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Tangent Lie Algebra of a Diffeomorphism Group and Application to Holonomy Theory

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Abstract
In this paper we introduce the notion of tangent space $\mathcal{T}_o\mathcal{G}$ of a (not necessary smooth) subgroup $\mathcal{G}$ of the diffeomorphism group $\mathcal{D}iff^{\infty}(M)$ of a compact manifold M. We prove that $\mathcal{T}_o\mathcal{G}$ is a Lie subalgebra of the Lie algebra of smooth vector fields on M. The construction can be generalized to subgroups of any (finite- or infinite-dimensional) Lie groups. The tangent Lie algebra $\mathcal{T}_o\mathcal{G}$ introduced this way is a generalization of the classical Lie algebra in the smooth cases. As a working example we discuss in detail the tangent structure of the holonomy group and fibered holonomy group of Finsler manifolds.

Keywords Diffeomorphism group · Infinite-dimensional Lie group · Holonomy group · Finsler geometry

Mathematics Subject Classification 22E65 · 17B66 · 53C29 · 53B40

1 Introduction

Important geometric objects, structures, or properties can often be investigated through algebraic structures. In many interesting cases, these algebraic structures are groups, where the group operations are smooth maps. Such groups became indispensable tools for modern geometry, analysis, and theoretical physics. Lie groups and diffeomorphism groups are the most important examples for such structures.

Considering a Lie group $\mathcal{G}_L$, it is well known that most of the important information about it is captured in its tangent object, the Lie algebra $\mathfrak{g}_L$. Naturally, if $\mathcal{G}$ is a Lie subgroup of $\mathcal{G}_L$, then its Lie algebra $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}_L$. The Lie subalgebra $\mathfrak{g} \subset \mathfrak{g}_L$

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can be used to obtain information or eventually to determine the subgroup $\mathcal{G}$. In many relevant geometric situations, however, this framework is not general enough because of two factor: Firstly, $\mathcal{G}_L$ is not a (finite-dimensional) Lie group but the (infinite-dimensional) diffeomorphism group $\mathcal{D}iff^\infty(M)$ of some manifold M. Secondly, the subgroup $\mathcal{G}$ is not necessarily a Lie subgroup of $\mathcal{D}iff^\infty(M)$. Nevertheless, natural questions arise: *can we introduce a tangential property and tangent objects to the subgroup $\mathcal{G}$ in this situation? Does the set of tangent elements possess a special algebraic stricture? Can this algebraic structure be used to get information about the subgroup and thus about geometric properties?* In this paper we answer these questions.

We introduce the notion of tangent vector fields to a subgroup $\mathcal{G}$ of the diffeomorphism group $\mathcal{D}iff^\infty(M)$, where M is a compact manifold. Denoting by $\mathcal{T}_o\mathcal{G}$ the set of tangent vector fields to $\mathcal{G}$ at the identity, we prove that $\mathcal{T}_o\mathcal{G}$ is a Lie subalgebra of the Lie algebra of smooth vector fields on M (Theorem 3.4). It follows that subalgebras of $\mathcal{T}_o\mathcal{G}$ inherit the tangential properties, and therefore the elements of a subalgebra generated by vector fields tangent to the subgroup $\mathcal{G}$ are tangent to $\mathcal{G}$ (Corollary 3.6). This property can be particularly interesting when the Lie bracket of two tangent vector fields to $\mathcal{G}$ generates a new direction: the tangential property will be satisfied in this new direction as well. As we show in Theorem 3.10, the group generated by the exponential image of $\mathcal{T}_o\mathcal{G}$ is a subgroup of the closure of $\mathcal{G}$ in $\mathcal{D}iff^\infty(M)$ which can give important information about the group $\mathcal{G}$ itself, especially in the infinite-dimensional cases.

We note that a similar tangential property was already introduced in [10, Definition 2], but we also remark that the concept had two major defects: the tangent property introduced in [10] is not preserved under the bracket operation, and therefore in that approach it is not true that tangent vector fields to a subgroup $\mathcal{G}$ generate a tangent Lie algebra to $\mathcal{G}$. Secondly, [10] was not able to guaranty the existence of the tangent Lie algebra $\mathcal{T}_o\mathcal{G}$ associated to $\mathcal{G}$. We note that with our approach we are able to overcome both deficiencies.

The main reason to investigate the tangent structure of a subgroup $\mathcal{G}$ of the diffeomorphism group is that it can provide valuable information about the group $\mathcal{G}$ itself. This method can be very effective when $\mathcal{G}$ is, for example, a symmetry group, the holonomy group, etc. We note that in many cases the determination of $\mathcal{T}_o\mathcal{G}$ or its subalgebras can be highly nontrivial, especially in the infinite-dimensional cases. As working examples, we consider the holonomy group and the fibered holonomy group of Finsler manifolds. The holonomy group is the transformation group generated by parallel translations with respect to the canonical connection along closed curves. For Riemannian manifolds it has been extensively studied and now the complete classification is known [1–3,6]. In particular, it is well known that the holonomy group of a simply connected Riemannian manifold is a closed Lie subgroup of the special orthogonal group $SO(n)$. Despite the analogues in the construction, Finslerian holonomy groups can be much more complex and up to now, we do not know much about them: For special spaces the holonomy can be a finite-dimensional Lie group (see [7,17]), but recent results show that there are Finsler manifolds with infinite-dimensional holonomy group [12–14]. These latter results show the difficulties: one cannot use the well-understood principal bundle machinery in the investigation because the structure group should be infinite dimensional. In [9] Michor proposed a general setting for the study of infinite-dimensional holonomy groups and holonomy algebras which was the motivation for Z. Muzsnay and P.T. Nagy to start investigating the tangent objects to a subgroup of the diffeomorphism group [10]. In this paper we are able to step forward: using the results of Sect. 3, we are able to introduce the notion of holonomy algebra and fibered holonomy algebra for Finslerian manifolds. By improving the results of [10] we also prove that the curvature and the infinitesimal holonomy algebras (resp. their restrictions) are Lie subalgebras of the fibered holonomy algebras (resp. the holonomy algebra). We are confident that in the future, the tools described above can be used successfully in the investigation of geometric structures in general and in the holonomy theory in particular.

2 Preliminaries

In this chapter we introduce the basic notions and concepts of Finsler geometry which are necessary to understand in Sects. 4.1 and 4.2 the nontrivial application of the theory discussing the tangent structure of a subgroup of the diffeomorphism group. These notions are not necessary to understand Sect. 3, and therefore the reader, who is not particularly interested in these applications, can jump directly to the next chapter.

In this paper, M denotes a C^∞-smooth n-dimensional manifold, $\mathfrak{X}^\infty(M)$ is the Lie algebra of C^∞ vector fields, and $\mathcal{D}iff^\infty(M)$ is the group of C^∞ diffeomorphisms of M. We will denote by TM the tangent manifold and by $\widehat{T}M = TM \setminus \{0\}$ the slit tangent manifold. Local coordinate charts (U, x^i) on M induce local coordinate charts $(\pi^{-1}(U), (x^i, y^i))$ on TM, where $\pi : TM \to M$ is the canonical projection. The *vertical distribution* $\mathcal{V}TM \subset TTM$ on TM is given by $\mathcal{V}TM = \operatorname{Ker} \pi_*$.

2.1 Finsler Manifold

A *Finsler manifold* is a pair $(M, \mathcal{F})$, where the norm $\mathcal{F}\colon TM \to \mathbb{R}_+$ is a positively 1-homogeneous continuous function, which is smooth on $\hat{T}M$ and the symmetric bilinear form

$$g_{x,y}\colon (u, v) \ \mapsto \ g_{ij}(x, y)u^i v^j = \frac{1}{2} \frac{\partial^2 \mathcal{F}_x^2(y + su + tv)}{\partial s\, \partial t}\Big|_{t=s=0}$$

is positive definite at every $y \in \hat{T}_x M$. The *indicatrix* $\mathcal{I}_x M$ at $x \in M$ is a hypersurface of $T_x M$ defined by

$$\mathcal{I}_x M = \{\, y \in T_x M : \mathcal{F}(y) = 1 \,\}. \tag{1}$$

Geodesics of $(M, \mathcal{F})$ are determined by a system of second-order ordinary differential equations $\ddot{x}^i + 2G^i(x, \dot{x}) = 0$, $i = 1, \dots, n$ in a local coordinate system (x^i, y^i) of TM, where $G^i(x, y)$ are determined by the formula

$$4G^i = g^{il} \left(2 \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right) y^j y^k.$$

2.2 Parallel Translation

A vector field $X(t) = X^i(t)\frac{\partial}{\partial x^i}$ along a curve $c\colon [0,1] \to M$ is called *parallel* if $D_{\dot c}X(t) = 0$ where the covariant derivative is defined as

$$D_{\dot c}X(t) = \left(\frac{\mathrm{d}X^i(t)}{\mathrm{d}t} + G^i_j(c_t, X(t))\,\dot{c_t}^j\right)\frac{\partial}{\partial x^i}, \tag{2}$$

with $G^i_j = \frac{\partial G^i}{\partial y^j}$. Clearly, for any $X_0 \in T_{c(0)}M$ there is a unique parallel vector field $X(t)$ along the curve c such that $X_0 = X(0)$. Moreover, if $X(t)$ is a parallel vector field along c, then $\lambda X(t)$ is also parallel along c for any $\lambda \geq 0$. Then the *homogeneous (nonlinear) parallel translation* along a curve $c(t)$

$$\mathcal{P}^t_c : T_{c_0}M \to T_{c_t}M \tag{3}$$

is defined by the positive homogeneous map $\mathcal{P}^t_c : X_0 \mapsto X_t$ given by the value $X_t = X(t)$ of the parallel vector field with initial value $X(0) = X_0$. We remark that (3) preserves the Finslerian norm, and therefore it can be considered as a map between the indicatrices

$$\mathcal{P}^t_c : \mathcal{I}_{c_0}M \to \mathcal{I}_{c_t}M. \tag{4}$$

Moreover, since the parallel translation is a homogeneous map, (3) and (4) determine each other.

2.3 Holonomy

The *holonomy group* $\mathcal{H}ol_p(M)$ of a Finsler manifold (M, F) at a point $p \in M$ is the group generated by parallel translations along piece-wise differentiable closed curves starting at p. Considering the parallel translation (4) on the indicatrix, a holonomy element is a diffeomorphism $\mathcal{P}_c : \mathcal{I}_p \to \mathcal{I}_p$, and therefore the holonomy group $\mathcal{H}ol_p(M) \subset \mathcal{D}iff^\infty(\mathcal{I}_p)$ is a subgroup of the diffeomorphism group of the indicatrix $\mathcal{I}_p$.

In the particular case, when $(M, \mathcal{F})$ is a simply connected Riemann manifold, the holonomy group is a closed Lie subgroup of the special orthogonal group $SO(n)$. Finslerian holonomy groups can, however, be much more complex: in [12–14] one can find examples of Finsler manifolds with infinite-dimensional holonomy groups. Until now it is not known if the Finsler holonomy groups are (finite- or infinite-dimensional) Lie groups or not.

2.4 Horizontal Lift and Curvature

The parallel translation on a Finsler manifold can also be introduced by considering the associated Ehresmann connection (cf. [18]): the horizontal distribution is determined by the horizontal lift $T_xM \to T_{(x,y)}TM$ defined in the local basis as

$$\left(\frac{\partial}{\partial x^i}\right)^h = \frac{\partial}{\partial x^i} - G_i^k(x, y)\frac{\partial}{\partial y^k}, \tag{5}$$

where $y \in T_xM$. Since the horizontal distribution is complementary to the vertical distribution, we have the decomposition $T_yTM = \mathcal{H}_y \oplus \mathcal{V}_y$ with canonical projectors $h\colon TTM \to \mathcal{H}$ and $v\colon TTM \to \mathcal{V}$. The image $\mathcal{H} \subset TTM$ is the *horizontal distribution* of the manifold. The horizontal lift of a curve $c\colon [0, 1] \to M$ with initial condition $X_0 \in T_{c_0}M$ is a curve $c^h\colon [0, 1] \to TM$ such that $\pi \circ c^h = c$, $\frac{dc^h}{\mathrm{d}t} = (\frac{dc}{\mathrm{d}t})^h$ and $c^h(0) = X_0$. Then the parallel translation can be geometrically obtained as $\mathcal{P}_c^t(X_0) = c^h(t)$. We remark that the horizontal lift φ_t^h of the flow φ_t of a vector field $X \in \mathfrak{X}^\infty(M)$ is the flow of the horizontal lift of the vector field $X^h \in \mathfrak{X}^\infty(TM)$. Therefore the parallel translation along the integral curves of X can be calculated in terms of the horizontal lift of the flow:

$$\mathcal{P}_\varphi^t = \varphi_t^h. \tag{6}$$

The horizontal distribution $\mathcal{H}TM$ is, in general, non-integrable. The obstruction to its integrability is given by the *curvature tensor* $R = \frac{1}{2}[h, h]$ which is the Nijenhuis torsion of the horizontal projector [4].

3 Tangent Lie Algebra of a Subgroup of the Diffeomorphism Group

In this paragraph we investigate the tangential property and tangential structure of subgroups of the diffeomorphism group. Let $\mathcal{G}$ be a subgroup of $\mathcal{D}iff^\infty(M)$ where M is a compact differentiable manifold. We do not suppose any special property on $\mathcal{G}$; in particular, we do not suppose that $\mathcal{G}$ is a Lie subgroup of $\mathcal{D}iff^\infty(M)$. Questions that we consider: can we introduce a tangential property and tangent object to the subgroup $\mathcal{G}$? Does the set of tangent elements possess a special algebraic structure? Can this algebraic structure be used to get information about the subgroup? In this paragraph, we answer all these questions.

A smooth curve $c\colon I \to M$ on the manifold M has a $(k-1)$st-order singularity at $t = 0$, if its derivatives vanish up to order $k-1$, $(k \geq 0)$. It is well known that if a curve c has a $(k-1)$st-order singularity at $0 \in \mathbb{R}$ then its kth-order derivative $c^{(k)}(0) = X_p$ is a tangent vector at $p = c(0)$. In that case, the curve c is called a kth-*order integral curve* of the vector $X_p \in T_pM$. Extending this concept to vector fields, we can introduce the following:

Definition 3.1 A C^∞−smooth curve in the diffeomorphism group $\varphi\colon I \to \mathcal{D}iff^\infty(M)$, $t \to \varphi_t$ is called an *integral curve of the vector field* $X \in \mathfrak{X}^\infty(M)$ if

(1) $\varphi_0 = id_M$,
(2) there exists $k \in \mathbb{N}$ such that for any point $p \in M$ the curve $t \to \varphi_t(p)$ is a kth-order integral curve of $X(p) \in T_pM$.

This $k \in \mathbb{N}$ is called the *order* of the integral curve φ_t of the vector field X.

In particular, the flow φ_t^X of $X \in \mathfrak{X}^\infty(M)$ is a 1st-order integral curve of X. Moreover, if $k > 1$ and $t \to \varphi_t$ is a k^{th}–order integral curve of the vector field X, then we have

$$\varphi_0 = id_M, \qquad \frac{\partial \varphi_t}{\partial t}\Big|_{t=0} = 0, \quad \ldots \quad \frac{\partial^{k-1} \varphi_t}{\partial t^{k-1}}\Big|_{t=0} = 0, \qquad \frac{\partial^k \varphi_t}{\partial t^k}\Big|_{t=0} = X. \tag{7}$$

Let $\mathcal{G} \subset \mathcal{D}iff^\infty(M)$ be an arbitrary subgroup of the diffeomorphism group $\mathcal{D}iff^\infty(M)$. Using the terminology of Definition 3.1 we introduce the following:

Definition 3.2 A vector field $X \in \mathfrak{X}^\infty(M)$ is called *tangent* to a subgroup $\mathcal{G} \subset \mathcal{D}iff^\infty(M)$ of the diffeomorphism group if there exists an integral curve of X in $\mathcal{G}$. The set of tangent vector fields of $\mathcal{G}$ is denoted by $\mathcal{T}_o\mathcal{G}$.

Remark 3.3 We have $X \in \mathcal{T}_o\mathcal{G}$ if and only if there exists a C^∞–smooth curve $\varphi\colon I \to Diff^\infty(M)$ such that

(*1*) $\varphi_t \in \mathcal{G}$,
(2) $\varphi_0 = id_M$,
(*3*) there exists $k \in \mathbb{N}$ such that Eq. (7) is satisfied.

One can observe that in Definition 3.2 we do not suppose that $\mathcal{G}$ is a Lie subgroup of $\mathcal{D}iff^\infty(M)$. Indeed, we use the differential structure of the later to formulate the smoothness condition on the curve in $\mathcal{G}$. Nevertheless, we have the following:

Theorem 3.4 *If $\mathcal{G}$ is a subgroup of $\mathcal{D}iff^\infty(M)$, then $\mathcal{T}_o\mathcal{G}$ is a Lie subalgebra of $\mathfrak{X}^\infty(M)$.*

Proof To prove the theorem, we have to show that

$$X, Y \in \mathcal{T}_o\mathcal{G} \Rightarrow [X, Y] \in \mathcal{T}_o\mathcal{G}, \tag{8a}$$
$$X, Y \in \mathcal{T}_o\mathcal{G} \Rightarrow X + Y \in \mathcal{T}_o\mathcal{G}, \tag{8b}$$
$$\lambda \in \mathbb{R},\ X \in \mathcal{T}_o\mathcal{G} \Rightarrow \lambda X \in \mathcal{T}_o\mathcal{G}. \tag{8c}$$

Indeed, let $X, Y \in \mathcal{T}_o\mathcal{G}$, that is $X, Y \in \mathfrak{X}^\infty(M)$ tangent to G. According to Definition 3.1, there exist $k, l \in \mathbb{N}$ such that $\varphi_t, \psi_t \in \mathcal{G}$ are integral curves of X and Y, respectively. Let us suppose that φ_t is a k^{th}–order integral curve of X and ψ_t is an lth-order integral curve of Y $(k, l \geq 1)$. Then

$$\varphi_0 = id_M, \qquad \left\{ \frac{\partial^i \varphi_t}{\partial t^i}\Big|_{t=0} = 0 \right\}_{1 \leq i < k} \qquad \frac{\partial^k \varphi_t}{\partial t^k}\Big|_{t=0} = X, \tag{9}$$

and

$$\psi_0 = id_M, \qquad \left\{ \frac{\partial^j \psi_t}{\partial t^j}\Big|_{t=0} = 0 \right\}_{1 \leq j < l} \qquad \frac{\partial^l \varphi_t}{\partial t^l}\Big|_{t=0} = Y. \tag{10}$$

• *Proof of* (8a). The computation is similar to that of [8]: Considering the group theoretical commutator

$$[\varphi_t, \psi_s] := \varphi_t^{-1} \circ \psi_s^{-1} \circ \varphi_t \circ \psi_s, \tag{11}$$

we get a two-parameter family of diffeomorphisms such that if one of the parameters s or t is zero then (11) is the identity transformation. From (9) and (10) we also know that the first, potentially nonzero derivative is the $(k+l)$th-order mixed derivative:

$$\begin{aligned} \frac{\partial^{(k+l)} [\varphi_t, \psi_s]}{\partial t^k \partial s^l}\Big|_{(0,0)}(p) &= \frac{\partial^l}{\partial s^l}\Big|_{s=0} \left(\frac{\partial^k \left(\varphi_s^{-1} \circ \psi_t^{-1} \circ \varphi_s \circ \psi_t(p)\right)}{\partial t^k}\Big|_{t=0} \right) \\ &= \frac{\partial^l}{\partial s^l}\Big|_{s=0} \left(d\left(\varphi_s^{-1}\right)_{\varphi_s(p)} \circ \frac{\partial^k \psi_t^{-1}}{\partial t^k}\Big|_{t=0} (\psi_s(p)) \right), \end{aligned} \tag{12}$$

where $d\left(\varphi_s^{-1}\right)_{\varphi_s(p)}$ denotes the tangent map (or Jacobi operator) of φ_s^{-1} at the point $\varphi_s(p)$. Since $d\left(\varphi_{s=0}^{-1}\right)_{\varphi_s(p)} = id$, the above formula can be written in the form

$$d\left(\frac{\partial^l \varphi_s^{-1}}{\partial s^l}\Big|_{s=0}\right)_p \frac{\partial^k \psi_t^{-1}(p)}{\partial t^k}\Big|_{t=0} + d\left(\frac{\partial^k \psi_t^{-1}}{\partial t^k}\Big|_{t=0}\right)_p \frac{\partial^l \varphi_s(p)}{\partial s^l}\Big|_{s=0}. \tag{13}$$

From $\varphi_t \circ \varphi_t^{-1} = id$ we get

$$0 = \frac{\partial^k}{\partial t^k}\Big|_{t=0} \left(\varphi_t \circ \varphi_t^{-1}\right) = X + \frac{\partial^k (\varphi_t^{-1})}{\partial t^k}\Big|_{t=0}$$

which yields

$$\frac{\partial^k (\varphi_t^{-1})}{\partial t^k}\Big|_{t=0} = -X. \tag{14}$$

Therefore we get that (13) can be written as

$$d\left(\frac{\partial^l \varphi_s}{\partial s^l}\Big|_{s=0}\right)_p \frac{\partial^k \psi_t(p)}{\partial t^k}\Big|_{t=0} - d\left(\frac{\partial^k \psi_t}{\partial t^k}\Big|_{t=0}\right)_p \frac{\partial^l \varphi_s(p)}{\partial s^l}\Big|_{s=0}, \tag{15}$$

which is the Lie bracket of the vector fields X and Y, that is

$$\frac{\partial^{k+l} [\varphi_t, \psi_s]}{\partial t^k \partial s^l}\Big|_{(0,0)} = [Y, X]. \tag{16}$$

From (16) we get that $t \to [\varphi_t, \psi_t]$ is a $(k+l)$th-order integral curve of $[X, Y]$ in $\mathcal{G}$. Therefore $[X, Y] \in \mathcal{T}_o\mathcal{G}$ which proves (8a).

• *Proof of* (8b).

For any $c_1, c_2, m_1, m_2 \in \mathbb{R}$, $\phi_t = \varphi_{c_1 t^{m_1}} \circ \psi_{c_2 t^{m_2}}$ is a smooth curve in $\mathcal{G}$ with $\phi_0 = \varphi_0 \circ \psi_0 = id_M$. Moreover, if $r =$ denotes the least common multiple of k and l and

$$m_1 = r/k, \quad m_2 = r/l, \quad c_1 = \left(m_1^k (r-k)!\right)^{-1/r}, \quad c_2 = \left(m_2^l (r-l)!\right)^{-1/r},$$

one gets

$$\frac{\partial^r \phi_t}{\partial t^r}\bigg|_{t=0} = \frac{\partial^r}{\partial t^r}\bigg|_{t=0} \left(\varphi_{c_1 t^{m_1}} \circ \psi_{c_2 t^{m_2}}\right) = X + Y, \tag{17}$$

showing that ψ_t is an rth-order integral curve of $X+Y$ in $\mathcal{G}$, and therefore $X+Y$ is tangent to $\mathcal{G}$.

• *Proof of* (8c).

It is clear that in the case when $\lambda \geq 0$, one can reparametrize the integral curve of X, and using that the lower order terms are zero, we get

$$\frac{\partial^k \varphi_{\sqrt[k]{\lambda} t}}{\partial t^k}\bigg|_{t=0} = \lambda X. \tag{18}$$

In the case when $\lambda < 0$ one can use (14) and we get

$$\frac{\partial^k}{\partial t^k}\bigg|_{t=0} \left(\varphi^{-1}_{\sqrt[k]{|\lambda|} t}\right) = -|\lambda| X = \lambda X. \tag{19}$$

From *(21)* and *(22)* we get that λX is tangent to G, that is $\lambda X \in \mathcal{T}_o\mathcal{G}$, and from *11b)* and *11c)* we get that any linear combinations of X and Y are in $\mathcal{T}_o\mathcal{G}$. □

Motivated by the results of Theorem 3.4 we propose the following:

Definition 3.5 $T_0\mathcal{G}$ is called the tangent Lie algebra of the subgroup $\mathcal{G} \subset \mathcal{D}iff^\infty(M)$.

As a direct consequence of Theorem 3.4 we get the following:

Corollary 3.6 *Let $\mathcal{G}$ be a subgroup of $\mathcal{D}iff^\infty(M)$ and $\mathcal{S}$ be a subset of $\mathfrak{X}^\infty(M)$ such that the elements of $\mathcal{S}$ are tangent to $\mathcal{G}$. Then the Lie subalgebra $\langle \mathcal{S} \rangle_{\mathcal{L}ie}$ of $\mathfrak{X}^\infty(M)$ generated by the elements of $\mathcal{S}$ is also tangent to $\mathcal{G}$, that is*

$$\mathcal{S} \subset \mathcal{T}_o\mathcal{G} \quad \Rightarrow \quad \left\langle \mathcal{S} \right\rangle_{Lie} \subset \mathcal{T}_o\mathcal{G}.$$

Remark 3.7 Slightly different tangent properties of vector fields to a subgroup $\mathcal{G}$ of the diffeomorphism group were already introduced in [10]. We will refer to the property [10, Definition 2.] as the *weak tangent property* and to [10, Definition 4.] as the *strong tangent property*. Our language is justified by the following proposition which is clarifying the relationship between the tangent property introduced in Definition 3.1 and the tangent properties introduced in [10]:

Proposition 3.8 *Let $\mathcal{G}$ be a subgroup of $\mathcal{D}iff^{\infty}(M)$ and $X \in \mathfrak{X}^{\infty}(M)$. Using the terminology of Remark 3.7:*

(i) if X is strongly tangent to $\mathcal{G}$, then $X \in \mathcal{T}_o\mathcal{G}$.
(ii) if $X \in \mathcal{T}_o\mathcal{G}$, then it is weakly tangent to $\mathcal{G}$.

Proof *(i)* If $X \in \mathfrak{X}^{\infty}(M)$ is a strongly tangent vector field to the subgroup $\mathcal{G} \subset \mathcal{D}iff^{\infty}(M)$, there exists a k-parameter commutator like family of diffeomorphisms $\phi_{t_1 \dots t_k} \in \mathcal{G}$ which is C^{∞}-smooth in $\mathcal{D}iff^{\infty}(M)$, $\phi_{t_1,\dots,t_k} = id_M$ whenever one of its parameters is zero and

$$X = \frac{\partial^k \phi_{t_1 \dots t_k}}{\partial t_1 \dots \partial t_k}\Big|_{(0 \dots 0)}.$$

Consequently, if we consider the map $t \to \varphi_t$ where $\varphi_t = \phi_{t,\dots,t}$, we get a 1-parameter family of diffeomorphisms which satisfies the conditions of Definition 3.2. Therefore, the vector field X is tangent to $\mathcal{G}$.

To prove *(ii)*, let us suppose that φ_t is a kth-order integral curve of X. Then we have (9) and one can write $\varphi_t(p)$ as

$$\varphi_t(p) = p + \tfrac{1}{k!}\, t^k \big(X(p) + \omega(p,t),\big) \tag{20}$$

where $\lim_{t \to 0} \omega(p,t) = 0$. The reparametrization $t \to \psi_t := \varphi_{k!\sqrt[k]{t}}$ gives a C^1-differentiable 1-parameter family of diffeomorphism in $\mathcal{D}iff^{\infty}(M)$ such that $\psi_0 = id_M$ and

$$\frac{\partial \psi_t}{\partial t}\Big|_{t=0}(p) = \frac{\partial \varphi_{k!\sqrt[k]{t}}}{\partial t}\Big|_{t=0}(p) = X(p),$$

which proves *(ii)*. □

Remark 3.9 One may wonder why to introduce a new tangent property when there are already two, the weak and the strong tangent property (using the terminology of Remark 3.7) introduced in the literature. As an answer we point out that the concept in [10] has a major defect: the weak tangent property is not preserved under the bracket operation, and therefore it is not true in general that weakly tangent vector fields to a subgroup $\mathcal{G}$ generate a weakly tangent Lie algebra to $\mathcal{G}$. To overcome this difficulty, the authors introduced the strongly tangent property but the strongly tangent property was not preserved under the linear combination. It follows that [10] and the succeeding papers using these techniques were not able to guaranty the existence of the tangent Lie algebra $\mathcal{T}_o\mathcal{G}$ associated to $\mathcal{G}$. With our approach we are able to overcome this major deficiency.

The main feature of $\mathcal{T}_o\mathcal{G}$ is that one can obtain information about the group $\mathcal{G}$. Indeed, one has the following:

Theorem 3.10 *Let $\mathcal{G}$ be a subgroup of $\mathcal{D}iff^{\infty}(M)$ and $\overline{\mathcal{G}}$ its topological closure with respect to the C^{∞} topology. Then the group generated by the exponential image of* *the tangent Lie algebra* $\mathcal{T}_o\mathcal{G}$ *with respect to the exponential map* $\exp\colon \mathfrak{X}^\infty(M) \to \mathcal{D}iff^\infty(M)$ *is a subgroup of* $\overline{\mathcal{G}}$.

Proof From the proof of Proposition 3.8 we know that for any element $X \in \mathcal{T}_o\mathcal{G}$ there exists a C^1-differentiable 1-parameter family $\{\psi_t\} \subset \mathcal{G}$ of diffeomorphisms of M such that $\psi_0 = id_M$ and $X = \frac{\partial \psi_t}{\partial t}\big|_{t=0}$. Then, using the argument of [15, Corollary 5.4, p. 84] on ψ_t we get that

$$\psi^n\left(\tfrac{t}{n}\right) = \psi\left(\tfrac{t}{n}\right) \circ \cdots \circ \psi\left(\tfrac{t}{n}\right)$$

in $\mathcal{G}$, as a sequence of $\mathcal{D}iff^\infty(M)$, converges uniformly in all derivatives to $\exp(tX)$. It follows that

$$\{\exp(tX) \mid t \in \mathbb{R}\} \subset \overline{\mathcal{G}},$$

for any $X \in \mathcal{T}_o\mathcal{G}$. Therefore, one has $\exp(\mathcal{T}_o\mathcal{G}) \subset \overline{\mathcal{G}}$ and if we denote by $\langle \exp(\mathcal{T}_o\mathcal{G})\rangle$ the group generated by the exponential image of $\mathcal{T}_o\mathcal{G}$ we get

$$\langle \exp(\mathcal{T}_o\mathcal{G})\rangle \subset \overline{\mathcal{G}},$$

which proves Theorem 3.10. □

We note that, assuming the manifold M is compact, we could avoid technical difficulties. Indeed, in this case, the diffeomorphism group $\mathcal{D}iff^\infty(M)$ is an (infinite-dimensional) manifold and the exponential image of the flow of vector fields exists everywhere on M. For a more general and deeper discussion of the subject see [19].

The concept worked out in Definition 3.2 and Theorem 3.4 can be adapted not only for subgroups of the diffeomorphism group but for any subgroup of any (finite- or infinite-dimensional) Lie group:

Definition 3.11 Let $\mathcal{G}_L$ be a Lie group, $e \in \mathcal{G}_L$ is the identity element of $\mathcal{G}_L$, and $\mathfrak{g}_L := T_e\mathcal{G}_L$ the Lie algebra of $\mathcal{G}_L$. If $\mathcal{G} \subset \mathcal{G}_L$ is a subgroup of $\mathcal{G}_L$, then $X \in \mathfrak{g}_L$ is called tangent to $\mathcal{G}$ if there exists a C^∞−smooth curve $\varphi\colon I \to \mathcal{G}_L$ such that

(1) $\varphi_t \in \mathcal{G}$,
(2) $\varphi_0 = e$,
(3) there exists $k \in \mathbb{N}$ such that $t \to \varphi_t$ is a kth-order integral curve of X.

The set of tangent vector of $\mathcal{G}$ is denoted by $\mathcal{T}_o\mathcal{G}$.

Then, adapting the proof of Theorems 3.4 and 3.10 we can get the following:

Theorem 3.12 *If* $\mathcal{G}$ *is a subgroup of a Lie group* $\mathcal{G}_L$, *then* $\mathcal{T}_o\mathcal{G}$ *is a Lie subalgebra of* $\mathfrak{g}_L$. *The group* $\langle \exp(\mathcal{T}_o\mathcal{G})\rangle$ *generated by the exponential image of* $\mathcal{T}_o\mathcal{G}$ *with respect to the exponential map* $\exp\colon \mathfrak{g}_L \to \mathcal{G}_L$ *is a subgroup of the topological closure* $\overline{\mathcal{G}}$ *of* $\mathcal{G}$ *in* $\mathcal{G}_L$.

It is clear that in the case when $\mathcal{G}$ is a Lie subgroup of $\mathcal{G}_L$, then $\mathcal{T}_o\mathcal{G} = \mathfrak{g}$ is just the usual Lie subalgebra of $\mathfrak{g}_L$ associated to the Lie subgroup $\mathcal{G}$. Therefore Definition 3.11 generalizes the classical notion of the Lie subalgebra associated to a Lie subgroup.

4 An Application: Holonomy Algebra

The notion of the holonomy group was already introduced in Sect. 2.3. It is well known that in the particular case when $(M, \mathcal{F})$ is a Riemann manifold, the holonomy group is a compact Lie subgroup of the orthogonal group $O(n)$ and its Lie algebra is a Lie subalgebra of $\mathfrak{o}(n)$. It is also clear that the holonomy group of a linear connection is a subgroup of the linear group $GL(n)$ and its Lie algebra is a Lie subalgebra of $\mathfrak{gl}(n)$. However, the situation for a Finsler manifold or in a more general context the holonomy of a homogeneous connection can be much more complex. Examples show that in some cases the holonomy group cannot be a finite-dimensional Lie group [11–13]. Until now it is not known if the Finsler holonomy groups are (finite- or infinite-dimensional) Lie groups or not. Nevertheless, the theory developed in Sect. 3 allows us to consider its tangent Lie algebra, the holonomy algebra.

4.1 The Fibered Holonomy Algebra and Its Lie Subalgebras

Let $(M, \mathcal{F})$ be a compact Finsler manifold. The notion of *fibered holonomy group* $\mathcal{H}ol_f(M)$ appeared in [10]: It is the group generated by fiber preserving diffeomorphisms Φ of the indicatrix bundle $(\mathcal{I}M, \pi, M)$, such that for any $p \in M$ the restriction $\Phi_p = \Phi|_{\mathcal{I}_p}$ is an element of the holonomy group $\mathcal{H}ol_p(M)$. It is obvious that

$$\mathcal{H}ol_f(M) \subset \mathcal{D}iff^{\infty}(\mathcal{I}M), \tag{21}$$

where $\mathcal{H}ol_f(M)$ is a subgroup of the diffeomorphism group of the indicatrix bundle. Until now it is not known whether or not $\mathcal{H}ol_f(M)$ is a Lie subgroup of $\mathcal{D}iff^{\infty}(\mathcal{I}M)$. The set of tangent vector fields to the group $\mathcal{H}ol_f(M)$ denoted as

$$\mathfrak{hol}_f(M) := \mathcal{T}_0\big(\mathcal{H}ol_f(M)\big). \tag{22}$$

Definition 4.1 $\mathfrak{hol}_f(M)$ is called the *fibered holonomy algebra* of the Finsler manifold $(M, \mathcal{F})$.

From Theorem 3.4 one can obtain the following:

Theorem 4.2 *The fibered holonomy algebra $\mathfrak{hol}_f(M)$ is a Lie subalgebra of the Lie algebra of smooth vector fields $\mathfrak{X}^{\infty}(\mathcal{I}M)$.*

In the sequel we will investigate the two most important Lie subalgebras of $\mathfrak{hol}_f(M)$ which can be introduced with the help of the curvature tensor (see Sect. 2.3) of a Finsler manifold: the curvature algebra and the infinitesimal holonomy algebra.

Definition 4.3 A vector field $\xi \in \mathfrak{X}^{\infty}(\mathcal{I}M)$ is called a *curvature vector field* if there exist vector fields $X, Y \in \mathfrak{X}^{\infty}(M)$ such that $\xi = R(X^h, Y^h)$. The Lie subalgebra $\mathfrak{R}$ of vector fields generated by curvature vector fields is called the *curvature algebra*.

It is easy to see that from the definition of the curvature tensor that a curvature vector field can be calculated as

$$\xi = R(X^h, Y^h) = \left[X^h, Y^h\right] - \left[X, Y\right]^h, \tag{23}$$

and from the definition we have also $\mathfrak{R} \subset \mathfrak{X}^\infty(\mathcal{I}M)$. Moreover, we have the following:

Proposition 4.4 *(1) The elements of the curvature algebra are tangent to the group $\mathcal{H}ol_f(M)$.*
(2) The curvature algebra $\mathfrak{R}$ is a Lie subalgebra of $\mathfrak{hol}_f(M)$.

To prove the first part of the proposition, we have to show that the curvature vector fields are tangent to the fibered holonomy group $\mathcal{H}ol_f(M)$, that is they are elements of $\mathfrak{hol}_f(M)$. Let $\xi \in \mathfrak{X}^\infty(\mathcal{I}M)$ be a curvature vector field and $X, Y \in \mathfrak{X}^\infty(M)$ such that $\xi = R(X^h, Y^h)$. We denote by φ and ψ the integral curves of X and Y, respectively. Define

$$\alpha_{t,s} := \begin{cases} \varphi_s, & 0 \le s \le t, \\ \psi_{s-t}\varphi_t, & t \le s \le 2t, \\ \varphi_{2t-s}\psi_t\varphi_t, & 2t \le s \le 3t, \\ \psi_{3t-s}\varphi_{-t}\psi_t\varphi_t, & 3t \le s \le 4t, \end{cases}$$

and

$$\beta_{t,s} := \psi_{-s}\varphi_{-s}\psi_s\varphi_s, \qquad 0 \le s \le t.$$

Then, for every $p \in M$ and fixed t the map $\alpha_t(p)\colon s \to \alpha_{t,s}(p)$ and $\beta_t(p)\colon s \to \beta_{t,s}(p)$ are parametrized curves: $\alpha_t(p)\colon s \to \alpha_{t,s}(p)$ is a (not necessarily closed) parallelogram and $\beta_t(p)$ joins the endpoints of $\alpha_t(p)$. Indeed, for every $p \in M$ and fixed t the endpoint of $\alpha_t(p)$ coincides with the endpoint of $\beta_t(p)$ and consequently the curve $\alpha_t(p) * \beta_t^{-1}(p)$ defined as going along the curve $\alpha_t(p)$ then continuing along $\beta_t^{-1}(p)$ (which is the curve $\beta_t(p)$ with opposed orientation) is a closed curve that starts and ends at $p \in M$. Let us consider

$$h_{t,p} := \mathcal{P}_{\alpha_t(p)*\beta_t^{-1}(p)} = \mathcal{P}_{\alpha_t(p)} \circ \mathcal{P}^{-1}_{\beta_t(p)}, \tag{24}$$

the parallel translation along $\alpha_t(p) * \beta_t^{-1}(p)$. We have the following:

Lemma 4.5 *For any $p \in M$*

(1) $h_{t,p} \in \mathcal{H}ol_p(M)$,
(2) $t \to h_{t,p}$ is a second-order integral curve of the vector field $\xi_p := \xi\big|_{\mathcal{I}_p}$ $\big(\xi_p \in \mathfrak{X}^\infty(\mathcal{I}_p)\big)$.

Proof Indeed, for every $p \in M$ and sufficiently small t the curve $\alpha_t(p) * \beta_t^{-1}(p)$ is a closed loop starting and ending at p, and therefore the parallel transport $h_{t,p} : \mathcal{I}_p \to \mathcal{I}_p$ is a holonomy transformation at p and we get *(1)* of the lemma.

To show *(2)* we first remark that $\alpha_0(p)$ and $\beta_0(p)$ are the trivial curves ($s \to \alpha_{0,s}(p) = \beta_{0,s}(p) \equiv p$), and therefore the parallel translation along them is the identity transformation and

$$h_{0,p} = id_{\mathcal{I}_p}. \tag{25}$$

On the other hand, as we have seen in Sect. 2, the parallel transport along a curve is determined by the horizontal lift of the curve. Consequently, the parallel transport along the integral curves of the vector fields X and Y can be expressed with the flows of the horizontal lifts X^h and Y^h. Let us consider first the parallel transport along the curve $\alpha_t(p)$: the parallel transport of a vector $v \in \mathcal{I}_p$ along the curve $\alpha_t(p)$ is

$$\mathcal{P}_{\alpha_t(p)}(v) = \begin{cases} \varphi_s^{X^h}(v), & 0 \le s \le t, \\ \varphi_{s-t}^{Y^h}\varphi_t^{X^h}(v), & t \le s \le 2t, \\ \varphi_{-(s-2t)}^{X^h}\varphi_t^{Y^h}\varphi_t^{X^h}(v), & 2t \le s \le 3t, \\ \varphi_{-(s-3t)}^{Y^h}\varphi_{-t}^{X^h}\varphi_t^{Y^h}\varphi_t^{X^h}(v), & 3t \le s \le 4t. \end{cases}$$

Therefore, $\mathcal{P}_{\alpha_t(p)}$ corresponds to the infinitesimal (not necessarily closed) parallelogram having as sides the integral curves of the horizontal lifts X^h and Y^h. From the well-known properties of the Lie brackets (see for example [16, p.162]) we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{P}_{\alpha_t}(v) = 0, \qquad \text{and} \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0}\mathcal{P}_{\alpha_t}(v) = 2\left[X^h, Y^h\right]_v. \tag{26}$$

On the other hand, the parallel transport of a vector $w \in \mathcal{I}_{\alpha_t(p)}$ along $\beta_t^{-1}(p)$ can be calculated with the help of its horizontal lift $\mathcal{P}_{\beta_t^{-1}}(w) = \mathcal{P}_{\beta_t}^{-1}(w) = ((\beta)^h(t))^{-1}(w)$, where by the definition of the horizontal lift $\pi \circ (\beta)^h(t) = \beta(t)$ and $(\beta^{-1})^h(0) = w$ are fulfilled. Since $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\beta_t(p) = 0$, and $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big|_{t=0}\beta_t(p)(v) = 2\,[X, Y]_p$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{P}_{\beta_t}^{-1} = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0}\mathcal{P}_{\beta_t}^{-1}(v) = -\big(2\,[X, Y]^h\big)_v; \tag{27}$$

thus, from the two equations of (26) and the two equations of (27) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}h_t(v) = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0}h_t(v) = 2\left(\left[X^h, Y^h\right] - \left[X, Y\right]^h\right)_v = 2\xi_p, \tag{28}$$

where we also used (23). To summarize, we get from (25) and (28):

$$h_{0,p} = \mathrm{id}\big|_{\mathcal{I}_p}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}h_{t,p} = 0, \qquad \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0}h_{t,p} = \xi_p, \tag{29}$$

which means that the reparametrized map $t \to h_{t/\sqrt{2},p}$ is a second-order integral curve of the curvature vector field $\xi_p \in \mathfrak{X}^\infty\big(\mathcal{I}_p\big)$ and proves point *(2)* of the lemma. □

Proof of Proposition 4.4 Let us consider the map $h_t : \mathcal{I}M \to \mathcal{I}M$ on the indicatrix bundle, where $h_t\big|_{\mathcal{I}_p} := h_{t,p}$. From Lemma 4.5 we get (by dropping the variable $p \in M$) that

(1) $h_t \in \mathcal{H}ol_f(M)$,
(2) $t \to h_t$ is a second-order integral curve of the vector field $\xi \in \mathfrak{X}^\infty(\mathcal{I})$,

which shows that the curvature vector field ξ is tangent to $\mathcal{H}ol_f(M)$ and proves the first part of the proposition. Applying Corollary 3.6, we get that the Lie algebra generated by the curvature vector field is tangent to $\mathcal{H}ol_f(M)$ which proves the second part of the proposition. □

We remark that *(1)* of Proposition 4.4 is an improvement of Proposition 3. and Corollary 2. of [10]. Indeed, the tangent property proved in [10] is weaker: C^1 instead of C^∞ smoothness. Moreover, [10] uses a very strong topological restriction on the manifold M supposing it is diffeomorphic to the n-dimensional euclidean space. In Proposition (4.4) we presented a natural geometric construction without any constraints on the topology of the manifold M.

Definition 4.6 The *infinitesimal holonomy algebra* $\mathfrak{hol}^*(M)$ of a Finsler manifold $(M, \mathcal{F})$ is the smallest Lie algebra on the indicatrix bundle which satisfies the following properties:

1) Every curvature vector field ξ is an element of $\mathfrak{hol}^*(M)$,
2) if $\xi, \eta \in \mathfrak{hol}^*(M)$, then $[\xi, \eta] \in \mathfrak{hol}^*(M)$,
3) if $\xi \in \mathfrak{hol}^*(M)$ and $X \in \mathfrak{X}^\infty(M)$, then the horizontal Berwald covariant derivative $\nabla_X \xi$ is also an element of $\mathfrak{hol}^*(M)$.

We have the following:

Proposition 4.7 *(1) The elements of the infinitesimal holonomy algebra* $\mathfrak{hol}^*(M)$ *are tangent to* $\mathcal{H}ol_f(M)$.
(2) The infinitesimal holonomy algebra $\mathfrak{hol}^*(M)$ *is a Lie subalgebra of* $\mathfrak{hol}_f(M)$.

Proof From Proposition 4.4 we know that the curvature vector fields are tangent to the fibered holonomy group. Moreover, from [10, Proposition 4] and from *(i)* of Remark 3.8 we get that the horizontal Berwald covariant derivative of tangent vector fields to $\mathcal{H}ol_f(M)$ are also tangent to $\mathcal{H}ol_f(M)$ which proves the first part of the proposition. As a consequence, the infinitesimal holonomy algebra is generated by tangent vector fields and, according to Corollary 3.6, it is tangent to $\mathcal{H}ol_f(M)$ proving the second part of the proposition. □

We remark that the first part of Proposition 4.7 is an improvement of [10, Theorem 2], because in Proposition 4.7 the strong topology condition on the manifold M is dropped.

4.2 Holonomy Algebra and Its Lie Subalgebras

Let (M, F) be an $n-$dimensional Finsler manifold. At any points $p \in M$ the indicatrix defined in (1) is an $(n-1)$-dimensional compact manifold in T_pM. Therefore, the diffeomorphism group $\mathcal{D}iff^{\infty}(\mathcal{I}_p)$ is an infinite-dimensional Fréchet Lie group whose Lie algebra is $\mathfrak{X}^{\infty}\big(\mathcal{I}_p\big)$, the Lie algebra of smooth vector fields on $\mathcal{I}_p$. As it was introduced in Sect. 2.3, the holonomy group

$$\mathcal{H}ol_p(M) \subset \mathcal{D}iff^{\infty}(\mathcal{I}_p M), \tag{30}$$

is a subgroup of the diffeomorphism group $\mathcal{D}iff^{\infty}(\mathcal{I}_p M)$. The set of tangent vector fields to the group $\mathcal{H}ol_p(M)$, denoted as

$$\mathfrak{hol}_p(M) := \mathcal{T}_0\big(\mathcal{H}ol_p(M)\big).$$

Definition 4.8 $\mathfrak{hol}_p(M)$ is called the *holonomy algebra* of the Finsler manifold $(M, \mathcal{F})$ at $p \in M$.

From Theorem 3.4 one can obtain

Theorem 4.9 *The holonomy algebra $\mathfrak{hol}_p(M)$ of a Finsler manifold $(M, \mathcal{F})$ at $p \in M$ is a Lie subalgebra of $\mathfrak{X}^{\infty}\big(\mathcal{I}_p\big)$.*

In the sequel we identify two important Lie subalgebras of the holonomy algebra of Finsler manifolds.

Definition 4.10 A vector field $\xi_p \in \mathfrak{X}^{\infty}\big(\mathcal{I}_p\big)$ on the indicatrix $\mathcal{I}_p \subset T_pM$ is called a curvature vector field at $p \in M$ if there exist tangent vectors $X_p, Y_p \in T_pM$ such that $\xi_p = R(X^h_p, Y^h_p)$. The Lie subalgebra $\mathfrak{R}_p$ of vector fields generated by curvature vector fields at $p \in M$ is called the *curvature algebra at* p.

The relationship between the curvature algebra $\mathfrak{R}_p$ at $p \in M$ and the curvature algebra $\mathfrak{R}$ introduced in Definition 4.3 is

$$\mathfrak{R}_p = \left\{ \xi_p = \xi|_{\mathcal{I}_p} \;\middle|\; \xi \in \mathfrak{R} \right\},$$

that is $\mathfrak{R}_p$ is the restriction of $\mathfrak{R}$ to the indicatrix $\mathcal{I}_p$. We have

Proposition 4.11 *The elements of the curvature algebra $\mathfrak{R}_p$ at $p \in M$ are tangent to the group $\mathcal{H}ol_p(M)$ and the curvature algebra $\mathfrak{R}_p$ is a Lie subalgebra of the holonomy algebra $\mathfrak{hol}_p(M)$.*

The proof is a direct consequence of the computation of Proposition (4.4). Moreover, by localizing the infinitesimal holonomy algebra at a point we can obtain

Definition 4.12 The Lie algebra $\mathfrak{hol}^*_p(M) := \left\{ \xi|_{\mathcal{I}_p} \mid \xi \in \mathfrak{hol}^*(M) \right\}$ of vector fields on the indicatrix $\mathcal{I}_p$ is called the infinitesimal holonomy algebra at the point $p \in M$.

From Proposition 4.7 we get

Proposition 4.13 *The elements of the infinitesimal holonomy algebra $\mathfrak{hol}^*_p(M)$ are tangent to the group $\mathcal{H}ol_p(M)$ and the infinitesimal holonomy algebra $\mathfrak{hol}^*_p$ is a Lie subalgebra of the holonomy algebra $\mathfrak{hol}_p(M)$.*

We note that by the construction of the infinitesimal holonomy algebra, the curvature vector fields are elements of $\mathfrak{hol}^*_p(M)$, and therefore we have the sequence of the Lie algebras

$$\mathfrak{R}_p(M) \subset \mathfrak{hol}^*_p(M) \subset \mathfrak{hol}_p(M) \subset \mathfrak{X}^\infty\big(\mathcal{I}_p\big)\,. \tag{31}$$

We also remark that the first parts of the statement of Proposition 4.11 and 4.13 are improvements of the results of [10] because the tangential property of the Lie algebra is improved: we can guaranty C^∞-smoothness instead of C^1-smoothness.

5 Concluding Remarks

Many interesting geometric results can be obtained on the holonomy structure from the Lie algebras (31) through the tangent property. Indeed, by using Theorem 3.10, one can find examples where, in contrast to the Riemannian case, the holonomy group $\mathcal{H}ol_p(M)$ is not a compact Lie group [11–13], or where the closure of the holonomy group is the infinite-dimensional Lie group $\mathcal{D}iff^\infty_+(\mathcal{I}_p)$ of the orientation preserving diffeomorphism group of the indicatrix [5,14]. All these results were obtained by using the tangent property of the curvature algebra $\mathfrak{R}_p(M)$ and the infinitesimal holonomy algebra $\mathfrak{hol}^*_p(M)$. The method developed in Sect. 3, however, allows us to introduce in a natural and canonical way a potentially larger Lie algebra, the holonomy algebra, which is the *tangent Lie algebra* of the holonomy group. This Lie algebra gives the best linear approximation of the holonomy group. The technique can be applied in other fields of geometry as well. We are convinced that the method, exploring the tangential property of a group associated with a geometric structure, can be used successfully to investigate various geometric properties.

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