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METRIZABILITY OF HOLONOMY INVARIANT PROJECTIVE DEFORMATION OF SPRAYS

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ABSTRACT. In this paper, we consider projective deformation of the geodesic system of Finsler spaces by holonomy invariant functions: Starting by a Finsler spray S and a holonomy invariant function $\mathcal{P}$, we investigate the metrizability property of the projective deformation $\widetilde{S} = S - 2\lambda\mathcal{P}\mathcal{C}$. We prove that for any holonomy invariant nontrivial function $\mathcal{P}$ and for almost every value $\lambda \in \mathbb{R}$, such deformation is not Finsler metrizable. We identify the cases where such deformation can lead to a metrizable spray: in these cases, the holonomy invariant function $\mathcal{P}$ is necessarily one of the principal curvatures of the geodesic structure.

1. INTRODUCTION

A system of second order homogeneous ordinary differential equations (SODE), whose coefficients do not depend explicitly on time, can be identified with a special vector field, called spray. The spray corresponding to the geodesic equation of a Riemann or Finslerian metric is called the geodesic spray of the corresponding metric.

The metrizability problem for a spray S seeks for a Riemannian or Finslerian metric whose geodesics coincide with the geodesics of S. For the projective metrizability problem, one seeks for a Riemannian or Finslerian metric whose geodesics coincide with the geodesics of S, up to an orientation preserving reparameterization. The two problems can be viewed as particular, and probably the most interesting cases of the inverse problem of the calculus of variation. For various approaches and results of the metrizability and projective metrizability problem, we refer to [1, 4, 6, 7, 10, 12, 16].

Two sprays on the same manifold are said to be projectively equivalent if they have the same geodesics as point sets. Two sprays S and $\widetilde{S}$ on the manifold M are projectively equivalent if there is a function $\widetilde{\mathcal{P}}\colon TM \to \mathbb{R}$ such that

$$\widetilde{S} = S - 2\widetilde{\mathcal{P}}\cdot\mathcal{C}, \tag{1.1}$$

where $\mathcal{C}$ is the Liouville vector field. The function $\widetilde{\mathcal{P}}$ is called projective factor of the projective deformation. In [17], Yang shows that for a projectively flat spray of constant flag curvature its projective class contains sprays which are not Finsler metrizable. In [3] the authors extend Yang's result, and show that for an arbitrary spray its projective class contains sprays which are not Finsler metrizable by considering the most natural projective deformation of the geodesic spray S of a Finsler metric F, where the projective factor $\widetilde{\mathcal{P}} = \lambda \cdot F$ in (1.1) is a scalar multiple of the Finsler function F of S. They showed that the deformed spray is not Finsler metrizable for almost any value of $\lambda \in \mathbb{R}$.

It would be very interesting to describe the general situation, that is the necessary and sufficient conditions for a projective deformation of a metrizable spray to be metrizable. This problem is however very complex and it contains, as a particular case, Hilbert's fourth problem. Therefore, even partial results, when the projective factor possesses special geometric or analytic properties can be interesting. In this paper we consider the case where the projective factor in (1.1) is invariant with respect the parallel translation, or in other

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words, a holonomy invariant function. We will call such transformation a *holonomic projective deformation*. Writing the projective factor in the form $\widetilde{\mathcal{P}} = \lambda \cdot \mathcal{P}$ with $\lambda \in \mathbb{R}$ we extend the results of [3] by proving the following theorem:

Theorem 1. *For any nontrivial holonomy invariant 1-homogeneous projective factor $\mathcal{P}$ and for almost any scalar $\lambda \in \mathbb{R}$ the projective deformation*

$$\widetilde{S} = S - 2\lambda \mathcal{P}\, \mathcal{C}, \tag{1.2}$$

of a Finsler metrizable spray S is not metrizable.

Only very special holonomy invariant projective factors can lead to metrizable projective deformation. As one can see in Corollary 4.2, these holonomy invariant projective factors must be related to the principal curvature of the deformed Finsler structure.

2. Preliminaries

Let M be an n-dimensional manifold and (TM, π, M) be its tangent bundle. $\mathcal{T}M := TM \setminus \{0\}$ denotes the set of nonzero tangent vectors. We denote by (x^i) local coordinates on the base manifold M and by (x^i, y^i) the induced coordinates on TM. We use in the sequel Frölicher-Nijenhuis formalism and notations.

A vector ℓ-form on M is a skew-symmetric $C^\infty(M)$-linear map $L\colon \mathfrak{X}^\ell(M) \to \mathfrak{X}(M)$. Every vector-valued ℓ-form L defines two graded derivations i_L and d_L of the exterior algebra $\Lambda(M)$ defined as follows: for any $f \in C^\infty(M)$ we have $i_L f = 0$ and $i_L df = df \circ L$ and

$$d_L := [i_L, d] = i_L \circ d - (-1)^{\ell-1} d i_L.$$

If $X \in \mathfrak{X}(M)$ is a vector field, then i_X is simply the interior product by X and $d_X = \mathcal{L}_X$ the Lie derivative with respect to X.

2.1. Projective deformation of geodesic structure.

There are two canonical objects on TM, the natural almost-tangent structure J and the the Liouville vector field $\mathcal{C} \in \mathfrak{X}(TM)$. Locally they are defined by the formulas

$$J = \frac{\partial}{\partial y^i} \otimes dx^i, \qquad \mathcal{C} = y^i \frac{\partial}{\partial y^i}.$$

A vector field $S \in \mathfrak{X}(\mathcal{T}M)$ is called a spray if $JS = \mathcal{C}$ and $[\mathcal{C}, S] = S$. Locally, a spray can be expressed as follows

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}, \tag{2.1}$$

where the *spray coefficients* $G^i = G^i(x,y)$ are 2-homogeneous functions in the $y = (y^1, \dots, y^n)$ variable. A regular curve $\sigma : I \to M$ on M is called *geodesic* of a spray S if $S \circ \sigma' = \sigma''$. Locally, $\sigma(t) = (x^i(t))$ is a geodesic of S if and only if it satisfies the equation

$$\frac{d^2 x^i}{dt^2} + 2G^i\Big(x, \frac{dx}{dt}\Big) = 0. \tag{2.2}$$

Consequently, a system of second order homogeneous ordinary differential equations (SODE), whose coefficients functions do not depend explicitly on time, can be identified with a special vector field, called spray.

Definition 2.1. *Two sprays S and $\widetilde{S}$ are projectively related if their geodesics coincide up to an orientation preserving reparameterization.*

An orientation preserving reparameterization $t \to \tilde{t}$ of the spray (2.1) leads to a new spray given by formula (1.1) with some 1-homogeneous scalar function $\widetilde{\mathcal{P}} \in C^\infty(\mathcal{T}M)$. This function is related to the new parametrization by

$$\frac{d^2 \tilde{t}}{dt^2} = 2\widetilde{\mathcal{P}}\Big(x, \frac{dx}{dt}\Big)\frac{d\tilde{t}}{dt}, \qquad \frac{d\tilde{t}}{dt} > 0. \tag{2.3}$$

Definition 2.2. *The spray* $\widetilde{S}$ *given by formula* (1.1) *is called the projective deformation of the spray* S *with the projective factor* $\widetilde{\mathcal{P}}$*. The projective deformation is called* holonomic *if* $\widetilde{\mathcal{P}}$ *is a holonomy invariant function.*

2.2. **Geometric quantities associated to a spray.**

A nonlinear connection is defined by an n-dimensional distribution $\mathcal{H}$ on TM which gives a direct decomposition of

$$T(\mathcal{T}M) = \mathcal{H} \oplus \mathcal{V}, \tag{2.4}$$

where $V = \operatorname{Ker}\pi_*$ is the vertical space. Every spray S induces a canonical nonlinear connection through the corresponding horizontal and vertical projectors,

$$h = \frac{1}{2}(Id + \Gamma), \quad v = \frac{1}{2}(Id - \Gamma) \tag{2.5}$$

where $\Gamma = [J, S]$ is the nonlinear connection induced by spray [8]. We remark that the spray S is horizontal with respect to (2.4), that is $S = hS$. Locally, the projectors (2.5) can be expressed as follows

$$h = \delta_i \otimes dx^i, \qquad v = \dot{\partial}_i \otimes \delta y^i,$$

where

$$\delta_i := \frac{\partial}{\partial x^i} - G_i^j \frac{\partial}{\partial y^j}, \quad \dot{\partial}_j := \frac{\partial}{\partial y^j}, \quad \delta y^i = dy^i + G^i_j dx^j,$$

with the 1-homogeneous $G^j_i := \frac{\partial G^j}{\partial y^i}$ functions. We note that

$$S = y^i \delta_i, \quad \text{and} \quad \mathcal{C} = y^i \dot{\partial}_i. \tag{2.6}$$

The Nijenhuis torsion of h measuring the integrability of the horizontal distribution

$$R = \frac{1}{2}[h, h] = \frac{1}{2} R^i_{jk} \frac{\partial}{\partial y^i} \otimes dx^j \wedge dx^k, \qquad R^i_{jk} = \frac{\delta G^i_j}{\delta x^k} - \frac{\delta G^i_k}{\delta x^j}$$

is called the curvature of S. From the curvature tensor one can obtain the Jacobi endomorphism [3], which is defined by

$$\Phi = R^i_j \, dx^j \otimes \frac{\partial}{\partial y^i}, \qquad R^i_j = 2\frac{\partial G^i}{\partial x^j} - S(G^i_j) - G^i_k G^k_j. \tag{2.7}$$

The two tensors are related by

$$\Phi = i_S R, \qquad 3R = [J, \Phi], \tag{2.8}$$

respectively. The spray S is called R-flat, if it Jacobi endomorphism vanishes.

2.3. **Finsler structure.**

Definition 2.3. *A Finsler function on a manifold* M *is a continuous function* $F : TM \to \mathbb{R}$ *such that*

i) F *is smooth and strictly positive on* $\mathcal{T}M$ *and* $F(x, y) = 0$ *if and only if* $y = 0$*,*
ii) F *is positively homogeneous of degree* 1 *in the directional argument* y*,*
iii) The metric tensor $g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$ *has maximal rank on* $\mathcal{T}M$*.*

The function $E := \frac{1}{2}F^2$ is called the energy function associated to F. From condition *iii)* one can obtain that the 2-form $dd_J E$ is non-degenerate, and the Euler-Lagrange equation

$$i_S dd_J E = -dE \tag{2.9}$$

uniquely determines a spray S on TM. This spray is called the *geodesic spray* of the Finsler function.

Definition 2.4. *A spray* S *on a manifold* M *is called* Finsler metrizable *if there exists a Finsler function* F *such that the geodesic spray of the Finsler manifold* (M, F) *is* S*.*

The *holonomy distribution* $\mathcal{D}_{hol}(S)$ of a spray S is the smallest involutive distribution generated by the horizontal distribution $\mathcal{H}$ (see [11]). This distribution is generated by the horizontal vector fields and their successive Lie-brackets, that is

$$\mathcal{D}_{hol}(S) := \Big\{ [X_1, [X_2, \dots [X_{m-1}, X_m]...]] \ \Big| \ X_i \in \mathfrak{X}^h(TM), \ m \in \mathbb{N} \Big\}, \tag{2.10}$$

where $\mathfrak{X}^h(TM)$ denotes the module of horizontal vector fields. A function $\mathcal{P} \in C^\infty(TM)$ is called *holonomy invariant*, if it is invariant with respect to parallel translation, that is, for any $v \in TM$ and for any parallel translation τ we have $\mathcal{P}(\tau(v)) = \mathcal{P}(v)$. Using the geometric construction of parallel transport through horizontal lifts, it is clear that a function $\mathcal{P}$ is holonomy invariant if and only if

$$d_h \mathcal{P} = 0, \tag{2.11}$$

that is for any horizontal vector field $X \in X^h(TM)$ we have $\mathcal{L}_X \mathcal{P} = 0$. Obviously, this property must be also true for the successive Lie-brackets of horizontal vector fields. Consequently, we get the following

Property 2.5. *$\mathcal{P} \in C^\infty(TM)$ is a holonomy invariant function if and only if $\mathcal{L}_X \mathcal{P} = 0$ for any $X \in \mathcal{D}_{hol}(S)$.*

The above property shows that the elements of the holonomy distribution are the infinitesimal symmetries of the holonomy invariant functions. Since $\operatorname{Im} R \subset \mathcal{D}_{hol}(S)$ and $\operatorname{Im} \Phi \subset \mathcal{D}_{hol}(S)$, that is the images of the curvature tensor and the Jacobi endomorphism are in the holonomy distribution we have the following

Corollary 2.6. *The derivatives of a holonomy invariant function with respect to any vector field in the image of R and Φ are identically zero.*

We note that if S is Finsler metrizable, then its Finsler function and its energy function are both holonomy invariant functions, therefore we have the

Corollary 2.7. *If S is Finsler metrizable and E is its energy function, then $\mathcal{L}_X E = 0$ for any $X \in \mathcal{D}_{hol}(S)$.*

2.4. **Principal curvatures of a Finsler metric.**

The Jacobi endomorphism (2.7) of the geodesic spray S of a Finsler metric is also called the Riemann curvature [14]. It is diagonalizable in the following sense: there exist $\kappa_\alpha \in C^\infty(\mathcal{T}M)$ and $X_\alpha \in \mathfrak{X}^h(\mathcal{T}M)$ for $\alpha = 1, \dots, n$ such that

$$\Phi(X_\alpha) = \kappa_\alpha \cdot JX_\alpha. \tag{2.12}$$

(The summation convention is not applied on the index α here and in the sequel). X_α is called an eigenvector field of Φ corresponding to the eigenfunction κ_α. In particular, using (2.8), we have

$$\Phi(S) = i_S R(S) = R(S, S) = 0, \tag{2.13}$$

that is $X_n := S$ is always an eigenvector of Φ corresponding to the eigenfunction $\lambda_n = 0$.

Definition 2.8. *The eigenfunctions $\kappa_1, \dots, \kappa_{n-1}$ of the Riemannian curvature are called the principal curvatures of the Finsler metric.*

The principal curvatures are the most important intrinsic invariants of the Finsler metric (see [15]).

3. Holonomic projective deformations

In this section we investigate the holonomic projective deformations, that is projective deformations by a holonomy invariant functions. We focus mainly on the properties of the holonomy distribution of the projective deformation (1.1). The rather technical results of this section are necessary to prove the metrizability results of Section 4.

Lemma 3.1. *Let S be the geodesic spray, $\mathcal{P}$ a holonomy invariant function and $\lambda \in \mathbb{R}$. Then some geometric quantities associated to the projectively deformed spray $\widetilde{S} = S - 2\lambda\mathcal{P}C$ are given by*

$$\widetilde{h} = h - \lambda(\mathcal{P}J + d_J\mathcal{P} \otimes \mathcal{C}), \tag{3.1a}$$

$$\widetilde{v} = v + \lambda(\mathcal{P}J + d_J\mathcal{P} \otimes \mathcal{C}), \tag{3.1b}$$

$$\widetilde{\Phi} = \Phi + \lambda^2(\mathcal{P}^2 J - \mathcal{P} d_J\mathcal{P} \otimes \mathcal{C}), \tag{3.1c}$$

Proof. In [3, Proposition 4.4] the geometric quantities of the projectively deformed spray (1.1) given by $\widetilde{S} = S - 2\widetilde{\mathcal{P}}\,\mathcal{C}$, were expressed in terms of that of the original spray S and the projective factor $\widetilde{\mathcal{P}}$:

$$\widetilde{h} = h - \widetilde{\mathcal{P}}J - d_J\widetilde{\mathcal{P}} \otimes \mathcal{C}, \tag{3.2a}$$

$$\widetilde{v} = v + \widetilde{\mathcal{P}}J + d_J\widetilde{\mathcal{P}} \otimes \mathcal{C}, \tag{3.2b}$$

$$\widetilde{\Phi} = \Phi + (\widetilde{\mathcal{P}}^2 - \mathcal{L}_S\widetilde{\mathcal{P}})J + (2d_h\widetilde{\mathcal{P}} - \widetilde{\mathcal{P}} d_J\widetilde{\mathcal{P}} - \nabla d_J\widetilde{\mathcal{P}}) \otimes \mathcal{C}, \tag{3.2c}$$

where ∇ is the dynamical covariant derivative [2, Definition 3.4]. Using the fact that the spray S is horizontal, that is $hS = S$ and $\widetilde{\mathcal{P}} := \lambda\mathcal{P}$ is holonomy invariant, form (2.11) we get

$$\mathcal{L}_S\mathcal{P} = \mathcal{L}_{hS}\mathcal{P} = d_h\mathcal{P}(S) = 0. \tag{3.3}$$

Finally, using the commutator formula $\nabla d_J - d_J\nabla = 4i_R - d_h$ ([3, eq.(4.11)]), we get

$$\nabla d_J\mathcal{P} = d_J\nabla\mathcal{P} - d_h\mathcal{P} + 4i_R\mathcal{P} = d_J\nabla\mathcal{P} = d_J\mathcal{L}_S\mathcal{P} = 0. \tag{3.4}$$

Using (2.11), (3.3), and (3.4) one can simplify the formulas of (3.2) and we get (3.1). □

3.1. Horizontal and vertical subdistributions adapted to holonomic projective deformation.

For further computation and analysis, it will be very useful to introduce a decomposition of the horizontal (resp. the vertical) distributions adapted to a holonomic projective deformation associated to the projective factor $\mathcal{P}$: we introduce the endomorphsims

$$h_{\mathcal{P}} = h - \frac{d_J\mathcal{P}}{\mathcal{P}} \otimes S, \qquad v_{\mathcal{P}} = v - \frac{d_v\mathcal{P}}{\mathcal{P}} \otimes \mathcal{C}. \tag{3.5}$$

and we set

$$\mathcal{H}_{\mathcal{P}} := Im\, h_{\mathcal{P}}, \qquad \mathcal{V}_{\mathcal{P}} := Im\, v_{\mathcal{P}}. \tag{3.6}$$

We have the following

Lemma 3.2.

(1) Properties of $v_{\mathcal{P}}$ and $\mathcal{V}_{\mathcal{P}}$:
  - *i)* $\ker(v_{\mathcal{P}}) = \mathcal{H} \oplus Span\{C\}$
  - *ii)* $\mathrm{Im}(v_{\mathcal{P}}) = \mathcal{V}_{\mathcal{P}}$ *is an $(n-1)$-dimensional involutive subdistribution of $\mathcal{V}$,*
  - *iii) any $X \in \mathcal{V}_{\mathcal{P}}$ is an infinitesimal symmetry of $\mathcal{P}$ that is $\mathcal{L}_X\mathcal{P} = 0$.*
  - *iv) the vertical distribution have the decomposition $\mathcal{V} = \mathcal{V}_{\mathcal{P}} \oplus Span\{\mathcal{C}\}$.*

(2) Properties of $h_{\mathcal{P}}$ and $\mathcal{H}_{\mathcal{P}}$:
  - *i)* $\ker(h_{\mathcal{P}}) = \mathcal{V} \oplus Span\{S\}$
  - *ii)* $\mathrm{Im}(h_{\mathcal{P}}) = \mathcal{H}_{\mathcal{P}}$ *is an $(n-1)$-dimensional subdistribution of $\mathcal{H}$,*

iii) *any* $X \in \mathcal{H}_{\mathcal{P}}$ *is an infinitesimal symmetry of* $\mathcal{P}$ *that is* $\mathcal{L}_X \mathcal{P} = 0$.

iv) *the horizontal distribution have the decomposition* $\mathcal{H} = \mathcal{H}_{\mathcal{P}} \oplus Span\{S\}$,

(3) $J(\mathcal{H}_{\mathcal{P}}) = \mathcal{V}_{\mathcal{P}}$.

Proof. We prove *(1)* in detail. The computations for *(2)* are similar.

ad i) We note that $\mathcal{H} = \operatorname{Ker} v$, therefore $\mathcal{H} \subset \operatorname{Ker} v_{\mathcal{P}}$. Moreover, if $V \in \ker v_{\mathcal{P}}$ is vertical, then using $v(V) = V$ we get

$$v_{\mathcal{P}}(V) = 0 \quad \Longleftrightarrow \quad V = \frac{V(\mathcal{P})}{P}\mathcal{C},$$

that is $V \in \operatorname{Span}\{\mathcal{C}\}$ and we get *i)*.

ad ii) We introduce the simplified notation $\mathcal{P}_i := \dot{\partial}_i \mathcal{P}$ and the vector fields

$$h_i := h_{\mathcal{P}}(\delta_i) = \delta_i - \frac{\mathcal{P}_i}{\mathcal{P}} S, \tag{3.7a}$$

$$v_i := v_{\mathcal{P}}(\dot{\partial}_i) = \dot{\partial}_i - \frac{\mathcal{P}_i}{P}\mathcal{C} \tag{3.7b}$$

for $i = 1, \dots, n$. We get

$$\mathcal{H}_{\mathcal{P}} = \operatorname{Span}\{h_1, \dots, h_n\}, \tag{3.8a}$$

$$\mathcal{V}_{\mathcal{P}} = \operatorname{Span}\{v_1, \dots, v_n\}. \tag{3.8b}$$

We note that the vector fields in (3.8a) (resp. in (3.8b)) are not independent since $y^i h_i = 0$ (resp. $y^i v_i = 0$). Because the 1-homogeneity property of $\mathcal{P}$ (and the 0-homogeneity property of $\mathcal{P}_i$) for any $v_i, v_j \in \mathcal{V}_{\mathcal{P}}$, their Lie bracket is

$$[v_i, v_j] = \Big[\dot{\partial}_i - \frac{\mathcal{P}_i}{\mathcal{P}} y^k \dot{\partial}_k,\ \dot{\partial}_j - \frac{\mathcal{P}_j}{\mathcal{P}} y^\ell \dot{\partial}_\ell\Big] = \frac{\mathcal{P}_i}{\mathcal{P}}\dot{\partial}_j - \frac{\mathcal{P}_j}{\mathcal{P}}\dot{\partial}_i = \frac{\mathcal{P}_i}{\mathcal{P}} v_j - \frac{\mathcal{P}_j}{\mathcal{P}} v_i$$

and hence from (3.8b) we get that $[v_i, v_j] \in \mathcal{V}_{\mathcal{P}}$ hence $\mathcal{V}_{\mathcal{P}}$ is involutive.

ad iii) One can check that the generators (3.8b) of the distribution are infinitesimal symmetry of $\mathcal{P}$. Indeed, using Euler's theorem of the homogeneous functions we get for the 1-homogeneous $\mathcal{P}$:

$$\mathcal{L}_C \mathcal{P} = \mathcal{P}, \tag{3.9}$$

and therefore

$$\mathcal{L}_{v_i} P = \dot{\partial}_i(\mathcal{P}) - \frac{\mathcal{P}_i}{\mathcal{P}}\mathcal{C}(\mathcal{P}) = \mathcal{P}_i - \frac{\mathcal{P}_i}{\mathcal{P}}\mathcal{P} = 0. \tag{3.10}$$

ad iv) Supposing $\mathcal{C} \in \mathcal{V}_{\mathcal{P}}$ we get form (3.8b) that $\mathcal{C} = C^i v_i$ with some coefficients C^i. Solving this equation, since $\mathcal{C}(\mathcal{P}) = \mathcal{P}$ and $v_i(\mathcal{P}) = 0$, we find that $\mathcal{C}(\mathcal{P}) = C^i v_i(\mathcal{P}) = 0$ which is a contradiction.

For *3)*, we note that for the generators (3.7a) of (3.8a) and (3.7b) of (3.8b), we get

$$J h_i = J\delta_i - \frac{\mathcal{P}_i}{\mathcal{P}} JS = \dot{\partial}_i - \frac{\mathcal{P}_i}{\mathcal{P}}\mathcal{C} = v_i, \tag{3.11}$$

$i = 1, \dots, n$, and this proves *3)*. □

3.2. **Curvature properties of the holonomy deformation.**

In the sequel, we investigate the curvature properties of the connections associated to a Finsler metrizable spray S and its holonomy invariant projective deformation $\widetilde{S} = S - 2\lambda\mathcal{P}C$. We focus on the Riemannian curvature.

Lemma 3.3. *[Riemann curvature of a Finsler spray S]*
Let $\mathcal{P}$ be a nontrivial holonomy invariant 1-homogeneous function with respect to the Finsler spray S. Then one can choose a basis $\mathcal{X} = \{X_i\}_{i=1\dots n}$, of the horizontal distribution $\mathcal{H}$ such that the elements of $\mathcal{X}$ are eigenvectors of Φ with $X_n = S$ and

$$\mathcal{H}_{\mathcal{P}} = Span\{X_1, \dots, X_{n-1}\}. \tag{3.12}$$

Proof. Using the notation of Section 2.4, there exists a basis $\{X_\alpha\}$ composed by eigenvectors of Φ where $X_n := S$ is an eigenvector of Φ corresponding to the eigenfunction $\kappa_n = 0$. We consider the decomposition $\mathcal{H} = \mathcal{H}_{\mathcal{P}} \oplus \mathrm{Span}\{S\}$ given in Lemma 3.2. For $\alpha \in \{1, \dots, n-1\}$ the eigenvector X_α can be written as a linear combination

$$X_\alpha = X^i_\alpha \cdot h_i + X^S_\alpha \cdot S, \tag{3.13}$$

of the vectors (3.7a) and the spray S. If $\kappa_\alpha \neq 0$ then, using Corollary 2.6 we get $\mathcal{L}_{\Phi(X_\alpha)}\mathcal{P} = 0$, and using (3.10) we get:

$$0 = \mathcal{L}_{\Phi(X_\alpha)}\mathcal{P} = \kappa_\alpha \, \mathcal{L}_{JX_\alpha}\mathcal{P} = \kappa_\alpha (X^i_\alpha \mathcal{L}_{v_i}\mathcal{P} + X^S_\alpha \mathcal{L}_{\mathcal{C}}\mathcal{P}) = \kappa_\alpha \, X^S_\alpha \, \mathcal{P}. \tag{3.14}$$

Since $\mathcal{P} \neq 0$, it follows that $X^S_\alpha = 0$, that is $X_\alpha \in \mathcal{H}_{\mathcal{P}}$. On the other hand, if $\kappa_\alpha = 0$, then using the notation (3.13), we can modify X_α to get $\hat{X}_\alpha := X_\alpha - X^S_\alpha \cdot S$, which will be an eigenvector of Φ in $\mathcal{H}_{\mathcal{P}}$ with eigenvalue $\kappa_\alpha = 0$.

□

Let $\mathcal{P}$ be a holonomy invariant function. If we fix an arbitrary point $(x, y) \in \mathcal{T}M$, then for almost any value of $\lambda \in \mathbb{R}$ the inequality

$$\kappa_\alpha(x, y) + \lambda^2 \mathcal{P}^2(x, y) \neq 0, \tag{3.15}$$

holds for any $\alpha = 1, \dots, n$. Using the continuity property of the eigenvaules κ_α, there is an open neighbourhood $U \subset \mathcal{T}M$ of (x, y) such that the condition (3.15) is satisfied on U. From now on, all geometric objects will be restricted to U.

Lemma 3.4 (Riemann curvature of the projectively deformed spray $\widetilde{S}$).
For $\lambda \in \mathbb{R}$ such that (3.15) *holds, the image of the Riemann curvature $\widetilde{\Phi}$ of $\widetilde{S}$ is $\mathcal{V}_{\mathcal{P}}$:*

$$\mathcal{V}_{\mathcal{P}} = \mathrm{Im}\, \widetilde{\Phi}.$$

Proof. $\widetilde{\Phi}$ is determined by (3.1c). Since it is semibasic, it is identically zero on vertical vector fields. Hence, its image can be calculated by using horizontal vectors. We will use the horizontal basis introduced in Lemma 3.3.

For $\alpha = n$ we have $X_n = S$ and $d_J\mathcal{P}(S) = d_{JS}\mathcal{P} = d_{\mathcal{C}}\mathcal{P} = \mathcal{P}$, hence from (2.13), (3.1c), and (3.9) we obtain

$$\widetilde{\Phi}(S) = \Phi(S) + \lambda^2\mathcal{P}^2 JS - \lambda^2\mathcal{P}\, d_J\mathcal{P}(S) \otimes \mathcal{C} = 0 + \lambda^2\mathcal{P}^2\mathcal{C} - \lambda^2\mathcal{P}^2\mathcal{C} = 0.$$

For $1 \leq \alpha < n$ we have $X_\alpha \in \mathcal{H}_{\mathcal{P}}$. Using *3)* of Lemma 3.2 we have $JX_\alpha \in \mathcal{V}_{\mathcal{P}}$ and from *(1/iii)* of the same lemma we get $d_J\mathcal{P}(X_\alpha) = \mathcal{L}_{JX_\alpha}\mathcal{P} = 0$. It follows that

$$\widetilde{\Phi}(X_\alpha) = \Phi(X_\alpha) + \lambda^2(\mathcal{P}^2 J - \mathcal{P} d_J\mathcal{P} \otimes \mathcal{C})(X_\alpha) = (\kappa_\alpha + \lambda^2\mathcal{P}^2) JX_\alpha. \tag{3.16}$$

Using (3.15) we get that $JX_\alpha \in \mathrm{Im}\, \widetilde{\Phi}$. Summarizing, we have

$$\mathrm{Im}\, \widetilde{\Phi} = \mathrm{Span}\{JX_1, \dots, JX_{n-1}\} = \mathcal{V}_{\mathcal{P}}.$$

□

Since the image of the Riemann curvature is a subspace of the holonomy distribution (see Corollary (2.6)) we get the following corollary.

Corollary 3.5. *Under the hypothesis of Lemma 3.4 we have*

$$\mathcal{V}_{\mathcal{P}} \subset \mathcal{D}_{hol}(\widetilde{S}). \tag{3.17}$$

Proposition 3.6. *If the projective factor $\mathcal{P}$ is nonlinear and $\lambda \neq 0$ satisfies* (3.15) *on $U \subset TM$, then the holonomy distribution of the non-trivial projectively deformed spray $\widetilde{S} = S - 2\lambda\mathcal{P}\mathcal{C}$ is the full TU, that is*

$$\mathcal{D}_{hol}(\widetilde{S})\big|_U = TU. \tag{3.18}$$

Proof. The holonomy distribution $\mathcal{D}_{hol}(\widetilde{S})$ of the spray $\widetilde{S}$ contains its horizontal space $\widetilde{\mathcal{H}}$ and the image of the the Riemann curvature $\widetilde{\Phi}$, therefore, from Lemma 3.4 we get that

$$\widetilde{\mathcal{H}} \oplus \mathcal{V}_{\mathcal{P}} \subset \mathcal{D}_{hol}(\widetilde{S}). \tag{3.19}$$

It follows that $\widetilde{h}_i := \widetilde{h}(h_i)$ and v_i are elements of $\mathcal{D}_{hol}(\widetilde{S})$. By the involutivity of $\mathcal{D}_{hol}(\widetilde{S})$ the Lie bracket $[\widetilde{h}_i, v_i]$ and its horizontal part are in $\mathcal{D}_{hol}(\widetilde{S})$, therefore so its vertical part:

$$\widetilde{v}[\widetilde{h}_i, v_j] \in \mathcal{D}_{hol}(\widetilde{S}). \tag{3.20}$$

On the other side, we get from (3.1a) $\widetilde{h}_i = h_i - \lambda\mathcal{P}v_i$, and hence, taking $\mathcal{L}_{v_i}\mathcal{P} = 0$ into account, we have

$$\widetilde{v}[\widetilde{h}_i, v_j] = \widetilde{v}[h_i, v_j] - \lambda\mathcal{P}\widetilde{v}[v_i, v_j]. \tag{3.21}$$

Since the distribution $\mathcal{V}_{\mathcal{P}}$ is integrable $\widetilde{v}$ is the identity on $\mathcal{V}_{\mathcal{P}}$ and we have

$$\widetilde{v}[v_i, v_j] = [v_i, v_j] \quad \in \mathcal{V}_{\mathcal{P}} \subset \mathcal{D}_{hol}(\widetilde{S}). \tag{3.22}$$

Therefore, from (3.20) and (3.22), using (3.21) we get that

$$\widetilde{v}[h_i, v_j] \in \mathcal{H}ol_{\widetilde{S}}. \tag{3.23}$$

On the other hand, using the identities

$$\delta_i\mathcal{P}_j = G^k_{ij}\mathcal{P}_k, \quad \delta_i y^j = -G^j_i, \quad S(\mathcal{P}_j) = G^k_j\mathcal{P}_k, \quad S(y^j) = -2G^j,$$

we have

$$v[h_i, v_j] = v\Big[\delta_i - \frac{\mathcal{P}_i}{\mathcal{P}}S,\ \dot{\partial}_j - \frac{\mathcal{P}_j}{\mathcal{P}}\mathcal{C}\Big] = \Big(G^k_{ij} - \frac{\mathcal{P}_i}{\mathcal{P}}G^k_j\Big)\, v_k, \tag{3.24}$$

from which we get that $v[h_i, v_j] \in \mathcal{V}_{\mathcal{P}}$ and

$$v[h_i, v_j] \in \mathcal{D}_{hol}(\widetilde{S}). \tag{3.25}$$

Now, by (3.1b), we have

$$\widetilde{v}[h_i, v_j] - v[h_i, v_j] = \lambda\mathcal{P}J[h_i, v_j] + \lambda\mathcal{L}_{J[h_i,v_j]}\mathcal{P}\,\mathcal{C}. \tag{3.26}$$

and because of (3.23) and (3.25) the left-hand side of (3.26) is in $\mathcal{D}_{hol}(\widetilde{S})$, so is the right-hand side:

$$\mathcal{P}\cdot J[h_i, v_j] + \mathcal{L}_{J[h_i,v_j]}\mathcal{P}\cdot\mathcal{C} \quad \in \mathcal{D}_{hol}(\widetilde{S}). \tag{3.27}$$

Calculating the second term on the right-hand side of (3.26) we get

$$J[h_i, v_j] = J\Big[\delta_i - \frac{\mathcal{P}_i}{\mathcal{P}}S,\ \dot{\partial}_j - \frac{\mathcal{P}_j}{\mathcal{P}}\mathcal{C}\Big] = \frac{\mathcal{P}_i}{\mathcal{P}}v_j + \frac{\mathcal{P}_{ij}}{\mathcal{P}}\mathcal{C},$$

where $\mathcal{P}_{ij} := \dot{\partial}_j\mathcal{P}_i$. Using *iii)* of *(1)* from Lemma 3.2 we get

$$\mathcal{P}\cdot J[h_i, v_j] + \mathcal{L}_{J[h_i,v_j]}\mathcal{P}\cdot\mathcal{C} = \mathcal{P}_i v_j + 2\mathcal{P}_{ij}\mathcal{C}. \tag{3.28}$$

The (3.27) and (3.17) show that the left-hand side and the first term in the right-hand side are in $\mathcal{D}_{hol}(\widetilde{S})$, therefore the $\mathcal{P}_{ij}\mathcal{C} \in \mathcal{D}_{hol}(\widetilde{S})$. Since $\mathcal{P}$ is non linear, then there exists at least pair of indices (i, j) such that $\mathcal{P}_{ij} \neq 0$. It follows that

$$\mathcal{C} \in \mathcal{D}_{hol}(\widetilde{S}). \tag{3.29}$$

Completing (3.19) with Span$\{C\}$ we get

$$\widetilde{\mathcal{H}} \oplus \mathcal{V}_{\mathcal{P}} \oplus \mathrm{Span}\{\mathcal{C}\} \subset \mathcal{D}_{hol}(\widetilde{S}). \tag{3.30}$$

According to *iv)* of *(1)* from Lemma 3.2 we have $\mathcal{V}_{\mathcal{P}} \oplus \mathrm{Span}\{\mathcal{C}\} = \mathcal{V} = \widetilde{\mathcal{V}}$, therefore

$$\widetilde{\mathcal{H}} \oplus \widetilde{\mathcal{V}} = \mathcal{D}_{_{hol}}(\widetilde{S}). \tag{3.31}$$

which proves the proposition.

□

4. Metrizability of holonomic projective deformations

In this section we investigate the metrizability property of the holonomic projective deformation $\widetilde{S} = S - 2\lambda \mathcal{P}\mathcal{C}$ of the spray S. Our goal is to prove Theorem 1 and to characterize the cases where such a deformation can lead to a metrizable spray.

Proposition 4.1. *Let $\lambda \in \mathbb{R}$ be such that* (3.15) *holds. If $\lambda \neq 0$, then the projectively deformed spray $\widetilde{S} = S - 2\lambda \mathcal{P}\mathcal{C}$ is not metrizable.*

Proof. Arguing by contradiction, let us suppose that $\widetilde{S}$ is Finsler metrizable and $\widetilde{E}$ is a Finsler energy function associated to $\widetilde{S}$. Depending on the linearity of the projective factor $\mathcal{P}$ we consider two cases. If the projective factor $\mathcal{P}$ *is nonlinear*, from Proposition 3.6 we get that $\mathcal{D}_{_{hol}}(\widetilde{S}) = TTM$. Hence, using Corollary 2.7 we get that the derivative of $\widetilde{E}$ of $\widetilde{S}$ should be identically zero with respect to any vector field $X \in \mathfrak{X}(TM)$, that is $\widetilde{E}$ is constant, which is impossible. On the other hand, if the projective factor $\mathcal{P}$ *is linear*, then using (3.17) and Corollary 2.7 we get

$$\mathcal{L}_{v_i}\widetilde{E} = 0 \quad \Longrightarrow \quad \dot{\partial}_i \widetilde{E} - \frac{\mathcal{P}_i}{P}\mathcal{L}_{\mathcal{C}}(\widetilde{E}) = 0 \quad \Longrightarrow \quad \frac{\dot{\partial}_i \widetilde{E}}{\widetilde{E}} = 2\frac{\dot{\partial}_i \mathcal{P}}{\mathcal{P}},$$

therefore locally there exists a function $\theta(x)$ on M such that $\widetilde{E} = \mathcal{P}^2 e^{\theta(x)}$. Writing the linear projective factor in the form $\mathcal{P} = a_i(x)y^i$ we get

$$g_{ij}(x,y) = \dot{\partial}_i \dot{\partial}_j \widetilde{E} = 2a_i(x)a_j(x)e^{\theta(x)},$$

hence g_{ij} has rank 1 and in the case $n \geq 2$, the energy function $\widetilde{E}$ is degenerate which is a contradiction. □

Proof of the Theorem 1. Let $\mathcal{P}$ be a nontrivial holonomy invariant 1-homogeneous function. Let us fix a point $x \in M$ and a direction $y \in \mathcal{T}_x M$. Then, using the eigenvalue κ_i of the Riemann curvature Φ at y, the set

$$\Lambda_{(x,y)} := \left\{ \lambda \in \mathbb{R} \mid \kappa_i + \lambda^2 \mathcal{P}^2 = 0, \ i = 1, \dots, n{-}1 \right\} \tag{4.1}$$

is a finite set, therefore its complement is an open dense subset of $\mathbb{R}$. For any element $\lambda \in \mathbb{R} \setminus \Lambda_{(x,y)}$ we have (3.15) and, using Theorem 4.1 one obtains that $\widetilde{S} = S - 2\lambda \mathcal{P}\,\mathcal{C}$, is not metrizable.

□

As the precedent results show, for a given Finsler structure (M, F), only very specific holonomy invariant projective factors can produce Finsler metrizable sprays. Such projective factor must be related to the principal curvature of the original Finsler structure. More precisely, we have the following

Corollary 4.2. *Let (M, F) be a Finsler manifold, S its geodesic spray and let $\widetilde{\mathcal{P}}$ be a holonomy invariant nonzero function. If the projective deformation $\widetilde{S} = S - 2\widetilde{\mathcal{P}}\,\mathcal{C}$, is metrizable, then*

$$\widetilde{\mathcal{P}}^2 + \kappa_\alpha = 0, \tag{4.2}$$

for some (nonzero) principal curvature κ_α, $\alpha \in \{1, \dots, n{-}1\}$.

In particular we obtain that if the principal curvatures are all non-negatives, then there is no non-trivial holonomy invariant metrizable projective deformation of the Finsler structure.

As Corollary 4.2 shows, the holonomy invariant projective deformations $\widetilde{S} = S - 2\widetilde{\mathcal{P}}\mathcal{C}$ leading to metrizable sprays are limited by the condition (4.2). We emphasize however, that (4.2) gives only a necessary condition as will be shown in coming examples where we consider Finsler functions F having constant flag curvature κ. It follows, that the principal curvatures

$$\kappa_\alpha = \kappa F^2, \tag{4.3}$$

for $\alpha = 1, \dots, n-1$ are equal [3].

Example 1. Let us consider the Klein metric

$$F = \sqrt{\frac{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}{(1-|x|^2)^2}}.$$

It is projectively flat metric of constant flag curvature $\kappa = -1$ and its geodesic spray S is given by the geodesic coefficients $G^i = \frac{\langle x,y\rangle}{1-|x|^2} y^i$. Since F is a holonomy invariant function, the

$$\widetilde{S} = S - 2F\mathcal{C} \tag{4.4}$$

is a holonomy invariant projective deformation of the Finsler spray S with $\widetilde{\mathcal{P}} = F$. From (4.3) we get $\kappa_\alpha = -F^2$ and (4.2) is satisfied. The the geodesic coefficients of (4.4) are

$$\widetilde{G}^i = \left(\sqrt{\frac{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}{(1-|x|^2)^2}} + \frac{\langle x, y\rangle}{1-|x|^2} \right) y^i. \tag{4.5}$$

It is clear that the above spray $\widetilde{S}$ is projectively flat. Moreover, one can show that (4.4) is also R-flat and by [9] it is locally Finsler metrizable. It should be noted that the (global) Finsler metrizability of (4.4) is questioned in [14, Chapter 10.3].

Example 2. Modifying the above example, let us consider for $\mu > 0$ the Finsler function

$$F = \sqrt{\frac{(1-\mu|x|^2)|y|^2 + \mu\langle x, y\rangle^2}{(1-\mu|x|^2)^2}}. \tag{4.6}$$

It is a projectively flat metric of constant flag curvature $\kappa = -\mu$ (see, [5]), and its geodesic spray S is given by $G^i = \mu\frac{\langle x,y\rangle}{1-|x|^2} y^i$. From (4.3) the principal curvatures are $\kappa_\alpha = -\mu F^2$. Then

$$\widetilde{S} = S - 2\lambda F \cdot \mathcal{C}, \tag{4.7}$$

with $\lambda \in \mathbb{R}$, $\lambda \neq 0$ is a nontrivial holonomy invariant projective deformation of the Finsler spray S with projective factor $\widetilde{\mathcal{P}} = \lambda F$. If $\lambda = \sqrt{\mu}$, then (4.2) is satisfied, the spray (4.7) is R-flat and hence it is locally Finsler metrizable. For any other nonzero value of λ the condition (4.2) is not satisfied, and (4.7) is not Finsler metrizable. Indeed, one can check that in a generic direction $y \in \mathcal{T}M$, the holonomy distribution $\mathcal{D}_{hol}(S)_y$ contains the full second tangent direction, that is $\mathcal{D}_{hol}(S)_y = T_y\mathcal{T}M$.

Open problem: Corollary 4.2 gives necessary conditions on the Finsler metrizability of holonomy invariant projective deformations in terms of the principal curvatures. It would be very interesting to find sufficient conditions of metrizability which can be expressed by these important geometric quantities.

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References

[1] J. C. Alvarez Paiva, *Symplectic geometry and Hilbert's fourth problem*, Journal of Differential Geometry, **69** (2005), 353–378.
[2] I. Bucataru and M. F. Dahl, *Semi basic 1-forms and Helmholtz conditions for the inverse problem of the calculus of variations*, J. Geom. Mech. **1(2)** (2009) 159–180.
[3] I. Bucataru and Z. Muzsnay, *Projective and Finsler metrizability: parameterization-rigidity of the geodesics*, Int. J. Math. **23**, 9 (2012).
[4] I. Bucataru and Z. Muzsnay, *Projective metrizability and formal integrability*, Symmetry, Integrability and Geometry: Methods and Applications, **7** (2011).
[5] S. S. Chern and Z. Shen, *Riemann-Finsler Geometry*, World Scientific Publishers 2004.
[6] M. Crampin, *On the inverse problem for sprays*, Publ. Math. Debrecen, **70** (2007), 319–335.
[7] M. Eastwood and V.S. Matveev, *Metric connections in projective differential geometry*, The IMA Volumes in Math. and its Appl., **144** (2008), 339–350.
[8] J. Grifone, *Structure presque-tangente et connexions,* I, Ann. Inst. Fourier, Grenoble, **22, 1** (1972), 287–334.
[9] J. Grifone and Z. Muzsnay, *Variational Principles for Second Order Differential Equations: Application of the Spencer Theory to Characterize Variational Sprays*, World Scientific, 2000.
[10] D. Krupka and A. E. Sattarov, *The inverse problem of the calculus of variations for Finsler structures*, Math. Slovaca, **35** (1985), 217–222.
[11] Z. Muzsnay, *The Euler-Lagrange PDE and Finsler metrizability*, Houston J. Math., **32** (2006), 79–98.
[12] O. Krupková, *Variational metric structures* Publ. Math. Debrecen **62** (2003), no. 3-4, 461–495.
[13] W. Sarlet, G. Thompson and G. Prince, *The inverse problem of the calculus of variations: the use of geometrical calculus in Douglas's analysis*, Trans. Amer. Math. Soc. **354** (2002), no. 7, 2897–2919
[14] Z. Shen, *Differential geometry of spray and Finsler spaces*, Springer, 2001.
[15] Z. Shen, *Geometric Meanings of Curvatures in Finsler Geometry*, Rendiconti del Circolo Matematico di Palermo, Serie II, Suppl. **66** (2001) pp. 165–178.
[16] J. Szilasi and S. Vattamany, *On the Finsler-metrizabilities of spray manifolds*, Period. Math. Hungar., **44** (2002), 81–100.
[17] G. Yang, *Some classes of sprays in projective spray geometry*, Diff. Geom. Appl., **29** (2011), 606–614.

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