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A classroom teaching experiment on the preparation of the function concept

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We present a teaching experiment that was conducted one year before the function concept is introduced, namely in 6th grade, and that points towards the objectification of the function concept. For the preparation of the function concept we use the model of function machine and different representation forms in line with this. We discuss whether introducing students to these representation methods will help them to familiarize with three constitutive parts of the function concept such as relationships, variables and rules.

Introduction

This paper reports findings from a stage of a longitudinal study which focuses on the investigation of the formation of the function concept. In the recent work we present a teaching experiment that was conducted one year before the function concept is introduced, namely in 6th grade. For the objectification of the concept of covariant quantities and for the preparation of the function concept we decided to use the model of function machine. Among representation methods the table, the rule expressed in a narrative or algebraic form were emphasized. The use of graphical representations has not been involved in this research material so far.

The chosen educational environment, an Ukrainian 6th grade class ensured that the students meet the idea of function machine and the table representation forms of covariant quantities for the first time. That is to say, the Ukrainian curriculum prescribes the process of familiarization of the expressions using literal symbols in 5th grade, and lays the basis for the introduction of the function concept in 7th grade. Results of our earlier research (Szanyi, 2015) suggested that having a practical understanding only of this form of representation is not sufficient when it comes to the foundation of the function concept, even with the addition of graphs.

The aim of the three-hour long teaching experiment was to find an answer to the question whether the chosen activities regarding the introduction of the representation methods mentioned above are suitable for familiarization with the constitutive parts of the function concept such as “relationships”, “variables” and “rules” (Sierpinska, 1992). The reflections and the experiences gained from this teaching experiment allow us to refine our ideas about the appropriate implementation of the preparation of the function concept.

Theoretical framework

In school, the introduction of a mathematical concept is preceded by long preparatory work. There are concepts that rely on students' spontaneously developing preliminary knowledge taken from everyday life. In cases like these, some particular examples are needed to facilitate the abstraction of the necessary mathematical concepts (Skemp, 1971). The concept of function is not like that. In this respect, there is no such informal basis on which we can build a new mathematical discourse (Nachlieli & Tabach, 2012). In agreement with the conclusions of the authors, we also believe that before introducing them to the function concept, students need to become familiar with the following forms of representation: graphs, tables of numbers, stories, rules, and the algebraic expressions. One of the conclusions of the paper quoted above is that before we point out that the function is one situation which could be represented in all of these five ways, students would need to gain some relevant experience of the acquaintance and practicing all of these representation forms. Therefore, it is obvious that – in a spiral manner – preparatory work should begin in the years that precede the introduction of the function concept, hence, typically before entering 7th grade.

Sierpinska (1992) described the necessary previous knowledge for understanding of the function concept. According to her statement, students have to familiarize with the following „worlds" which contain the constitutive parts of the function concept:

- World of changes or changing objects – In the definition of the function the two variables (dependent and independent) represent the changing objects.
- World of relationships or processes – The relationship between changing objects or the process which transform objects into other objects can be described for example by table, graph, formula or can be expressed in verbal form.
- World of rules, patterns, and laws – The relationship have to be well defined by a rule which specifies how the first or independent variable affects the second or dependent variable.

The detection of change and the recognition of the relations between variables is important in the concept formation process. The "quality" of change can be determined by the rules describing the relationship between the variables.

Tall (1992, p. 497) defined a cognitive root to be "an anchoring concept which the learner finds easy to comprehend, yet forms a basis on which a theory may be built." So, this is a meaningful cognitive unit of core knowledge for the student at the beginning of the learning sequence (as cited in Tall, McGowen & DeMarois, 2000, p. 257). Didactic analyses of function concept draws up the cognitive roots which facilitate the preparation of the concept and connected them with other criteria of understanding, like the ability to use different representations. One of this cognitive root is the *function machine* which objectifies the covariant quantities and helps to get to know the constitutive parts of function (variables, relationships and rules).

According to the well-known theory of Bruner (Bruner, 1968), there are three modes of representation that are the way in which information or knowledge are stored and encoded in memory. Enactive mode involves encoding action based information, iconic mode involves storing information visually in the form of images and symbolic mode involves storing information in the form of a code or mathematical symbol, such as language. Typically, enactive then iconic modes are the ones that dominate in usage at the very beginning of encoding however it can happen that in certain situations we decide to prefer one of the aforementioned modes over the other one only because it seems less complicated. As Rivera explains in his book, "… the representational path toward the construction of generalizations proceeds from using concrete models, the numbers as quasi-variables, followed by words, and finally literal symbols and variables in algebra. That path seems to resemble a movement from the enactive and iconic to the symbolic." (as cited in Rivera, 2013, p. 76). So, we start the preparation of the function concept with activities from the enactive mode (matching objects, introduction and application of the model of function machine). After these activities we use the picture of the function machine to evoke the model which can be consider as iconic form of representation. We associate the description of the operation of the machine with the introduction of the table form, this way transitioning to the representation form which is located between the iconic and the symbolic. Creating a table of input and output values involves generalizations "in terms of specific numbers and even to an example of any number before they can provide a generalization in language or symbols" or "thinking of numbers themselves as variables" (as cited in Rivera, 2013, p. 76.). Finally, students produce a formula expressing the functional relation by recognizing a connection between covariant quantities arranged in a table.

In order to have students engaged in comparing dependent and independent values instead of successive output values when applying a recursive strategy, it is reasonable to provide input values not in an ascending order but unordered, randomly. This way, students can focus on covariant values instead of the set of output elements (Moss & London, 2011).

The background of the teaching experiment

In the Ukrainian education system, function concept is defined at the 7th grade of the school. Analysing the curriculum for the 5th and 6th grade, the rule recognising and rule following skills between development requirements are not mentioned. In our previous research we investigated these skills of a group of 6th grade Ukrainian students (Szanyi, 2015). Our result shows that the students are able to recognise and follow a rule which expresses the connection between the elements of a table using only their previously acquired mathematical knowledge. However, argue in favour of a well-recognized rule either in a narrative form or with formulas remained a difficulty for them.

According to these findings, we developed a teaching experiment, which was conducted among another group of 6th grade Ukrainian students in November 2015. By the time, they were already familiar with common and decimal fractions and had a clear understanding of the operations carried out with them as well as of the methods of calculating the perimeter and area of rectangles and squares. The group consisted of 20 students. Based on the result of an assessment that was conducted at the beginning of the school year their mathematical knowledge can be consider as average.

The teaching experiment was planned on the curriculum topic *Proportion, proportion pairs*. Three activities were elaborated with different duration in time and were introduced in three 45 minutes classes. During the classes, students were either instructed by the teacher or had to work individually. In case of individual work, they recorded their responses in their exercise book. Notes, voice recording and photos were made about all lessons.

Descriptions of the teaching experiment and results

Lesson 1: Matching elements manually (35 minutes, 20 students)

The aim of this activity was to have students recognize a function-like relationship between the elements of two sets. The familiarization with the "world of relationships" we started through enactive mode of representation (associating objects with manual activity). Students received three envelopes containing elements to be matched. We drew a distinction between the domain set and the codomain set by marking their elements with different colours. *Envelope 1* contained green squares as well as red rectangles. The areas of the red rectangles equalled the half of the areas of the squares (object to object assignment). There was a single rectangle that could not be matched. *Envelope 2* contained four given size rectangles and six numbers written on blue cards (18; 18; 18.5; 24; 20; 80). These numbers (expect two numbers) determined the areas of rectangles (number to object assignment). In *Envelope 3* there were black and red numbers. Red numbers (4; 3; 1; 0.25; 0) equalled the quarter of the black ones (16; 12; 4; 1; 0) (number to number assignment). After matching the elements of the envelopes, students were required to verbalize the rule based upon how they performed the task. We constructed the tasks in a way that the connection between the given elements was easily definable, because we didn't want to encourage students to recognize more and more rules. Our primary focus was on the concept of relationship.

It took 3 minutes for the students to complete the task of *Envelope 1*. One student realized that placing the rectangles on the squares, the half of the areas of the squares equalled the areas of the matching rectangles and the rule he phrased reflected this approach as well (Figure 1). Regarding the rest of the students, they placed those rectangles next to the squares where rectangles' one side length equalled the side length of the squares (Figure 2). Even though there were inaccuracies, each student managed to find and verbalize a rule that reflected their work process well.

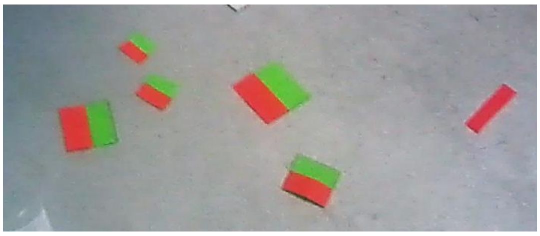


Figure 1

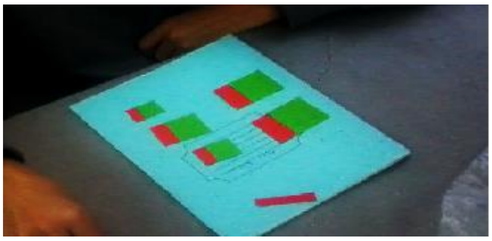


Figure 2

Students could accomplish the task of *Envelope 2* effectively; they could match elements easily (Figure 3). 17 students managed to phrase a rule that reflected perfectly the right matching process. By having observed the work process of students we realized that 2 students calculated the perimeters of the rectangles first however in the moment they become aware of not being able to match each rectangle to a number, they modified their computations and calculated the areas of the rectangles. 3 students found a rule that was not applicable to the process of matching elements of the two sets ("I have calculated the perimeter of the rectangles and I matched them to the numbers"). These students might have presumed that the numbers are related to the areas or the perimeters of the rectangles however were unable to confirm this assumption.

Students had the most trouble with matching numbers (*Envelope 3*). In this case, despite the colour codes we provided them with, students experienced difficulties in making a distinction between the domain set and the codomain set. 2 of the students found it hard to take even after emphasizing: "The task is to match the red numbers to the black ones according to a particular rule that will apply to all of the matches." (Figure 4). Here we detected the previous phenomena again that is the ignorance of the fact that the rule must be valid for all the constructed pairs. 16 of the students applied the following rule when matching elements of the two sets: "The black numbers can be divided by the red ones". There were only 2 students who noticed that the elements of one set were the quarters of the elements of the other set ("We divide the black numbers by 4 and so we can find their matches").

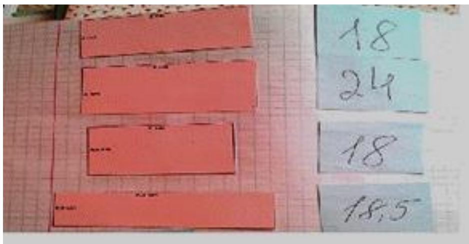


Figure 3

Figure 4

Lesson 2: Introduction of the concept of variable with the help of the model function machine (25 minutes, 20 students)

The aim of this activity was to introduce the table form and the concept of covariant quantities. We focused on the “world of changes” and used one of the cognitive roots of the function concept, the function machine. We prepared a concrete “machine” that allowed students to gain some experience of its “operation”: the machine “did” something to the objects that was placed inside then it “produced” the result. It did obviously the same to each object that was placed inside. We placed various things inside the machine (caps, dices), then it produced the result after doubling the number of objects that had been inputted (there was a hole on the rear side of the machine; the teacher used it for doubling objects that were put inside). (Figure 5). Students were required to make notes of the things we inputted and of the results the machine outputted. This way, students unintentionally created a two-column table (Figure 6).

Figure 5

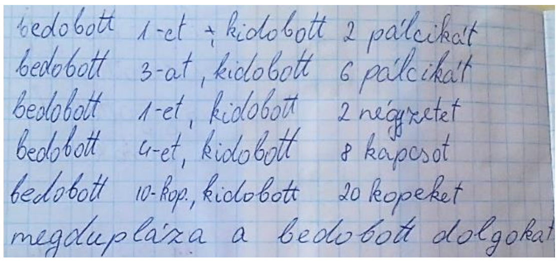


Figure 6[1]

We asked students: “Can we substitute the things we inputted with numbers?” They answered immediately and listed the numbers corresponded to the objects. After this, students were required to fill in Rows 1-6 of the table based upon the operational rule of the machine. (Students received worksheets that contained the table and it was also written on the board.) (Table 1).

This was followed by phrasing the rule applicable to the operation principle of the machine, transitioning from a verbal expression to an algebraic expression. We filled in Row 7 with “number” and with “the number doubled”. We asked students if there was a way to substitute the word “number” with an even shorter

[1] Translation of the notes:
Inputted 1, outputted 2 sticks, … 6 sticks, … 2 squares, … 8 clips, … 20 coins
It doubles the inputted things.

form, since mathematicians like to abbreviate things. Students then abbreviated the word “number” and indicated it with *s*, *n* or *c* (initial letter of the Hungarian, English or Ukrainian word “number”).

Input	Output
2	
8	
0	
	12
	7
81	
number	*the number doubled*
number	*number·2*
s	$s\cdot 2$
n	$n\cdot 2$
c	$c\cdot 2$

Table 1

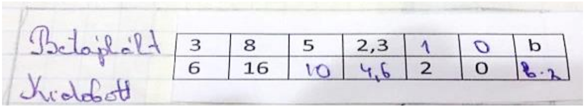
We also had a discussion about the rule that would have been applicable in the case if the column on the right had contained the input data and the left one had contained the output data. The purpose of this discussion was to facilitate the understanding of inverse functions. Students responded clearly: “The machine would then divide the input numbers by 2”. Also, this rule was defined symbolically such as “$n:2$”. Finally, students were required to identify the pairs and describe the connection between them. We received quick and correct responses that led us to the conclusion that manual activities can be considered effective in the aspect of matching elements as well as in following and then verbalizing a rule.

At the end of this lesson, students received a new table that had to be filled in individually based on a rule they recognize. They were also required to determine the operation of the machine of which inputted and outputted data could be represented by this table (Figure 8); they had to do this by verbalizing a rule and by expressing it with mathematical symbols. The rule was the same as the previous one however covariant quantities were aligned vertically this time. Also we did not indicate which numbers are the inputted and which numbers are outputted. With this attempt we tried to encourage students to interpret both type of tables (horizontal and vertical) correctly as well as focusing on the connection between covariant quantities regardless of how they are aligned in the tables. The reason behind indicating the variable with a new letter (*b*) was to avoid associating only a few letters to the concept of variable. 18 students managed to fill in the table correctly and they could translate the recognized rule into words (“The input number is multiplying by 2, this way we get the output number, or vice versa but then we have to divide it by 2”) and into mathematical symbols

("$b \cdot 2$") as well. 4 of them deemed it important to highlight that the table represents connection between the elements of two sets. They added header to the table in order to indicate which set did those numbers belong to: input (domain set) or output (codomain set) (Figure 9). There were 2 students who filled in the blank cells of the second row of the table based on the rule they recognized (10; 4.6), however they applied the same rule to the empty cells of the first row as well, instead of applying the inverse of it (4; 0). They probably were not aware of the possible reversibility of the machine model.

3	8	5	2,3			b
6	16			2	0	

Figure 8



Betáplált	3	8	5	2,3	1	0	b
Kiadott	6	16	10	4,6	2	0	b·2

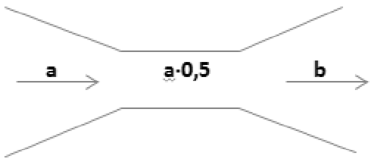
Figure 9

Lesson 3: Determining values of algebraic expressions (15 minutes, 16 students)

With this activity, our aim was to develop students' rule recognizing and rule following skills in the "world of rules" as well as the transitioning from one form of representation to another, namely from algebraic expression to table of numbers.

Unlike in *Lesson 2*, the machine was not accompanied physically but with illustrations. Students had to interpret the given operational rule of the machine, follow it, and create a table of values. Students had to work on the worksheet they received (Figure 10).

A function machine based on the following rule:



a) Verbalize a rule according to the machine works. (What does the machine do?)

b) How much is the value of b, if 1) $a = 1$; 2) $a = 10$; 3) $a = 0.5$?

c) Create a table based on the elements you have matched.

d) How much is the value of a, if $b = 14$?

Figure 10

Seeing the hesitancy of some students we found it important to complete the written instruction orally: "The machine is doing something with the input element a, which is written inside of the machine, then it outputs the result b". 10 students, so more than half of them managed to answer this question correctly. In their cases, the determination and application of the inverse rule caused no great difficulty. 3 students did not take into account that the machine drawn in the task works by different rule than the previous one, so they applied the "multiplication by 2" rule again. The picture probably reminded them of the concrete machine presented on the previous lesson. They may not have interpreted the role of the the model of function machine completely, they insisted using enactive mode while encoding. 3 of the students made no attempt in answering this question.

Conclusion

Activities presented above aim at the preparation of the function concept and are appropriate for studying the mental maturity of “average” 6^{th} grade students.

Students easily recognise function-like relationships between the elements of two sets through manipulative activities if the sets consist well distinguishable elements like different type of objects or objects and numbers. Otherwise it is difficult to separate the domain and codomain sets from each other. This led us to the conclusion that the first introductory tasks in the “world of relationships” should be done with sets of different type.

We detected several times that being able to find the proper relationship which is satisfied by ***every*** pairs is the most crucial point in the process of understanding the function concept. In order to understand how to recognize and follow a rule that applies to matching elements, students need to find a rule that is true for ***every*** ordered pair. Most of them are able to interpret this idea, furthermore the idea of variable, i.e. the variable is a numeral which represents a definite though unspecified number from a given set. However, a number of students need to perform some extra activities in line with that. We gained similar experiences form the findings of our previous research, where some of the 6^{th} grade students applied a different rule to the same table (Szanyi, 2015).

The model of function machine really helps to recognize and understand that to every inputted element are assigned the outputted elements by the same rule. The table of values connected to the working process of the machine meets the students’ level of development however, this one occasion proved to be not enough for about the half of the students to be able to abstract from the model and, based on a rule, to be able to focus on the concept of covariant quantities. The provided illustration and description of the operation of one machine was not sufficient in itself for them to understand any machine illustrated. If we add a “button” to the function machine model which able to “switch” between different working rules students probably can objectify and understand the function concept more accurately. Obviously, more function-machine-exercises would be needed for this.

To transit a rule from a narrative form to an algebraic expression, presented no difficulty thanks to the machine related activities. Description of rules using literal symbols and variables in algebra is closely related to the simple algebraic expressions learned at 5^{th} grade. So, the activities showed in this paper contribute to the development of students’ algebraic thinking too.

Acknowledgement

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