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Topological loops with three-dimensional solvable left translation group

Ágota Figula

Abstract. We classify all connected topological loops having a three-dimensional solvable Lie group G as the group topologically generated by their left translations. It is surprising that to the non-nilpotent Lie group G having precisely one one-dimensional normal subgroup there are topological but no differentiable strongly left alternative loops.

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1. Introduction

A concrete classification of two-dimensional loops L having a solvable Lie group G of dimension 3 as the group topologically generated by the left translations is given in [6], Sect. 23, for the case that the multiplication of L is differentiable and L is strongly left alternative. There the loops L are consistently considered as sharply transitive sections $\sigma : G/H \to G$, where H is the stabilizer of the identity element of L in G.

In this paper we determine all connected *topological* loops as sections in three-dimensional solvable Lie groups. These loops are two-dimensional solvable loops (see Proposition 3.8) which are uniquely determined by one continuous real function of one or two variables (see Propositions 3.4, 3.5, 3.6, 3.7 and the discussion after these propositions). Unlike the methods of [6], we cannot use the tangent space of the image $\sigma(G/H)$ of the section σ and we must remain with $\sigma(G/H)$ in G. This has the advantage that for our classification the condition to be strongly left alternative becomes dispensable.

Our classification shows that up to essentially two exceptions any connected three-dimensional solvable Lie group occurs as the group topologically

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generated by the left translations of a connected topological loop. These exceptions are locally isomorphic either to the connected component of the group of motions or to the connected component of the group of homotheties of the Euclidean plane (see Proposition 23.6 in [6] and Proposition 3.2).

The most surprising fact is that there are topological but no differentiable strongly left alternative loops having the three-dimensional non-nilpotent Lie group with precisely one one-dimensional normal subgroup as the group topologically generated by their left translations.

2. Preliminaries

A binary system $(L, \cdot)$ is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution, which we denote by $y = a \backslash b$ and $x = b/a$. A loop L is proper if it is not a group. The left and right translations $\lambda_a : y \mapsto a \cdot y : L \times L \to L$ and $\rho_a : y \mapsto y \cdot a : L \times L \to L, a \in L$, are bijections of L. The permutation group $\mathrm{Mult}(L)$ generated by all left and right translations of the loop L is called the multiplication group of L.

A loop L is called topological if L is a topological space and the binary operations $(x, y) \mapsto x \cdot y$, $(x, y) \mapsto x \backslash y, (x, y) \mapsto y/x : L \times L \to L$ are continuous. Every connected topological loop L having a Lie group as the group topologically generated by the left translations is obtained on a homogeneous space G/H, where G is a connected Lie group, H is a closed subgroup containing no non-trivial normal subgroup of G and $\sigma : G/H \to G$ is a continuous sharply transitive section with $\sigma(H) = 1 \in G$ such that the subset $\sigma(G/H)$ generates G. The multiplication of L on the manifold G/H is defined by $xH * yH = \sigma(xH)yH$ and the group G is the group topologically generated by the left translations of L. Moreover, the subgroup H is the stabilizer of the identity element $e \in L$ in the group G.

A connected topological loop L is called strongly left alternative, if there exists a neighbourhood U of $e \in L$ which is simply covered by one-parameter subgroups and for all $x, y \in L$ and all one-parameter subgroups $\{x^r; r \in \mathbb{R}\}$ of L the identity $x^t \cdot (x^s \cdot y) = x^{t+s} \cdot y$ holds.

A connected topological loop L defined on a differentiable manifold is called almost differentiable if the mappings $(x, y) \mapsto x \cdot y$, $(x, y) \mapsto x \backslash y$ of L are differentiable. Let L be an almost differentiable loop and $\sigma : G/H \to G$ be the section corresponding to L such that G is a Lie group. Then σ is a differentiable map. Moreover, L is strongly left alternative precisely if $\exp\,[T_1\sigma(G/H)] \subset \sigma(G/H)$ holds, where $T_1\sigma(G/H)$ is the tangent space of $\sigma(G/H)$ at $1 \in G$.

A loop L is called a Bol loop, if for any two left translations λ_a, λ_b the product $\lambda_a\lambda_b\lambda_a$ is again a left translation of L.

The kernel of a homomorphism $\alpha : (L, \cdot) \to (L', *)$ of a loop L into a loop L' is a normal subloop N of L, i.e. a subloop of L such that

$$x \cdot N = N \cdot x, \quad (x \cdot N) \cdot y = x \cdot (N \cdot y), \quad x \cdot (y \cdot N) = (x \cdot y) \cdot N.$$

A loop L is solvable if it has a series $1 = L_0 \leq L_1 \leq \cdots \leq L_n = L$, where L_{i-1} is normal in L_i and L_i/L_{i-1} is an abelian group, $i = 1, \ldots, n$. The centre $Z(L)$ of a loop L consists of all elements z which satisfy the equations $zx \cdot y = z \cdot xy$, $x \cdot yz = xy \cdot z$, $xz \cdot y = x \cdot zy$, $zx = xz$ for all $x, y \in L$. If we put $Z_0 = 1, Z_1 = Z(L)$ and $Z_i/Z_{i-1} = Z(L/Z_{i-1})$, then we obtain a series of normal subloops of L. If Z_{n-1} is a proper subloop of L but $Z_n = L$, then L is centrally nilpotent of class n.

Let L be a connected topological loop corresponding to the section $\sigma : G/H \to G$. Let $\tilde{L}$ be the universal covering of L corresponding to the section $\tilde{\sigma} : \tilde{G}/\tilde{H} \to \tilde{G}$, where $\tilde{G}$ is the group topologically generated by the left translations of $\tilde{L}$ and $\tilde{H}$ is the stabilizer of $e \in \tilde{L}$ in $\tilde{G}$. If we identify $\tilde{L}$ with the image set $\tilde{\sigma}(\tilde{G}/\tilde{H})$, then $\tilde{L}$ contains a discrete central subgroup $\tilde{Z}$ isomorphic to the fundamental group Z of L (cf. [4, p. 216]). Moreover, the group $\tilde{G}$ is a covering group of G such that for the covering map $p : \tilde{G} \to G$ one has $p\big(\tilde{\sigma}\big(\tilde{G}/\tilde{H}\big)\big) = \sigma(G/H), p\big(\tilde{H}\big) = H$. The kernel of p is the subgroup $\tilde{Z}, \tilde{H} \cap \tilde{Z} = \{1\}$ and $\tilde{H}$ is isomorphic to H (cf. [3] and Lemma 1.34 in [6, p. 34]). If Z' is a subgroup of $\tilde{Z}$, then the factor loop $\tilde{L}/Z'$ is a covering loop of L and any covering loop of L is isomorphic to a factor loop $\tilde{L}/Z'$ with a suitable subgroup Z' (cf. [3]).

3. Two-dimensional topological loops

Lemma 3.1. *Every connected topological proper loop L, such that the group G topologically generated by the left translations is a three-dimensional Lie group, has dimension at most 2. In particular, if G is solvable, then one obtains* $\dim L = 2$.

Proof. Since L is proper, one has $\dim L < \dim G$ and the first assertion follows. If $\dim L = 1$, then G is a covering of the group $PSL_2(\mathbb{R})$ (cf. Proposition 18.2 in [6, p. 235]). This gives the second assertion. □

A three-dimensional non-abelian connected solvable Lie group G has either a one-dimensional centre or its centre is equal $\{1\}$. In the first case the group G is either nilpotent or it is the direct product of a one-dimensional Lie group and the two-dimensional non-abelian Lie group. If the centre of G is trivial, then G has either no or precisely one, or two or infinitely many one-dimensional normal subgroups. By Lemma 3.1 and Proposition 23.6 in [6, p. 298], a group having infinitely many one-dimensional normal subgroups cannot be the group topologically generated by the left translations of a connected topological proper loop.

In [6, Lemma 23.15, p. 312], it is proved that there is no two-dimensional almost differentiable strongly left alternative loop L such that the group topologically generated by the left translations is locally isomorphic to the group of orientation preserving motions of the Euclidean plane. Now we prove that also for topological loops this is the case. The proof of this fact substantially differs from the proof of Lemma 23.15 in [6], since there the exponential map is used.

Proposition 3.2. *Let G be a connected Lie group locally isomorphic to the group of orientation preserving motions of the Euclidean plane. There is no topological proper loop L having G as the group topologically generated by its left translations.*

Proof. We assume that L exists. Then there is also a simply connected loop $\tilde{L}$. Moreover, one has $\dim \tilde{L} = 2$ (see Lemma 3.1). As all one-dimensional subgroups which are not contained in the commutator subgroup of G are conjugate, the group G topologically generated by the left translations of $\tilde{L}$ is simply connected. We can identify G with the linear group of matrices

$$\left\{ g(t,u,v) = \begin{pmatrix} 1 & u & v & 0 \\ 0 & \cos t & \sin t & 0 \\ 0 & -\sin t & \cos t & 0 \\ 0 & 0 & 0 & e^t \end{pmatrix},\ t,u,v \in \mathbb{R} \right\}.$$

According to the proof of Lemma 23.15 in [6] we may choose the stabilizer H of $e \in L$ in G as the subgroup $\{g(0,u,0); u \in \mathbb{R}\}$. Since all elements of G have a unique decomposition as $g(x,0,y)g(0,u,0)$, any continuous function $f : \mathbb{R}^2 \to \mathbb{R}; (x,y) \mapsto f(x,y)$ determines a continuous section $\sigma : G/H \to G$ given by

$$\sigma : g(x,0,y)H \mapsto g(x,0,y)g(0,f(x,y),0) = g(x,f(x,y),y).$$

The section σ is sharply transitive if and only if for every $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$ there exists precisely one pair $(x,y) \in \mathbb{R}^2$ such that

$$g(x,f(x,y),y)g(x_1,0,y_1) = g(x_2,0,y_2)g(0,u,0)$$

for a suitable $u \in \mathbb{R}$. This gives the equations

$$\begin{aligned} &x = x_2 - x_1,\ u = f(x_2-x_1,y)\cos x_1 - y \sin x_1 \quad \text{and} \\ &0 = y_1 - y_2 + f(x_2-x_1,y)\sin x_1 + y\cos x_1. \end{aligned} \tag{3.1}$$

For given $x_1 \in \left\{\frac{\pi}{2} + k\pi;\ k \in \mathbb{Z}\right\}$ and $y_1,y_2,x_2 \in \mathbb{R}$ (3.1) has a unique solution y precisely when the equation

$$\pm (y_2 - y_1) = f(x_2 - x_1, y) \tag{3.2}$$

is uniquely solvable. For $x_1 \in \mathbb{R}\setminus\left\{\frac{\pi}{2}+k\pi;\ k\in\mathbb{Z}\right\}$ the existence of the unique solution y of (3.1) is equivalent to the condition that for every $x_0 = x_2 - x_1 \in \mathbb{R}$ the function $g : y \mapsto y + \tan x_1 f(x_0, y) : \mathbb{R} \to \mathbb{R}$ is a bijective mapping. Let $\psi_1 < \psi_2 \in \mathbb{R}$. Then $g(\psi_1) \neq g(\psi_2)$ and we may assume that $g(\psi_1) < g(\psi_2)$. We consider the inequality

$$0 < g(\psi_2) - g(\psi_1) = \psi_2 - \psi_1 + \tan x_1 \left[f\left(x_0, \psi_2\right) - f\left(x_0, \psi_1\right)\right].$$

In the case that $f(x_0, \psi_2) \neq f(x_0, \psi_1)$ there exists $x_1 \in \mathbb{R}\setminus\left\{\frac{\pi}{2}+k\pi;\ k\in\mathbb{Z}\right\}$ such that $g(\psi_2) = g(\psi_1)$, which is a contradiction. Hence for the function f one has $f(x, y) = f(x)$ for all $x, y \in \mathbb{R}$, i.e. f is independent of y. This is a contradiction to (3.2). Hence the section σ is not sharply transitive. Then the loop $\tilde{L}$ does not exist. But in this case L does not exist either and the assertion follows. □

Remark 3.3. In contrast to the assertion of Proposition 3.2, in a Lie group G locally isomorphic to the group of orientation preserving motions of the Euclidean plane there exist local sections $\sigma : U \to G$, where U is a neighbourhood of the origin H in G/H such that $\sigma(U)$ determines a local Bol loop on U (cf. [6], Lemma 23.15).

We show that, with the exceptions of the connected component of the group of homotheties of the Euclidean plane and the groups treated in Proposition 3.2, any three-dimensional connected solvable Lie group occurs as the group topologically generated by the left translations of a connected topological proper loop. Hence we have to treat three-dimensional Lie groups having finitely many one-dimensional normal subgroups.

In [6, Theorem 23.1, p. 290], a classification of two-dimensional almost differentiable strongly left alternative loops L is given, if their groups topologically generated by the left translations are three-dimensional Lie groups with trivial centre and with precisely two one-dimensional normal subgroups. In the next proposition we determine all connected topological proper loops as sections in these Lie groups. Among them there are loops which are almost differentiable but not strongly left alternative and which are neither almost differentiable nor strongly left alternative. Unlike the proof of Theorem 23.1 in [6] we cannot use the exponential image of the tangent space of the loops L.

Proposition 3.4. *Let G be a three-dimensional connected solvable Lie group with trivial centre having precisely two one-dimensional normal subgroups and choose for G the representation*

$$G = \left\{ g(v,u,w) = \begin{pmatrix} 1 & v & u \\ 0 & e^{aw} & 0 \\ 0 & 0 & e^{bw} \end{pmatrix},\ u,v,w \in \mathbb{R} \right\}$$

with fixed numbers $a, b \in \mathbb{R}\backslash\{0\}, a \neq b$*. Let* H *be a one-dimensional subgroup of* G *which does not contain any non-trivial normal subgroup of* G*. Then using automorphisms of* G *we may choose* $H = \{g(t,t,0),\ t \in \mathbb{R}\}$ *and any continuous sharply transitive section* $\sigma : G/H \to G$*, such that* $\sigma(H) = 1 \in G$ *and* $\sigma(G/H)$ *generates* G*, has the form*

$$\sigma_f : g(v,0,w)H \mapsto g(v + f(v,w), f(v,w), w),$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ *is a continuous function with* $f(0,0) = 0$ *such that* f *is different from* $f(v,w) = C\left(1 - e^{bw}\right)$ *and from* $f(v,w) = K\left(1 - e^{aw}\right) - v, C, K \in \mathbb{R}$ *and such that for all pairs* $(v_1, w_1), (v_2, w_2) \in \mathbb{R}^2$ *there is a unique* $v \in \mathbb{R}$ *satisfying the equation*

$$v + f(v, w_2 - w_1)\left(1 - e^{(b-a)w_1}\right) = \frac{v_2 - v_1}{e^{aw_1}}.$$

The multiplication of the topological loop L_f *corresponding to* σ_f *is given by*

$$(v_1, w_1) * (v_2, w_2) = \left(v_2 + e^{aw_2} v_1 + f(v_1, w_1)\left(e^{aw_2} - e^{bw_2}\right), w_1 + w_2\right). \quad (3.3)$$

The group topologically generated by all left and right translations of L_f *is not a finite dimensional Lie group.*

The loop L_f *is almost differentiable if the section* σ_f *is differentiable. The almost differentiable loop* L_f *is strongly left alternative precisely if the image set* $\sigma_f(G/H)$ *has the form* $\left\{g\left(x\frac{e^{az}-1}{az}, \alpha x\frac{e^{bz}-1}{bz}, z\right);\ x, z \in \mathbb{R}\right\}$*, where* $\alpha < 0$*.*

Proof. The form of the subgroup H follows from the proof of Theorem 23.1, p. 291, in [6]. Since all elements $g(v,u,w)$ of G have a unique decomposition as $g(v-u,0,w)g(u,u,0)$ any continuous function $f : \mathbb{R}^2 \to \mathbb{R}; (v,w) \mapsto f(v,w)$ determines a continuous section $\sigma : G/H \to G$ given by the map

$$g(v,0,w)H \mapsto g(v,0,w)g(f(v,w), f(v,w), 0) = g(v + f(v,w), f(v,w), w).$$

This section is sharply transitive if and only if for any $(v_1, w_1), (v_2, w_2) \in \mathbb{R}^2$ the equation

$$g(v + f(v,w), f(v,w), w)g(v_1, 0, w_1) = g(v_2, 0, w_2)g(t,t,0) \quad (3.4)$$

has a unique solution $(v,w) \in \mathbb{R}^2$ with a suitable $t \in \mathbb{R}$. From this we obtain $w = w_2 - w_1, t = e^{bw_1} f(v, w_2 - w_1)$ and

$$v + f(v, w_2 - w_1)\left(1 - e^{(b-a)w_1}\right) = \frac{v_2 - v_1}{e^{aw_1}}. \quad (3.5)$$

Equation (3.4) has a unique solution $(v,w) \in \mathbb{R}^2$ precisely if (3.5) has a unique solution $v \in \mathbb{R}$. The set $\sigma(G/H)$ contains the subsets

$$F_1 = \{g(v + f(v,0), f(v,0), 0);\ v \in \mathbb{R}\} \subset G' = [G,G] \quad \text{and}$$
$$F_2 = \{g(f(0,w), f(0,w), w);\ w \in \mathbb{R}\}.$$

We have $F_1 \cap F_2 = \{1\}$. Therefore $\sigma(G/H)$ generates G if the set F_1 generates the whole two-dimensional commutator subgroup $G' = \{g(v, u, 0); u, v \in \mathbb{R}\}$ of G. If F_1 does not generate G', then F_1 is a one-parameter subgroup which is equivalent to the condition that $f(v, 0) = \lambda v$ for some $\lambda \in \mathbb{R}$. Assuming this, the set $\sigma(G/H)$ generates G, if there exists an element $h \in F_2$ such that $h^{-1}F_1 h \neq F_1$ holds. For $h = g(f(0, w), f(0, w), w) \in F_2$ with $w \neq 0$ we have

$$h^{-1} = g\left(-e^{-aw} f(0, w), -e^{-bw} f(0, w), -w\right)$$

and

$$h^{-1}g(v + f(v, 0), f(v, 0), 0)h = g\left(e^{aw}(v + f(v, 0)), e^{bw} f(v, 0), 0\right).$$

Hence we obtain $h^{-1}F_1 h = F_1$ if and only if $f(v, 0) = 0$ or $f(v, 0) = -v$ for all $v \in \mathbb{R}$. In the first case the set F_1 has the form $\tilde{F_1} = \{g(v, 0, 0);\ v \in \mathbb{R}\}$, in the second case $F_1^* = \{g(0, v, 0);\ v \in \mathbb{R}\}$. The sets $\tilde{F_1}$ and F_1^* are normal subgroups of G. The set $\tilde{F_1} \cup F_2$, or $F_1^* \cup F_2$ respectively, does not generate G precisely if the set $F_2\tilde{F_1}/\tilde{F_1}$, or $F_2F_1^*/F_1^*$ respectively, is a one-parameter subgroup of $G/\tilde{F_1}$, or G/F_1^* respectively. Because of

$$g(\mathbb{R}, f(0, w_1), w_1)g(\mathbb{R}, f(0, w_2), w_2) = g\left(\mathbb{R}, f(0, w_2) + e^{bw_2} f(0, w_1), w_1 + w_2\right)$$

and

$$g(f(0, w_1), \mathbb{R}, w_1)g(f(0, w_2), \mathbb{R}, w_2) = g\left(f(0, w_2) + e^{aw_2} f(0, w_1), \mathbb{R}, w_1 + w_2\right)$$

the set $\tilde{F_1} \cup F_2$, or $F_1^* \cup F_2$ respectively, does not generate G if and only if for all $w_1, w_2 \in \mathbb{R}$ one has

$$f(0, w_2) + e^{bw_2} f(0, w_1) = f(0, w_1 + w_2),$$

or

$$f(0, w_2) + e^{aw_2} f(0, w_1) = f(0, w_1 + w_2).$$

Interchanging the variables w_1 and w_2 we obtain in the first case

$$f(0, w_1) + e^{bw_1} f(0, w_2) = f(0, w_1 + w_2),$$

in the second case

$$f(0, w_1) + e^{aw_1} f(0, w_2) = f(0, w_1 + w_2).$$

Hence one gets in the first case

$$f(0, w_1)\left(1 - e^{bw_2}\right) = f(0, w_2)\left(1 - e^{bw_1}\right),$$

in the second case

$$f(0, w_1)\left(1 - e^{aw_2}\right) = f(0, w_2)\left(1 - e^{aw_1}\right).$$

Consequently, for any $w_1 \neq 0, w_2 \neq 0$ we have in the first case

$$\frac{f(0, w_2)}{1 - e^{bw_2}} = \frac{f(0, w_1)}{1 - e^{bw_1}} = C$$

for a suitable constant $C \in \mathbb{R}$, in the second case

$$\frac{f(0,w_2)}{1-e^{aw_2}} = \frac{f(0,w_1)}{1-e^{aw_1}} = K$$

for a suitable constant $K \in \mathbb{R}$.

Since $g(f(0,w), f(0,w), w)g(v+f(v,0), f(v,0), 0) = g(v+f(v,0)+f(0,w), f(v,0)+f(0,w), w)$ the set $\sigma(G/H) = \{g(v+f(v,w), f(v,w), w); v, w \in \mathbb{R}\}$ does not generate G, if the function f has one of the following forms: $f(v,w) = C\left(1-e^{bw}\right)$ or $f(v,w) = K\left(1-e^{aw}\right) - v$, where $C, K \in \mathbb{R}$.

Now we represent the loop multiplication in the coordinate system $(v,w) \mapsto g(v,0,w)H$. Then the product $(v_1,w_1)*(v_2,w_2)$ will be determined, if we apply

$$\sigma_f(g(v_1,0,w_1)H) = g(v_1+f(v_1,w_1), f(v_1,w_1), w_1)$$

to the left coset $g(v_2,0,w_2)H$ and find in the image coset the element of G which lies in the set $\{g(v,0,w)H;\, v,w \in \mathbb{R}\}$. A direct computation yields

$$(v_1,w_1)*(v_2,w_2) = \left(v_2 + e^{aw_2}v_1 + f(v_1,w_1)\left(e^{aw_2} - e^{bw_2}\right), w_1+w_2\right).$$

According to Theorem 1 in [2], the group topologically generated by all left and right translations of the loop L_f cannot be a Lie group of finite dimension.

The loop L_f is almost differentiable, if the section σ_f is differentiable (cf. [6, p. 32]). In this case L_f is strongly left alternative if and only if the image set $\sigma(G/H)$ has the form $\left\{g\left(x\frac{e^{az}-1}{az}, \alpha x\frac{e^{bz}-1}{bz}, z\right); x, z \in \mathbb{R}\right\}$, where $\alpha < 0$ (cf. [6, Theorem 23.1, p. 290]). □

In contrast to Lemma 23.16 in [6, p. 314], there are connected topological proper loops L having the three-dimensional solvable Lie group with trivial centre and with only one one-dimensional normal subgroup as the group topologically generated by the left translations.

Proposition 3.5. *Let G be the three-dimensional connected non-nilpotent Lie group containing only one one-dimensional normal subgroup and choose for G the representation*

$$G = \left\{ g(u,v,w) = \begin{pmatrix} 1 & u & v \\ 0 & e^w & we^w \\ 0 & 0 & e^w \end{pmatrix}, u,v,w \in \mathbb{R} \right\}.$$

Let H be a one-dimensional connected subgroup of G which does not contain any non-trivial normal subgroup of G, then using automorphisms of G we may choose $H = \{g(t,0,0); t \in \mathbb{R}\}$ and every continuous sharply transitive section $\sigma : G/H \to G$ with the properties that $\sigma(G/H)$ generates G and $\sigma(H) = 1$ is determined by the map $\sigma_f : g(0,v,w)H \mapsto g(f(w),v,w)$, where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function which is different from $C(1-e^w), C \in \mathbb{R}$ and which has the property $f(0) = 0$.

The multiplication of the loop L_f corresponding to σ_f can be written as

$$(v_1,w_1)*(v_2,w_2) = \left(v_2 + f(w_1)w_2e^{w_2} + v_1e^{w_2}, w_1+w_2\right). \tag{3.6}$$

None of the loops L_f has the strongly left alternative property.

The multiplication group of L_f has infinite dimension.

Proof. By Lemma 23.16 in [6, p. 314], the subgroup H has the form as in the assertion. Since all elements of G have a unique decomposition as a product $g(0,v,w)g(u,0,0)$, any continuous function $f:\mathbb{R}^2\to\mathbb{R}; (v,w)\mapsto f(v,w)$ determines a continuous section $\sigma: G/H\to G$ with $H=\{g(t,0,0); t\in\mathbb{R}\}$ given by

$$\sigma: g(0,v,w)H\mapsto g(0,v,w)g(f(v,w),0,0)=g(f(v,w),v,w).$$

This section acts sharply transitively on the factor space G/H if for every $(v_1,w_1),(v_2,w_2)\in\mathbb{R}^2$ there exists precisely one pair $(v,w)\in\mathbb{R}^2$ such that

$$g(f(v,w),v,w)g(0,v_1,w_1)=g(0,v_2,w_2)g(t,0,0)$$

for a suitable $t\in\mathbb{R}$. This gives the equations $w=w_2-w_1, t=f(v,w)e^{w_1}$ and

$$0=\frac{v_1-v_2}{e^{w_1}}+f(v,w_2-w_1)w_1+v.$$

These are equivalent to the condition that for every $w_0=w_2-w_1$ and $w_1\in\mathbb{R}$ the function $g: v\mapsto v+w_1 f(v,w_0):\mathbb{R}\to\mathbb{R}$ is a bijective mapping. Let $\psi_1<\psi_2\in\mathbb{R}$. Then $g(\psi_1)\neq g(\psi_2)$ and we may assume that $g(\psi_1)<g(\psi_2)$. We consider

$$0<g(\psi_2)-g(\psi_1)=\psi_2-\psi_1+w_1\left[f(\psi_2,w_0)-f(\psi_1,w_0)\right]$$

as a linear function of $w_1\in\mathbb{R}$. If $f(\psi_2,w_0)\neq f(\psi_1,w_0)$, then there exists a $w_1\in\mathbb{R}$ such that $g(\psi_2)-g(\psi_1)=0$, which is a contradiction. Hence for the function f we have $f(v,w)=f(w)$, i.e. f does not depend on v. The set $\sigma(G/H)$ contains the normal subgroup $F_1=\{g(0,v,0); v\in\mathbb{R}\}<G'=[G,G]$ and the subset $F_2=\{g(f(w),0,w); w\in\mathbb{R}\}$. We have $F_1\cap F_2=\{1\}$. As $g(f(w),0,w)g(0,v,0)=g(f(w),v+f(w),w)$ the set $\sigma(G/H)$ generates G, if the set $(F_2F_1)/F_1$ is not a one-parameter subgroup of G/F_1. Because of

$$g(f(w_1),\mathbb{R},w_1)g(f(w_2),\mathbb{R},w_2)=g\left(f(w_1)e^{w_2}+f(w_2),\mathbb{R},w_1+w_2\right),$$

the set $\sigma(G/H)$ does not generate G precisely if for all $w_1,w_2\in\mathbb{R}$ the equality

$$f(w_1)e^{w_2}+f(w_2)=f(w_1+w_2) \tag{3.7}$$

holds. Interchanging the variables w_1 and w_2 we obtain

$$f(w_1)e^{w_2}+f(w_2)=f(w_2)e^{w_1}+f(w_1).$$

For $w_1\neq 0$ and $w_2\neq 0$ it is equivalent to $\frac{f(w_2)}{1-e^{w_2}}=\frac{f(w_1)}{1-e^{w_1}}=C$ for a suitable constant $C\in\mathbb{R}$. Therefore, the loop L_f is proper if and only if $f(w)\neq C\left(1-e^{w}\right)$.

The multiplication of the loop L_f can be expressed in the coordinate system $(v,w)\mapsto g(0,v,w)H$ if we apply $\sigma_f(g(0,v_1,w_1)H)=g(f(w_1),v_1,w_1)$ to the left coset $g(0, v_2, w_2)H$ and find in the image coset the element of G which lies in the set $\{g(0, v, w)H; v, w \in \mathbb{R}\}$. A direct computation yields

$$(v_1, w_1) * (v_2, w_2) = (v_2 + f(w_1)w_2 e^{w_2} + v_1 e^{w_2}, w_1 + w_2)\,.$$

For all $x, y \in L_f$ the identity $x * (x * y) = x^2 * y$ holds precisely if for all $w \in \mathbb{R}$ the continuous function f satisfies the identity $f(w)\,(1 + e^w) = f(2w)$. This is a contradiction to (3.7). Hence the loop L_f is not strongly left alternative.

The assertion that the multiplication group of the topological loop L_f has infinite dimension follows from Theorem 1 in [2]. □

Let G be the three-dimensional simply connected non-nilpotent Lie group having a one-dimensional centre. A classification of the continuous, sharply transitive sections $\sigma : G/H \to G$ with $\dim H = 1$ is given in Theorem 23.7 in [6]. If $H = \{g(1, t, t); t \in \mathbb{R}\}$ we find under which circumstances the set $\sigma(G/H)$ generates G. In the case that $H = \{g\,(e^t, 0, 0)\,;\ t \in \mathbb{R}\}$ we show that the loops corresponding to σ are not strongly left alternative.

Proposition 3.6. *Let G be a three-dimensional simply connected, non-nilpotent Lie group with one-dimensional centre and choose for G the representation by matrices*

$$g(e^u, v, z) = \begin{pmatrix} 1 & v & z \\ 0 & e^u & 0 \\ 0 & 0 & 1 \end{pmatrix},\ u, v, z \in \mathbb{R}. \tag{3.8}$$

Let H be a one-dimensional subgroup containing no non-trivial normal subgroup of G.

(a) *If H is not contained in the product of the centre Z with the commutator subgroup of G, then using automorphisms of G we may choose $H = \{g\,(e^t, 0, 0)\,; t \in \mathbb{R}\}$ and each continuous, sharply transitive section $\sigma : G/H \to G$ such that $\sigma(H) = 1 \in G$ and $\sigma(G/H)$ generates G is given by the map $\sigma_t : g(1, v, z)H \mapsto g\left(e^{t(z)}, e^{t(z)}v, z\right)$, where $t : \mathbb{R} \to \mathbb{R}$ is a non-linear continuous function with $t(0) = 0$.*
The multiplication of the loop L_t determined by σ_t can be written as

$$(v_1, z_1) * (v_2, z_2) = \left(v_1 + v_2 e^{-t(z_1)}, z_1 + z_2\right). \tag{3.9}$$

The multiplication group of the topological loop L_t has infinite dimension.
None of the loops L_t satisfies the strongly left alternative property.

(b) *If H is contained in the product of the centre Z with the commutator subgroup of G, then using automorphisms of G we may choose $H = \{g(1, t, t); t \in \mathbb{R}\}$ and every continuous, sharply transitive section $\sigma : G/H \to G$ with the properties $\sigma(H) = 1 \in G$ and $\sigma(G/H)$ generates G has the form*

$$\sigma_f : g\,(e^u, 0, z)\,H \mapsto g\,(e^u, f(u, z), z + f(u, z))\,,$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ *is a continuous function with* $f(0,0) = 0$ *such that for all pairs* $(u_1, z_1), (u_2, z_2) \in \mathbb{R}^2$ *there is a unique* $z \in \mathbb{R}$ *satisfying the equation*

$$z + f(u_2 - u_1, z)\,[1 - e^{u_1}] = z_2 - z_1.$$

and such that

$$f(u,z) \text{ is different from } C\,(e^u - 1) \text{ and from } \lambda u - z, \text{ where } C, \lambda \in \mathbb{R}. \tag{3.10}$$

The multiplication of the loop L_f *corresponding to* σ_f *is defined by*

$$(u_1, z_1) * (u_2, z_2) = (u_1 + u_2, z_1 + z_2 + f(u_1, z_1)\,(1 - e^{u_2}))\,. \tag{3.11}$$

The group topologically generated by all translations of the topological loop L_f *has infinite dimension.*
The loop L_f *is almost differentiable if* $f(u,z)$ *is a real* C^r*-function. In this case* L_f *is strongly left alternative if and only if the tangent space* $T_1(\sigma(G/H))$ *of the image of the section* σ_f *has the form*

$$\left\{ \begin{pmatrix} 0 & \alpha x + (1+\beta)y & \alpha x + \beta y \\ 0 & x & 0 \\ 0 & 0 & 0 \end{pmatrix} ; x, y \in \mathbb{R} \right\},$$

where $\alpha, \beta \in \mathbb{R}$ *with* $-1 < \beta < 0$.

Proof. According to Theorem 23.7(i) in [6], in the case (a) it remains to prove that the loop L_t is not strongly left alternative. For all $x, y \in L_t$ the identity $x*(x*y) = x^2*y$ holds if and only if the function t is linear. This contradiction yields the assertion.

According to Theorem 23.7(ii) and (iii) in [6], in the case (b) it remains to investigate under which circumstances the set $\sigma(G/H)$ generates the group G. Let $F_1 = \{g(1, f(0,z), z + f(0,z)); z \in \mathbb{R}\}$ and $F_2 = \{g\,(e^u, f(u,0), f(u,0))\,; u \in \mathbb{R}\}$ be two subsets of $\sigma(G/H)$ (see Proof of Theorem 23.7 in [6]). The set F_1 is a one-parameter subgroup if and only if the function $f(0,z) = \mu z$ for some $\mu \in \mathbb{R}$. Now let $f(0,z) = \mu z$. Then the set $F_1 \cup F_2$ generates G, if there exists an element $h \in F_2$ such that $h^{-1} F_1 h \neq F_1$ holds. For $h = g\,(e^u, f(u,0), f(u,0))$ with $u \neq 0$ we have $h^{-1} = g\,(e^{-u}, -e^{-u} f(u,0), -f(u,0))$. Moreover, one has $h^{-1} g(1, \mu z, (1+\mu)z) h = g\,(1, \mu z e^{-u}, (1+\mu)z)$. Hence $h^{-1} F_1 h = F_1$ if and only if one of the following cases holds: $f(0,z) = 0$ or $f(0,z) = -z$ for all $z \in \mathbb{R}$. In the first case, the set F_1 has the form $\tilde{F_1} = \{g(1,0,z); z \in \mathbb{R}\}$, in the second case $F_1^* = \{g(1,z,0); z \in \mathbb{R}\}$. The sets $\tilde{F_1}$ and F_1^* are normal subgroups of G. The set $\tilde{F_1} \cup F_2$, respectively $F_1^* \cup F_2$, does not generate G precisely if the set $F_2 \tilde{F_1}/\tilde{F_1}$, or $F_2 F_1^*/F_1^*$ respectively, is a one-parameter subgroup of $G/\tilde{F_1}$, or G/F_1^*, respectively. The set $F_2 \tilde{F_1}/\tilde{F_1}$ is a one-parameter subgroup of $G/\tilde{F_1}$

precisely if $f(u,0) = C(e^u - 1)$ for a suitable constant $C \in \mathbb{R}$ (cf. [6, p. 302]). Because of

$$g(e^{u_1}, \mathbb{R}, f(u_1,0))\, g(e^{u_2}, \mathbb{R}, f(u_2,0)) = g\left(e^{u_1+u_2}, \mathbb{R}, f(u_1,0) + f(u_2,0)\right)$$

the set $F_2F_1^*/F_1^*$ is a one-parameter subgroup of G/F_1^* if and only if for all $u_1, u_2 \in \mathbb{R}$ the equality $f(u_1,0)+f(u_2,0) = f(u_1+u_2,0)$ holds. This is the case precisely if $f(u,0) = \lambda u$ for some $\lambda \in \mathbb{R}$. As $g(e^u, f(u,0), f(u,0))\, g(1, f(0,z), z + f(0,z)) = g(e^u, f(u,0) + f(0,z), z + f(u,0) + f(0,z))$ the set $\sigma(G/H) = \{g(e^u, f(u,z), z + f(u,z)); u, z \in \mathbb{R}\}$ does not generate G, if the function $f(u,z)$ has one of the following form: $f(u,z) = C(e^u - 1)$ or $f(u,z) = \lambda u - z, C, \lambda \in \mathbb{R}$.

The assertion about the multiplication group of the loop L_t as well as of the loop L_f follows from Theorem 1 in [2]. □

In Theorem 23.7 in [6], the condition that $\sigma(G/H)$ generates G is due to a small slip marginal stronger than (3.10).

The continuous, sharply transitive sections $\sigma : G/H \to G$ in the non-abelian nilpotent three-dimensional Lie group G are given by Theorem 23.12 in [6, p. 308]. Now we give, in addition, necessary and sufficient conditions for the image set $\sigma(G/H)$ to generate G and show that the loops corresponding to σ do not satisfy the strongly left alternative property. We recall that the elementary filiform Lie group $\mathcal{F}_{n+2}$ is the simply connected Lie group of dimension $n+2 \geq 3$ whose Lie algebra has a basis $\{e_1, e_2, \ldots, e_{n+2}\}$ such that $[e_1, e_i] = (n+2-i)e_{i+1}$ for $2 \geq i \geq n+1$ and all other Lie brackets are zero.

Proposition 3.7. *Let G be the simply connected non-commutative nilpotent Lie group the multiplication of which is given by*

$$g(u_1, v_1, z_1) g(u_2, v_2, z_2) = g\left(u_1 + u_2, v_1 + v_2, z_1 + z_2 + \frac{1}{2}(u_1v_2 - u_2v_1)\right) \tag{3.12}$$

and let H be a one-dimensional connected subgroup which does not contain any non-trivial normal subgroup of G. Then using automorphisms of G we may choose $H = \{g(0,t,0);\ t \in \mathbb{R}\}$ and any continuous sharply transitive section $\sigma : G/H \to G$ such that $\sigma(G/H)$ generates G and $\sigma(H) = 1$ is given by the map

$$\sigma_v : g(u,0,z)H \mapsto g\left(u, v(u), z + \frac{1}{2}uv(u)\right),$$

where $v : \mathbb{R} \to \mathbb{R}$ is a non-linear continuous function with $v(0) = 0$.

The multiplication of the loop L_v corresponding to σ_v is given by

$$(u_1, z_1) * (u_2, z_2) = (u_1 + u_2, z_1 + z_2 - u_2 v(u_1)). \tag{3.13}$$

The multiplication group of the topological loop L_v is isomorphic to the elementary filiform Lie group $\mathcal{F}_{n+2}, n \geq 2$, precisely if v is a polynomial of degree n.

None of the loops L_v is strongly left alternative.

Proof. In view of Theorem 23.12 in [6, p. 308], it remains to prove that the set $\sigma(G/H)$ generates G precisely if the function v is non-linear. The set $\sigma(G/H)$ contains the commutator subgroup $G' = \{g(0,0,z); z \in \mathbb{R}\}$ of G and the subset $F = \left\{g\left(u, v(u), \frac{1}{2}uv(u)\right); u \in \mathbb{R}\right\}$. As

$$g\left(u_1, v(u_1), \frac{1}{2}u_1v(u_1)\right) g\left(u_2, v(u_2), \frac{1}{2}u_2v(u_2)\right)$$
$$= g\left(u_1+u_2, v(u_1)+v(u_2), \frac{1}{2}u_1v(u_1)+\frac{1}{2}u_2v(u_2)+\frac{1}{2}u_1v(u_2)-\frac{1}{2}u_2v(u_1)\right),$$

the group G' and the subgroup $\langle F\rangle$ topologically generated by the set F generate G precisely if the mapping assigning to the first component of any element of $\langle F\rangle$ its second component is not a homomorphism. This occurs if and only if the function v is non-linear. The multiplication of the loop L_v in the coordinate system $(u,z) \mapsto g(u,0,z)H$ is given by Proposition 3.1 in [7, p. 5101].

The assertion about the multiplication group of the topological loop L_v follows from Theorems 1 and 3 in [2].

A necessary condition for the strongly left alternative property for loops L is that for all $x, y \in L$ the identity $x \cdot (x \cdot y) = x^2 \cdot y$ holds. Using the multiplication of L_v we obtain that this identity is satisfied precisely if the function v is linear. This contradiction gives the assertion. □

Propositions 3.4, 3.5, 3.6 and 3.7 give the classification of all connected topological loops L having a three-dimensional solvable simply connected Lie group G as the group topologically generated by the left translations. All these loops are simply connected. Hence if $\check{L}$ is a non simply connected topological loop having L as the universal covering, then the group topologically generated by the left translations of $\check{L}$ is a homomorphic image G/Σ, where $\Sigma \neq 1$ is a suitable discrete central subgroup of G. It follows that the non-abelian Lie group G is either nilpotent or the direct product of $\mathbb{R}$ and the two-dimensional non-abelian Lie group.

If G is the non-abelian nilpotent Lie group with multiplication defined by (3.12), then any discrete central subgroup of G has the form $\mathbb{Z}_r = \{g(0,0,rk); k \in \mathbb{Z}\}$, where r is a fixed real number. The multiplication of the group $G_r = G/\mathbb{Z}_r$ is given by

$$g(u_1, v_1, z_1 + r\mathbb{Z})g(u_2, v_2, z_2 + r\mathbb{Z})$$
$$= g\left(u_1+u_2, v_1+v_2, z_1+z_2+\frac{1}{2}(u_1v_2-u_2v_1)+r\mathbb{Z}\right),$$

where $u_i, v_i, z_i \in \mathbb{R}, i = 1, 2$. The image of the section σ_v in Proposition 3.7 contains the group $\mathbb{Z}_r$ as a discrete central subgroup which corresponds to a discrete central subgroup of L_v defined by the multiplication (3.13). Hence every loop L_v^r having L_v as the universal covering is given by the multiplication

$$(u_1, z_1 + r\mathbb{Z}) * (u_2, z_2 + r\mathbb{Z}) = (u_1 + u_2, z_1 + z_2 - u_2 v(u_1) + r\mathbb{Z}). \qquad (3.14)$$

If G is the simply connected Lie group defined by (3.8), then any discrete central subgroup of G has the form $\mathbb{Z}_r = \{g(1, 0, rk); k \in \mathbb{Z}\}$, where r is a fixed real number. The multiplication of the group $G_r = G/\mathbb{Z}_r$ is given by

$$g\left(e^{u_1}, v_1, z_1 + r\mathbb{Z}\right) g\left(e^{u_2}, v_2, z_2 + r\mathbb{Z}\right) = g\left(e^{u_1+u_2}, e^{u_2} v_1 + v_2, z_1 + z_2 + r\mathbb{Z}\right),$$

where $u_i, v_i, z_i \in \mathbb{R}, i = 1, 2$. The group $\mathbb{Z}_r$ is contained as a discrete central subgroup in the image of the section σ_t in Proposition 3.6 (a) precisely if the function $t : \mathbb{R} \to \mathbb{R}$ satisfies $t(z) = t(z + rk)$ for all $z \in \mathbb{R}$ and $k \in \mathbb{Z}$. Hence the multiplication of a loop L_t^r having L_t defined by (3.9) as the universal covering is given by

$$(v_1, z_1 + r\mathbb{Z}) * (v_2, z_2 + r\mathbb{Z}) = \left(v_1 + v_2 e^{-t(z_1)}, z_1 + z_2 + r\mathbb{Z}\right), \qquad (3.15)$$

where $t : \mathbb{R} \to \mathbb{R}$ is a non-linear continuous periodic function with period r (see also Proposition 3.5 in [7, p. 5105]).

The image of the section σ_f in Proposition 3.6 (b) contains the group $\mathbb{Z}_r$ as a discrete central subgroup if and only if the function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies the condition $f(u, z) = f(u, z + rk)$ for all $u, z \in \mathbb{R}$ and $k \in \mathbb{Z}$. Hence every loop L_f^r having L_f defined by (3.11) as the universal covering is given by

$$(u_1, z_1 + r\mathbb{Z}) * (u_2, z_2 + r\mathbb{Z}) = (u_1 + u_2, z_1 + z_2 + f(u_1, z_1)(1 - e^{u_2}) + r\mathbb{Z}), \qquad (3.16)$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is the continuous function defined in Proposition 3.6 (b) satisfying the additional condition $f(u, z) = f(u, z + rk)$ for all $u, z \in \mathbb{R}, k \in \mathbb{Z}$ (see also Proposition 3.5 in [7, p. 5105]).

Proposition 3.8. *Every connected topological proper loop L such that the group topologically generated by the left translations is a 3-dimensional solvable Lie group is solvable.*

Proof. If L is simply connected, then its multiplication is given by (3.3), (3.6), (3.9), (3.11) and (3.13) in Propositions 3.4, 3.5, 3.6, 3.7. The set $N_1 = \{(v, 0);\ v \in \mathbb{R}\}$ is a normal subgroup of the loops defined by (3.3), (3.6) and (3.9). The set $N_2 = \{(0, z);\ z \in \mathbb{R}\}$ is a normal subgroup of the loops given by (3.11) and (3.13). A direct computation shows that for the multiplications (3.3) and (3.6) the identity $(0, w_1)N_1 * (0, w_2)N_1 = (0, w_1 + w_2)N_1$, for the multiplication given by (3.9) the identity $(0, z_1)N_1 * (0, z_2)N_1 = (0, z_1 + z_2)N_1$ and for the multiplications (3.11) and (3.13) the identity $(u_1, 0)N_2 * (u_2, 0)$

$N_2 = (u_1 + u_2, 0)N_2$ is satisfied. Hence every factor loop $L/N_i, i = 1, 2$, is isomorphic to the one-dimensional Lie group $\mathbb{R}$. Therefore, every simply connected loop L is an extension of the group $\mathbb{R}$ by the group $\mathbb{R}$ and hence it is solvable.

If L is defined either by the multiplication (3.14) or by (3.16), then L is an extension of the group $SO_2(\mathbb{R})$ by the group $\mathbb{R}$. If L is given by the multiplication (3.15), then L contains a normal subgroup N isomorphic to $\mathbb{R}$ such that the factor loop L/N is isomorphic to the group $SO_2(\mathbb{R})$ and the assertion follows. □

Remark 3.9. Among the connected topological proper loops having a solvable three-dimensional Lie group as the group topologically generated by their left translations, only the loops corresponding to a nilpotent group are central extensions of a one-dimensional Lie group N by the group $(u_1, 0)N \cong \mathbb{R}$. These loops are centrally nilpotent loops of class 2 (cf. [1, Chap. VI]). Moreover, the group N is isomorphic to $\mathbb{R}$ or to $SO_2(\mathbb{R})$ depending on whether L is simply connected or not.

References

[1] Bruck, R.H.: A Survey of Binary Systems. Springer, Berlin, Heidelberg, New York (1971)
[2] Figula, Á.: The multiplication groups of 2-dimensional topological loops. J. Group Theory **12**, 419–429 (2009)
[3] Hofmann, K.H.: Topological loops. Math. Z. **70**, 125–155 (1958)
[4] Hofmann, K.H., Strambach, K.: Topological and analytical loops. In: Chein, O., Pflugfelder, H.O., Smith, J.D.H. (eds.) Quasigroups and Loops: Theory and Applications, pp. 205–262. Heldermann-Verlag, Berlin (1990)
[5] Jacobson, N.: Lie Algebras. Wiley Interscience, New York (1962)
[6] Nagy, P.T., Strambach, K.: Loops in groups theory and Lie theory. In: De Gruyter Expositions in Mathematics, vol. 35. Berlin, New York (2002)
[7] Nagy, P.T., Strambach, K.: Coverings of Topological Loops. J. Math. Sci. **137**, 5098–5116 (2006)

Ágota Figula
Institute of Mathematics
University of Debrecen
P.O. Box 12
4010 Debrecen
Hungary
e-mail: figula@math.klte.hu

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