happy to do the tasks and they were also delighted to have access to the scissors. They made an effort to cover the large sheets of paper economically, however as the teacher did not point out that they should cut the tiles only it is required. Thus they did not really paid attention to how many of the original tiles they cut into pieces. When finally the pieces pasted to the sheets were counted the results were rather various such as 19, 23, 20, 15 pieces right angle triangles to cover the hexagon. Of course it was not really conducive to the establishment of the concept of the area measurement unit, but it contributed to the covering the polygons without gaps and overlapping.

Two strategies could be observed: in the first case they aimed at systematic arrangement, whereas in the other case they tried and made use of every little pieces of cutting (Figure 3)

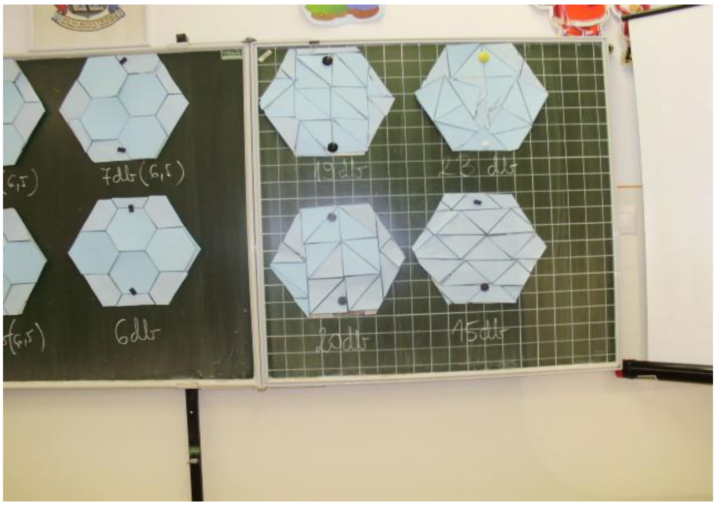


Figure 3: Hexagons on the board (Class 6)

This first task meant to make a connection between tiling and cutting and rearranging, although cutting the tiles into pieces not necessarily leads to the concept of the area measurement unit, still in everyday life cutting the tiles into pieces is quite often used in tiling the bathroom for instance.

Children of Class 6 applied three different strategies in the second task, in the comparison of the hexagon and the L-shape, the hexagon was cut into L-shape (Figure 4), the L-shape was cut into hexagon, and they tried to turn both polygons into rectangle (Figure 5)

Figure 4: Hexagon into L-shape

Figure 5: Both polygons into rectangle

In Class 5 children did not manage to turn polygons into rectangle.

Cutting and rearranging one polygon into another seems to be useful to compare their areas.

Rectangle has emerged as a kind of transmission polygon for the comparison of areas.

The task has actually focused on the main thing in comparison: if one of the polygons can be cut and rearranged in a way that it covers the other one, then the two areas are equal, but if there is gap, the areas are not equal.

CONCLUSIONS

The two approaches to cover the subject matter, the frontal classroom and group work were appropriate.

After the first two sessions both the children and the teachers came up with the idea to continue the experiment. Cutting seemed to be a useful way to solve the problems, however we realized that in class 4 more activities of tiling are required prior to cutting and rearranging.

Activities are enjoyable in class 5 and 6 and they are not boring at all. Learners were highly interested and creative.

In Class 6 during the activities learners did not rely on their knowledge about rectangles and squares gained earlier. For instance they did not want to tile the rectangle only with squares, or to use the number of squares for the estimation. For them the tasks of cutting and rearranging were as much as novelty as for younger learners.

The teachers, who were involved in the experiment, came to realize the complexity of the problem and also the benefits of the extended elaboration of the topic.

REFERENCES

Baturo, A., & Nason, G.: 1996, Student teachers' subject matter knowledge within the domain of area measurement, *Educational Studies in Mathematics*, **31**, 235-268.

Chamorro, M.C.: 2001/2002, Le difficoltá nell'insegnamento-apprendimento delle grandezze nella scuola, *La Matematica e la sua Didattica*, Parte I, 2001, 4, 332-351, Parte II, 2002, 1, 58-77.

Dickson, L.: 1989, Area of a Rectangle, in: D. Hohnson (Ed.), *Children's Mathematical frameworks 8-13: A Study of Classroom Teaching*, Windsor, Berkshire: NFER-Nelson, pp. 89-125.

Freudenthal, H.: 1983, *Didactical phenomenology of mathematical structures*, D. Reidel Publishing Company, Holland.

Murphy, C.: 2010, The role of subject knowledge in primary student teachers' approaches to teaching the topic of area, *Proceedings Cerme 6*, 1821-1830.

Nitabach, E. & Lehrer, R.: 1996, Developing Spatial Sense through Area Measurement Relational Learning, *Journal for Research in Mathematics Education*, **18**, 653-667.

Santi, G., & Sbaragli, S.: 2007, Semiotic representations, "avoidable" and "unavoidable" misconceptions, *La matematica e la sua didattica*, **1**, 105-110.

Zacharos, K.: 2006, Prevailing Educational Practices for Area Measurement and Students' Failure in Measuring Areas, *Journal of Mathematical Behaviour*, **25**, 224-239.