# Loops which are semidirect products of groups

Á. Figula (Erlangen,Debrecen) and K. Strambach (Erlangen)

Dedicated to Péter T. Nagy on the occasion of his 60th birthday, in friendship.

**Abstract**

We construct loops which are semidirect products of groups of affinities. As their elements in many cases one may take transversal subspaces of an affine space. In particular we obtain in this manner smooth loops having Lie groups of affine real transformations as the groups generated by left translations, whereas the groups generated by right translations are smooth groups of infinite dimension. We also determine the Akivis algebras of these loops.

## 0. Introduction

In [2], [3] and [11] constructions of proper loops are discussed which are semidirect products of groups. Whereas in [2] there are few constructions of such loops related to Example 3 in [4], p. 128, in [3] a general theory for loops which are semidirect products of groups is developed. In [11] examples of proper analytic Bol loops are presented which are „twisted semidirect products“ of two Lie groups.

In our paper we show that a wide class of proper loops $L$ can be represented within the group of affinities of an affine space $\mathcal{A}$ of dimension $2n$ over a commutative field $\mathbb{K}$. They are semidirect products of groups of translations of $\mathcal{A}$ by suitable subgroups $\Gamma_0'$ of $GL(2n,\mathbb{K})$. For many of them we may take as elements affine $n$-dimensional transversal subspaces of $\mathcal{A}$. This representation of the loops $L$ depends in an essential manner on the existence of a regular orbit in the hyperplane at infinity of $\mathcal{A}$ for the group $\Gamma_0'$.

To realize our examples it is important to know the eigenvalues for certain products of matrices in $GL(n,\mathbb{K})$. Since there is no unique procedure for the calculation of the eigenvalues of the product $AB$ from the eigenvalues of the matrices

This paper was supported by DAAD.

*Key words and phrases:* splitting loops extensions of groups, sharply transitive sections, loops in groups of affinities, eigenvalues of products of matrices, Akivis algebras.

2000 *Mathematics Subject Classification:* 20N05, 51A99, 22A30, 15A18, 17D99.

$A$ and $B$ we have devoted Section 3 to this problem and give answers in special cases.

If the field $\mathbb{K}$ is a topological field then we obtain topological loops, for real or complex numbers the constructed loops are analytic. For smooth proper loops obtained in this paper the group topologically generated by the left translations is a Lie group. The difference between the multiplication for semidirect products in our paper and the multiplication for „twisted semidirect products“ in [11] seems to be negligible. But the groups topologically generated by all translations of analytic loops treated in [11] are Lie groups, whereas the analytic loops considered here have smooth transformation groups of infinite dimension as the groups generated by all translations. Moreover, we prove that already the groups topologically generated by the right translations of these loops are smooth groups having a normal abelian subgroup of infinite dimension.

The Akivis algebras of smooth loops contructed in this paper are semidirect products of Lie algebras. Moreover, there are non-connected proper smooth loops having Lie algebras as their Akivis algebras.

## 1. Some basic notions of loop theory

A set $L$ with a binary operation $(x, y) \mapsto x \cdot y$ is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution which we denote by $y = a\backslash b$ and $x = b/a$. The left translation $\lambda_a : y \mapsto a \cdot y : L \to L$ as well as the right translation $\varrho_a : y \mapsto y \cdot a : L \to L$ is a bijection of $L$ for any $a \in L$.
A loop $(L, \cdot)$ is a semidirect product of $H$ by $K$ if $H$ and $K$ are subloops of $(L, \cdot)$ such that: (i) $H$ is a normal subloop of $(L, \cdot)$, (ii) $L = HK$, (iii) $H \cap K = \{e\}$, where $e$ is the identity of $(L, \cdot)$ (cf. [3], p. 81).

Let $L$ be a topological space, respectively a $\mathcal{C}^\infty$-differentiable manifold. Then $(L, \cdot)$ is a topological, respectively a differentiable loop if the maps $(x, y) \mapsto x \cdot y$, $(x, y) \mapsto x\backslash y$, $(x, y) \mapsto y/x : L^2 \to L$ are continuous, respectively differentiable.

To any differentiable loop $L$ we may associate an Akivis algebra which is realized in the tangent space of $L$ at the identity $e \in L$ and which plays a similar role as the Lie algebra in the case of a Lie group (cf. [1], [7]). An Akivis algebra $(A, [.,.], \langle ., ., . \rangle)$ is a real vector space with a bilinear skew-symmetric map $(x, y) \mapsto [x, y] : A \times A \to A$ (called the commutator map) and a trilinear map $(x, y, z) \mapsto \langle x, y, z \rangle : A \times A \times A \to A$ (called the associator map) such that the following identity (called the Akivis identity) holds:

$$\big[[x, y], z\big] + \big[[z, x], y\big] + \big[[y, z], x\big] =$$

$$\langle x, y, z \rangle + \langle y, z, x \rangle + \langle z, x, y \rangle - \langle z, y, x \rangle - \langle x, z, y \rangle - \langle y, x, z \rangle$$

for all $x, y, z \in A$.

Let $(A_1, [.,.], \langle ., ., . \rangle)$ and $(A_2, [.,.], \langle ., ., . \rangle)$ be Akivis algebras. A homomorphism $\alpha : A_1 \to A_2$ is a linear map such that $[x, y]^\alpha = [x^\alpha, y^\alpha]$ and $\langle x, y, z \rangle^\alpha = \langle x^\alpha, y^\alpha, z^\alpha \rangle$ for all $x, y, z \in A_1$ holds.

The Akivis algebra $A$ is a semidirect product $A = N \rtimes M$ of a Lie algebra $N$ by a Lie algebra $M$ if there exist in $A$ Lie subalgebras $N$ and $M$ together with an endomorphism $\alpha : A \to M$ such that $N$ is the kernel of $\alpha$ and the vector space $A$ is the direct sum $A = N \oplus M$.

Let $G$ be the group generated by the left translations of $L$ and let $H$ be the stabilizer of $e \in L$ in the group $G$. The left translations of $L$ form a subset of $G$ acting on the cosets $xH, x \in G$, such that for any given cosets $aH$ and $bH$ there exists precisely one left translation $\lambda_z$ with $\lambda_z aH = bH$.
Conversely, let $G$ be a group, let H be a subgroup of $G$ and let $\sigma : G/H \to G$ be a section with $\sigma(H) = 1 \in G$ such that the subset $\sigma(G/H)$ generates $G$ and acts sharply transitively on the factor space $G/H = \{xH; x \in G\}$ (cf. [9], p. 18). We call such a section sharply transitive. Then the multiplication defined by $xH * yH = \sigma(xH)yH$ on $G/H$ or by $x * y = \sigma(xyH)$ on $\sigma(G/H)$ yields a loop $L(\sigma)$. If $N$ is the largest normal subgroup of $G$ contained in $H$ then the factor group $G/N$ is isomorphic to the group generated by the left translations of $L(\sigma)$.

The loop $L$ is a group if and only if the set $\{\lambda_x;\ x \in L\}$ of left translations is a group. A loop $L$ has the left inverse, respectively the right inverse property if the identity $x^{-1}(xy) = y$, respectively $(yx)x^{-1} = y$ holds for all $x, y \in L$.

## 2. A general construction

Let $\Gamma_0$ be a subgroup of the general linear group $GL(n, \mathbb{K})$ which is different from the identity $I$ and acts on the $n$-dimensional vector space $\mathbb{K}^n$ of column vectors, where $\mathbb{K}$ is a commutative field. Let $\Gamma$ be the group of matrices

$$g(u, v, M) = \begin{pmatrix} 1 & 0 & 0 \\ u & M & 0 \\ v & 0 & M^\delta \end{pmatrix}, u, v \in \mathbb{K}^n,\ M \in \Gamma_0,$$

where $\delta : \Gamma_0 \to \Gamma_0$ is an endomorphism.

**Theorem 1.** *Let $\Xi = T_B \Gamma_0'$ be the complex product of the groups*

$$T_B = \{g(u, Bu, I);\ u \in \mathbb{K}^n\} \text{ with } B \in GL(n, \mathbb{K}) \text{ and } \Gamma_0' = \{g(0, 0, M);\ M \in \Gamma_0\}.$$

*If no element of the set $\{M^{-1}B^{-1}M^\delta A;\ M \in \Gamma_0\}$ has eigenvalue $1$ then for the subgroup $H = \{g(u, Au, I);\ u \in \mathbb{K}^n\} \in \Gamma$ with $A \in GL(n, \mathbb{K})$ the mapping $\sigma : \Gamma/H \to \Gamma$ defined by*

$$g(0, v, M)H \mapsto g(M(BM - M^\delta A)^{-1}v, BM(BM - M^\delta A)^{-1}v, M)$$

*is a sharply transitive section with image $\Xi$.*

*The loop $L_\Xi$ corresponding to $\sigma$ is isomorphic to the semidirect product $\mathbb{K}^n \rtimes \Gamma_0$ of the normal subgroup $\mathbb{K}^n$ by the group $\Gamma_0$ under the multiplication defined by*

$$(u_1, M_1) * (u_2, M_2) = (u_1 + u_2^\psi, M_1 M_2)$$

*for all* $u_i \in \mathbb{K}^n$, $M_i \in \Gamma_0$, *where* $\psi$ *is the invertible linear map*

$$\text{(I)} \qquad x \mapsto [B - (M_1M_2)^\delta A(M_1M_2)^{-1}]^{-1}[M_1^\delta(B - M_2^\delta AM_2^{-1})]x.$$

*The loop* $L_\Xi$ *is a group if and only if the following property*

$$\text{(II)} \qquad \{M^{-1}B^{-1}M^\delta;\ M \in \Gamma_0\} = \{B^{-1}\}$$

*holds. This is equivalent to the condition that* $T_B$ *is a normal subgroup of* $\Gamma$.

*Let* $\Delta'$ *be the semidirect product of the group* $T'_B$, *which is the minimal normal subgroup of* $\Gamma$ *containing* $T_B$, *by the group* $\Gamma'_0$. *If in* $\Gamma_0$ *there is an element* $M_0$ *such that the matrix* $M_0^{-1}B^{-1}M_0^\delta B$ *has no eigenvalue* 1 *then one has* $T'_B = \{g(u, v, I);\ u, v \in \mathbb{K}^n\}$ *and* $\Delta' = \Gamma$.

*Let* $\Theta$ *be the maximal normal subgroup of* $\Gamma$ *contained in* $H$. *Then* $\Theta$ *consists of matrices* $g(v, Av, I)$, $v \in V$, *where* $V$ *is the maximal subspace of* $\mathbb{K}^n$ *for which* $M^\delta v = (AMA^{-1})v$ *for all* $v \in V$ *and* $M \in \Gamma_0$ *holds. The group* $\Delta$ *generated by the set* $\Xi$ *of left translations is the factor group* $\Delta'/(\Theta \cap \Delta')$. *The stabilizer of the identity of* $L_\Xi$ *is the group* $H/(\Theta \cap \Delta')$.

*No loop* $L_\Xi$ *satisfies the left inverse as well as the right inverse property.*

*If* $\mathbb{K}$ *is a topological field then any loop* $L_\Xi$ *is a topological loop. For real or complex numbers any loop* $L_\Xi$ *is analytic.*

*Proof.* Any matrix $g(x, y, M)$ has a unique decomposition

$$g(x, y, M) = g(0, y - M^\delta AM^{-1}x, M)g(M^{-1}x, AM^{-1}x, I).$$

Therefore the set $\{g(0, v, M);\ v \in \mathbb{K}^n, M \in \Gamma_0\}$ forms a system of representatives of the left cosets of $H$ in $\Gamma$. A mapping $\sigma : \Gamma/H \to \Xi$ is a sharply transitive section if and only if for given $g(0, v_1, M_1)$ and $g(0, v_2, M_2)$ there is precisely one matrix $g(u, Bu, M)$ in $\sigma(\Gamma/H)$ and one matrix $g(z, Az, I)$ in $H$ such that

$$(1) \qquad g(u, Bu, M)g(0, v_1, M_1) = g(0, v_2, M_2)g(z, Az, I)$$

holds. Since $M_2^{-1}B^{-1}M_2^\delta A$ has no eigenvalue 1 the matrix $BM_2(I - M_2^{-1}B^{-1}M_2^\delta A) = BM_2 - M_2^\delta A$ is invertible and the unique solution of (1) is given by

$$M = M_2M_1^{-1},\ z = (BM_2 - M_2^\delta A)^{-1}(v_2 - M_2^\delta M_1^{-\delta}v_1),$$

$$u = M_2(BM_2 - M_2^\delta A)^{-1}(v_2 - M_2^\delta M_1^{-\delta}v_1).$$

The set $\Xi = T_B\Gamma'_0$ is the set of the left translations of the loop $L_\Xi$ defined by the sharply transitive section $\sigma$.

For the elements $g(u_i, Bu_i, M_i) \in \sigma(G/H)$, $i = 1, 2$, one has

$$g(u_1, Bu_1, M_1)g(u_2, Bu_2, M_2) \in g(z, Bz, M')H$$

with a unique $z \in \mathbb{K}^n$ and a unique $M' \in \Gamma_0$. This yields $M' = M_1M_2$ and

$$[M_1^\delta(B - M_2^\delta AM_2^{-1})]u_2 = [B - (M_1M_2)^\delta A(M_1M_2)^{-1}](z - u_1).$$

The matrix $B-(M_1M_2)^\delta A(M_1M_2)^{-1}$ is invertible since

$$B-(M_1M_2)^\delta A(M_1M_2)^{-1} = B[(M_1M_2 - B^{-1}(M_1M_2)^\delta A)(M_1M_2)^{-1}] =$$

$$B[(M_1M_2(I-(M_1M_2)^{-1}B^{-1}(M_1M_2)^\delta A))(M_1M_2)^{-1}]$$

and $(M_1M_2)^{-1}B^{-1}(M_1M_2)^\delta A$ has no eigenvalue 1. It follows

$$z = u_1 + [B-(M_1M_2)^\delta A(M_1M_2)^{-1}]^{-1}[M_1^\delta(B-M_2^\delta A M_2^{-1})]u_2. \tag{2}$$

Hence (cf. [9], p. 18) the loop $L_\Xi$ is isomorphic to the loop defined on $\Xi$ by the multiplication

$$g(u_1,Bu_1,M_1)\circ g(u_2,Bu_2,M_2) = g(z,Bz,M_1M_2),$$

where $z$ is given in (2). Moreover, $L_\Xi$ is isomorphic to the loop $\tilde{L}_\Xi$ defined on $\mathbb{K}^n\Gamma_0$ by $(u_1,M_1)*(u_2,M_2) = (u_1+u_2^\psi, M_1M_2)$, where $\psi$ is a linear map given in $(I)$. Since $(\mathbb{K}^n, I)$ is a normal subgroup of $\tilde{L}_\Xi$ the loop $\tilde{L}_\Xi$ is a semidirect product of the group $\mathbb{K}^n$ by the group $\Gamma_0$.

The loop $L_\Xi$ is a group if and only if the set $\Xi$ is a group. This is equivalent to the fact that $T_B\Gamma_0' = \Gamma_0' T_B$ or that for given $g(u,Bu,I)$ and $g(0,0,M)$ there are elements $g(u',Bu',I)$ and $g(0,0,M')$ such that

$$g(u,Bu,I)g(0,0,M) = g(0,0,M')g(u',Bu',I).$$

This is the case if and only if $M=M'$ and $B = M^\delta B M^{-1}$ which is equivalent to $\{M^{-1}B^{-1}M^\delta;\ M\in\Gamma_0\} = \{B^{-1}\}$ or to the normality of $T_B$ in $\Gamma$.

If the set $\Xi$ is not a group then $\Xi = T_B\Gamma_0' \neq \Gamma_0' T_B$ and the loop $L_\Xi$ does not satisfy the left inverse property since there is an element $\xi\in\Xi$ such that $\xi^{-1}$ is not contained in $\Xi$.

If the loop $L_\Xi$ satisfies the right inverse property then using

$$(u,M)*(-(B-M^{-\delta}AM)^{-1}M^{-\delta}(B-A)u, M^{-1}) = (0,I)$$

we have

$$(u,X) = [(u,X)*(u',M)]*(-(B-M^{-\delta}AM)^{-1}M^{-\delta}(B-A)u', M^{-1}) = (u+$$

$$[(B-(MX)^\delta A(MX)^{-1})^{-1}X^\delta(B-M^\delta AM^{-1}) - (B-X^\delta AX^{-1})^{-1}X^\delta(B-A)]u', X)$$

for all $u,u'\in\mathbb{K}^n$ and $M,X\in\Gamma_0$. For $X=M^{-1}$ this yields

$$(B-A)^{-1}M^{-\delta}(B-M^\delta AM^{-1}) = (B-M^{-\delta}AM)^{-1}M^{-\delta}(B-A) \tag{3}$$

and for $X=M$ we obtain

$$(B-M^{2\delta}AM^{-2})^{-1}M^\delta(B-M^\delta AM^{-1}) = (B-M^\delta AM^{-1})^{-1}M^\delta(B-A) \tag{4}$$

for all $M \in \Gamma_0$. Taking the inverses of both sides of (3) we get

$$(B - M^{\delta} A M^{-1})^{-1} M^{\delta} (B - A) = (B - A)^{-1} M^{\delta} (B - M^{-\delta} A M). \tag{5}$$

Since the left side of (5) is equal to the right side of (4) we have for a proper loop $L_{\Xi}$ the contradiction $(B - A) = (B - M^{2\delta} A M^{-2})$ for all $M \in \Gamma_0$.

Let $T'_B$ be the minimal normal subgroup of $\Gamma$ containing $T_B$ and let $\Delta'$ be the semidirect product $T'_B \rtimes \Gamma'_0$. If $g(u_0, Bu_0, I)$ is an element of

$$T_B \cap \{g(M_0 u, M_0^{\delta} B u, I);\ u \in \mathbb{K}^n\} = T_B \cap g(0,0,M_0) T_B g(0,0,M_0)^{-1}$$

then one has $u_0 = M_0 u$ and $Bu_0 = M_0^{\delta} B u$ or $B M_0 u = M_0^{\delta} B u$. But in this case the matrix $M_0^{-1} B^{-1} M_0^{\delta} B$ would have an eigenvalue 1. This contradiction yields that

$$T'_B = \{g(u, v, I);\ u, v \in \mathbb{K}^n\} = T_B \times [g(0,0,M_0) T_B g(0,0,M_0)^{-1}]$$

and one has $\Delta' = \Gamma$.

The maximal normal subgroup $\Theta$ of $\Gamma$ contained in $H$ consists of the matrices $g(v, Av, I)$, where $v$ is an element of a subspace $V$ such that for all $v \in V$ and $M \in \Gamma_0$ one has

$$g(0,0,M) g(v, Av, I) g(0,0,M^{-1}) = g(Mv, M^{\delta} A v, I) = g(v', Av', I).$$

This is equivalent to $v' = Mv$ and $M^{\delta} A v = AMv$ for all $M \in \Gamma_0$ and $v \in V$. Hence for the restrictions of $M^{\delta} A$ and $AM$ to $V$ we have $M^{\delta} A|_V = AM|_V$ or equivalently $M^{\delta}|_V = AMA^{-1}|_V$. According to Prop. 1.13 in [9] the group generated by the left translations of the loop $L_{\Xi}$ is the group $\Delta = \Delta'/(\Theta \cap \Delta')$ and the stabilizer of the identity $e \in L_{\Xi}$ is the group $H/(\Theta \cap \Delta')$.

If $\mathbb{K}$ is a topological field, respectively the field of real or complex numbers then $\Gamma$ is a topological group, respectively a Lie group, and the section $\sigma$ is continuous, respectively analytic. Then the multiplication of $L_{\Xi}$ as well as the left divison $(a, b) \mapsto a \backslash b : L_{\Xi} \times L_{\Xi} \to L_{\Xi}$ are continuous, respectively analytic. Looking at the solution of the equation (1) we see that also the right divison $(a, b) \mapsto a/b : L_{\Xi} \times L_{\Xi} \to L_{\Xi}$ is continuous, respectively analytic. □

The group $\Gamma$ may be regarded as a subgroup of the group of affinities of the $2n$-dimensional affine space $\mathcal{A}_{2n}$ acting on the set $\{(1, x, y);\ x, y \in \mathbb{K}^n\}$ by

$$g(u, v, M)(1, x, y) = (1, u + Mx, v + M^{\delta} y).$$

Then $\Gamma'_0 = \{g(0,0,M);\ M \in \Gamma_0\}$ is the stabilizer of the point $(1,0,0) \in \mathcal{A}_{2n}$ in $\Gamma$ and $\{g(u, v, I);\ u, v \in \mathbb{K}^n\}$ is the translation group of $\mathcal{A}_{2n}$. The $(2n-1)$-dimensional projective space

$$E = \{\mathbb{K}^*(0, x, y);\ x, y \in \mathbb{K}^n, (x, y) \neq (0,0), \mathbb{K}^* = \mathbb{K} \backslash \{0\}\}$$

is the hyperplane at infinity of the affine space $\mathcal{A}_{2n}$. The group $\Gamma$ acts on $E$ by $g(u,v,M)(0,x,y) = (0, u + Mx, v + M^{\delta}y)$.

The group $\Gamma_0'$ leaves the subspace $\mathcal{Q}_1 = \{(1,x,0);\ x \in \mathbb{K}^n\}$ as well as the subspace $\mathcal{Q}_2 = \{(1,0,x);\ x \in \mathbb{K}^n\}$ invariant. We call a subspace $\mathcal{Q}_C$ of the form $\{(1,x,Cx);\ x \in \mathbb{K}^n\}$ with $C \in GL(n,\mathbb{K})$ an $n$-dimensional transversal subspace with respect to $\Gamma_0'$. Any transversal subspace $\mathcal{Q}_C$ intersects $\mathcal{Q}_1$ and $\mathcal{Q}_2$ only in the point $(1,0,0)$. The projective subspace $\mathcal{Q}_C^* = \{\mathbb{K}^*(0,x,Cx);\ x \in \mathbb{K}^n\}$ of $E$ may be seen as the trace of $\mathcal{Q}_C$ in $E$.

In the next Lemma we give necessary and sufficient conditions for the existence of regular orbits for the group $\Gamma_0'$ in the set $\mathcal{T}$ of $n$-dimensional transversal subspaces of $\mathcal{A}_{2n}$. These conditions are needed for a geometric representation of loops $L_\Xi$ within the affine space $\mathcal{A}_{2n}$.

**Lemma 2.** *The group $\Gamma_0'$ has in the set $\mathcal{T}$ of $n$-dimensional affine transversal subspaces of $\mathcal{A}_{2n}$ a regular orbit $\mathcal{O} = \{\varphi(\mathcal{Q}_A);\ \varphi \in \Gamma_0'\}$ if and only if one of the following equivalent conditions is satisfied:*

*(i) There exists an inner automorphism $\alpha$ of $GL(n,\mathbb{K})$ given by $X \mapsto AXA^{-1}$ with $A \in GL(n,\mathbb{K})$ such that $M^{\delta}M^{-\alpha} \neq I$ for all $M \in \Gamma_0 \setminus \{I\}$.*

*(ii) There exists an orbit $\widehat{\mathcal{O}} = \{\varphi(\mathcal{Q}_A);\ \varphi \in \Gamma\}$ such that the stabilizer of $\mathcal{Q}_A$ in $\Gamma$ is the group $H = \{g(u,Au,I);\ u \in \mathbb{K}^n\}$.*

*Proof.* Let $\mathcal{Q}_A = \{(1,x,Ax);\ x \in \mathbb{K}^n\}$ be an $n$-dimensional transversal subspace of $\mathcal{A}_{2n}$. The orbit $\mathcal{O}$ containing $\mathcal{Q}_A$ under $\Gamma_0'$ consists of all subspaces

$$\{(1, Mx, M^{\delta}Ax);\ x \in \mathbb{K}^n\} = \{(1, x, M^{\delta}AM^{-1}x);\ x \in \mathbb{K}^n\},$$

where $M$ varies over the elements of $\Gamma_0$. The orbit $\mathcal{O}$ is a regular orbit of $\Gamma_0'$ if and only if $A \neq M^{\delta}AM^{-1}$ or $I \neq M^{\delta}M^{-\alpha}$ for all $M \in \Gamma_0 \setminus \{I\}$, where $\alpha$ is the map $X \mapsto AXA^{-1} : GL(n,\mathbb{K}) \to GL(n,\mathbb{K})$.

The stabilizer $\Gamma_{\mathcal{Q}_A}$ of $\mathcal{Q}_A$ in $\Gamma$ is the group $H = \{g(u,Au,I);\ u \in \mathbb{K}^n\}$ if and only if the relation $g(u,v,M)(1,x,Ax) = (1,y,Ay)$ for all $x \in \mathbb{K}^n$ and suitable $y \in \mathbb{K}^n$ holds. Since $g(u,v,M)(1,x,Ax) = (1, u + Mx, v + M^{\delta}Ax)$ we obtain for $x = 0$ that $v = Au$. Hence $H \leq \Gamma_{\mathcal{Q}_A}$. Moreover, one has $M^{\delta}Ax = AMx$ for all $x \in \mathbb{K}^n$. Therefore $H$ is the stabilizer of $\mathcal{Q}_A$ in $\Gamma$ if and only if for each $I \neq M \in \Gamma_0$ there is an $0 \neq x_0 \in \mathbb{K}^n$ such that $M^{\delta}Ax_0 \neq AMx_0$. This is equivalent to $M^{\delta}AM^{-1} \neq A$ which is the condition $(i)$. It follows that the conditions $(i)$ and $(ii)$ are equivalent for the existence of a regular orbit of $\Gamma_0'$ in the set $\mathcal{T}$. □

Using the geometric interpretation for $\Gamma$ we prove that the loops $L_\Xi$ have realizations on the orbit $\widehat{\mathcal{O}}$ if the conditions of Lemma 2 are satisfied (cf. [6], Theorem 1, p. 153).

**Theorem 3.** *Let $L_\Xi$ be the loop determined by the matrices $A, B \in GL(n,\mathbb{K})$, the group $\Gamma_0$ and the endomorphism $\delta : \Gamma_0 \to \Gamma_0$. Let $H = \{g(u,Au,I);\ u \in \mathbb{K}^n\}$ be* *the stabilizer of $\mathcal{Q}_A$ in $\Gamma$. Then $L_\Xi$ is isomorphic to a loop $L'_\Xi$ the elements of which are elements of the orbit $\widehat{\mathcal{O}} = \{\psi(\mathcal{Q}_A);\ \psi \in \Xi\} = \{\varphi(\mathcal{Q}_A);\ \varphi \in \Gamma\}$. The loop $L'_\Xi$ has $\mathcal{Q}_A$ as the identity and the multiplication of $L'_\Xi$ is defined by*

$$X \circ Y = \tau_{\mathcal{Q}_A,X}(Y) \quad \text{for all} \quad X, Y \in \widehat{\mathcal{O}},$$

*where $\tau_{\mathcal{Q}_A,X}$ is the unique element of $\Xi$ mapping $\mathcal{Q}_A$ onto $X$.*

*The group $\Gamma'_0$ acts sharply transitively on the traces of elements of $\mathcal{O}$ in the hyperplane $E$ at infinity, and the subspace $\{(1, x, Bx);\ x \in \mathbb{K}^n\}$ intersects any subspace in $\mathcal{O}$ in precisely one point.*

*Proof.* Since the subgroup $H$ is the stabilizer of $\mathcal{Q}_A$ in $\Gamma$ and the set $\Xi$ acts sharply transitively on the cosets $g(0, v, M)H$ the loop $L_\Xi$ is isomorphic to the loop $L'_\Xi$ defined on $\widehat{\mathcal{O}}$, with $\mathcal{Q}_A$ as identity and with the multiplication defined in the assertion.

According to Lemma 2 the group $\Gamma'_0$ acts sharply transitively on the orbit $\mathcal{O} = \{\varphi(\mathcal{Q}_A);\ \varphi \in \Gamma'_0\}$ and hence also sharply transitively on the set of traces of elements of $\mathcal{O}$ in the hyperplane $E$.

The subspace $\{(1, x, Bx);\ x \in \mathbb{K}^n\}$ intersects any element of $\mathcal{O}$ only in the point $(1, 0, 0)$ if and only if $(1, x, Bx) \neq (1, x, M^\delta A M^{-1} x)$ for all $M \in \Gamma_0 \backslash \{I\}$ and $x \neq 0$. By Theorem 1 no of the matrices $M^{-1}B^{-1}M^\delta A$ with $M \in \Gamma_0$ has an eigenvalue 1. Hence $BM(I - M^{-1}B^{-1}M^\delta A) = BM - M^\delta A = (B - M^\delta A M^{-1})M = B[(I - B^{-1}M^\delta A M^{-1})M]$, and the last claim of the Theorem follows. □

## 3. Applications

In this Section we give concrete examples for matrices $A, B \in GL(n, \mathbb{K})$, groups $\Gamma_0$ and endomorphisms $\delta : \Gamma_0 \to \Gamma_0$ such that the loop $L_\Xi = L_{A,B,\Gamma_0,\delta}$ exists. To archieve this goal we have in particular to show that no matrix $M^{-1}B^{-1}M^\delta A$ has an eigenvalue 1 for all $M \in \Gamma_0$.

**3.1** Let $\mathbb{K}$ be a commutative field and let $\Gamma_0$ be a subgroup of $GL(n, \mathbb{K})$.

a) We assume that the group $\Gamma_0$ is not commutative and that $\delta$ is the inner automorphism $X \mapsto C^{-1}XC$ of $\Gamma_0$ different from the identity. We choose $A = C^{-1}$ and $B \in GL(n, \mathbb{K})$ such that $B^{-1}C^{-1}$ does not centralize $\Gamma_0$ and has no eigenvalue 1. Then the loop $L_{A,B,\Gamma_0,\delta}$ is a proper loop. But because of $M^\delta C^{-1} M^{-1} C = I$ for all $M \in \Gamma_0$ this loop has no geometric realization in sense of Theorem 3.

b) We suppose that $\delta$ is the identity, the matrix $A$ centralizes $\Gamma_0$ but the matrix $B$ does not centralize $\Gamma_0$. If $B^{-1}A$ has no eigenvalue 1 then the eigenvalues of the matrices $M^{-1}B^{-1}AM$ are also different from 1 for all $M \in \Gamma_0$. Hence the proper loop $L_{A,B,\Gamma_0,id}$ exists.

c) Let $P_m$ be an $(m \times m)$-matrix and let $Q_{n-m}$ be an $(n-m \times n-m)$-matrix. We denote by $P_m \oplus Q_{n-m}$ the matrix $\begin{pmatrix} P_m & 0 \\ 0 & Q_{n-m} \end{pmatrix}$. Let $A = diag(a, \cdots, a) \oplus A'$ with

$diag(a,\cdots,a)\in GL(m,\mathbb{K})$ and $A'\in GL(n-m,\mathbb{K})$, let $B=B'\oplus diag(b,\cdots,b)$ with $diag(b,\cdots,b)\in GL(n-m,\mathbb{K})$ and $B'\in GL(m,\mathbb{K})$. Let $\widehat{\Gamma_0}\neq I$ be a subgroup of $GL(m,\mathbb{K})$ and $\Gamma_0=\{M\oplus I_{n-m};M\in\widehat{\Gamma_0}\}$, where $I_{n-m}$ is the identity of $GL(n-m,\mathbb{K})$. If $\delta$ is the identity the loop $L_{A,B,\Gamma_0,id}$ exists if and only if $A'$ has no eigenvalue $b$, the matrix $B'$ has no eigenvalue $a$ and $a\neq b$ holds. Futhermore, $L_{A,B,\Gamma_0,id}$ is a proper loop if $B'^{-1}$ does not centralize $\widehat{\Gamma_0}$.
The loops $L_{A,B,\Gamma_0,id}$ treated in b) and c) have no geometric realizations in sense of Theorem 3 since $A$ centralizes $\Gamma_0$.

**3.2** Let $\mathbb{K}$ be a commutative field. Let $\Gamma_0$ be a non-abelian subgroup of the group of upper triagonal matrices whose entries are elements of $\mathbb{K}$. Let $\chi_i$, $i=1,\cdots,n$, be the map which assigns to the matrix $M=(m_{ij})$ the element $m_{ii}$. Then $\chi_i$ is a homomorphism from $\Gamma_0$ into the multiplicative group $\mathbb{K}^*$ of $\mathbb{K}$. Let $\delta=0$ be the endomorphism which maps $\Gamma_0$ onto $I$. If $B^{-1}A=\text{diag }(t_1,t_2,\cdots,t_n)$ such that $\Phi_i=\{\chi_i(M);\ M\in\Gamma_0\}$ is a proper subgroup of $\mathbb{K}^*$ and $t_i\notin\Phi_i$ for $i=1,\cdots,n$, then $L_{A,B,\Gamma_0,0}$ is a proper loop and has a geometric realization on the set $\{\varphi(\mathcal{Q}_A);\ \varphi\in\Gamma\}$ (Theorem 3 and Lemma 2).

**3.3** Let $\Gamma_0$ be the group $SU_2(\mathbb{C})=\left\{\begin{pmatrix} z & w\\ -\bar{w} & \bar{z}\end{pmatrix};z,w\in\mathbb{C},z\bar{z}+w\bar{w}=1\right\}$. Every element of $\Gamma_0$ has eigenvalues $\{e^{i\Theta},e^{-i\Theta}\}$ with $0\leq\Theta<2\pi$. Let $\delta$ be the inner automorphism $X\mapsto C^{-1}XC$ of $\Gamma_0$, let $A$ and $B$ be elements of $\Gamma_0$. We assume that $B^{-1}C^{-1}$ has an eigenvalue $e^{i\Theta_1}$ and $CA$ has an eigenvalue $e^{i\Theta_2}$ with $0<\Theta_i$, $i=1,2$, $\Theta_1>\Theta_2$ and $\Theta_1+\Theta_2\leq\pi$. Since for any $M\in\Gamma_0$ the matrix $M^{-1}B^{-1}C^{-1}M$ has an eigenvalue $e^{i\Theta_1}$ we have (see [8], Prop. 3.1, p. 601) the inequalities

$$\cos(\Theta_1+\Theta_2)\leq\cos\Theta_3\leq\cos(\Theta_1-\Theta_2),$$

where $e^{i\Theta_3}$ is an eigenvalue of the matrix $[M^{-1}B^{-1}C^{-1}M]CA$. It follows that no matrix $[M^{-1}B^{-1}C^{-1}M]CA$ has the eigenvalue 1 and the differentiable proper loop $L_{A,B,\Gamma_0,\delta}$ exists. Since $-I$ is contained in the centre of $SU_2(\mathbb{C})$ the loop $L_{A,B,\Gamma_0,\delta}$ has no geometric realization.

**3.4** (i) Let $\Gamma_0\neq I$ be a compact subgroup of $GL(n,\mathbb{R})$.

(ii) Let $\mathbb{K}$ be a commutative field with an exponential (ultrametric) valuation $v:\mathbb{K}\to\mathbb{R}\cup\{\infty\}$ (cf. [5], p. 20, [10], p. 65) and let $\mathcal{A}$ be the corresponding valuation ring. The field $\mathbb{K}$ is a topological field with respect to $v$ ([5], p. 2) and $GL(n,\mathbb{K})$ carries the topology induced by the topology of $\mathbb{K}$. Let $M$ be a matrix which topologically generates a compact subgroup $\Upsilon$ of $GL(n,\mathbb{K})$. The matrix $M$ is conjugate to an upper triangular matrix in $GL(n,\mathbb{L})$, where $\mathbb{L}$ is a finite algebraic extension of $\mathbb{K}$. Let $\hat{v}$ be an extension of the valuation of $\mathbb{K}$ to $\mathbb{L}$ and let $\hat{\mathcal{A}}$ be the corresponding valuation ring. Assume that $\lambda\neq 0$ is an eigenvalue of $M$. Since $\Upsilon$ is compact there exist natural numbers $n_i$ with $\lim\limits_{i\to\infty}n_i=\infty$ such that $M^{n_i}$ converges to $S$. The matrix $S$ has $\lim\limits_{i\to\infty}\lambda^{n_i}$ as an eigenvalue. Because of $\hat{v}(\lim\limits_{i\to\infty} \lambda^{n_i}) = \lim\limits_{i\to\infty} \hat{v}(\lambda^{n_i}) = \hat{v}(\lambda) \lim\limits_{i\to\infty} n_i$ and $\hat{v}(x) = \infty$ if and only if $x = 0$ it follows that $\hat{v}(\lambda) = 0$. This means that $\lambda$ is a unit in the valuation ring $\hat{\mathcal{A}}$.

Let $\Gamma_0 \neq I$ be a closed subgroup of the group $GL(n, \mathcal{A})$. According to [10], p. 104, the group $GL(n, \mathcal{A})$ is compact and hence $\Gamma_0$ is also compact.

Let $A = diag(a, \cdots, a)$ and $B = diag(b, \cdots, b)$ be diagonal matrices in the centre of $GL(n, \mathbb{F})$. If $\mathbb{F}$ is the field of real numbers then we suppose that $|b^{-1}a| \neq 1$. If $\mathbb{F}$ has an exponential valuation then we assume that $v(ab^{-1}) \neq 0$. Then any matrix $M^{-1}B^{-1}M^{\delta}A = M^{-1}M^{\delta}B^{-1}A$ with $M \in \Gamma_0$ has no eigenvalue 1 (cf. [12], p. 288 and Satz 8, p. 196).

If $\delta$ is not the identity then $L_{A,B,\Gamma_0,\delta}$ is a proper loop. It has a geometric realization if and only if $M^{\delta} \neq M$ for all $M \in \Gamma_0 \backslash \{I\}$. This is for instance the case if there exists a natural number $k$ such that $\delta^k = 0$. (If for a matrix $I \neq M \in \Gamma_0$ one has $M^{\delta^k} = I$ then $M^{\delta} \neq M$.) Let $\Gamma_0 = H_1 \times H_2 \times \cdots \times H_k \leq GL(n, \mathbb{F})$ such that $H_i$ is isomorphic to the same compact Lie group, respectively closed subgroup of $GL(n, \mathcal{A})$ for $i \in \{1, \cdots, k\}$. Then $\delta : (x_1, \cdots, x_k) \mapsto (1, x_1, \cdots, x_{k-1})$ is an endomorphism of $\Gamma_0$ and $\delta^k = 0$.

**3.5** (i) Let $\mathbb{K}$ be a commutative field and let $\mathbb{F}$ be a subfield of $\mathbb{K}$. Moreover, let

$$\Gamma_0 = \left\{ M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix};\ det(M) = 1 \right\}$$

be the group $SL(2, \mathbb{F})$. We assume that $\delta$ is the identity.

Let $B = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} a & 0 \\ c & a \end{pmatrix}$ be matrices with $t, c \in \mathbb{F}$ such that $ct$ is a square in $\mathbb{F}$ and $a \in \mathbb{K}\backslash\mathbb{F}$. The trace $t(S_M)$ of the matrix $S_M = M^{-1}B^{-1}MA$, where $M \in \Gamma_0$, has the value $ctm_{22}^2 + 2a = \lambda_1 + \lambda_2$, where $\lambda_i$, $i = 1, 2$, are the eigenvalues of $S_M$. If $\lambda_1 = 1$ then $det(S_M) = a^2 = \lambda_2$ and the equation $a^2 - 2a + 1 - ctm_{22}^2 = 0$ yields that $a = 1 \pm m_{22}\sqrt{ct} \in \mathbb{F}$, which is a contradiction. Hence the matrix $S_M$ has no eigenvalue 1 for all $M \in SL(2, \mathbb{F})$.

(ii) Let $\mathbb{K}$ be a formally real field (i.e. the sums of squares are squares in $\mathbb{K}$, but $-1$ is not a square in $\mathbb{K}$) and let $\Gamma_0$ be the group $SL(2, \mathbb{K})$. Moreover, let $B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $A = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ be matrices in $SL(2, \mathbb{K})$ such that $1 - a$ is a square but $-tb$ is not a square in $\mathbb{K}$. The trace $t(S_M)$ of the matrix $S_M = M^{-1}B^{-1}MA$ has the value $-tb(m_{11}^2 + m_{12}^2) + 2a$. As $2(1 - a)$ is a square, but $-tb(m_{11}^2 + m_{12}^2)$ is not a square we have $-tb(m_{11}^2 + m_{12}^2) \neq 2(1 - a)$ for all $M \in SL(2, \mathbb{K})$.

Any loop $L_{A,B,\Gamma_0,id}$ of (i) as well as of (ii) is a proper loop because the condition $(II)$ in Theorem 1 is not satisfied. But it has no geometric realization since $-I \in \Gamma_0$.

**3.6** Let $\mathbb{K}$ be a commutative field and let $\Gamma_0$ be the group

$$\left\{ M := g(m, n) = \begin{pmatrix} m & n \\ 0 & m^{-1} \end{pmatrix};\ n \in \mathbb{K}, m \in \Omega, \right\}$$

where $\Omega$ is a subgroup of $\mathbb{K}^*$ which does not contain $-1 \neq 1$. (If the characteristic of $\mathbb{K}$ is 2 then we may take $\Omega = \mathbb{K}^*$.) Let $\delta$ be the mapping $g(m,n) \mapsto g(m,dn)$ with $d \in \mathbb{K}$. If $d \neq 0$ then $\delta$ is an automorphism of $\Gamma_0$, if $d = 0$ then $\delta$ is a proper endomorphism of $\Gamma_0$. Let $B = \begin{pmatrix} a^{-1} & -b \\ 0 & a \end{pmatrix}$ and $A = \begin{pmatrix} r & s \\ 0 & r^{-1} \end{pmatrix}$ be matrices in $SL(2,\mathbb{K})$. We assume that $ar \neq 1$. The trace $t(S_M)$ of the matrix $S_M = M^{-1}B^{-1}M^{\delta}A$ has the value $ar + a^{-1}r^{-1}$. Hence $t(S_M) \neq 2$ for all $M \in \Gamma_0$.

The loop $L_{A,B,\Gamma_0,\delta}$ is a proper loop if and only if $a^2 d \neq 1$ or $b \neq 0$ because of the condition $(II)$ in Theorem 1. Moreover, it has a geometric realization on the set $\{\varphi(\mathcal{Q}_A);\ \varphi \in \Gamma\}$ if and only if $M^{\delta}A \neq AM$ for all $M \in \Gamma_0 \setminus \{I\}$ or $0 \neq sr(1-m^2) + nm(r^2 - d)$ for all $(m,n) \neq (1,0)$ and $m \neq 0$. This is for instance the case if $r^2 = d$ and $s \neq 0$.

**3.7** Let $\Gamma_0$ be the group of matrices $M := g(m) = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}$, $0 < m \in \mathbb{R}$, and let $\delta$ be the automorphism $g(m) \mapsto g(m^c)$ with $0 \neq c \in \mathbb{R}$. Let $B^{-1} = \begin{pmatrix} k & l \\ n & s \end{pmatrix}$ and $A = \begin{pmatrix} p & q \\ r & v \end{pmatrix}$ be matrices of $SL(2,\mathbb{R})$. For the trace $t(S_M)$ of the matrix $S_M = M^{-1}B^{-1}M^{\delta}A$ one has $d(m^{c-1} + m^{-(c-1)}) + lrm^{-(c+1)} + nqm^{(c+1)}$, where $d := kp = sv$. If $d > 1$, $lr \geq 0$ and $nq \geq 0$ or $d < -1$, $lr \leq 0$ and $nq \leq 0$ then $t(S_M) \neq 2$ for all $M \in \Gamma_0$ and $S_M$ has no eigenvalue 1.

For $c \neq 1$ the loop $L_{A,B,\Gamma_0,\delta}$ is always a proper loop which has a geometric realization on the set $\{\varphi(\mathcal{Q}_A);\ \varphi \in \Gamma\}$. If $c = 1$ then the loop $L_{A,B,\Gamma_0,\delta}$ is a proper loop if and only if $l \neq 0$ or $n \neq 0$ and has a geometric realization precisely if $q \neq 0$ or $r \neq 0$ (cf. condition $(II)$ in Theorem 1 and Lemma 2).

**3.8** Let $\Gamma_0$ be the group consisting of the matrices

$$M := g(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix},\ \varphi \in [0, 2\pi),$$

and let $\delta$ be the mapping $g(\varphi) \mapsto g(n\varphi)$ with $n \in \mathbb{Z}$. If $n \notin \{-1,1\}$ then $\delta$ is a proper endomorphism of $\Gamma_0$, if $n \in \{-1,1\}$ then $\delta$ is an automorphism of $\Gamma_0$. Let $B^{-1} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, $a,b,c \in \mathbb{R}$, $-a^2 - bc = 1$, and let $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. If $b > 0$, $c < 0$ and $-\sqrt{2} < c - b < 0$ then for the trace $t(S_M)$ of $S_M = M^{-1}B^{-1}M^{\delta}A$ one has

$$t(S_M) = \frac{c-b}{\sqrt{2}}[\cos((n-1)\varphi) + \sin((n-1)\varphi)] < \frac{2(b-c)}{\sqrt{2}} < 2.$$

Hence the eigenvalues of $S_M$ for all $M \in \Gamma_0$ are different from 1.

Any loop $L_{A,B,\Gamma_0,\delta}$ is a proper loop since the condition $(II)$ in Theorem 1 is not satisfied. Moreover, it has a geometric realization on the set $\{\varphi(\mathcal{Q}_A);\ \varphi \in \Gamma\}$ precisely if $M^{\delta}A \neq AM$ for all $M \in \Gamma_0 \setminus \{I\}$. This is the case if and only if $n \neq 1$.

**3.9** Let $a, b$ be elements of the multiplicative group $\mathbb{K}^*$ of a commutative field $\mathbb{K}$. A proper loop $L_{a,b,\Gamma_0,\delta}$ exists if one of the following conditions holds:

a) $\Gamma_0 \neq 1$ is a proper subgroup of $\mathbb{K}^*$, the endomorphism $\delta$ is fixed point-free (i.e. $x^\delta \neq x$ for all $x \in \Gamma_0 \backslash \{1\}$) and $b^{-1}a \notin \Gamma_0$. This is for instance the case if $\Gamma_0$ does not contain $-1 \neq 1$ and $\delta$ is the automorphism $x \mapsto x^{-1}$. If $\mathbb{K}$ has the characteristic 2 then any proper subgroup of $\mathbb{K}^*$ is suitable as $\Gamma_0$.

b) $\mathbb{K} = \mathbb{R}$, $\Gamma_0$ is the multiplicative group $\mathbb{R}^*$, $b^{-1}a < 0$ and $\delta : x \mapsto x^{2k+1}$ for an integer $k \neq 0$.

c) $\mathbb{K} = \mathbb{C}$ and $\Gamma_0$ is the multiplicative group $\mathbb{C}^*$. If $b^{-1}a$ is a real number then let $\delta$ be the endomorphism $re^{it} \mapsto re^{-it}$. If $b^{-1}a$ is not real then let $\delta$ be the endomorphism $re^{it} \mapsto r^{-1}e^{it}$.

The loop $L_{a,b,\Gamma_0,\delta}$ has a geometric realization on the set $\{\varphi(\mathcal{Q}_A);\ \varphi \in \Gamma\}$ in the case (a), but no geometric realization in the cases (b) or (c).

## 4. Groups generated by right translations

Let $L_\Xi$ be a loop realized on the semidirect product $\mathbb{K}^n \rtimes \Gamma_0$ and let $\varphi$ be the epimorphism $L_\Xi \to \Gamma_0$ mapping $(u, M)$ onto $M$. If $\varrho_a$ is a right translation of $L_\Xi$ then $\varrho_{\varphi(a)}$ denotes the corresponding right translation of $\Gamma_0$ by $\varphi(a)$. Because of $\varrho_{\varphi(ab)} = \varrho_{\varphi(a)\varphi(b)} = \varrho_{\varphi(a)}\varrho_{\varphi(b)}$, there is an epimorphism $\omega$ from the group $\Sigma$ generated by the right translations of $L_\Xi$ onto $\Gamma_0$, such that $\varrho_{(u,M)} = \varrho_M$. The kernel $N$ of $\omega$ consists of all elements of $\Sigma$ which leave any subset $P_M = \{(u, M);\ u \in \mathbb{K}^n\}$ invariant. Since $\Sigma$ acts sharply transitively on the set $\{P_M;\ M \in \Gamma_0\}$ the group $\Sigma$ is a semidirect product $N \rtimes \Gamma_0$ of $N$ by $\Gamma_0$.

If $\mathbb{K} = \mathbb{R}$ then $\Sigma$ is a smooth group and the manifold $P_M$ is diffeomorphic to $\mathbb{R}^n$.

**Proposition 4.** *The group $\Sigma$ topologically generated by all right translations of the proper smooth loop $L_\Xi$ is a smooth group which contains an infinite dimensional abelian subgroup $\mathcal{D}$. The subgroup $\mathcal{D}$ leaves any manifold $P_M$ invariant.*

*Proof.* The right translation of $L_\Xi$ by $(u_2, M_2)$ is the smooth map

$$\varrho_{(u_2,M_2)} : (u_1, M_1) \mapsto (u_1 + u_2^\psi, M_1M_2),$$

where $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is the linear map defined by (I) in Theorem 1. Hence $\Sigma$ contains the subgroup $S = \{\varrho_{(u,I)};\ u \in \mathbb{R}^n\}$ of the mappings

$$\varrho_{(u,I)} : (u_1, M_1) \mapsto (u_1 + [B - M_1^\delta A M_1^{-1}]^{-1} M_1^\delta (B - A)u, M_1).$$

The group $S$, which is contained in the normal subgroup $N$ of $\Sigma$, is diffeomorphic to $\mathbb{R}^n$. The conjugate subgroup

$$\Sigma_M = \varrho_{(0,M)}^{-1} S \varrho_{(0,M)} = \varrho_{(0,M^{-1})} S \varrho_{(0,M)}$$

consists of the mappings

$$(u_1, M_1) \mapsto (u_1 + [B - (M_1 M)^{\delta} A (M_1 M)^{-1}]^{-1} (M_1 M)^{\delta} (B - A) u, M_1).$$

The set of subgroups $\Sigma_M, M \in \Gamma_0$, generates in the group $\Sigma$ an abelian subgroup $\mathcal{D}$, which is a real vector space contained in $N$. We assume that $\mathcal{D}$ has finite dimension. Let $0 \neq u \in \mathbb{R}^n$ be a fixed vector. Then there exist elements

$$([B - (M_1 M^{(i)})^{\delta} A (M_1 M^{(i)})^{-1}]^{-1} (M_1 M^{(i)})^{\delta} (B - A) u, M_1), \ i = 1, \cdots, m,$$

such that from matrix equation

$$\sum_{i=1}^{m} \nu_i [B - (M^{(i)})^{\delta} A (M^{(i)})^{-1}]^{-1} (M^{(i)})^{\delta} = 0, \ \nu_i \in \mathbb{R}, \tag{6}$$

it follows $\nu_i = 0$ for all $i = 1, \cdots, m,$. Moreover, for any $M^* \in \Gamma_0$ there are real numbers $\lambda_i, \ i = 1, \cdots, m$, satisfying the identity

$$\sum_{i=1}^{m} \lambda_i [B - (M_1 M^{(i)})^{\delta} A (M_1 M^{(i)})^{-1}]^{-1} (M_1 M^{(i)})^{\delta} = \tag{7}$$

$$[B - (M_1 M^*)^{\delta} A (M_1 M^*)^{-1}]^{-1} (M_1 M^*)^{\delta}$$

for all $M_1 \in \Gamma_0$. For $M^* \in \{M_1^{-1}, I, M_1\}$ the equation (7) yields

$$(B - A)^{-1} = \sum_{i=1}^{m} \lambda_i [B - (M_1 M^{(i)})^{\delta} A (M_1 M^{(i)})^{-1}]^{-1} (M_1 M^{(i)})^{\delta}$$

$$[B - M_1^{\delta} A M_1^{-1}]^{-1} M_1^{\delta} = \sum_{i=1}^{m} \mu_i [B - (M_1 M^{(i)})^{\delta} A (M_1 M^{(i)})^{-1}]^{-1} (M_1 M^{(i)})^{\delta}$$

and

$$[B - M_1^{2\delta} A M_1^{-2}]^{-1} M_1^{2\delta} = \sum_{i=1}^{m} \nu_i [B - (M_1 M^{(i)})^{\delta} A (M_1 M^{(i)})^{-1}]^{-1} (M_1 M^{(i)})^{\delta}$$

for suitable $\lambda_i, \mu_i, \nu_i \in \mathbb{R}$. Putting in these equations $M_1 = I$ and using (6) we obtain $\lambda_i = \mu_i = \nu_i$ for all $i = 1, \cdots, m$, and

$$(B - A)^{-1} = [B - M_1^{\delta} A M_1^{-1}]^{-1} M_1^{\delta} = [B - M_1^{2\delta} A M_1^{-2}]^{-1} M_1^{2\delta}.$$

For $\delta = 0$ one has $(B - A)^{-1} = [B - A M_1^{-1}]^{-1}$ or $(B - A) = [B - A M_1^{-1}]$ which gives the contradiction $0 = A(I - M_1^{-1})$ for all $M_1 \in \Gamma_0$.

If $\delta \neq 0$ then we obtain $[B - M_1^{\delta} A M_1^{-1}]^{-1} M_1^{\delta} = [B - M_1^{2\delta} A M_1^{-2}]^{-1} M_1^{2\delta}$ or $(I - M_1^{-2\delta}) B = A(I - M_1^{-1})$. Hence $M_1^{\delta} = M_1^{2\delta}$ for all $M_1 \in \Gamma_0$ which is a contradiction. □

## 5. The Akivis algebra of the smooth loop $L_{A,B,\Gamma_0,\delta}$

Let $\Gamma_0$ be a Lie subgroup of positive dimension in $GL(n,\mathbb{R})$. Then the Akivis algebra $\mathfrak{a}_{L_\Xi} = (\mathfrak{a}_{L_\Xi}, [.,.], \langle .,.,. \rangle)$ of a smooth loop $L_\Xi = L_{A,B,\Gamma_0,\delta}$ can be obtained in the following way. Let $\exp m$ be the exponential image of the element $m$ in the Lie algebra $\mathfrak{m}$ of $\Gamma_0$. Let

$$C_{i,j} = (x_i, \exp m_i) * (x_j, \exp m_j) =$$
$$\big(x_i + (B - (\exp m_i \exp m_j)^\delta A(\exp m_i \exp m_j)^{-1})^{-1}$$
$$(\exp m_i)^\delta (B - (\exp m_j)^\delta A(\exp m_j)^{-1}) x_j, \exp m_i \exp m_j\big),$$

where $i, j \in \{1, 2\}$. One has

$$C_{1,2}/C_{2,1} = \big(I - [B - (\exp m_1 \exp m_2)^\delta A(\exp m_1 \exp m_2)^{-1}]^{-1}$$
$$(\exp m_1 \exp m_2 \exp m_1{}^{-1})^\delta (B - \exp m_1{}^\delta A \exp m_1{}^{-1})\big) x_1 +$$
$$[B - (\exp m_1 \exp m_2)^\delta A(\exp m_1 \exp m_2)^{-1}]^{-1}$$
$$\{\exp m_1{}^\delta (B - \exp m_2{}^\delta A \exp m_2{}^{-1}) - (\exp m_1 \exp m_2 \exp m_1{}^{-1} \exp m_2{}^{-1})^\delta$$
$$(B - (\exp m_2 \exp m_1)^\delta A(\exp m_2 \exp m_1)^{-1})\} x_2, \exp m_1 \exp m_2 \exp m_1{}^{-1} \exp m_2{}^{-1}\big).$$

Let

$$D_1 = \big((x_1, \exp m_1) * (x_2, \exp m_2)\big) * (x_3, \exp m_3) = (x_1 +$$
$$[B - (\exp m_1 \exp m_2)^\delta A(\exp m_1 \exp m_2)^{-1}]^{-1} \exp m_1{}^\delta (B - \exp m_2{}^\delta A \exp m_2{}^{-1}) x_2 +$$
$$[B - (\exp m_1 \exp m_2 \exp m_3)^\delta A(\exp m_1 \exp m_2 \exp m_3)^{-1}]^{-1}$$
$$(\exp m_1 \exp m_2)^\delta (B - \exp m_3{}^\delta A(\exp m_3)^{-1}) x_3,\ \exp m_1 \exp m_2 \exp m_3)$$

and

$$D_2 = (x_1, \exp m_1) * \big((x_2, \exp m_2)) * (x_3, \exp m_3)\big) =$$
$$(x_1 + [B - (\exp m_1 \exp m_2 \exp m_3)^\delta A(\exp m_1 \exp m_2 \exp m_3)^{-1}]^{-1}$$
$$\exp m_1{}^\delta (B - (\exp m_2 \exp m_3)^\delta A(\exp m_2 \exp m_3)^{-1}) x_2 +$$
$$[B - (\exp m_1 \exp m_2 \exp m_3)^\delta A(\exp m_1 \exp m_2 \exp m_3)^{-1}]^{-1} (\exp m_1 \exp m_2)^\delta$$
$$(B - \exp m_3{}^\delta A(\exp m_3)^{-1}) x_3,\ \exp m_1 \exp m_2 \exp m_3).$$

Then one has

$$D_1/D_2 =$$
$$\big([B - (\exp m_1 \exp m_2)^\delta A(\exp m_1 \exp m_2)^{-1}]^{-1} \exp m_1{}^\delta (B - \exp m_2{}^\delta A \exp m_2{}^{-1}) x_2 -$$
$$[B - (\exp m_1 \exp m_2 \exp m_3)^\delta A(\exp m_1 \exp m_2 \exp m_3)^{-1}]^{-1} \exp m_1{}^\delta$$
$$(B - (\exp m_2 \exp m_3)^\delta A(\exp m_2 \exp m_3)^{-1}) x_2,\ I).$$

To obtain the binary, respectively the ternary operation of the Akivis algebra $\mathfrak{a}_{L_\Xi}$, which is realized on the vector space $\mathbb{R}^n \oplus \mathfrak{m}$, we replace in $C_{1,2}/C_{2,1}$, respectively in $D_1/D_2$ the elements $\exp m_k, k = 1, 2$, by one parameter subgroups $\exp tm_k$, the elements $x_k$ by one parameter subgroups $tx_k$ and form the following limits:

$$\lim_{t\to 0} \frac{1}{t^2}(C_{1,2}(t)/C_{2,1}(t)) =: [(x_1, m_1), (x_2, m_2)],$$

$$\lim_{t\to 0} \frac{1}{t^3}(D_1(t)/D_2(t)) =: \langle (x_1, m_1), (x_2, m_2), (x_3, m_3)\rangle$$

(cf. [7], Prop. 3.3, p. 323). Using often the fact

$$\frac{d}{dt}(F(t))^{-1} = -(F(t))^{-1}\frac{d}{dt}(F(t))(F(t))^{-1}$$

we obtain by straightforward calculation that

$$[(x_1, m_1), (x_2, m_2)] = \tag{8}$$

$$\big((B - A)^{-1}\{(m_1^{\tilde\delta} B - Am_1)x_2 + (Am_2 - m_2^{\tilde\delta} B)x_1\}, [m_1, m_2]\big),$$

as well as

$$\langle (x_1, m_1), (x_2, m_2), (x_3, m_3)\rangle =$$

$$\big((B - A)^{-1}\big\{(m_3^{\tilde\delta} A - Am_3)(B - A)^{-1}(Am_1 - m_1^{\tilde\delta} B) - (m_3^{\tilde\delta} Am_1 - Am_3m_1)\big\}x_2, 0\big),$$

where in both cases $\tilde\delta$ is the endomorphism of $\mathfrak{m}$ corresponding to $\delta$.

A straightforward but tedious calculation shows that for the Akivis algebra $\mathfrak{a}_{L_\Xi}$ the left as well as the right side of the Akivis identity equals to

$$\begin{aligned}
&\big((B - A)^{-1}\big\{(m_2^{\tilde\delta} A - Am_2)(B - A)^{-1}\big[Am_3 - m_3^{\tilde\delta} B\big]+ \\
&(Am_3 - m_3^{\tilde\delta} A)(B - A)^{-1}\big[Am_2 - m_2^{\tilde\delta} B\big] + (m_2^{\tilde\delta} Am_3 - m_3^{\tilde\delta} Am_2 - Am_2m_3 + Am_3m_2)\big\}x_1+ \\
&(B - A)^{-1}\big\{(m_3^{\tilde\delta} A - Am_3)(B - A)^{-1}\big[Am_1 - m_1^{\tilde\delta} B\big]+ \\
&(Am_1 - m_1^{\tilde\delta} A)(B - A)^{-1}\big[Am_3 - m_3^{\tilde\delta} B\big] + (m_3^{\tilde\delta} Am_1 - m_1^{\tilde\delta} Am_3 - Am_3m_1 + Am_1m_3)\big\}x_2+ \\
&(B - A)^{-1}\big\{(m_1^{\tilde\delta} A - Am_1)(B - A)^{-1}\big[Am_2 - m_2^{\tilde\delta} B\big]+ \\
&(Am_2 - m_2^{\tilde\delta} A)(B - A)^{-1}\big[Am_1 - m_1^{\tilde\delta} B\big] + (m_1^{\tilde\delta} Am_2 - m_2^{\tilde\delta} Am_1 - Am_1m_2 + Am_2m_1)\big\}x_3, 0\big).
\end{aligned} \tag{9}$$

If the loop $L_\Xi$ is a group then the Akivis algebra $\mathfrak{a}_{L_\Xi}$ is a Lie algebra. The derivation of the condition (II) in Theorem 1 yields $m^{\tilde\delta} B = Bm$ for all $m \in \mathfrak{m}$. Putting this in (8) we obtain for the multiplication in the Lie algebra $\mathfrak{a}_{L_\Xi}$ the rule

$$[(x_1, m_1), (x_2, m_2)] = (m_1x_2 - m_2x_1, [m_1, m_2]).$$

The mapping $\gamma : (x, m) \mapsto m : \mathbb{R}^n \times \mathfrak{m} \to \mathfrak{m}$ is an endomorphism from the Akivis algebra $\mathfrak{a}_{L_\Xi}$ onto the Lie algebra $\mathfrak{m}$ since

$$\Big[(x_1, m_1), (x_2, m_2)\Big]^\gamma = [m_1, m_2] = \Big[(x_1, m_1)^\gamma, (x_2, m_2)^\gamma\Big]$$

and in $\mathfrak{m}$ the Jacobi identity holds. Hence we have the following

**Proposition 5.** *The Akivis algebra $\mathfrak{a}_{L_\Xi}$ of the loop $L_\Xi = L_{A,B,\Gamma_0,\delta}$ is a semidirect product $\mathbb{R}^n \rtimes \mathfrak{m}$ of the commutative Lie algebra $\mathbb{R}^n$ by the Lie algebra $\mathfrak{m}$ of the group $\Gamma_0$.*

Let $L_{a,b,\mathbb{R}^*,\delta}$ be a proper loop constructed in **3.9** b) of Section 3. Then the map $\tilde{\delta} : \mathbb{R} \to \mathbb{R}$ corresponding to $\delta$ is the automorphism $x \mapsto (2k+1)x$ with $k \neq 0$. We have

$$[(x_1, m_1), (x_2, m_2)] = \big([(2k+1)b - a](b-a)^{-1}(m_1 x_2 - m_2 x_1), 0\big)$$

and

$$\langle (x_1, m_1), (x_2, m_2), (x_3, m_3) \rangle = \big(-4k^2 ba(b-a)^{-2} m_1 m_3 x_2, 0\big).$$

Using these expressions we see that both sides (9) of the Akivis identity are equal to $(0,0)$.

*These examples show that there are proper non-connected smooth loops $L_\Xi$ of positive dimension having Lie algebras as their Akivis algebras.*

# References


[1] M. A. Akivis, *The local algebras of a multidimensional three-web* (Russian), Sibirsk. Mat. Z., **17**, No. 1 (1976) 5-11. English translation: Sib. Math. J., **17**, No. 1 (1976) 3-8.

[2] G. F. Birkenmeier, C. B. Davis, K. J. Reeves, S. Xiao, *Is a semidirect product of groups necessarily a group?*, Proc. Amer. Math. Soc., **118(3)** (1993), 689-692.

[3] G. F. Birkenmeier, S. Xiao, *Loops which are semidirect products of groups*, Comm. Algebra, **23(1)** (1995), 81-95.

[4] R. H. Bruck, *A survey of binary systems*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958.

[5] O. Endler, *Valuation theory*, Universitext, Springer-Verlag, New York-Heidelberg, 1972.

[6] Á. Figula, K. Strambach, *Affine extensions of loops*, Abh. Math. Sem. Univ. Hamburg, **74** (2004), 151-162.

[7] K. H. Hofmann, K. Strambach, *Lie's fundamental theorems for local analytic loops*, Pacific J. Math., **123**, No. 2 (1986), 301-327.

[8] L. C. Jeffrey, J. Weitsman, *Bohr-Sommerfeld orbits in the moduli space of flat connections and the Verlinde dimension formula*, Comm. Math. Phys., **150**, No. 3 (1992), 593-630.

[9] P. T. Nagy, K. Strambach, *Loops in group theory and Lie theory*, de Gruyter Expositions in Mathematics, 35. Berlin, 2002.

[10] J-P. Serre, *Lie algebras and Lie groups*, Lecture Notes in Mathematics, 1500. Springer-Verlag, Berlin, 1992.

[11] E. Winterroth, *Right Bol loops with a finite dimensional group of multiplications*, Publ. Math. Debrecen, **59**, No. 1-2 (2001), 161-173.

[12] R. Zuhrmühl, S. Falk, *Matrizen und ihre Anwendungen für angewandte Mathematiker, Physiker und Ingenieure*, Teil 1, (German) Springer-Verlag, Berlin, 1984.

Mathematisches Institut
der Universität Erlangen-Nürnberg
Bismarckstr. 1 $\frac{1}{2}$
D-91054 Erlangen
Germany
E-MAIL:figula@mi.uni-erlangen.de
and
Institute of Mathematics
University of Debrecen
P.O.B. 12
H-4010 Debrecen
Hungary
E-MAIL:figula@math.klte.hu

Mathematisches Institut
der Universität Erlangen-Nürnberg
Bismarckstr. 1 $\frac{1}{2}$
D-91054 Erlangen
Germany
E-MAIL:strambach@mi.uni-erlangen.de