ON OBSERVATION OF PROBLEM SOLVING IN A SWEDISH AND A HUNGARIAN 5TH YEAR CLASS[1]

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The method of teaching problem solving greatly depends on the teaching methodologies and traditions used in the given countries. In our research, we compare a math lesson between a Swedish and a Hungarian 5th year class. In both classes, the problem to be solved was the same, and it was left to the teachers how to conduct the lessons. In our analysis we primarily focus on the teachers' method of processing the problem during the lesson, but we also look at the teacher's methodology as well as the presentation of the students' trains of thought and mathematical attitude.

INTRODUCTION

In our qualitative research we investigate the differences and similarities between the teachers' method of processing the same problem during the lesson. The problem and the year of class were identical, while circumstances in the schools, traditions regarding mathematics teaching and practices of training of teachers were different. Our hypothesis was the following: The method of teaching problem solving greatly depends on the teaching methodologies and traditions used in the given countries. Our aim was to collect different and similar aspects observing the classes, and not to make an exact, representative comparison between the teaching practices of the two countries.

THEORETICAL BACKGROUND AND THE TEACHING PRACTICE

In this section we mention two relevant theories connected to the teaching of problem solving: Pólya's ideas and Stein's method. Furthermore we briefly describe the recent teaching situation in Hungary and Sweden.

Pólya's method of problem solving

Pólya's method of problem solving is being taught in the teacher training in Hungary. The four phases of problem solving are the following: understand the problem; devise a plan; carry out the plan; look back (Pólya, 1957). Pólya suggests appropriate questions for the teachers to help their learners, like "What are

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the data?", "What is the unknown?", "Draw a figure.", "If you can't solve the proposed problem, look around for an appropriate related problem." etc. He emphasizes the importance of the fourth phase i.e., the reflection and generalization. "Perceive points of general interest which deserve further study."

In his book "Mathematical discovery" Pólya writes on teaching problem solving, and describes three principles of learning and teaching (Pólya, 1981, p. 103-104.):

- Active learning: "The best way to learn anything is to discover it by yourself."
- Best motivation: the interest of the material, the pleasure of intensive mental activity. The problem should appear as meaningful and relevant from the student's standpoint; it should be related (if possible) to the everyday experience of the students.
- Consecutive phases: exploration, verbalization and concept formation, assimilation, (application and/or generalization). "For efficient learning an exploratory phase should precede the phase of verbalization and concept formation, and eventually, the materials learned should be merged in, and contribute to, the integral mental attitude of the learner."

Keeping these principles in mind we chose the problem we investigated.

We agree with Pólya, that speaking about the teacher's mental attitude is relevant part of the professional training. Here we mention only a few points from the "Ten commandments for teachers" (Pólya, 1981, p. 116.), which are especially important regarding our investigation.

"1. Know your subject.
3. … The best way to learn anything is to discover it by yourself.
4. Try to read the faces of your students …
7. Let them learn proving.
8. … try to disclose the general pattern that lies behind the present concrete situation.
9. Do not give away your whole secret at once …"

The professional attitude of the two selected teachers made us sure that their teaching is organized in accordance with Pólya's commandments.

Stein's method of orchestrating whole-class discussion

The pedagogical model specifies five key practices teachers can learn in order to use students' responses to inquiry-based and student centered instructional tasks more effectively in discussions (Smith & Stein, 2011):

- Anticipating: 1. Do the problem yourself. 2. What are students likely to produce? 3. Which problems will most likely be the most useful in addressing the mathematics?

- Monitoring: 1. Listen, observe, identify key strategies. 2. Keep track of approaches. 3. Ask questions of students to get them back on track or to think more deeply.
- Selecting: 1. Crucial step – what do you want to highlight? 2. Purposefully select those that will advance mathematical ideas.
- Sequencing: 1. In what order do you want to present the student work samples? 2. Do you want the most common? Present misconceptions first? 3. How will students share their work? Draw on board? Put under doc cam?
- Connecting: 1. Craft questions to make the mathematics visible. 2. Compare and contrast 2 or 3 students' work – what are the mathematical relationships? 3. What do parts of student's work represent in the original problem? The solution? Work done in the past?

Teaching problem solving in Hungarian schools

In a 45-minutes-lesson usually the part of problem solving usually takes 10-15 minutes. It's divided into two parts: individual work and class discussion guided by the teacher. Pólya's four stages is well known and used by teachers, although sometimes only mechanically, without deeper understanding. The problem solving is part of the curriculum; it is integrated in the other main topics. Students often have to justify their solution or idea. The teachers often ask for feedback to know whether students understand the explanation or not. Sometimes the questions are formal, like: "Who got the same result? Who got something else? What was the mistake? Pay attention next time!" The Hungarian teacher was an expert teacher, who applies Pólya's method for a long time.

Practice, changes in Sweden during the past few years

Problem solving is even more emphasized at all levels in the new 2011 curriculum and the problem solving competence appears at all levels. All mathematics teachers in the country had to attend a course for in-service teachers called Matematiklyft. The course material is accessible for anyone via the web in Sweden. The Stein model, as almost the only model of problem solving, is especially important in years 6-9. The Swedish teacher was an expert teacher too, who already took part in the course "Matematiklyft" and applies the Stein model in her problem-solving lessons regularly.

METHODOLOGY

In order to investigate the differences and similarities between the teachers' method of processing the same problem we observed two lessons, one in Hungary and one in Sweden. The two selected classes are 5th grade classes at schools where student teachers from the universities are sent to practice. 15 students are in the 5^{th} class of the practice school of Mälardalen University in Eskilstuna, Sweden. 21 students belong to the 5^{th} class of the practice school of University Debrecen, in Debrecen, Hungary.

The two selected teachers are considered (by the colleagues and parents and also by the university teachers) one of the best in their respective schools. In both classes, the problem to be solved was the same, and it was left to the teachers how to conduct the lesson. The teacher that taught the Swedish lesson had already attended the course Matematiklyft, and the problem used during the lesson is from the course material (Hagland, Sundberg, & Hårrskog, 2014, p. 28).

The tradition of education in the two countries widely differs due to their history and geographical position. However we tried to select two teachers and two classes having many common properties.

The problem

Some teams organize a football/bandy tournament. Every team plays against every other exactly once. How many matches are they playing all together

a) if 3 teams attend to the tournament?
b) if 4 teams attend to the tournament?
c) if you may decide how many teams attend to the tournament?
d) Find out a similar problem. Solve it.

The classes took place at the end of May 2015. In accordance with Swedish and Hungarian ethical requirements, audio recordings, photographs and reports were compiled from the classes.[2]

DISCUSSION AND RESULTS

In the discussion of our experiences we concentrate on the following aspects: time-table and intensity of the lessons; fitting the lesson within the progression of math lessons; presentation of the problem. We analyze the individual phase of the work and the class discussions furthermore investigate to what extent grants the teacher the student's independence. The description of the lessons relies on the comparative study of Andrews (2007).

The structure and time-table of the lessons

The duration of the Hungarian lesson was set, 45 minutes while the duration of the Swedish lesson is flexible; the observed lesson lasted 60 minutes. More than a quarter of the Swedish lesson was spent without any mathematical content arising. In the Hungarian lesson, this number was 4.4%. The problem at hand was dealt with in various ways during both lessons: independent work, class discussion, summary by the teacher. This took 31 minutes, during the second half of the Swedish lesson and 25 minutes during the first half of the Hungarian (Figure 1).

[2] We possessed the parental consents and the permissions of the head teachers.

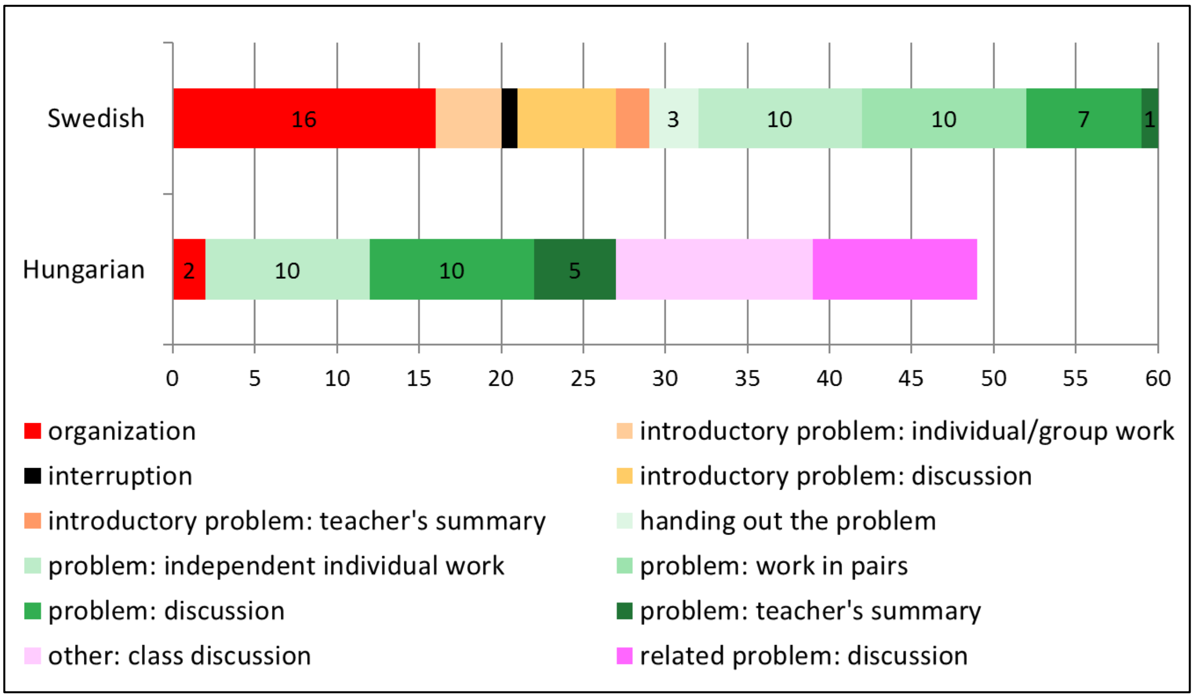

Figure 1

The intensity of the lesson

The Hungarian lesson is more fluent, focuses on learning all the way, the students are working continuously. The teacher is often urging on the students (”Let it go.”; ”Are you done, are you done?”; ”I'd like to move on... As if you were a 5th grader.”). Several mathematical concepts are mentioned (tables, graphs, multiplication, polygon, square, rectangle, side, diagonal, and vertex).

There is more idleness during the Swedish lesson. More time is spent organizing the work, too. On several occasions, the flow of the lesson was disturbed by someone from the outside. Only a few mathematical concepts were mentioned (currency conversion, patterns, matching of numbers, common formula, method of calculation and addition).

Fitting the lesson within the progression of math lessons

The Hungarian lesson fits into the ”Functions, Tables, Graphs” chapter, since the number of teams and matches was presented in table form, and the drawing the graph of related values was given as homework. The related problem (number of diagonals of a polygon) gave an opportunity to refresh knowledge in geometry.

The Swedish lesson is seemingly independent of the math studied during that week. Its goal was solely to improve problem solving skills. The attempt at generalization was directed at the determination of the sum of 1 + 2 + 3 + ... + n.

Presentation of the problem

The Hungarian students receive the problem in writing. They analyze and solve it independently, and they write down their solution. At the beginning of the class discussion, one of the students reads out the problem aloud.

The Swedish teacher projects the text of the problem on the document camera. She reads it out loud and explains it, thereby avoiding any misinterpretation of the problem. She makes sure everyone understands the problem by asking the class. The problem description is not brought up again prior to the discussion.

How much independence does the teacher grant the students?

During the independent work phase, the Hungarian teacher restrains from helping the students. The teacher encourages the students, while presenting the problem as something easy (”Look at it! Easy as play.”; „We'll be discussing it in ten minutes. Heaven forbid... What would become of us, if we messed around with the problem for the entire duration of the lesson?”).

The Swedish teacher continuously keeps an eye on the students. When needed, she helps them with a leading question or short answer. Recommendations are used to start the solving of the problem. („Do something clever, discuss it among yourselves, and maybe even try it!”). Constant short positive comments are used to help the work flow. („Exciting! Ok! Let's see, explain it! Tell me, how do you think? Very good, you solved the problem!”). She encourages the students to write down their solutions. („Consider even now how you will present your solution to the rest of the class! A single number is not good enough as solution. Instead, show how you arrived at the result!”). She initiates generalization („Very good, Olle, and what if we included all 5th graders?”).

The individual phase of the work

Hungarian students had 10 minutes of individual work. They take advantage of all time available in this phase. The class-wide discussion starts when most of them are done with both the a) and b) parts of the problem.

Ten minutes of individual work followed by 10 minutes of working in pairs in Sweden. Not all students use all the allotted 20 minutes, which was enough for all students who wanted to solve the problem.

The students' independent work

The number of correct solutions concerning 3, 4, 5, 6, 7, 9, 20, 2 and 1 teams (Table 1):

	a) 3t	*b) 4t*	*c) 5t*	*c) 6t*	*c) 7t*	*c) 9t*	*c) 20t*	*c) 2t*	*c) 1t*
Hungarian (21)	15	14	9	0	0	0	0	2	1
Swedish (15)	14	12	3	3	3	1	1	0	0

Table 1

We observed various way of enumeration of the matches in both countries, just as table-representation (Figure 2), or drawing a graph with collinear (Figure 3) or planar vertices (Figure 4).

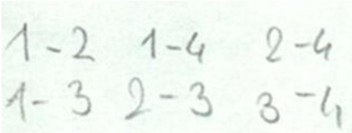

Figure 2

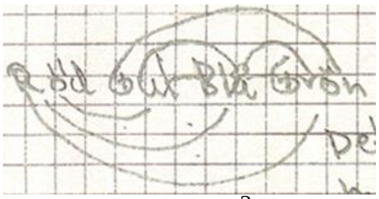

Figure 3[3]

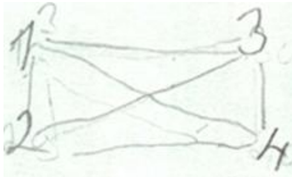

Figure 4

Figure 5 summarizes the way of argumentations.

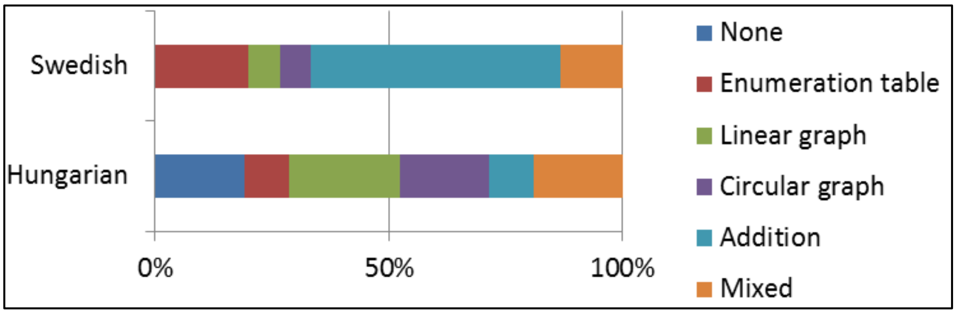

Figure 5

The usage of symbols i.e. the method of marking the teams may express the student's level of mathematical thinking which also marks the boundaries for generalization. Small circles indicate that the name of the teams are irrelevant in this problem, while sensible words may resonate better with the students on an emotional level. Table 2 shows the number of students used different notations.

	Small circles (Figure 6)	*Numbers or letters (Figure 7)*	*Sensible words (Figure 8)*
Hungarian (21 students)	3	8	0
Swedish (15 students)	2	2	4

Table 2

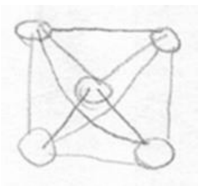

Figure 6

Figure 7

[3] In English: Red; Yellow; Blue; Green

Figure 8[4]

Class discussion

1. Presenting the solution

In Hungary volunteers walk to the blackboard, draw and explain their solution, bringing their notes with them, but not using them as they argue their case. The teacher repeats some things as well as asks questions, in order to emphasize important stages of the solution.

In Sweden two students from different groups present their solutions at the blackboard while the teacher projects their solutions on the wall for the rest of the class. The teacher repeats parts of the solutions, emphasizing some things. The solution of the third student is presented in its entirety by the teacher.

2. Erroneous reasoning

The Hungarian teacher allows the student to finish, and then emphasizes the critical point. When the student makes the error, the teacher corrects him naturally and kindly, and turns out that several others were reasoning the exact same way.

> S1: And since all teams play twice, meaning 2 + 2 + 2 = 6 ... (*The other students try to intervene, but the teacher does not allow it.*)
> T: Hold it! Let's have a look at this. Hands up, those of you who got a 6!" (*Two students put up their hands.*) "I see. But then what's the problem? What is wrong?
> S2: When we calculated the 6, only half of it...
> T: Even though this looked like a great idea, didn't it?
> S2: If three plays with two, and with one, then they can't play again …
> T: ... because each team can only play once... Clearly. Alright, let me see hands: who got the 3? (*Fifteen people did.*) Yay, how nice! There were so many fewer before ...

[4] In English: Lemon; Orange; Strawberry
Lemon against Orange 1 time
Orange against Strawberry
Lemon against Strawberry; Answer: 9 matches

No erroneous lines of reasoning were presented on the Swedish lesson. These were avoided on purpose – their presence can only be seen in the notebooks of the students.

3. Several good solutions

In the case of four teams, several Hungarian students voluntarily present correct solutions. To start with, the teacher leaves the question of evaluating the various representations open.

Whole Swedish class sees three different solutions during the discussion phase. For each solution, any arising questions are answered in turn. The solutions are not, however, compared with each other. On one occasion, the teacher asks if there are any students who solved the problem the same way as presented in the first group's solution, but the students' response is unclear.

4. Guiding the discussion

In the case of five teams, the Hungarian teacher recommends graphic representation instead of enumeration. In cases with large number of teams, this makes it easier to count the possibilities. (”Five teams … I say, five teams... no (interruption) … ten games. You can gather these cleverly, using these techniques, pairing things up. But some prefer these small symbols (circles, lines).” He directs attention to the difficulty of the counting, as the number of teams grows (”Alright, we have to be alert now as we could the number of lines...”).

The Swedish teacher uses repetitions and questions to help the students present their solutions to the class. The presenter of the first solution uses the letters of the Swedish word for three (”tre”) to mark the three teams. Similarly, the Swedish word for four (”fyra”) is used in the case of four teams. In the case of six teams, this method fails, as the Swedish word for six is only three letters (”sex”), and so ”orange” is used instead. It would've been worth it to clarify that this method will fail for certain (and increasingly larger) number of teams.

5. Attempt at generalization

Students in Hungary with the guidance of their teacher investigated the case of 7 teams (6 is left out). At the end of the following transcription we can recognize the analogous line of reasoning of a student.

T: How many did we get in the case of 7?
S1: I got 42. (*Others want to interrupt, but T does not allow it.*)
T: 42 at 7. Shall we wrestle 7?
S1: Yes. (*The student draws the graph at the blackboard.*)
T: Oh, there won't be any ink left... (*The teacher implies that there is a shorter solution.*)
S2: Everyone plays with 6, that's 21. (*S2 interrupts.*)
T: What was that again? Every team plays...
S2: With 6. Multiplying is enough; we don't have to count...

T: That's it! Very good! I thought you were going to count them all... Sweaty work indeed. Now, once again. Let's listen to this line of reasoning. There are 7 teams. Each team plays against 6 other teams. What is 7 times 6?
S': 42. (*together*)
T: Is it true, that there will be 42 games? Who agrees with this? Let me see some hands! (*About half the class raises their hands.*)

The teacher uses an interspersed question and the response of the class to confirm that many do not understand S2's line of reasoning. He has a student explain it again.

T: Very good. Are there any counterarguments? (*Students raise their hands.*)
S2: It was the same case in the beginning, when one team played two; three times two is six, but only half of that. So half of it here too.

The presenter of the first solution in Sweden used colors to mark the teams, and wrote them down on one line, and repeated it two times (Figure 9).

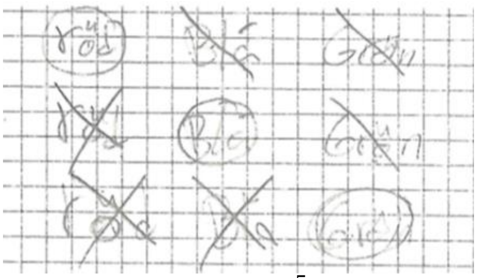

Figure 9[5]

This method of solution, which uses a table, can be generalized, but the teacher did not discuss this. When the student chooses 6 teams, and utilizes the relation provided by the teacher at the introductory task (5 + 4 + 3 + 2 + 1 = 15), the teacher neither reacted to this nor linked the two approaches of the problem.

The Swedish author of the third solution chooses 7 teams in the c) part of the problem. When presenting the solution, the student correctly writes down all matches, but does not arrive at a final result. Granted, he is not making any mistakes either. The teacher asks who else picked 7 teams for c), and how many matches they arrived at. Twenty one, say the students, but this is not expanded upon. Already during the teacher's interpretation of the problem, it becomes apparent that a student immediately considers the case of only one team participating, but the teacher's only reaction is that a tournament requires several teams. This was an important mathematical moment that should have been revisited.

6. Teacher's summary, conclusion of the class discussion

[5] In English: Red; Blue; Green

In Hungary a table was created with the known pairs of values on the blackboard (Table 3):

Number of teams	3	4	5	6	7	10	2	1	0
Number of games	3	6	10		21				

Table 3

During the discussion, the method of calculation devised from the seven team case is applied to the six team case. The teacher explains the method at the blackboard again. The case of ten teams is calculated as well. Several students arrive at 45 individually, but the teacher stops here with the generalization and does not state the general formula. The case of less than four teams was brought up earlier by some students, now it is integrated into the table. During the discussion the addition and multiplication methods were both suggested by students. The teacher accepts both methods, and in the end creates a scenario that implies that for large number of teams, the multiplication method is more practical.

The Swedish teacher summarizes the discussion of the solutions, mentions the different notations used by the student without going into specific details. She briefly mentions the case of only one team. She also says that there is another way to generalize, when two teams play two games, one home and one away. The students recognize this, as they have participated in similar tournaments. The addition method was suggested by the teacher.

T: What did you learn today?
S: The method, the way to count the games…
T: Similar task, drawing, or writing, discussion, exchanging thoughts about your solutions, it was all good.

”Come up with a similar problem and solve it!”

The goal of that problem was not reached in either of the lessons. It turned out that the Hungarian students interpreted the expression “similar problem” in a different way: problems or difficulties that arise from the situation. (”It was raining!”; ”None attends the tournament.”; ”The stadium collapsed due to an earthquake. It had to be rebuilt.”) In some cases they came up with a mathematically completely different problem. Due to the lengthy presentations and discussions in Sweden there was no time for this part. The students did not write anything about it in their notebooks either, with some exceptions.

Concluding remarks

We observed two different ways of teaching problem solving. We can consider that the applied method greatly depends on the teaching methodologies and traditions used in the given countries. We find some similarities and many differences, which can be summarized as follows.

– During each lesson, only about half the allotted time was used for the actual problem.
– Both teachers also discussed a related problem. The Swedish teacher used this as an additional task (number of handshakes) because this could be acted out, and discussed it first. The Hungarian teacher used a different mathematical theme after discussion of the main problem (number of diagonals in a polygon).
– Processing in both cases consisted of two phases: individual work and group discussion. The discussion was followed by the teacher's summary and explanation. During the Hungarian lesson, individual work consisted of independent written interpretation and solution of the problem. The teacher made sure that the students could concentrate. During the Swedish lesson, independent work seemingly was of two kinds: individual work and work in pairs. In practice, there was no clear difference between these two phases: some of the students worked in the whole time individually while the others worked in pairs. Twice as much time was used and there were interruptions.
– The amount of time allotted for discussion was the same, but the method was different. During the Hungarian lesson, students volunteered to present their solutions at the blackboard. The teacher allowed them to present erroneous lines of reasoning. The students commented each other's line of reasoning freely. The correct solution and useful representations were presented by the students. During the Swedish lesson three students presented their solutions by having their note projected. Their explanations required interspersed helpful questions from the teacher and they had difficulties expressing themselves with connected sentences. The last solution was presented by the teacher, based on the work of the student.
– It was clear that both teachers were adequately prepared for possible solutions from the students. The Hungarian teacher was also aware of possible typical mistakes that could occur. He was also more thorough regarding the mathematical background of the problem: the case of less than three teams, integration of acquires knowledge into the theme he was teaching (tables, diagrams). He took the opportunity to use several notations appropriate for the problem. The Swedish teacher did not emphasize this and did not integrate the theme of the lesson into the broader mathematical subject.
– Neither teacher forced the generalization and abstract solution of the problem. The Swedish teacher used one way to solve the problem, determining the sum of the series, but seeing that the students did not understand it in generally, she left the question open. The Hungarian teacher explained how one can determine the number of games for 6, 7 or 10 teams based on the students' solutions, but did not go further, even though one student brought up the case of 100 teams.
– The Hungarian students have more background knowledge of different representations than the Swedish ones, because in lower grades they deal with combinatorial problems as well as way to find all solutions to problems. Regarding notations, it is mentioned during the Hungarian lesson that it is not necessary to

name the different teams, as they're equally important for the problem. These points towards a more formalized line of reasoning, and signals multiplicative thinking. During the Swedish lesson, the teacher's explanation of the additional problem clearly signals additive thinking. This was reflected in the students' individual work as well. They gave names to the teams, or used numbers or letters (in a mathematically irrelevant way), meaning that their thinking is more dependent on concrete ideas.

– The Pólya's problem solving phases and their appearance in the lessons (Table 4).

	Hungarian	*Swedish*
Understand the problem	The students interpreted the problem themselves.	The teacher presented and interpreted the problem.
Devise a plan	Preliminary knowledge. Different representations.	Auxiliary problem. Representations of the teams and the games.
Carry out the plan	Different methods, both multiplicative and additive.	Application of the additive method discussed earlier.
Look back	Present at all phases. Summarizing table of cases. Discussion of a related problem.	Present at all phases, but touched on only briefly and in concrete terms.

Table 4

– Although the Hungarian teacher was not familiar with the phases of the Stein model, theses phases explicitly appeared in his lesson, except "Sequencing", which was rather spontaneously. The Swedish lesson followed clearly the first four phases, while the last one, "Connecting" was cancelled.

In our qualitative research we observed many differences between the two lessons. We found that besides the educational tradition, the existence or the lack of the combination of the teachers' mathematical and didactic knowledge has also an important role. It means that it is necessary for conscious development of the students' mathematical abilities and to recognize opportunities for development in unexpected situations. This includes not going into further details when they realize that most of the students do not follow any more. This kind of awareness could only be seen indirectly during the Swedish lesson, by observing which students the teacher chooses to present their solutions.

Acknowledgement

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