LET'S EXPLORE THE SOLUTION: LOOK FOR A PATTERN!

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In this paper we analyse some aspects of students' cognitive factors in problem-based learning. The problem we chose is closely related to the mathematical concept of sequence and offers also multiple solution strategies, multiple representations of the subject and possibility for mathematical communication. We report results of a study in the age group of Grades 5th and 6th, focusing on their problem solving strategies and the characteristics of their inductive reasoning.

INTRODUCTION

"Make the subject problematic!" – it is a conceivable way the teachers approach the curriculum. Hiebert et al. argue that "… instruction should be based on allowing students to problematize the subject. Rather than mastering skills and applying them, students should be engaged in resolving problems" (1996, p. 12). In the subject's research, professional and ethical issues are constantly emerging about problem-based learning. Is it possible to expect independent (or directed) discovery from every learner? Is the role of examples and counter-examples understandable to everyone? May the problem-based learning lead to meaningless learning in some cases? These dilemmas can only be resolved if the problems raised are examined with scientific certainty. The more we understand the problem-solving thinking of students at different ages, and the more thoroughly we analyse the effectiveness of problem-based mathematics teaching in classroom environments, the more secure we can apply this method. In this paper we analyse some aspects of students' cognitive factors in problem-based learning (including complex thinking and reasoning strategies, e.g. conjecturing or justifying), in order to understand students' problem solving thinking better.

CONCEPTUAL BACKGROUND

The key concept of our paper is problem-based learning, which has a continuously enriching conceptual structure in the literature of mathematics education; therefore, we first clarify why we use this concept. In a problem-based learning environment, a problem drives the learning material (Roh, 2003). The problem or task should be an activity that focuses students' attention on a particular mathematical concept that matches the goals of the curriculum. Students can also make connections between mathematical concepts and processes that are familiar to them. Good problems for problem-based learning offer also multiple solution strategies, multiple representations of the subject and possibility for mathematical communication that includes proof-based activities or justification (Erickson, 1999). Good problem solving skills are prerequisites of problem-based learning; additionally problem-based learning in mathematics classes would provide students more opportunities to think critically. In our opinion, in the mathematics field, problem-based learning means that a learner must analyse a mathematical problem situation; he or she must approach critically the thinking of their own and their classmates. Furthermore, students explain and justify their thinking (Csíkos, 2010). The problem solving process we are studying in this paper is characterized by all the three elements of the above definition, thus providing a suitable conceptual framework for describing our research.

The purpose of a problem appearing in the classroom is focusing students' attention on a particular mathematical concept, idea or skill. The model by Stein, Grover and Henningsen (1996) based on the fact that mathematical tasks pass through three phases in the classroom: as written by curriculum developers, as set up by the teacher in the classroom, and as implemented by students during the lesson. Teachers' goals, knowledge of subject matter and knowledge of students influence the setup of the mathematical task as represented in the curricular materials. Factors influencing student's implementation are classroom norms, task conditions, teachers' instructional dispositions and students' learning dispositions.

Mason, Burton & Stacy argue that "The process of conjecturing hinges on being able to recognize pattern or an analogy, in other words on being able to make generalizations" (2010, p. 73). More generally, this cognitive process is involved in the inductive reasoning activity. Haverty, Koedinger, Klahr and Alibali (2000) argue that fundamental areas of inductive reasoning are data gathering, pattern finding and hypothesis generation. Within the process of inductive reasoning Polya (1954) distinguishes stages, such as observation of particular cases, formulating a conjecture (generalization), testing the conjecture with other particular cases. Following these sources, we use a five-levels model for describing the inductive reasoning process (Kónya & Kovács, 2017).

(1) *Observation of particular cases* including looking for possible patterns as well.
(2) *Following the observed pattern*, i.e. applying it for other cases. It often happens without formulation of a general statement.
(3) *Formulating a conjecture*.
(4) *Testing* it by other particular cases.
(5) The result is a general statement at this stage, but the mathematical problem solving process requires the *deductive closure*. The form of deductive closure could be either a rigorous proof or justification using the underlying mathematical structure.

Patterns in school mathematics often are represented either numerically or figurally (Rivera, 2013). In this study we use a figurally given pattern. The underlying mathematical structure can be represented numerically by a sequence. Students are expected to continue the pattern figurally, and they are also expected to formulate generalizations concerning this sequence, e.g. to determine a "near", "far" or "arbitrary" element of the sequence. We also look for mathematically valid explanations or non-proof arguments, i.e. empirical arguments or rationales in the sense of Stylianides (2009).

For describing the background of our research, we outline the Hungarian traditions of problem-based learning. Problem-based learning is an essential element of the Hungarian mathematics-teaching traditions, which is closely related to heuristics, inductive reasoning or to Polya's principle of active learning (Polya, 1965, pp. 102-106). Problem-based learning is one of the fundamental principles of the "Complex Mathematics Teaching Experience" conducted by Tamás Varga in the Sixties and Seventies in Hungary (Varga, 1988). One of the important effects of the Complex Mathematics Teaching Experiment is that this principle has always been present in the everyday practice of Hungarian mathematics teaching and learning. In this place, we emphasize C. Neményi Eszter's pedagogical work (C. Neményi, 1999), where one of the focal points is the pattern recognition in a sequence which is uniquely defined by some activity, drawing, or procedure. C. Neményi argues that pattern recognition activities support

- recognizing the modelling function of sequences (i.e. a sequence is the mathematical model of a problem),
- identifying functional relationships between quantities,
- understanding mathematical concepts, and ideas.

METHODOLOGY

We conducted a cross-sectional study, where we used a textbook-problem for fifth-graders, but we used five different setups of this problem for various ages. Table 1 presents the number of students in each grade who took part in the investigation.

Grade	5-6	7-8	9-10	11-12	12+	Total
N	47	21	60	33	71	238

Table 1: Number of students in the sample of investigations (12+ refers on teacher trainees.)

The sample

In this paper we report results of the study in the age group of Grades 5th and 6th. We implemented the task in the classroom using pair work method. 47 pupils (24 pairs altogether) involved in the classroom observation, which took place in 2016 in an urban school in Hungary. The sixth-graders had results in the lower part of the top tierce of the National Assessment of Students' Mathematical Competences (similar to the international PISA test) in the year of our research; and we suppose that this result is exhibitive also for fifth-graders. (The National Assessment is conducted only for 6-, 8- and 10-graders.) It means that based on their achievement in mathematics, they are average or slightly higher than average students.

The "House of cards" problem

In a Hungarian textbook for fifth graders the following problem appears: "Build a house of cards shown in the picture. Discuss how many cards you need to make 1, 2, 3, … level house!" (Gedeon, Korom, Számadó, Tóthné Szalontay, & Wintsche, 2016)

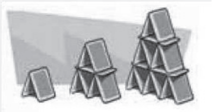

Figure 1: The picture in the textbook

We used this problem in the cross-sectional survey, but we have changed the text by age. There was a notable change in the fact that in the different age groups we asked students about different storeys: about "low" house (e.g. 5-storey), "high house" (30-storey), or generally about an n-storey building. Appendix 1 contains the worksheet prepared for 5th and 6th graders. Tasks 4, and 5 contain questions about "low" houses, i.e. "near" elements of the sequence. In Task 6 there is the option of "far" element of the sequence. In fact most of the groups built the problem for "high" houses. (We consider a house high when it is difficult to draw it accurately and counting the cards; e.g. a 30-storey house is a "high" house.)

We think that this problem has all the features of a "good problem" and gives the possibility of a problem-based learning activity. It proved oneself to be a real problem situation in all grades. It points to curriculum material connected with sequences, but in different depth in different grades. For the 5th and 6th graders the focus is on recognition of functional relationship. It gives the possibility for different representations, i.e. enactive, iconic and symbolic in the sense of Bruner (1971). Accordingly, in the survey we made 5th and 6th graders build the house. The problem is also suited for deep mathematical communication and reasoning: the pupils should formulate a generalization, and he or she is expected to explain it. Also, critical thinking is relevant in this problem, because it contains possibilities of typical misconceptions. For example, while the number of cards grows as the house becomes higher; many students thought that the number of cards is linearly proportional to levels. Another misconception is that the function in question is additive.

Moreover, several approaches are possible, because the problem can be modelled by different sequences:

A. number of cards in the sequence of houses: $2, 7, 15, 26, 40, \ldots$
B. number of new cards one needs to complete the previous house in the sequence: $5, 8, 11, 14, \ldots$
C. number of slanted cards in rows in a particular house (from up to down): $2, 4, 6, 8, \ldots$
D. number of horizontal cards in rows in a particular house (from up to down): $1, 2, 3, 4, \ldots$
E. number of triangles in rows in a particular house (from up to down): $3, 6, 9, 12, \ldots$ (excluding the last row, i.e. the "basement").

The textbook proposes that group work should be implemented for this problem. We agreed with the cooperative method, because the problem-based learning style requires students' critical and active attitude to the problem and to their own and their classmates' thoughts. Furthermore communication is an essential part of this learning approach. Taking all of this into consideration, we implemented the task in pair work in our survey.

RESEARCH QUESTION

1. What kinds of problem solving strategy are used by the 5th and 6th Graders?

2. What are the characteristics of their patterning?

RESULTS AND DISCUSSION

In order to answer to the questions we investigated the written works. We analysed the works of 9 pairs of 5th Graders and 15 pairs of 6th Graders. They worked on the "House of cards" problem during the class together and were asked to complete the tasks on their worksheet (see Appendix 1). First they built the 3-storey house from cards (enactive representation) then completed the figural sequence (iconic representation) with the next two elements (3- and 4-storey house). In the third task they counted the number of cards and wrote it under the figures of the houses (symbolic representation). With the exception of 1 pair everybody solved the Task 1-3 correctly. This means that 23 pairs understood the problem itself and were able to identify and use its iconic representation form. The correct figure of the 4-storey house shows us that they saw the structure of the card-building too.

Special attention was paid to the solution of the Task 4-6. Concerning the first research question, we examine the strategies occurring in the solutions.

We developed our coding system for problem solving strategies inductively. The authors performed a pilot coding and gave a coding system for coders. Every written work was coded by two different coders, and in the last step we consolidated the corpus. Task 4, 5, and 6 in the worksheet were the coding units. The coding system for problem solving strategies as follows:

- *Counting.* The students draw the house and count the cards without any sign of looking for patterns. (Figure 2)

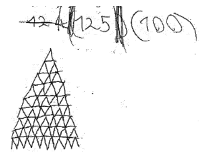

Figure 2: Example of the *Counting* strategy

- *Patterning*. The students refer to sequences A-E in problem solving. (Figure 3)

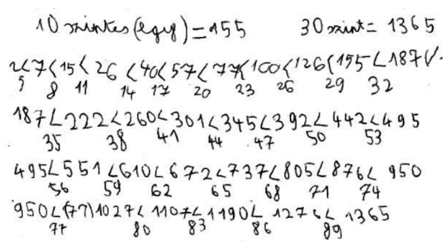

Figure 4: Example of the *Patterning* strategy (Sequence A and B; 10- and 30-storey house)

- *Recursion*. The students recall the one storey lower house while counting the cards of a particular house. (Figure 4)

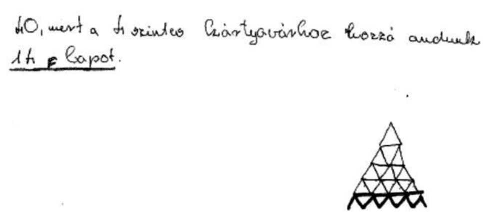

Figure 4: Example of *Recursion* (40, because we added 14 cards to the 4-storey house.)

- *False scheme*. The students refer to linear proportionality (Figure 5) or additivity (Figure 6).

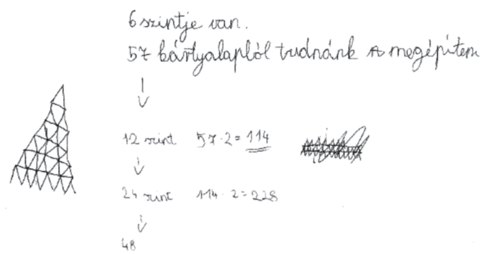

Figure 5: Example of *False scheme*, proportionality (It has 6 storeys. We can build it from 57 cards. → 12 storeys: 57·2=114 → 24 storeys: 114·2=228→48)

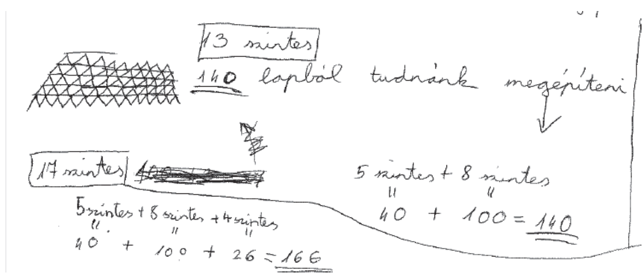

Figure 6: Example of *False scheme*, additivity (13-storey house; we can build it from 140 cards. 5-storey+8-storey = 40 +100 = 140)

– No answer

Table 2 gives an overview about the distribution of the applied strategies in the three tasks.

Number of works	5-storey house	8-storey-house	arbitrary house
5	Counting	Counting	Counting
1	Counting	Counting	Proportional scheme
1	Counting	Counting	Additive scheme
1	Counting	Counting	Patterning
2	Counting	Patterning	Patterning
1	Counting	Patterning	Additive scheme
7	Patterning	Patterning	Patterning
2	Recursion	Recursion	Recursion
2	Recursion	Recursion	Patterning
1	Recursion	Additive scheme	Recursion
1	No answer	No answer	No answer

Table 2: Strategies applied in the tasks

We can conclude that 14 pairs use the same strategy during their work and the most popular was the *Patterning* (7 works) then the *Counting* (5 works). 9 pairs used different strategies in the three tasks. 6 of them started with *Counting* in the case of 5-storey house, and then half of them recognized a pattern in the 8-storey house, while the others continued with *Counting* again. In the last task, where the drawing was difficult, *False scheme* appeared besides of the *Patterning*.

Figure 7 summarizes the applied strategies by tasks. We can see clearly, that the number of *Counting* strategy decreases, while the number of *Patterning* increases as the house gets higher. *False scheme* appears only in the case of high houses, when the *Counting* strategy does not work.

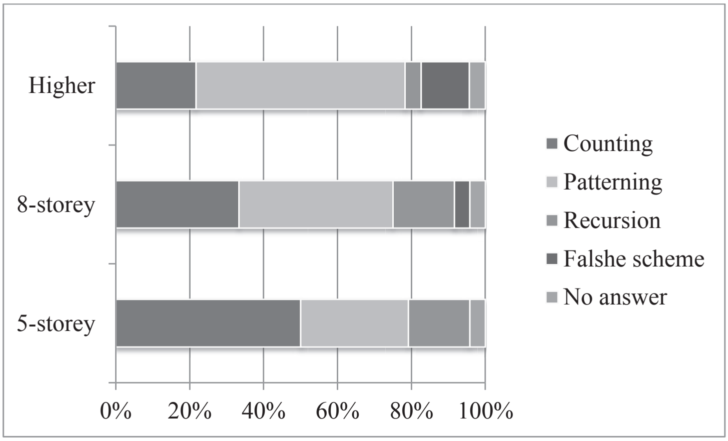

Figure 7: Distribution of the strategies by tasks

Concerning our second research question we investigate students' inductive reasoning, so we focus on those solutions which applied the *Patterning* strategy. The first phase of the inductive reasoning process i.e. *Observation of particular cases* was obvious in 23 works, because of the completing the Task 1-3. The next phase, namely *Following the observed pattern*, appeared in those works, where the *Patterning* strategy was applied. We detected all of the five sequences. Sequences A and B was used in 14 solutions (see Figure 4 as an example), sequences C and D, similarly in 14 solutions (Figure 8). Sequence E also appeared in 1 solution (Figure 9).

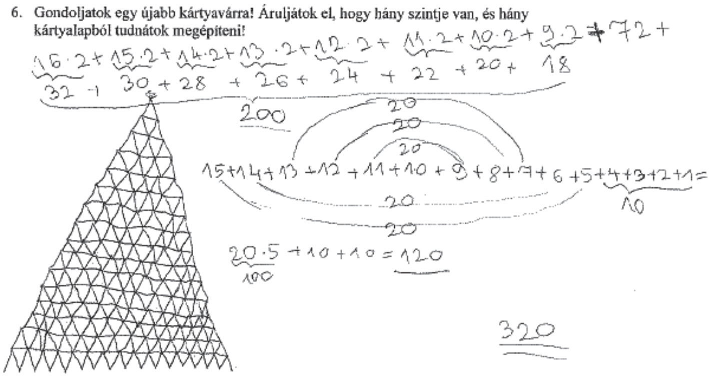

Figure 9: Example of the sequence C and D

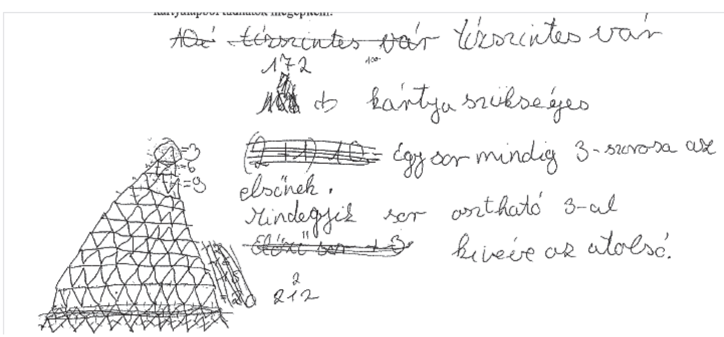

Figure 10: Using the sequence E (10-storey house, 172 cards. One row is 3 times more as the first one. Every row is divisible by 3, except the last one.)

We can conclude that in 14 works from the 24 *Patterning* strategy i.e. *Following the observed pattern* was detected at least in one task. Furthermore, the third phase of inductive reasoning *Formulating a conjecture* was observed only in some cases. We should make a difference between describing the way of counting they use and formulating a conjecture. The formulated conjecture contains typical phrases, like "always" we found it in 9 works, for example: "As much as the previous one has increased, you have to add 3 more to it." (Sequence A-B) or "Going from the top there is always 2 more cards in the rows; the cards that separate the rows always increase by 1." (Sequence C-D)

The control, namely testing the conjecture by other particular cases, didn't appear in this form. However, we observed in 9 works that the students drew the figure of the house because of the control of the patterning activity. Another way of the control appeared in one work only: they checked their result gained by using the Sequences C-D with the help of Sequences A-B, which was also recognized.

We couldn't find any clue of any kind for the argument for the discovered rule, except one work (Figure 10), where it was explained by marking the triangle on the top with a circle.

CONCLUSION

The simple *Counting* strategy was the most frequent one, especially in the case of 3-, 4-, 5- and 8-storey houses. The *Patterning* strategy also occurred in many cases, thanks to the possibility of using sequences in this problem's situation. The *Recursion*, i.e. the recursive thinking is closely related to the patterning activity. The lack of the generalization ability causes the appearance of the *False schemes*. The linear proportionality and the additive thinking are very common in the mathematics classrooms; the students use them automatically without any doubt about their compliance.

Following and observing a pattern is a well-known and often used strategy in the investigated age group. However, the further phases of the inductive thinking process are not realised at all. After the teacher requested it, the students were able to formulate and explain a "rule" or argue for it, using the real context that defined the pattern, but they didn't feel the need for such an explanation.

Our problem is closely related to the mathematical concept sequence. The problem solving activity contributed to the better understanding of that concept and to practice the flexible transition between the iconic and numeric representations.

Acknowledgement

The research was funded by the Content Pedagogy Research Program of the Hungarian Academy of Sciences.

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APPENDIX 1

Cards (Grade 5)

Name:...

Mark in mathematics after grade 4:

Name:...

Mark in mathematics after grade 4:

1. This is a three-story house. Build the house after the photo!

2. We drew the one-storey and two-story house. Continue drawing with the three-storey and four-storey house!

1-storey **2-storey** **3-storey** **4-storey**

3. Calculate the number of cards needed to build the houses and write in the rectangle under the figure.

4. How many cards are needed for the five-story house? Explain your answer!

5. How many cards are needed for the eight-story house? Explain your answer!

6. Think of another house! Tell us how many levels it has! How many cards are need to build it?