How can high school students solve problems based on the concept of area measurement?

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Abstract

Mathematical activity like problem solving involves work with concepts. Referring to the work of Usiskin, understanding of a concept in mathematics has different aspects. These aspects are skill-algorithm understanding, property-proof understanding, use-application understanding, representation-metaphor understanding and history-cultural understanding.

The aim of this study is to investigate how these aspects appear in problems based on the concept area measurement, and to look for the reasons of unsuccessful problem solving connected to this topic. The research reports the typical solving strategies and thinking mistakes concerning 5 tasks developed for high school students of class 9.

Introduction

The term mathematical understanding includes understanding both of concepts and of problems.

Mathematics as an activity "consists of concepts and problems or questions: mathematicians employ and invent concepts to answer questions and problems; mathematicians pose questions and problems to delineate concepts." (Usiskin, 2012)

We investigate the interaction between concept-formation and problem-solving. Problem-solving requires preliminary knowledge and understanding of concepts connected to the problem: „Do you understand all the words used in stating the problem?" (Polya, 1957) On the other hand the level of understanding of a concept is recognizable through solving problems connected to this concept. The analysis of the process of problem-solving contributes to the exploration of the concept deficiencies.

The aim of this study is to answer the question how the different aspects of understanding the same object namely the concept of area measurement appear in problems, and to look for the reason of unsuccessful problem solving connected to this topic.

Theoretical background

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Skemp (1976) distinguish two meanings of the world “understanding”. The relational understanding means knowing what to do and why, and the instrumental understanding means knowing and using rules. Usiskin (2012) views this meanings as different aspects or dimension of understanding the same subject furthermore he speaks about more than two aspects.

The dimensions of understanding according to Usiskin:

1. Skill-algorithm dimension (instrumental or procedural understanding): Knowing how to get an answer. Obtaining the correct answer in an efficient manner.
2. Property-proof dimension: Knowing why your way of obtaining the answer worked.
3. Use-application dimension (modeling): Knowing when to do something.
4. Representation-metaphor dimension: Knowing represent the concept in some way (with concrete object, picture or metaphor)
5. History-culture dimension (genetic approach, ethno mathematics): Knowing the history of the concept and its treatment in different cultures.

„The dimensions of understanding are relatively independent in the sense that they can be, and are often, learned in isolation from each other, and no particular dimension need precede any of the others…..Ordering ideas or concepts in terms of difficulty is only appropriate if these items are in the same dimension.” (Usiskin, 2012)

In my paper I apply Usiskin’s multidimensional view of understanding, because it helps to clarify the meaning of concepts and broadens options for developing these concepts.

After studying his examples for clarification the meaning of these dimensions, I delineate the first four dimensions of understanding the concept of area measurement.

1. Skill-algorithm understanding is choosing an appropriate algorithm to calculate the area depending on the plane figure and the given sizes.
2. Property-proof understanding includes derivations of the basic formulas for the areas of triangles and other polygons, relations between area and perimeter of the same figure etc.
3. Use-application understanding includes area measurement in everyday life, applications in complex problems etc.
4. Representation-metaphor understanding includes area measurement with congruent tiles, cutting and rearranging polygons, area representation with an array of dots etc.

If we would like to know the level of learners understanding concerning to a concrete concept, it is necessary to pose them problems connected to this concept. In many cases the

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simple routine tasks can't highlight the gaps and misunderstandings well. So the process of problem solving serves for detection of typical mistakes of the concept-formation too.

In his famous work *How to solve it?* Polya write down the main four steps of problem solving: understanding the problem; devising a plan; carrying out the plan; looking back. If somebody want to devise a plan, it is necessary – after understanding the problem – to activate his or her preliminary knowledge. The main point of the plan is to find the connection between the problem and the appropriate elements of the preliminary knowledge.

"A well-stocked and well-organized store of knowledge is an important asset of the problem solver…In any subject matter there are some key-facts which should be stored somehow in the forefront of your memory. When you are starting a problem, you should have some key facts around you close at hand, just as an expert workman lays out his most frequently used tools around him when he starts working." (Polya, 1962)

Making and carrying out a good plan also means to find the connections between the unknown and the givens. Polya use a graphical representation to demonstrate these connections, where the unknown (what we are looking for); the data (what we have); the auxiliary unknown (what we are looking for to solve an appropriate related problem) are symbolized as points, and the relations connecting the objects are indicated by lines (*Figure 1*). If we can't make direct relation between the unknown and data (we can't solve the proposed problem), we introduce auxiliary unknowns and join these to the original one and so indicate the relation between the quantities. The aim is to establish direct or indirect connection between the unknown and the data through some auxiliary unknowns. The graphical representation seems to be a suitable tool for investigating learners' different solving strategies.

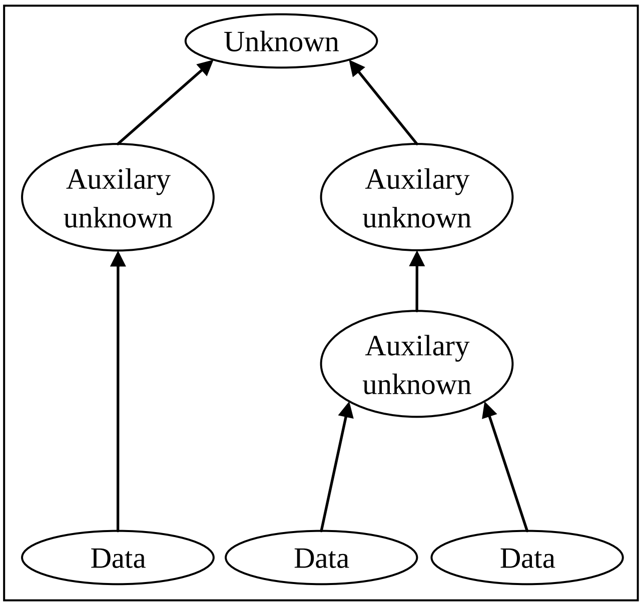

Figure 1

There are many studies on solving problems with focus on area measurement. The researchers agree that measurement of area is more complex than measurement of length since length is directly measured by ruler while area is indirectly measured through the lengths appearing in the formula for calculating it (Zacharos 2006, Murphy 2009). According to Santi&Sbaragli (2007) the early use of formulas has been criticized on the grounds that it generates misconceptions about area measurement and it could bring children to confuse area and perimeter. Another possible cause of the conflict perimeter vs. area that the intuitive rule “more A, more B” (Stavy&Tirosh 1996) means in case of geometrical shapes that as the perimeter increases so the area increase. Kospentaris at al. (2011) investigates students’ strategies in area conservation geometrical tasks and highlights the role of visual estimation.

Research questions

In this research I investigate the understanding of the concept of area measurement through the solution of five associated problems.

1. Which dimensions of understanding the problems refer to? How perform students in class 9 regarding to these dimensions?
2. What kinds of methods use the students if they haven’t got the preliminary knowledge necessary to solve the problem?

Methodology

The 27 students participating in this study are 9th graders in a Hungarian high school, in the same class. The class is a special „language-class”, they have mainly English lessons in the whole school year and only two mathematics lessons per week. The goal of this year is to preserve their previous mathematical knowledge. The students involved in this study had varied backgrounds in mathematics. In the previous school year they learned in different upper secondary schools. Students don’t show up particular motivation, talent and interest in mathematics.

A 45-minute written test was designed in a way that the 5 tasks are built on each other, because we wanted to help students to find a right idea for the solution. For example Task 1 can be understood as an auxiliary problem of Task 2. The solution of the Task 1 may become the part of the solution of the Task 2 and it may suggest the direction in which students should start working. It was considered to be also important to have a real-size picture to every task.

The test was written in March of 2013, earlier in the high school nothing has been taught from the topic of area measurement.

Analysis of students answers

In connection with the first research question I present and identify the area-tasks with dimensions of understanding, then discuss on the students’ responses.

Task 1

Determine the area of the following plane figures!

Figure 2

The dimensions of understanding:

Task 1a requires only skill-algorithm understanding in a very simple way.

Task 1b requires three dimensions: representation-metaphor understanding, since the area means covering with tiles; use-application understanding, namely using the formula for the area of the rectangle and skill-algorithm understanding that is breaking down the irregular polygon into pieces that are easier to manage, then add or subtract the areas of the pieces.

Table 1a and *Table 1b* show the distributions of the answers of students:

A=a·a	$A=a^2$	A=a·b	A=4·4	A=4·a	A=a·b·2	T=4·4·a	----
8	7	5	3	1	1	1	1
correct: 23				wrong: 4			

Table 1a

Remark:

26 students used some formula to calculate the area of the square, 23 of them were correct. 1 student couldn't answer.

addition of areas of pieces	subtraction of areas of 2 squares	A=12	A=a·b·2	A=a·b	multiplication of sides	perimeter	A=48 or A=16	----
9	5	1	1	1	1	1	2	6
correct: 15			wrong: 12					

Table 1b

Remarks:

From the 15 correct answers 9 students added and 5 subtracted the areas of known pieces i.e. squares or rectangles and 1 student answered well without reasoning. Between the wrong calculations appeared the multiplication or addition of all the sides (1-1 students).

Task 2

Determine the area of the colored shape, if the length of the square side is 6 cm!

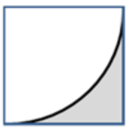

Figure 3

Task 1b is an auxiliary problem of this task especially if somebody solves it applying subtraction.

The dimensions of understanding:

Representation-metaphor us.: the same as in *Task 1b*.

Use-application us.: using the formula for the area of the square and the circle

Skill-algorithm us.: the same as in *Task 1b.*

The expected steps of solution: 1. calculation of the area of the square (A_S); 2. calculation of the area of a quarter of a circle (A_{QC}); 3. the area of the colored shape (A) is A_S-A_{QC}.

Table 2 shows the number of students related to the steps of solution:

subtraction $A=A_S-A_{QC}$	$A_S=a^2$	$A_{QC}=A_C/4$	$A_C=a^2\pi$	$A\approx A_S/4$, $A\approx A_S/3$, $A\approx A_S/2$	-----

3	13	1	0	**7**	12

Table 2

Remarks:

Nobody gave correct answer. 13 students determined the area of the square, but only 3 students indicated the subtraction of the areas. 1 student recognized that the colored shape is a quarter of a circle (*Figure 4*); however he didn't remember the formula of the area of a circle (A_C). There were 7 students who used visual estimation to determine the area A. 12 student didn't deal with this task.

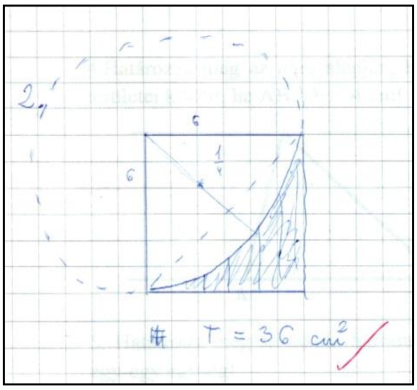

Figure 4

Task 3

Calculate the area of the attached polygon! (Numbers are in cm.)

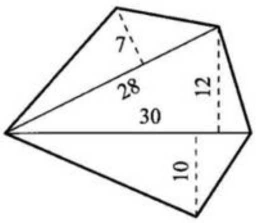

Figure 5

The dimensions of understanding:

Skill-algorithm us.: calculating the area of an irregular polygon (cutting into triangles, calculating the areas, then adding them).

Use-application us.: using the formula for the area of the triangle.

Representation-metaphor us.: recognizing the necessary data (side and the corresponding height) to use the formula.

Table 3 shows the performance of the students.

correct answer	see 3 triangles on the picture	determine sides of pentagon	add all the data	-----
5 students	3 students	3 students	1 student	15

Table 3

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Remarks:

15 students (more then the half of them) couldn't work on this task. There were only 5 correct answer, and further 3 students saw the 3 triangles as useful parts of the irregular pentagon. 3 students determined or tryed to determine the sides of the pentagon, for example *Levente*, who calculated the lenghts of the missing sides with a „formula" $ac=b^2$ (*Figure 6*).

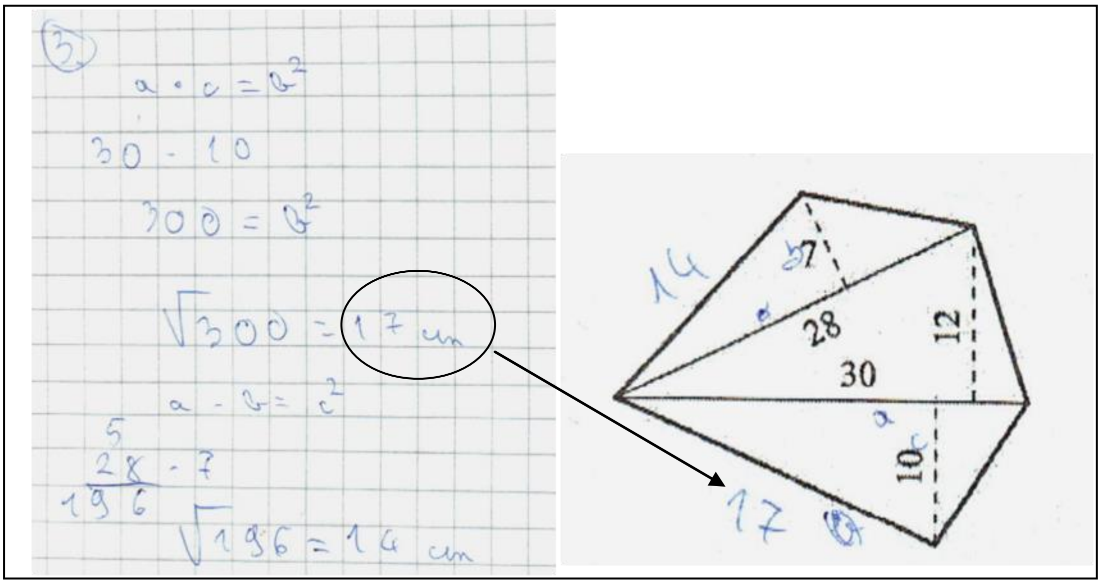

Figure 6

Task 4

Determine the relation between the areas of triangles ABC and DEF if AB=DE=4 cm!

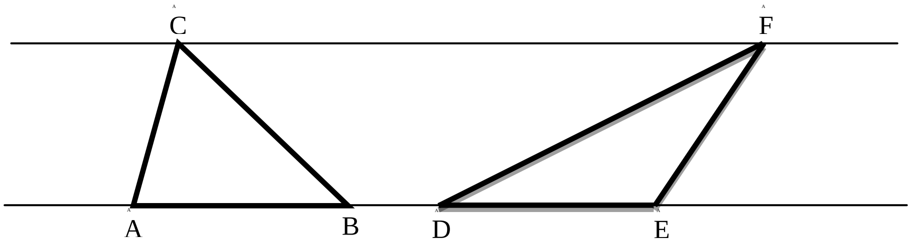

Figure 7

The dimensions of understanding:

Property-proof us.: What is the formula for triangles and why? What are the conditions of area conservation?

Representation-metaphor us.: pictorial understanding of the concepts of area, recognizing base and corresponding height in the picture. In this case cutting and rearranging one triangle into other is not easy.

The skill-algorithm understanding doesn't work, because we know one side exactly, but the height not.

Table 4 shows the performance of our students.

$A_{ABC\Delta}=A_{DEF\Delta}$ (without reasoning)	$A_{ABC\Delta}<A_{DEF\Delta}$ (without reasoning)	determined the length of sides	compared triangles to parallelograms	---
4 students	8 students	10 students	1 student	13

Table 4

Remarks

4 students answered that the areas are equal ($A_{ABC\Delta}=A_{DEF\Delta}$) but nobody gave correct reasoning. Only one solution (*Figure 8*) indicated the property-proof understanding: *Evelin* completed triangles into parallelograms however the statement that the areas are equal was missing.

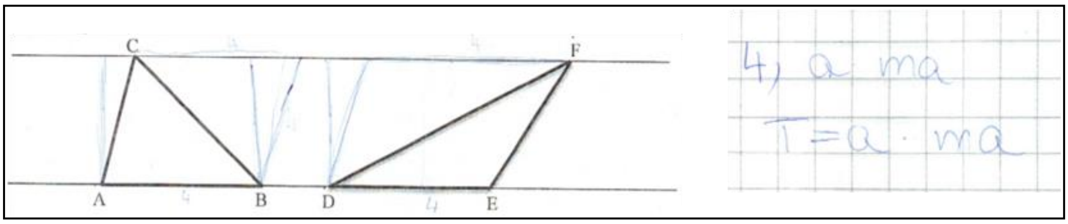

Figure 8

8 students noticed that the area of the triangle *DEF* is larger than the area of the triangle *ABC* ($A_{ABC\Delta}<A_{DEF\Delta}$). 10 students tried to calculate the length of the 3 sides to determine the areas of triangles.

This task is a typical area invariance problem concerning triangles with the same base and equal heights between parallel lines (Kospentaris at all 2011). Kospentaris, Spyrou and Lappas investigated the solutions of 12[th] graders. 47.3% of the student gave correct answer, but only 8% of them wrote a correct argument. Among the wrong answers 77% noticed that triangle with obtuse vertex has larger area. The explanations, if there were any, corresponded to visual estimations or false reasoning, like “smaller sides, smaller area”.

Task 5

Determine the area of the triangle and quadrilateral, if the distance of two adjacent grid-points is 1 unit!

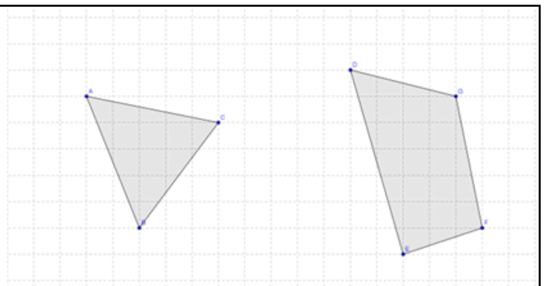

Figure 9

The dimensions of understanding:

Skill-algorithm us.: calculating area with the help of grid; completing the polygon into rectangle then subtracting the calculated area of the right triangles and square.
Representation-metaphor us: the area means covering tiles (squares).
Use-application us.: Using the formula of the area of right triangle and rectangle.
Table 5 shows the performance of the students.

completed into rectangle	determined the number of units (squares)	determined the length of sides	----
5 students	4 students	12 students	8 students

Table 5

Remarks:

Task 1b and *2* are related to this task, but here the additional rectangles aren't drawn, only the grid.

Nobody solved the task correctly. 4 students counted the number of units approximately, so they gave a quite good estimation for the area of the polygons. 5 students drawn the smallest rectangle which enclosed the triangle or quadrilateral, but they didn't continue the work, and didn't determine the area of the complemented triangles. 12 students tried to determine the lengthy of sides again, and gave "formula" to calculate the area from the sides. For example *Anna* thought that the area of every polygon is calculated by multiplication of its sides (*Figure 10*).

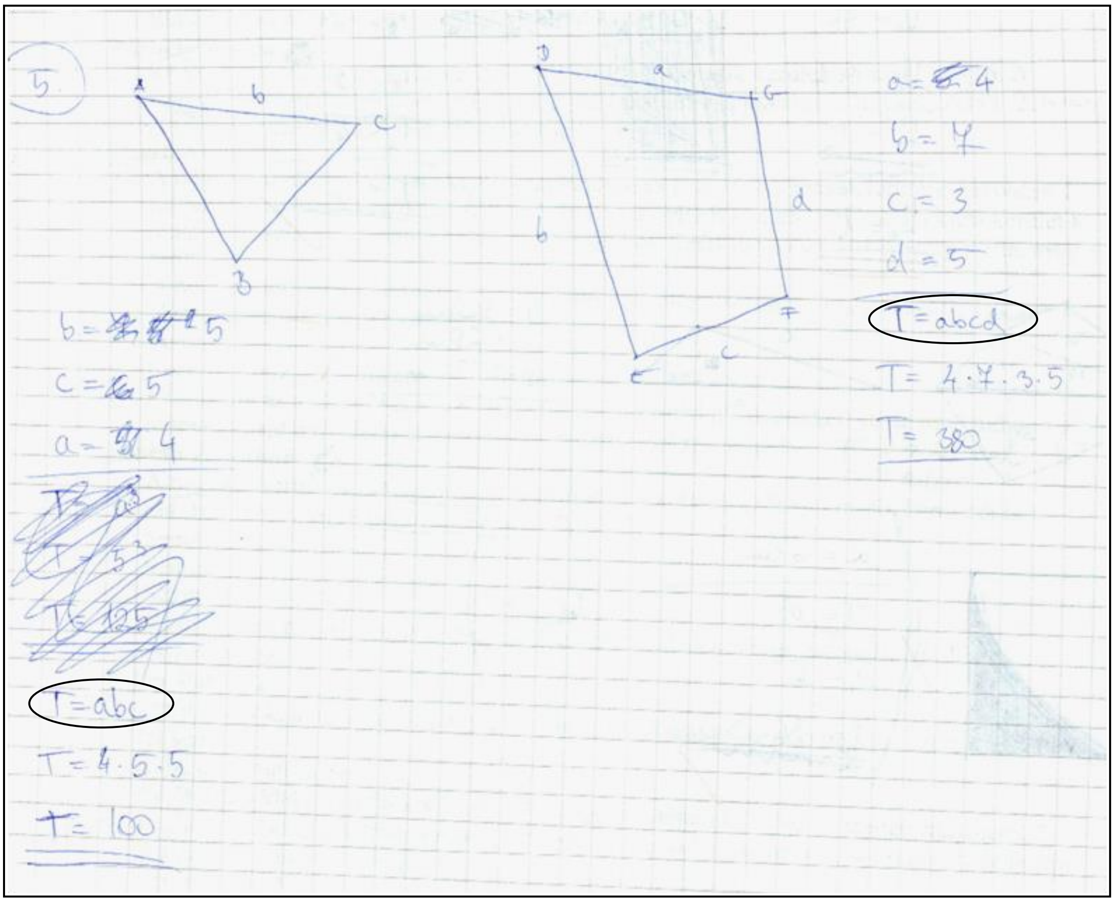

Figure 10

Solving strategies if the preliminary knowledge is missing

"If you are familiar with the domain to which your problem belongs, you know its "key facts", the facts you had most opportunity to use." (Polya, 1962) However the appropriate elements of our formerly acquired knowledge are missing, we have to use alternative solving methods. I adopt Polya's diagram, a graphical tool to describe students solving strategies.

I give two examples to present two different solving strategies in both cases.

In *Task 2* we have the problem to find the area of the colored figure. The area of an irregular shape can be computed as the difference of areas of shapes we are more familiar with. In this task we have to subtract the area of a quarter of a circle (A_{QC}) from the area of the square (A_S) if the side of the square (a) is given. We can see in *Dominika*'s solution, that the relation between a and A_{QC} is missing, so at first she gave up the work (*Figure 11*).

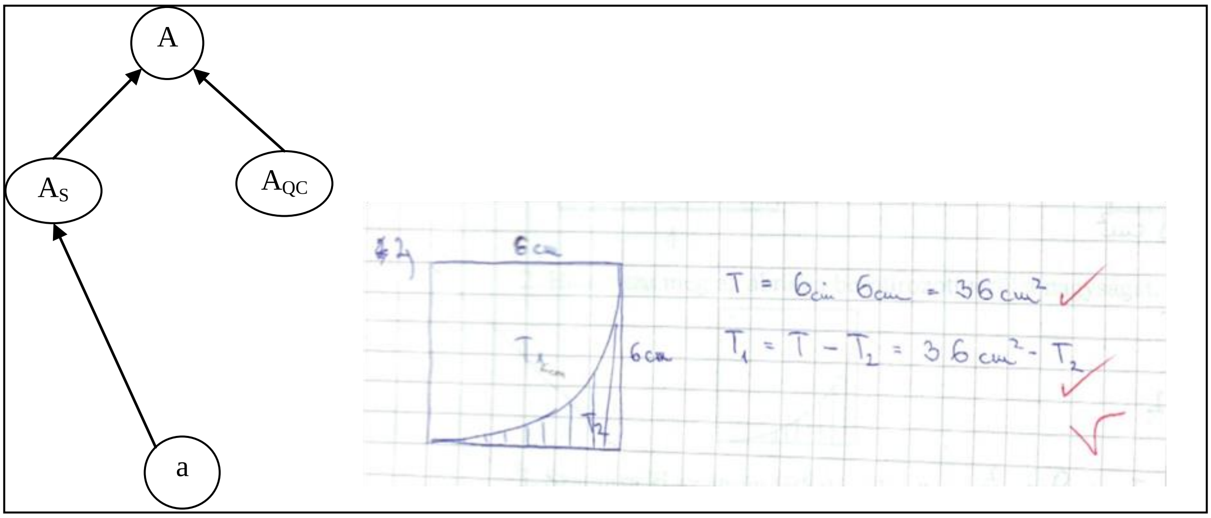

Figure 11

Dominika tried to solve the problem once again: Instead of the missing connection between *a* and A_{QC} she used **visual estimation** (*Figure 12*). She recognized that the area of the triangle *DBC* is the quarter of the area of the square ($A_{DBC}=9\ cm^2$). She assumed that the unknown area is the double of the half of this area, so $A=9\ cm^2$. It's an acceptable approximation for *A*.

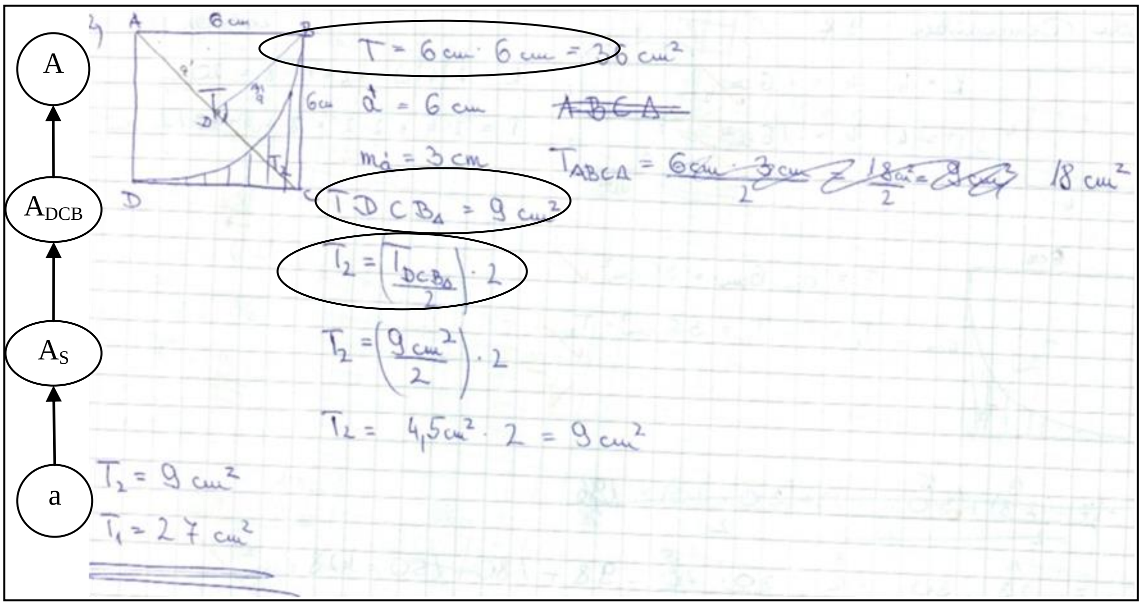

Figure 12

In *Task 4* we have to recognize the equality of the areas of the two triangles. The "key facts" of the solution is to realize that the parallel lines make sure the equality of the heights: $d=h_1=h_2$. *Dominika* couldn't use this visual data, so there are missing lines on the diagram describing her solving strategy. (*Figure 14*)

She tried to overlap the triangles and ***assumed special properties*** incorrectly instead: *ABC* is equilateral triangle (1); *DEF* is isosceles triangle (2). She interpreted the concept of „base and height” not a flexible way, because she wanted to calculate the height starting from the obtuse-vertex (*Figure 13*).

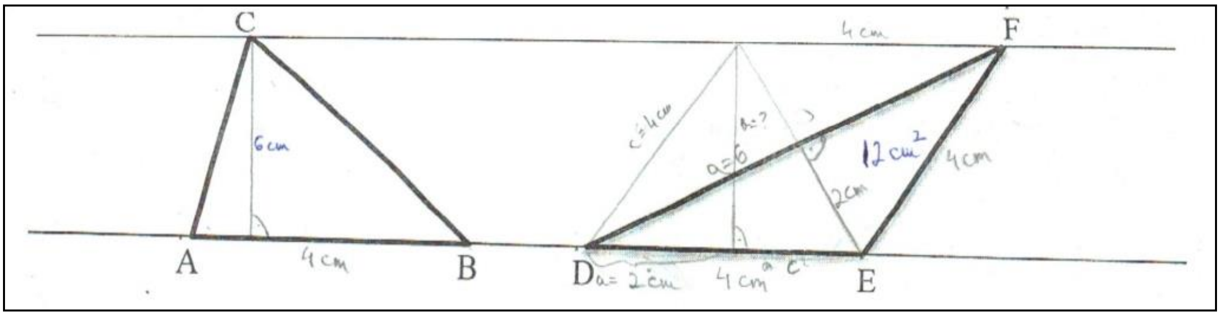

Figure 13

Following *Dominika*’s solution we can see that the diagram is much more complicated, because three new points (*1*, *2*, *DF*) and seven new lines appear (*Figure 15*).

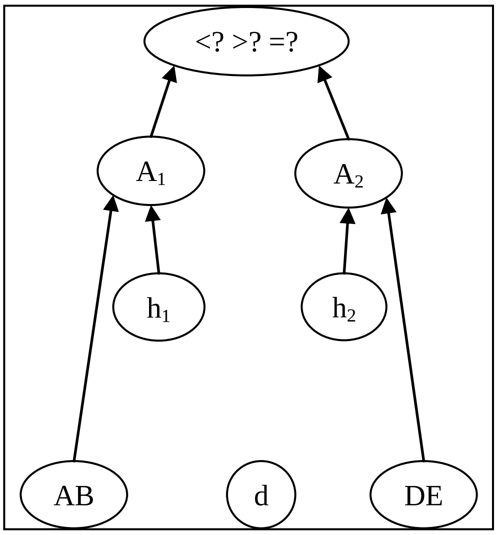

Figure 14

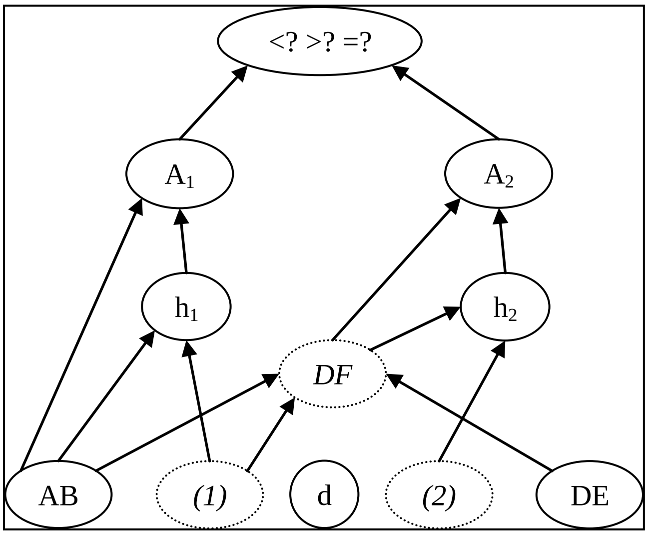

Figure 15

Conclusion

We can examine all the 4 dimensions of the understanding of the concept of area measurement in simple and in complex problems too. We find the representation-metaphor understanding in many problems, which is related to the correct concept image of the area. However the property-proof understanding appears only in a few cases.

Problem-solving is useful to examine the level of the concept formation especially in the cases where the necessary preliminary knowledge is in the long term memory. There are missing preliminary knowledge by most students. They often replace these concepts with false analogy, false assumptions or visual estimation as one can see well from the diagrams based on their solving strategies.

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Studied the problem solving process of students we detected some frequent mistakes according to the concept of area:

1) The area of a triangle is connected to the length of sides.
2) Triangles with larger perimeter have larger area.
3) The formula for the area and perimeter of a rectangle is well-known, however the students expend it in an inadequate way, for example multiply all the sides of a polygon to determine its area.
4) In obtuse angled triangle the „base” has to be the side opposite to the vertex of obtuse angle.

The detected mistakes in points 1) and 2) are in keeping with findings of researchers mentioned in the theoretical background section, however they in points 3) and 4) are not. I'm convinced that the analysis of further problems posing for students makes the process of concept formation more clearly.

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