# Characterization of projective Finsler manifolds of constant curvature having infinite dimensional holonomy group

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*Dedicated to Professor Lajos Tamássy on his 90th birthday*

**Abstract.** In this paper we prove that the holonomy group of a simply connected locally projectively flat Finsler manifold of constant curvature is a finite dimensional Lie group if and only if it is flat or it is Riemannian. In particular, the holonomy group of non-Riemannian projective Finsler manifolds of nonzero constant curvature is infinite dimensional.

## 1. Introduction

In this paper we will consider some aspects of Hilberts fourth problem which "asks to construct and study the geometries in which the straight line segment is the shortest connection between two points", [2], or in other words: construct and study metrics on an open subset in an affine space whose geodesics are straight lines. Such metrics are called *projectively flat* Finsler metrics. According to Beltrami's theorem, a Riemannian metric is locally projectively flat if and only if it has constant sectional curvature. In the Finslerian case the flag curvature is the analogue of the Riemannian sectional curvature [5]. Contrary to the Riemannian case, there are many locally projectively flat Finsler metrics which are not

of constant flag curvature, and there are many Finsler metrics with constant flag curvature which are not locally projectively flat. The first example of projectively flat non-Riemannian Finsler metrics with constant negative flag curvature is the famous Funk metric [6], and the first locally projectively flat non-Riemannian Finsler metric with positive flag curvature was given by R. Bryant [4]. Z. Shen made a systematic study of locally projectively flat Finsler metrics of constant flag curvature in [13]. In this paper we are investigating some geometric properties (the dimension of the holonomy group) of locally projectively flat Finsler manifolds of constant curvature.

In order to set the problem properly let us introduce some notation. A Finsler manifold is a pair $(M, \mathcal{F})$, where $M$ is an $n$-manifold and $\mathcal{F} : TM \to \mathbb{R}$ is a non-negative function, smooth and positive away from the zero section of $TM$, positively homogeneous of degree 1, and strictly convex on each tangent space. A Finsler manifold of dimension 2 is called *Finsler surface*.

The concept of Finsler manifold is a direct generalization of the Riemannian one. The fundamental tensor $g = g_{ij}dx^i \otimes dx^j$ associated to $\mathcal{F}$ is formally analogous to the metric tensor in Riemannian geometry. It is defined by

$$g_{ij} := \frac{1}{2}\frac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^j}, \tag{1}$$

in an induced standard coordinate system $(x, y)$ on $TM$. As in Riemannian geometry, a canonical connection $\Gamma$ can be defined for a Finsler space [7]. However, since the energy function $E = \frac{1}{2}\mathcal{F}^2$ is not necessarily quadratic and only homogeneous, the connection is in general non-linear. In the case, when the connection $\Gamma$ is linear, the Finsler space is called *Berwald space*. In particular, every Riemannian manifold is a Berwald space.

Due to the existence of the canonical connection, the holonomy group of a Riemann or Finsler manifold, can be defined in a very natural way: it is the group generated by parallel translations along closed curves. In the Riemannian case, since the Levi–Civita connection is linear and preserves, the Riemannian metric, the holonomy group is a Lie subgroup of the orthogonal group $O(n)$ (see [3]). The Riemannian holonomy theory has been extensively studied, and by now, its complete classification is known.

The holonomy properties of Finsler spaces are essentially different from the Riemannian one. It is proved in [9] that the holonomy group of a Finsler manifold of nonzero constant curvature with dimension greater than 2 is not a compact Lie group. In [11] large families of projectively flat Finsler manifolds of constant

curvature are constructed such that their holonomy groups are not finite dimensional Lie groups. There are explicitly given examples of Finsler 2-manifolds having maximal holonomy group. In these examples the closure of the holonomy group is isomorphic to the orientation preserving diffeomorphism group of the 1-dimensional sphere [12].

In this paper we are investigating the holonomy group of locally projectively flat Finsler manifolds of constant curvature. A Finsler function $\mathcal{F}$ on an open subset $D \subset \mathbb{R}^n$ is called *projectively flat*, if all geodesic curves are straight lines in $D$. The Finsler manifold $(M, \mathcal{F})$ is said to be locally projectively flat, if for any point there exists a local coordinate system in which $\mathcal{F}$ is projectively flat. Our aim is to characterize all locally projectively flat Finsler manifolds with infinite dimensional holonomy group. To obtain such a characterization, we will investigate the dimension of the infinitesimal holonomy algebra which was introduced in the Finsler case in [10]. In Proposition 3.2 we prove that if $(M, \mathcal{F})$ is a non-Riemannian locally projectively flat Finsler manifolds of nonzero constant curvature, then its infinitesimal holonomy algebra is infinite dimensional. Using this result and the tangent property of the infinitesimal holonomy algebra proved in [10] we obtain the characterization given by Theorem 3.6: *The holonomy group of a locally projectively flat Finsler manifold of constant curvature is finite dimensional if and only if it is a Riemannian manifold or a flat Finsler manifold.* From Theorem 3.6 we obtain that *the holonomy group of non-Riemannian projective Finsler manifolds of nonzero constant curvature is infinite dimensional.*

## 2. Preliminaries

Throughout this article, $M$ is a $C^\infty$ smooth simply connected manifold, $\mathfrak{X}^\infty(M)$ is the vector space of smooth vector fields on $M$ and $\mathsf{Diff}^\infty(M)$ is the group of all $C^\infty$-diffeomorphism of $M$. The first and the second tangent bundles of $M$ are denoted by $(TM, \pi, M)$ and $(TTM, \tau, TM)$, respectively.

**2.1. Finsler manifolds.** A *Finsler manifold* is a pair $(M, \mathcal{F})$, where the Finsler function $\mathcal{F} : TM \to \mathbb{R}$ is a continuous function, smooth on $\hat{T}M := TM \setminus \{0\}$, its restriction $\mathcal{F}_x = \mathcal{F}|_{T_xM}$ is a positively homogeneous function of degree one and the symmetric bilinear form

$$g_{x,y} : (u, v) \mapsto g_{ij}(x, y)u^i v^j = \frac{1}{2} \frac{\partial^2 \mathcal{F}_x^2(y + su + tv)}{\partial s \, \partial t} \bigg|_{t=s=0} \tag{2}$$

is positive definite. The Finsler manifold $(M, \mathcal{F})$ is Riemannian, if $\mathcal{F}^2$ induces a quadratic form on any tangent space $T_x M$. Hence we say that $(M, \mathcal{F})$ is *non-Riemannian Finsler manifold* if there exists a point $x \in M$ such that $\mathcal{F}_x^2$ is not quadratic.

A vector field $X(t) = X^i(t)\frac{\partial}{\partial x^i}$ along a curve $c(t)$ is said to be parallel with respect to the associated *homogeneous (nonlinear) connection* if it satisfies

$$D_{\dot{c}}X(t) := \left(\frac{dX^i(t)}{dt} + G_j^i(c(t), X(t))\dot{c}^j(t)\right)\frac{\partial}{\partial x^i} = 0, \tag{3}$$

where the *geodesic coefficients* $G^i(x, y)$ are given by

$$G^i(x, y) := \frac{1}{4}g^{il}(x, y)\left(2\frac{\partial g_{jl}}{\partial x^k}(x, y) - \frac{\partial g_{jk}}{\partial x^l}(x, y)\right)y^j y^k. \tag{4}$$

and $G_j^i = \frac{\partial G^i}{\partial y^j}$, (cf. formulas (4.7) and (4.8) in [5]). The *horizontal Berwald covariant derivative* $\nabla_X \xi$ of $\xi(x, y) = \xi^i(x, y)\frac{\partial}{\partial y^i}$ by the vector field $X(x) = X^i(x)\frac{\partial}{\partial x^i}$ is expressed locally by

$$\nabla_X \xi = \left(\frac{\partial \xi^i(x, y)}{\partial x^j} - G_j^k(x, y)\frac{\partial \xi^i(x, y)}{\partial y^k} + G_{jk}^i(x, y)\xi^k(x, y)\right)X^j\frac{\partial}{\partial y^i}, \tag{5}$$

where $G_{jk}^i(x, y) := \frac{\partial G_j^i(x, y)}{\partial y^k}$.

A Finsler manifold $(M, \mathcal{F})$ is said to be *projectively flat*, if there exists a diffeomorphism of $M$ to an open subset $D \subset \mathbb{R}^n$ such that the images of geodesic curves are straight lines in $D$. A Finsler manifold $(M, \mathcal{F})$ is said to be *locally projectively flat*, if for every $x \in M$ there exists a local coordinate system $(U, x)$ such that $x = (x^1, \ldots, x^n)$ is mapping the neighbourhood $U$ into the Euclidean space $\mathbb{R}^n$ such that the straight lines of $\mathbb{R}^n$ correspond to the geodesics of $(M, \mathcal{F})$ on $U$. Then there exists a function $\mathcal{P}(x, y)$, such that the geodesic coefficients are given by

$$G^i(x, y) = \mathcal{P}(x, y)y^i, \qquad i = 1, \ldots, n \tag{6}$$

The function $\mathcal{P} = \mathcal{P}(x, y)$ is called the *projective factor* of $(M, \mathcal{F})$ on $U$. Since it is 1-homogeneous in the $y$-variable, we have also the following relations:

$$G_k^i = \frac{\partial \mathcal{P}}{\partial y^k}y^i + \mathcal{P}\delta_k^i, \quad G_{kl}^i = \frac{\partial^2 \mathcal{P}}{\partial y^k \partial y^l}y^i + \frac{\partial \mathcal{P}}{\partial y^k}\delta_l^i + \frac{\partial \mathcal{P}}{\partial y^l}\delta_k^i. \tag{7}$$

It follows from equation (6) that the associated homogeneous connection (3) is linear if and only if the projective factor $\mathcal{P}(x, y)$ is linear in $y$. According to Lemma 8.2.1 in [5] p.155, if $(M \subset \mathbb{R}^n, \mathcal{F})$ is a projectively flat Finsler manifold with constant flag curvature $\lambda$, then we have

$$\mathcal{P} = \frac{1}{2\mathcal{F}}\frac{\partial \mathcal{F}}{\partial x^i}y^i, \qquad \mathcal{P}^2 - \frac{\partial \mathcal{P}}{\partial x^i}y^i = \lambda\mathcal{F}^2. \tag{8}$$

*Remark 2.1.* From (8) we obtain that if $\lambda \neq 0$ and $\mathcal{P}(x, y)$ is linear in $y$ at $x \in M$ then $\mathcal{F}^2(x, y)$ is a quadratic form in $y$ at $x$.

**2.2. Holonomy group, curvature, infinitesimal holonomy algebra.** For a Finsler manifold $(M, \mathcal{F})$ of dimension $n$ the *indicatrix* at $x \in M$ is

$$\mathcal{I}_x M := \{y \in T_x M \mid \mathcal{F}(y) = 1\}$$

in $T_x M$ which is an $(n-1)$-dimensional submanifold of $T_x M$. We denote by $(\mathcal{I}M, \pi, M)$ the *indicatrix bundle* of $(M, \mathcal{F})$. We remark that, although the homogeneous (nonlinear) parallel translation is in general not metrical, that is it does not preserve the Finsler metric tensor (2), but it preserves the value of the Finsler function. That means that for any curves $c : [0, 1] \to M$, the induced parallel translation $\tau_c : T_{c(0)} M \to T_{c(1)} M$ induces a map $\tau_c : \mathcal{I}_{c(0)} M \to \mathcal{I}_{c(1)} M$ between the indicatrices.

The *holonomy group* $\mathsf{Hol}_x(M)$ of $(M, \mathcal{F})$ at a point $x \in M$ is the subgroup of the group of diffeomorphisms $\mathsf{Diff}^\infty(\mathcal{I}_x M)$ generated by (nonlinear) parallel translations of $\mathcal{I}_x M$ along piece-wise differentiable closed curves initiated at the point $x \in M$.

The *Riemannian curvature tensor*

$$R = R^i_{jk}(x, y) dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^i}$$

has the expression

$$R^i_{jk}(x, y) = \frac{\partial G^i_j(x, y)}{\partial x^k} - \frac{\partial G^i_k(x, y)}{\partial x^j} + G^m_j(x, y) G^i_{km}(x, y) - G^m_k(x, y) G^i_{jm}(x, y).$$

The manifold has *constant flag curvature* $\lambda \in \mathbb{R}$, if for any $x \in M$ the local expression of the Riemannian curvature is

$$R^i_{jk}(x, y) = \lambda \big(\delta^i_k g_{jm}(x, y) y^m - \delta^i_j g_{km}(x, y) y^m\big).$$

For any vector fields $X, Y \in \mathfrak{X}^\infty(M)$ on $M$ the vector field $\xi = R(X, Y) \in \mathfrak{X}^\infty(\mathcal{I}M)$ is called a *curvature vector field* of $(M, \mathcal{F})$ (see [9]). The Lie algebra $\mathfrak{R}(M)$ of vector fields generated by the curvature vector fields of $(M, \mathcal{F})$ is called the *curvature algebra* of $(M, \mathcal{F})$. The restriction $\mathfrak{R}_x(M) := \{\xi|_{\mathcal{I}_x M} \xi \in \mathfrak{R}(M)\} \subset \mathfrak{X}^\infty(\mathcal{I}_x M)$ of the curvature algebra to an indicatrix $\mathcal{I}_x M$ is called the *curvature algebra at the point* $x \in M$.

We remark that the indicatrix of a Finsler surface is 1-dimensional at any point $x \in M$, hence the curvature vector fields at $x \in M$ are proportional to

any given non-vanishing curvature vector field. Therefore the curvature algebra $\mathfrak{R}_x(M)$ is at most a 1-dimensional commutative Lie algebra.

The *infinitesimal holonomy algebra* of $(M, \mathcal{F})$ is the smallest Lie algebra $\mathfrak{hol}^*(M)$ of vector fields on the indicatrix bundle $\mathcal{I}M$ containing the curvature algebra and invariant with respect to the horizontal Berwald covariant differentiation. The $\mathfrak{hol}^*(M)$ is characterized by the following properties:

(i) any curvature vector field $\xi$ belongs to $\mathfrak{hol}^*(M)$,

(ii) if $\xi, \eta \in \mathfrak{hol}^*(M)$ then $[\xi, \eta] \in \mathfrak{hol}^*(M)$,

(iii) if $\xi \in \mathfrak{hol}^*(M)$ and $X \in \mathfrak{X}^\infty(M)$ then $\nabla_X \xi \in \mathfrak{hol}^*(M)$.

The restriction

$$\mathfrak{hol}_x^*(M) := \{\xi|_{\mathcal{I}_x M} \ \xi \in \mathfrak{hol}^*(M)\} \subset \mathfrak{X}^\infty(\mathcal{I}_x M)$$

of the infinitesimal holonomy algebra to an indicatrix $\mathcal{I}_x M$ is called the *infinitesimal holonomy algebra at the point* $x \in M$.

Clearly, we have $\mathfrak{R}(M) \subset \mathfrak{hol}^*(M)$ and $\mathfrak{R}_x(M) \subset \mathfrak{hol}_x^*(M)$ for any $x \in M$ (see [10]).

## 3. Dimension of the holonomy group

Let $(M, F)$ be a Finsler manifold and $x \in M$ an arbitrary point in $M$. According to Proposition 3 of [10], the infinitesimal holonomy algebra $\mathfrak{hol}_x^*(M)$ is tangent to the holonomy group $\mathsf{Hol}_x(M)$. Therefore the group generated by the exponential image of the infinitesimal holonomy algebra at $x \in M$ with respect to the exponential map

$$\exp_x : \mathfrak{X}^\infty(\mathcal{I}_x M) \to \mathsf{Diff}^\infty(\mathcal{I}_x M)$$

is a subgroup of the closed holonomy group $\overline{\mathsf{Hol}_x(M)}$ (see Theorem 3.1 of [12]). Consequently, we have the following estimation on the dimensions:

$$\dim \mathfrak{hol}_x^*(M) \le \dim \mathsf{Hol}_x(M). \tag{9}$$

Using the result of S. Lie claiming that the dimension of a finite dimensional Lie algebra of vector fields on a connected 1-dimensional manifold is less than 4 (cf. [1], Theorem 4.3.4) we can obtain the following

**Lemma 3.1.** *If the infinitesimal holonomy algebra* $\mathfrak{hol}_x^*(M)$ *of a Finsler surface* $(M, \mathcal{F})$ *contains* 4 *simultaneously non-vanishing* $\mathbb{R}$-*linearly independent vector fields, then* $\mathfrak{hol}_x^*(M)$ *is infinite dimensional.*

PROOF. If $\mathfrak{hol}_x^*(M)$ is a finite dimensional Lie algebra, then the corresponding Lie group $\mathrm{Hol}_x^*(M)$ is also finite dimensional. Since $\mathfrak{hol}_x^*(M)$ contains 4 simultaneously non-vanishing $\mathbb{R}$-linearly independent vector fields we obtain that $\mathrm{Hol}_x^*(M)$ acts locally effectively on the (1-dimensional) indicatrix $I_x M$, and its dimension is at least 4. This is in contradiction to Lie's theorem.    □

Using Lemma 3.1 we can prove the following

**Proposition 3.2.** *The infinitesimal holonomy algebra of any locally projectively flat non-Riemannian Finsler surface $(M, \mathcal{F})$ of constant curvature $\lambda \neq 0$ is infinite dimensional.*

PROOF. Assume that the locally projectively flat Finsler surface $(M, \mathcal{F})$ of non-zero constant curvature $\lambda$ is non-Riemannian at a fixed point $x \in M$. Let $(x^1, x^2)$ be a local coordinate system centered at $x$, corresponding to the canonical coordinates of the Euclidean plane which is projectively related to $(M, \mathcal{F})$, and let $(y^1, y^2)$ be the induced coordinate system in the tangent planes $T_x M$.

Consider the curvature vector field $\xi \in \mathfrak{X}(I_x M)$ at the point $x \in M$ defined as

$$y \rightarrow \xi(x, y) := R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)(x, y) = \lambda\big(\delta_2^i g_{1m}(x, y) y^m - \delta_1^i g_{2m}(x, y) y^m\big)\frac{\partial}{\partial x^i}.$$

Since $\lambda \neq 0$, the vector field $\xi$ is non-vanishing. Moreover, since $(M, \mathcal{F})$ is of constant flag curvature, the horizontal Berwald covariant derivative $\nabla_W R$ of the curvature tensor field $R$ vanishes and one has

$$\nabla_W \xi = R\left(\nabla_k\left(\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}\right)\right) W^k.$$

Since

$$\nabla_k\left(\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}\right) = \left(G_{k1}^1 + G_{k2}^2\right)\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}$$

we obtain $\nabla_W \xi = (G_{k1}^1 + G_{k2}^2)W^k \xi$. According to (7) we have

$$G_{km}^m = 3\frac{\partial \mathcal{P}}{\partial y^k}$$

and hence $\nabla_k \xi = 3\frac{\partial \mathcal{P}}{\partial y^k}\xi$, where $\nabla_k = \nabla_{\frac{\partial}{\partial x^k}}$. Moreover we have

$$\nabla_j\left(\frac{\partial \mathcal{P}}{\partial y^k}\right) = \frac{\partial^2 \mathcal{P}}{\partial x^j \partial y^k} - G_j^m \frac{\partial^2 \mathcal{P}}{\partial y^m \partial y^k} = \frac{\partial^2 \mathcal{P}}{\partial x^j \partial y^k} - \mathcal{P}\frac{\partial^2 \mathcal{P}}{\partial y^k \partial y^j},$$

and hence

$$\nabla_j \left( \nabla_k \xi \right) = 3 \left( \frac{\partial^2 \mathcal{P}}{\partial x^j \partial y^k} - \mathcal{P} \frac{\partial^2 \mathcal{P}}{\partial y^k \partial y^j} + 3 \frac{\partial \mathcal{P}}{\partial y^k} \frac{\partial \mathcal{P}}{\partial y^j} \right) \xi.$$

According to Lemma 8.2.1, equation (8.25) in [5], p. 155, we have

$$\frac{\partial^2 \mathcal{P}}{\partial x^j \partial y^k} = \frac{\partial \mathcal{P}}{\partial y^j} \frac{\partial \mathcal{P}}{\partial y^k} + \frac{\partial^2 \mathcal{P}}{\partial y^j \partial y^k} - \lambda \, g_{jk}, \tag{10}$$

hence

$$\nabla_j \left( \nabla_k \xi \right) = 3 \left( 4 \frac{\partial \mathcal{P}}{\partial y^j} \frac{\partial \mathcal{P}}{\partial y^k} - \lambda \, g_{jk} \right) \xi.$$

It follows

**Lemma 3.3.** *For any fixed $1 \leq j, k \leq 2$, the vector field on $\mathcal{I}_x M$ defined as*

$$y \to \xi(x, y), \quad y \to \nabla_1 \xi(x, y), \quad y \to \nabla_2 \xi(x, y), \quad y \to \nabla_j \left( \nabla_k \xi \right)(x, y), \tag{11}$$

*are $\mathbb{R}$-linearly independent if and only if*

$$1, \qquad \frac{\partial \mathcal{P}}{\partial y^1}, \qquad \frac{\partial \mathcal{P}}{\partial y^2}, \qquad \frac{\partial^2 \mathcal{P}}{\partial y^j \partial y^k} - \frac{\lambda}{4} \, g_{jk} \tag{12}$$

*are $\mathbb{R}$-linearly independent $\left(\text{as elements of } C^\infty(T_x M)\right)$ functions on $T_x M$.*

Indeed, since we assumed that the Finsler function $\mathcal{F}$ is non-Riemannian at the point $x$, then $\mathcal{F}^2(x, y)$ is non-quadratic in $y$ and according to Remark 2.1, the function $\mathcal{P}(x, y)$ is non-linear in $y$ on $T_x M$. Let us choose a direction $y_0 = (y_0^1, y_0^2) \in T_x M$ with $y_0^1 \neq 0$, $y_0^2 \neq 0$ and having property that $\mathcal{P}$ is non-linear 1-homogeneous function in a conic neighbourhood $U$ of $y_0$ in $T_x M$. By restricting $U$ if it is necessary we can suppose that for any $y \in U$ we have $y^1 \neq 0$, $y^2 \neq 0$.

To avoid confusion between coordinate indexes and exponents, we rename the fiber coordinates of vectors belonging to $U$ by $(u, v) = (y^1, y^2)$. Using the values of $\mathcal{P}$ on $U$ we can define a 1-variable function $f = f(t)$ on an interval $I \subset \mathbb{R}$ by

$$f(t) := \frac{1}{v} \mathcal{P}(x_1, x_2, tv, v). \tag{13}$$

Then we can express $\mathcal{P}$ and its derivatives with $f$:

$$\mathcal{P} = v \, f(u/v), \qquad \frac{\partial \mathcal{P}}{\partial y^1} = f'(u/v), \qquad \frac{\partial \mathcal{P}}{\partial y^2} = f(u/v) - \frac{u}{v} f'(u/v),$$

$$\frac{\partial^2 \mathcal{P}}{\partial y^1 \partial y^1} = \frac{1}{v} f''(u/v), \quad \frac{\partial^2 \mathcal{P}}{\partial y^1 \partial y^2} = -\frac{u}{v^2} f''(u/v), \quad \frac{\partial^2 \mathcal{P}}{\partial y^2 \partial y^2} = \frac{u^2}{v^3} f''(u/v). \tag{14}$$

**Lemma 3.4.** *The functions* $1$, $\frac{\partial \mathcal{P}}{\partial y^1}$, $\frac{\partial \mathcal{P}}{\partial y^2}$ *are* $\mathbb{R}$-*linearly independent.*

PROOF. A nontrivial relation $a + b\frac{\partial \mathcal{P}}{\partial y^1} + c\frac{\partial \mathcal{P}}{\partial y^2} = 0$ yields the differential equation

$$a + bf' + c(f - tf') = 0.$$

It is clear that both $b$ and $c$ cannot be zero. If $c \neq 0$ we get the differential equation

$$\frac{(a + cf)'}{a + cf} = \frac{1}{t - \frac{b}{c}}.$$

The solutions is $f(t) = t - (a + b)/c$ and therefore the corresponding $\mathcal{P}(u, v) = u - v(a + b)/c$ is linear which is a contradiction. If $c = 0$, then $b \neq 0$ and $f = -\frac{a}{b}t + K$. The corresponding $\mathcal{P}(u, v) = -\frac{a}{b}u + Kv$ is again linear which is a contradiction. $\square$

Let us turn back to the proof of Proposition 3.2. Assume, that the infinitesimal holonomy algebra is finite dimensional. We will show that this assumption leads to contradiction which will prove then, that the infinitesimal holonomy algebra is actually infinite dimensional.

Since $\mathcal{I}_x M$ is 1-dimensional, according to the Lemma 3.1, the 4 vector fields in (11) are $\mathbb{R}$-linearly dependent for any $j, k \in \{1, 2\}$. Using Lemma 3.3 we obtain that the functions

$$1, \quad \mathcal{P}_1, \quad \mathcal{P}_2, \quad \mathcal{P}_j\mathcal{P}_k - \frac{\lambda}{4}g_{jk} \tag{15}$$

$\left(\mathcal{P}_i = \frac{\partial \mathcal{P}}{\partial y^i}, \mathcal{P}_{jk} = \frac{\partial^2 \mathcal{P}}{\partial y^j \partial y^k}\right)$ are $\mathbb{R}$-linearly dependent for any $j, k \in \{1, 2\}$. However, from Lemma 3.4 we know, that the first three functions in (15) are $\mathbb{R}$-linearly independent. Therefore by the assumption, the fourth function must be a linear combination of the first three, that is there exist constants $a_i, b_i, c_i \in \mathbb{R}$, $i = 1, 2, 3$, such that

$$\frac{\lambda}{4} g_{11} = \mathcal{P}_1\mathcal{P}_1 + a_1 + b_1\mathcal{P}_1 + c_1\mathcal{P}_2,$$

$$\frac{\lambda}{4} g_{12} = \mathcal{P}_1\mathcal{P}_2 + a_2 + b_2\mathcal{P}_1 + c_2\mathcal{P}_2,$$

$$\frac{\lambda}{4} g_{22} = \mathcal{P}_2\mathcal{P}_2 + a_3 + b_3\mathcal{P}_1 + c_3\mathcal{P}_2. \tag{16}$$

Using (1) we have to have $\partial_1 g_{21} - \partial_2 g_{11} = 0$ and $\partial_1 g_{22} - \partial_2 g_{12} = 0$ which yield

$$\mathcal{P}_2\mathcal{P}_{11} - \mathcal{P}_1\mathcal{P}_{12} + b_2\mathcal{P}_{11} + (c_2 - b_1)\mathcal{P}_{12} - c_1\mathcal{P}_{22} = 0,$$

$$\mathcal{P}_1\mathcal{P}_{22} - \mathcal{P}_2\mathcal{P}_{12} - b_3\mathcal{P}_{11} + (b_2 - c_3)\mathcal{P}_{12} + c_2\mathcal{P}_{22} = 0. \tag{17}$$

Using the expressions (14) we obtain from (17) the equations

$$\left(f - \frac{u}{v}f'\right)\frac{1}{v}f'' + f'\frac{u}{v^2}f'' + b_2\frac{1}{v}f'' - (c_2 - b_1)\frac{u}{v^2}f'' - c_1\frac{u^2}{v^3}f'' = 0,$$

$$f'\frac{u^2}{v^3}f'' + (f - \frac{u}{v}f')\frac{u}{v^2}f'' - b_3\frac{1}{v}f'' - (b_2 - c_3)\frac{u}{v^2}f'' + c_2\frac{u^2}{v^3}f'' = 0. \qquad (18)$$

Since by the non-linearity of $\mathcal{P}$ on $U$ we have $f'' \neq 0$, equations (18) can divide by $f''/v$ and we get

$$f + b_2 + (b_1 - c_2)\frac{u}{v} - \frac{c_1 u^2}{v^2} = 0$$

$$\frac{u}{v}f - b_3 + (c_3 - b_2)\frac{u}{v} + \frac{c_2 u^2}{v^2} = 0. \qquad (19)$$

for any $t = u/v$ in an interval $I \subset \mathbb{R}$. The solution of this system of quadratic equations for the function $f$ is $f(t) = -c_2\,t - b_2$ with $c_1 = b_3 = 0$, $b_1 = 2c_2$, $c_3 = 2b_2$. But this is a contradiction, since we supposed that by the non-linearity of $P$ we have $f'' \neq 0$ on this interval. Hence the functions $1, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_j\mathcal{P}_k - \frac{\lambda}{4}g_{jk}$ can not be linearly dependent for any $j, k \in \{1, 2\}$, from which follows the assertion. $\qquad \square$

*Remark 3.5.* From Proposition 3.2 we get that if $(M, \mathcal{F})$ is non-Riemannian and $\lambda \neq 0$, then the holonomy group has an infinite dimensional tangent algebra.

Indeed, according to Theorem 6.3 in [10] the infinitesimal holonomy algebra $\mathfrak{hol}_x^*(M)$ is tangent to the holonomy group $\mathsf{Hol}_x(M)$, from which follows the assertion.

Now, we can prove our main result:

**Theorem 3.6.** *The holonomy group of a locally projectively flat simply connected Finsler manifold $(M, \mathcal{F})$ of constant curvature $\lambda$ is finite dimensional if and only if $(M, \mathcal{F})$ is Riemannian or $\lambda = 0$.*

PROOF. If $(M, \mathcal{F})$ is Riemannian then its holonomy group is a Lie subgroup of the orthogonal group and therefore it is a finite dimensional compact Lie group. If $(M, \mathcal{F})$ has zero curvature, then the horizontal distribution associated to the canonical connection in the tangent bundle is integrable and hence the holonomy group is trivial.

If $(M, \mathcal{F})$ is non-Riemannian having non-zero curvature $\lambda$, then for each tangent 2-plane $S \subset T_xM$ the manifold $M$ has a totally geodesic submanifold $\widetilde{M} \subset M$ such that $T_x\widetilde{M} = S$. This $\widetilde{M}$ with the induced metric is a locally projectively

flat Finsler surface of constant curvature $\lambda$. Therefore from Proposition 3.2 we get that $\mathfrak{hol}_x^*(\widetilde{M})$ is infinite dimensional. Moreover, according to Theorem 4.3 in [11], if a Finsler manifold $(M, \mathcal{F})$ has a totally geodesic 2-dimensional submanifold $\widetilde{M}$ such that the infinitesimal holonomy algebra of $\widetilde{M}$ is infinite dimensional, then the infinitesimal holonomy algebra $\mathfrak{hol}_x^*(M)$ of the containing manifold is also infinite dimensional. Using (9) we get that $\mathsf{Hol}_x(M)$ cannot be finite dimensional. Hence the assertion is true.                                                  □

Theorem 3.6 can be reformulated in the following way

**Theorem 3.7.** *The holonomy group of a locally projectively flat Finsler manifold of constant curvature is infinite dimensional if and only if it is non-Riemannian with nonzero curvature.*

We note that there are examples of non-Riemannian type locally projectively flat Finsler manifolds with $\lambda = 0$ and $\lambda \neq 0$ curvature, (cf. [13], [8]).

*Remark 3.8.* In the discussion before Theorem 3.6, the key condition for the Finsler metric tensor was not the positive definiteness but its non-degenerate property. Therefore Theorem 3.6 can be generalized as follows.

A pair $(M, \mathcal{F})$ is called *semi-Finsler manifold* if in the definition of Finsler manifolds the positive definitness of the Finsler metric tensor is replaced by the nondegenerate property. Then we have

**Corollary 3.9.** *The holonomy group of a locally projectively flat simply connected semi-Finsler manifold $(M, \mathcal{F})$ of constant curvature $\lambda$ is finite dimensional if and only if $(M, \mathcal{F})$ is semi-Riemannian or $\lambda = 0$.*

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