Lie groups as multiplication groups of topological loops

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Abstract

In this short survey we give some new results about the question whether or not a Lie group can be represented as the multiplication group of a 3-dimensional topological loop. We deal with the classes of quasi-simple Lie groups and nilpotent Lie groups.

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1. Introduction

A loop $(L, \cdot)$ is a quasigroup with identity element $e \in L$. The left translations $\lambda_a : y \mapsto a \cdot y$ and the right translations $\rho_a : y \mapsto y \cdot a$, $a \in L$, are bijections of L. The group generated by all left and right translations of L is called the multiplication group $Mult(L)$ of L. The stabilizer of $e \in L$ in the group $Mult(L)$ is the inner mapping group $Inn(L)$ of L. The multiplication group and the inner mapping group of a loop L are important tools for research in loop theory since there are strict relations between the structure of the groups $Mult(L)$, $Inn(L)$ and that of L. If the group $Mult(L)$ is simple, then the loop L is also simple and the group $Inn(L)$ is a maximal subgroup of $Mult(L)$ (cf. [1], [4], [15]). The nilpotency of the group $Mult(L)$ implies the centrally nilpotency of L (cf. [4]).

The subgroup G_l of $Mult(L)$ generated by all left translations of L is the group of left translations of L. Topological loops L such that the groups G_l of L are Lie groups have been studied in [14] and they are treated as continuous sharply transitive sections $\sigma : G_l/H_l \to G_l$, where H_l is the stabilizer of the identity element $e \in L$ in G_l.

For infinite groups in any studied category there is till now only one feasible criterion for the decision whether a group Γ is the multiplication group of a loop L, namely the criterion of Niemenmaa and Kepka ([15]).

Proposition 1. *A group Γ is the multiplication group of L if and only if there exists a maximal subgroup K of Γ containing no non-trivial proper normal subgroup of Γ and two left transversals A and B of K in Γ such that $a^{-1}b^{-1}ab \in K$ for every $a \in A$ and $b \in B$ and the set $\{A, B\}$ generates Γ.*

This criterion has been successfully applied in particular in the case of Lie groups. The multiplication group $Mult(L)$ of a topological loop L is mostly a differentiable transformation group of infinite dimension. This is the case for every 1-dimensional topological loop ([14], Theorem 18.18, p. 248). *For 2-dimensional topological loops L the group $Mult(L)$ is a Lie group precisely if it is an elementary filiform Lie group $\mathcal{F}_n$ with $n \geq 4$, i. e. if the Lie algebra* $\mathbf{mult(L)}$ *of $Mult(L)$ has a basis $\{e_1, \cdots, e_n\}$ such that $[e_1, e_i] = e_{i+1}$ for $2 \leq i \leq n-1$* ([9], Theorem 1, p. 420).
In contrast to this there does not exist 3-dimensional connected simply connected topological loop L having an elementary filiform Lie group as its multiplication group (cf. Proposition 5). The 4-dimensional indecomposable Lie groups are not multiplication groups of L. There are only two classes of the 5-dimensional nilpotent Lie groups which are multiplication groups of L. The corresponding loops L are centrally nilpotent of class 2 (cf. Proposition 6).
In the last section we treat quasi-simple Lie groups acting transitively and effectively on 3-dimensional manifolds. None of the at most 8-dimensional quasi-simple Lie groups occurs as the multiplication group of a connected topological proper loop. However, the group $SL_4(\mathbb{R})$ is the multiplication group of connected topological loops homeomorphic to S^3.

2. Preliminaries

A binary system $(L, \cdot)$ is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution, which we denote by $y = a \backslash b$ and $x = b/a$. A loop L is proper if it is not a group.
The kernel of a homomorphism $\alpha : (L, \cdot) \to (L', *)$ of a loop L into a loop L' is a normal subloop N of L. The centre $Z(L)$ of a loop L consists of all elements z which satisfy the equations $zx \cdot y = z \cdot xy$, $x \cdot yz = xy \cdot z$, $xz \cdot y = x \cdot zy$, $zx = xz$ for all $x, y \in L$. If we put $Z_0 = e$, $Z_1 = Z(L)$ and $Z_i/Z_{i-1} = Z(L/Z_{i-1})$, then we obtain a series of normal subloops of L. If Z_{n-1} is a proper subloop of L but $Z_n = L$, then L is centrally nilpotent of class n. In [4] it was proved that if $Mult(L)$ is a nilpotent group of class n, then L is centrally nilpotent of class at most n.
The connections between the normal subgroup structure of $Mult(L)$ and the normal subloop structure of L are the following. Let L be a loop with multiplication group $Mult(L)$ and identity element e. Let N be a normal subloop of L and $M(N)$ be the set $\{m \in Mult(L);\ xN = m(x)N \text{ for all } x \in L\}$. Then $M(N)$ is a normal subgroup of $Mult(L)$ containing the multiplication group $Mult(N)$ of the loop N and the multiplication group of the factor loop L/N is isomorphic to $Mult(L)/M(N)$. Conversely, for every normal subgroup $\mathcal{N}$ of $Mult(L)$ the orbit $\mathcal{N}(e)$ is a normal subloop of L. Moreover, $\mathcal{N} \leq M(\mathcal{N}(e))$. (cf. [1], Theorems 3, 4 and 5 and in [5], IV.1, Lemma 1.3).

Let L be a loop and let G_l, respectively G_r the group of left, respectively of right translations of L. We have $G_l = G_r = Mult(L)$ if and only if for the stabilizers H_l, respectively H_r of $e \in L$ in G_l, respectively in G_r one has $H_l = H_r = Inn(L)$ and for all $x \in L$ the map $f(x) : y \mapsto \lambda_x^{-1} \lambda_y x : L \to L$ is an element of $Inn(L)$.

A loop L is called topological if L is a topological space and the binary operations $(x, y) \mapsto x \cdot y$, $(x, y) \mapsto x \backslash y, (x, y) \mapsto y/x : L \times L \to L$ are continuous. Every connected topological loop L having a Lie group G_l as the group of left translations of L is obtained on a homogeneous space G_l/H_l, where H_l is a closed subgroup of G_l with $Co_{G_l}(H_l) = 1$ and $\sigma : G_l/H_l \to G_l$ is a continuous section such that $\sigma(H_l) = 1 \in G_l$, the subset $\sigma(G_l/H_l)$ generates G_l and the set $\sigma(G_l/H_l)$ operates sharply transitively on G_l/H_l, which means that to any xH_l and yH_l there exists precisely one $z \in \sigma(G_l/H_l)$ with $zxH_l = yH_l$. The multiplication of L on the manifold G_l/H_l is defined by $xH_l * yH_l = \sigma(xH_l)yH_l$.
For any connected topological loop there exists universal covering which is simply connected (cf. [12], IX.1.).
A 2-dimensional connected simply connected loop $L_{\mathcal{F}}$ is called an elementary filiform loop if its multiplication group is an elementary filiform group $\mathcal{F}_n$, $n \geq 4$ ([10]).
A Lie group is called indecomposable if it is not the direct product of two proper ideals of positive dimension.
A quasi-simple connected Lie group is a connected Lie group G such that any normal subgroup of G is discrete and central in G. A connected loop L is quasi-simple if any normal subloop of L is discrete in L. According to [12], p. 216, all discrete normal subloops of a connected loop are central.

3. Nilpotent Lie groups as multiplication groups of topological loops

Lemma 3.3 in [9], p. 390, says the following.

Lemma 2. *Let L be a 3-dimensional proper connected topological loop such that its multiplication group is a solvable Lie group. If L is simply connected, then it is homeomorphic to $\mathbb{R}^3$.*

Assume that the group $Mult(L)$ of a 3-dimensional proper connected topological loop L is nilpotent. For the centre Z of $Mult(L)$ one has dim $Z \in \{1, 2\}$ (cf. Lemma 3.5. in [9], p. 391). By Theorem 11 in [1] the orbit $Z(e)$ is the centre $Z(L)$ of L. From Proposition 3.7. in [9], p. 392, one gets:

Proposition 3. *Let L be a 3-dimensional proper connected simply connected topological loop such that its multiplication group $Mult(L)$ is a nilpotent Lie group and the centre Z of $Mult(L)$ has dimension 2. Then $Mult(L)$ is a semidirect product of a group $Q \cong \mathbb{R}$ by the abelian group $M = Z \times Inn(L) \cong \mathbb{R}^m$, $m \geq 3$, where $\mathbb{R}^2 = Z \cong Z(L)$.*

For every 1-dimensional connected subgroup N of the centre Z of $Mult(L)$ the orbit $N(e)$ is a 1-dimensional connected normal subloop of L containing in $Z(L)$ (cf. Lemma 3.6. (a) in [9], p. 392).

Proposition 4. *Let L be a 3-dimensional proper connected simply connected topological loop such that its multiplication group $Mult(L)$ is an indecomposable nilpotent Lie group. Then there exists a 1-dimensional central subgroup $N(e)$ of L isomorphic to $\mathbb{R}$. Moreover, one of the following possibilities holds:*
a) If the factor loop $L/N(e)$ is isomorphic to the abelian group $\mathbb{R}^2$, then the centre Z of $Mult(L)$ has dimension 1 and $N(e) = Z(e) = Z(L)$. Moreover, $Mult(L)$ is a semidirect product of a group $Q \cong \mathbb{R}^2$ by the abelian group $P = Z \times Inn(L) \cong \mathbb{R}^m$, $m \geq 2$.
b) If the factor loop $L/N(e)$ is isomorphic to a 2-dimensional elementary filiform loop $L_{\mathcal{F}}$, then there is a normal subgroup S of $Mult(L)$ containing $N \cong \mathbb{R}$ such that the factor group $Mult(L)/S$ is an elementary filiform Lie group $\mathcal{F}_n$ with $n \geq 4$.

Proposition 5. *There does not exist any 3-dimensional connected simply connected topological loop having an elementary filiform Lie group as its multiplication group.*

The classification of the indecomposable nilpotent Lie algebras of dimension 5 can be found in [11], pp. 167-168.

Proposition 6. *Let L be a connected simply connected topological proper loop of dimension 3 such that its multiplication group $Mult(L)$ is a 5-dimensional indecomposable nilpotent Lie group. Then L contains a central subgroup $C \cong \mathbb{R}$ such that the factor loop $L/C \cong \mathbb{R}^2$. Moreover, the following Lie groups are the multiplication groups $Mult(L)$ and the following subgroups are the inner mapping groups $Inn(L)$ of L:*
1) $Mult(L)_1$ is the direct product $\mathcal{F}_3 \times_Z \mathcal{F}_3$ with amalgamated centre Z the multiplication of which is given by $g(q_1, z_1, w_1, x_1, y_1)g(q_2, z_2, w_2, x_2, y_2) =$

$$g(q_1 + q_2 + z_1x_2 + w_1y_2, z_1 + z_2, w_1 + w_2, x_1 + x_2, y_1 + y_2).$$

$Inn(L)$ is the subgroup $\{g(0, z, w, 0, 0), z, w \in \mathbb{R}\}$.
2) $Mult(L)_2$ is represented on $\mathbb{R}^5$ by the multiplication

$$g(x_1, y_1, q_1, z_1, w_1)g(x_2, y_2, q_2, z_2, w_2) =$$

$$g(x_1 + x_2 + q_1z_2 + w_1y_2 + \frac{w_1^2 q_2}{2}, y_1 + y_2 + w_1q_2, q_1 + q_2, z_1 + z_2, w_1 + w_2).$$

$Inn(L)_2$ is one of the subgroups $Inn(L)_{2,1} = \{g(0, y, q, 0, 0), y, q \in \mathbb{R}\}$, $Inn(L)_{2,2} = \{g(0, y, 0, z, 0), y, z \in \mathbb{R}\}$.

4. Quasi-simple Lie groups as multiplication groups of topological loops

The following lemma is proved in [15], Lemma 2.6.

Lemma 7. *Let S be a proper subgroup of a simple group K and let A and B be S-connected transversals in K. Then S is maximal in K.*

Proposition 8. *If L is a 3-dimensional connected simply connected topological loop having a quasi-simple Lie group as the multiplication group $Mult(L)$ of L, then one of the following cases can occur:*
(a) L is homeomorphic to S^3 and the group $Mult(L)$ is one of the following Lie groups: $SL_2(\mathbb{C})$, $SU_3(\mathbb{C},1)$, $SL_4(\mathbb{R})$, $SO_5(\mathbb{R},1)$, $Sp_4(\mathbb{R})$, the universal covering of $SL_3(\mathbb{R})$.
(b) L is homeomorphic to $\mathbb{R}^3$ and the group $Mult(L)$ is the group $PSL_2(\mathbb{C})$.

The proof of this Proposition can be found in [10], Proposition 3.2, p. 389.

Theorem 9. *There does not exist connected topological loop L such that its multiplication group $Mult(L)$ is locally isomorphic to the group $PSL_2(\mathbb{C})$.*

We consider the Lie groups which are locally isomorphic to $SL_3(\mathbb{R})$.

Lemma 10. *If there exists a connected topological proper loop L such that its multiplication group $Mult(L)$ is locally isomorphic to $SL_3(\mathbb{R})$, then we have the following possibility: the group $Mult(L)$ of L as well as the group G_l of L is the group $SL_3(\mathbb{R})$, the stabilizer $Inn(L) = H_l$ of $e \in L$ in the group $Mult(L) = G_l$ is the subgroup $SO_3(\mathbb{R})$ and L is homeomorphic to $\mathbb{R}^5$.*

Theorem 11. *There does not exist any connected topological loop L such that its multiplication group $Mult(L)$ is locally isomorphic to the Lie group $SL_3(\mathbb{R})$.*

Now we treat the Lie groups G which are locally isomorphic to $PSU_3(\mathbb{C},1)$. From [3], Satz 1, p. 251 and [6], Section 5, p. 276, we obtain the following:

Proposition 12. *Any connected closed maximal subgroup of $PSU_3(\mathbb{C},1)$ is one of the following groups*
(1) H_1 is isomorpic to the group $Spin_3 \times SO_2(\mathbb{R})$,
(2) H_2 is isomorphic to the 5-dimensional solvable group NG, where

$$N = \left\{ \begin{pmatrix} 1 & zi & z \\ \bar{z}i & 1+it-\frac{z\bar{z}}{2} & t+\frac{z\bar{z}}{2} \\ \bar{z} & t+\frac{z\bar{z}}{2} & 1-it+\frac{z\bar{z}}{2} \end{pmatrix}; z \in \mathbb{C}, t \in \mathbb{R} \right\} \textit{ and}$$

$$G = \left\{ \begin{pmatrix} e^{-ik} & 0 & 0 \\ 0 & \frac{1}{2}(e^{-u}+e^{u})e^{\frac{1}{2}ik} & \frac{1}{2}(e^{u}-e^{-u})ie^{\frac{1}{2}ik} \\ 0 & \frac{1}{2}(e^{-u}-e^{u})ie^{\frac{1}{2}ik} & \frac{1}{2}(e^{-u}+e^{u})e^{\frac{1}{2}ik} \end{pmatrix}; k, u \in \mathbb{R} \right\},$$

(3) H_3 is isomorphic to the group $SU_2(\mathbb{C},1) \times SO_2(\mathbb{R}) \cong SL_2(\mathbb{R}) \times SO_2(\mathbb{R})$,
(4) H_4 is isomorphic to the group $SO_0(2,1) \cong PSL_2(\mathbb{R})$.

Proposition 13. *If there exists a connected topological proper loop L such that its multiplication group $Mult(L)$ is locally isomorphic to $PSU_3(\mathbb{C}, 1)$, then the following possibilities can occur:*
(a) L is homeomorphic to $\mathbb{R}^4$, the group $Mult(L)$ is the group $PSU_3(\mathbb{C}, 1)$, the subgroup $Inn(L) = H_l$ of $Mult(L)$ is the subgroup H_1 given in Proposition 12 (1).
(b) L is homeomorphic to S^3, the group $Mult(L)$ as well as the group G_l of L is the group $PSU_3(\mathbb{C}, 1)$, the subgroup $Inn(L) = H_l$ of $Mult(L)$ is the subgroup H_2 given in Proposition 12 (2).

Theorem 14. *There does not exist a connected topological loop L such that the group $Mult(L)$ is the Lie group $PSU_3(\mathbb{C}, 1)$.*

Remark 15. Till now the only known quasi-simple Lie group which is the multiplication group of a 3-dimensional connected topological loop L is the group $SL_4(\mathbb{R})$. Such loops L are the multiplicative loops of locally compact connected topological quasifields Q such that the kernel of Q is the field of complex numbers and Q has dimension 2 over its kernel. These quasifields Q coordinatize non-desarguesian 8-dimensional topological translation planes and are determined by N. Knarr ([13], Section 6). Using the results of [13] we have proved that the multiplicative loops Q^* are the direct products of $\mathbb{R}$ and a compact loop $\mathcal{S}$ homeomorphic to S^3 and having the group $SL_4(\mathbb{R})$ as its multiplication group (cf. [7]). There are two classes of such compact loops $\mathcal{S}$. One is related to Rees algebras (cf. [14], Section 29.2).
The simple Lie groups $PSL_2(\mathbb{C})$, $PSU_3(\mathbb{C}, 1)$, $SL_3(\mathbb{R})$ are the groups G_l topologically generated by all left translations for differentiable loops L. If the stabilizer H_l of $e \in L$ in G_l is a maximal compact subgroup of G_l, then L is a Bruck loop of hyperbolic type (cf. [8]).

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