On the gambler's ruin problem and higher moments of some absorbing Markov chains

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Abstract

We present a simple formula that provides an upper bound for the expected runtime of the gambler's ruin process on an arbitrary finite graph and arbitrary initial wealth of the players. The estimation is proportionate to the harmonic mean of the expected runtimes of the two-player games played by the edges of the graph. We show that the same proof techniques can also be applied to provide asymptotic estimates to the higher moments of the absorption time of similar absorbing Markov chains, namely the discordant oblivious, push and pull protocols on cycle graphs.

Keywords: gambler's tuin, absorption, Markov chain, moments

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1. Introduction

Games on large graphs model several different complex socio-economic concepts. Out of these related areas including stock market simulation and income distribution [1], population statistics and human mobility[2] or communication in peer-to-peer networks [3], the present paper focuses on two applications.

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One is the gambler's ruin problem, a classical stochastic process with many variants in the literature [4, 5, 6, 7, 8]. In its simplest form, also known as the drunkard's walk: two players play a series of fair games betting 1 unit of their money until one of them goes bankrupt. It is well-known that this leads to a fair game with winning probabilities of each player proportionate to their initial wealth, and the expected runtime is the product of the two initial wealths. This can be generalized to arbitrary finite graphs in a number of ways. In [8], there is one winner in each round and the losers contribute an equal amount to pay the winner of the round. In [7], the game is played until all but one players lose their money. It was conjectured in [7] that given the special initial case with all vertices having an equal amount of money ν, and where the graph is complete with m edges, the runtime is proportionate to $m\nu^2$. This was verified in a number of special cases, including the original version of the problem [4]. For further examples, see also the monopolist game [9, 8, 4].

In the current paper, we study a natural variant of the gambler's ruin problem on arbitrary graphs. Given a graph together with an assigned positive integer wealth to each vertex, a series of fair games is played by randomly selected edges in each round with a 1 unit bet at stake, until the first bankruptcy occurs. According to the first main result of the paper (Theorem 3.4), the expected runtime of this process is at most the harmonic mean of the runtimes of the same game played by the separate edges of the graph multiplied by the number of edges. In particular, if the graph has m edges and all vertices have the same initial wealth ν, the theorem provides the upper bound $m\nu^2$ for the runtime of the above variant of the game.

The other related area under consideration is the spreading of rumor or infectious diseases [10], or more generally speaking, mathematical modeling of voting [11, 12, 13, 14, 15, 16]. The so-called linear voting model was introduced in [17] as a common generalization of many well-studied voting protocols. Three of the most common special cases of asynchronous linear voting are the oblivious, push, and pull protocols. In order to avoid a vast amount of idle rounds, the discordant versions were introduced in [18]. In a given round, an edge is discordant if the endpoints have different opinions, and a vertex is discordant if it lies on a discordant edge, that is, if it has a discordant neighbor.

- Discordant oblivious protocol: in each round a discordant edge uv is chosen uniformly at random, and then either u adopts the opinion of v or the other way around, with equal probability.
- Discordant push protocol: in each round a discordant vertex u is chosen uniformly at random, and that vertex forces a randomly chosen discordant neighbor to adopt the opinion of u.
- Discordant pull protocol: in each round a discordant vertex u is chosen uniformly at random, and that vertex is forced by a randomly chosen discordant neighbor v to adopt the opinion of v.

We note that the original linear variants are obtained from the above definitions by omitting the word discordant everywhere; see [17] for further details. In our model, there are only two possible opinions; the results could be generalized to a voting process with a given finite number of options to choose from. For simplicity, we refer to vertices with opinion 1 as red, and those with opinion 0 as blue.

All the above voting protocols are absorbing Markov chains. The absorbing states, representing the goal of the particular voting process, are the consensus states, that is, states where all participants have the same opinion. If the graph is a cycle, such a process can be viewed as a certain gambler's ruin problem on a (different) cycle graph. Namely, by replacing runs, i.e., maximal sets of consecutive vertices of the same color, by a vertex, and giving it the length of the corresponding run as initial wealth, the voting process is a series of games (always fair in case of the discordant oblivious protocol, not necessarily fair in the other two cases when a singleton run is involved) where 1 unit wealth is exchanged by neighbors, until one vertex has all the money. In this case, when a vertex goes broke, it is deleted, its two neighbors are identified, and their wealths are merged. This approach already provides a polynomial bound in terms of the number of vertices on the expected runtime of all three discordant linear voting protocols for cycle graphs. A more refined argument in [18] lead to a quadratic bound. It was further improved by the author of the present paper to asymptotically sharp formulas in [19, Theorem 5]. Namely, all three protocols on the n-cycle has an expected runtime $E(T) = \beta\varrho + O(rn)$, where β and ϱ are the number of blue and red vertices, respectively, and r is the number of blue runs in the initial state. Note that in all non-consensus states, the number of blue and red runs coincide, and by definition, we put $r = 0$ in both consensus states. It was also shown in [19, Theorem 5] that $E(T)$ is at most $n^2/4 + O(n^{3/2})$, an upper estimate that is independent of all parameters of the initial state. Furthermore, these games are nearly fair, provided a tame initial state, namely the probability of reaching the blue consensus is $\beta/n + O(r/n)$; see [19, Theorem 7].

Although some parts of the proof of the positive results require elaborate combinatorial and probabilistic arguments, the core is an elementary linear algebraic lemma; see Lemma 2.1, or [19, Lemma 1]. This paper demonstrates how the same technique can be used to provide asymptotic estimates to the higher moments of the runtime of each of the above three discordant linear protocols on cycle graphs; see Theorem 4.5. Roughly speaking, the moments are asymptotically the same as those of the simplest gambler's ruin problem, the drunkard walk. The same core idea is used in the proof of the first main result Theorem 3.4 on the expected runtime of the gambler's ruin problem, showing that the elementary lemma is quite robust with potential applications in proving general results.

2. General tools

We give a brief summary of the basic notation and techniques, using the same setup as [19]. The transient states of an absorbing Markov chain with transition matrix P is denoted by Tran. The set of penultimate states Pen $\subseteq$ Tran consists of those states $t \in$ Tran such that the transition probability from

t to an absorbing state is positive. As usual, Q is the upper left minor of the canonical form of $P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$, see [20]. So Q is the restriction of P to the transient states. Then $N = (I - Q)^{-1}$ denotes the fundamental matrix of the Markov chain. In this paper, vectors are column vectors of length $|\,\mathrm{Tran}\,|$, usually denoted by $\underline{u}, \underline{v}, \underline{\varepsilon} \in \mathbb{R}^{\mathrm{Tran}}$. We denote by $\underline{c}_0 \in \mathbb{R}^{\mathrm{Tran}}$ the vector all of whose entries equal to 1; this is a special case of the definition $\underline{c}_k \in \mathbb{R}^{\mathrm{Tran}}$ introduced in Section 4. The entry corresponding to the coordinate t in the vector $\underline{u}$ is denoted by $\underline{u}[t]$. If we sum up the entries $\underline{u}[t]$ while randomly walking on the coordinates starting from $t_0 \in \mathrm{Tran}$, then the expected value of this sum before the walk is absorbed is $(N\underline{u})[t_0]$. In particular, the expected times to absorption from each transient state as initial state are the coordinates of the vector $N\underline{c}_0$; see [20] for further details.

In general, it is hard to compute the fundamental matrix, or even to properly estimate its entries, especially if the transition matrix is not concrete but rather defined by using numbers or other objects (e.g., graphs) as parameters. To sidestep this major technical difficulty, the following elementary lemma was introduced in [19]: the idea is that the product $Q\underline{x}$ is easy to compute or estimate without computing the fundamental matrix. The entry $\underline{x}[t]$ is a "guesstimate" of the expected value of the sum of the entries of $\underline{u}$ during a random walk with initial state t before reaching an absorbing state. In the special case $\underline{u} = \underline{c}_0$, the vector $\underline{x}$ is the guesstimate for the time to absorption starting from each transient state.

Lemma 2.1. *[[19, Lemma 1]] Let $\underline{u}, \underline{x}, \underline{\varepsilon} \in \mathbb{R}^{\mathrm{Tran}}$ be vectors such that $Q\underline{x} = \underline{x} - \underline{u} + \underline{\varepsilon}$. Then $N\underline{u} = \underline{x} + N\underline{\varepsilon}$. In particular, if $Q\underline{x} \le \underline{x} - \underline{u}$, then $N\underline{u} \le \underline{x}$ (coordinate-wise).*

As a slight abuse of the big-O notation, we allow expressions of the form $\underline{u} = \underline{v} + O(f(n))$, where $\underline{u}, \underline{v}$ are vectors with entries depending on n and f is a function. It means that in each coordinate i, we have $\underline{u}[i] = \underline{v}[i] + O(f(n))$. In other words, we extend the notation to vectors and omit $\underline{c}_0$ from the expression

$\underline{u} = \underline{v} + O(f(n)\underline{c}_0)$.

3. Gambler's ruin

Lemma 3.1. *For any* $T \in]0, +\infty[$ *the function* $]0, \min(T, 1)[\to \mathbb{R}$

$$z \mapsto \left((1-z)^{-2} - 1\right)\left(T + \frac{z^2}{1-z}\right)^{-1}$$

is strictly monotone increasing.

Proof. The function is $z \mapsto \left(z^2 - 2z\right)\left(z^3 - (T+1)z^2 + 2Tz - T\right)^{-1}$, hence its derivative is

$$\frac{(2z-2)(z^3-(T+1)z^2+2Tz-T)-(z^2-2z)(3z^2-(2T+2)z+2T)}{\left(z^3-(T+1)z^2+2Tz-T\right)^2} = \frac{-z^4+4z^3-2z^2-2Tz+2T}{\left(z^3-(T+1)z^2+2Tz-T\right)^2}.$$

We need to show that the numerator is positive on $]0, \min(T, 1)[$. We can write the numerator in the form $z^3(2-z)+(2-2z)(T-z^2)$. The first summand is positive, thus we only need to argue that the second summand is non-negative. Since $2 - 2z \geq 0$, it suffices to show that $T - z^2 \geq 0$. If $T \geq 1$, then $T \geq z \geq z^2$ for all $z \in]0, 1[$, and if $T < 1$, then $T \geq z \geq z^2$ for all $z \in]0, T[$.

□

Theorem 3.2. *For* $m \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$ *and* $z_1, \ldots, z_m \in]0, 1]$, *we have*

$$(z_1+\cdots+z_m+k)^{-1}-(m+k)^{-1}\cdot\sum_{i=1}^{m}\left(\frac{z_i^2}{1-z_i}+z_1+\cdots+z_m+k\right)^{-1} \geq (m+k)^{-2}.$$

The theorem is proved by an inductive argument. The next lemma not only covers the initial step of the induction, but also the important special case of Theorem 3.2 where all the z_i are equal.

Lemma 3.3. *For* $m \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$ *and* $z \in]0, 1]$, *we have*

$$(mz+k)^{-1} - (m+k)^{-1}m \cdot \left(\frac{z^2}{1-z} + mz + k\right)^{-1} \geq (m+k)^{-2}.$$

Proof. We multiply both sides by $(m+k)^2(mz+k)((1-m)z^2+(m-k)z+k)$, a positive expression, and rearrange the inequality to obtain the equivalent assertion

$$(1-z)(m^3+m^2(n-1)(2+z)+m(n-1)(n+(2n-1)z)+(n^3-n^2)z)+ \\ +2mnz^2+(m^2-m)z+n(n-1)z^3 \geq 0,$$

which is clearly true since $m^2 \geq m, n \geq 1, z \geq 0, 1-z \geq 0$. ☐

Proof of Theorem 3.2. We prove by induction on m. Note that the initial case $m=1$ coincides with the special case of Lemma 3.3 when $m=1$. Let $m \geq 2$ and assume that the assertion holds for $m-1$. It is enough to show that the minimum of the function $f(z_1,\dots,z_m) = (z_1+\dots+z_m+k)^{-1} - (m+k)^{-1} \cdot \sum_{i=1}^{m} \left(\frac{z_i^2}{1-z_i} + z_1 + \dots + z_m + k\right)^{-1}$ is at least $(m+k)^{-2}$ if we restrict the domain to the set of all $(z_1,\dots,z_m) \in]0,1]^m$ with a fixed sum $S>0$. The advantage of considering this constrained optimization problem is that the restricted domain is compact, thus the minimum exists. If the minimum is on the boundary of the domain, then some z_i, which we may assume is z_m by symmetry, is 0 or 1.

If $z_m = 0$, then the induction hypothesis for $m' = m-1, k' = k$ and the tuple $(z_1,\dots,z_{m-1})$ yields

$$(z_1+\dots+z_{m-1}+k)^{-1}-(m+k-1)^{-1}\cdot\sum_{i=1}^{m-1}\left(\frac{z_i^2}{1-z_i}+z_1+\dots+z_{m-1}+k\right)^{-1} \geq \\ \geq (m+k-1)^{-2} \geq (m+k-1)^{-1}(m+k)^{-1}.$$

We multiply both ends of the chain of inequalities by $(m+k-1)(m+k)^{-1} = 1-(m+k)^{-1}$ to obtain

$$(z_1+\dots+z_{m-1}+k)^{-1}-(m+k)^{-1}\cdot(z_1+\dots+z_{m-1}+k)^{-1}- \\ -(m+k)^{-1}\cdot\sum_{i=1}^{m-1}\left(\frac{z_i^2}{1-z_i}+z_1+\dots+z_{m-1}+k\right)^{-1} \geq (m+k)^{-2},$$

which is exactly the inequality to prove if $z_m = 0$.

If $z_m = 1$, then the induction hypothesis for $m' = m-1, k' = k+1$ and the tuple $(z_1, \dots, z_{m-1})$ yields

$$(z_1+\dots+z_{m-1}+k+1)^{-1}-(m+k)^{-1}\cdot\sum_{i=1}^{m-1}\left(\frac{z_i^2}{1-z_i}+z_1+\dots+z_{m-1}+k+1\right)^{-1} \geq$$

$$\geq (m+k)^{-2},$$

which is exactly the inequality to prove if $z_m = 1$.

If the minimum is not on the boundary of the domain but at an inner point, then it must be a stationary point of the Lagrange function $L(z_1, \dots, z_n, \lambda) = f(z_1, \dots, z_n) - \lambda(z_1 + \dots + z_m - S)$.

As usual, the vanishing of the partial derivative $\frac{\partial}{\partial\lambda}L(z_1, \dots, z_n, \lambda)$ is equivalent to the constraint $z_1 + \dots + z_m = S$. Let $T := S + k$. Then the partial derivative $\frac{\partial}{\partial z_j}L(z_1, \dots, z_n, \lambda)$ equals to

$$-(z_1+\dots+z_m+k)^{-2}+(m+k)^{-1}\cdot\left(\sum_{i\neq j}\left(\frac{z_i^2}{1-z_i}+z_1+\dots+z_m+k\right)^{-2}+\right.$$

$$\left.+\left(\frac{z_j^2}{1-z_j}+z_1+\dots+z_m+k\right)^{-2}\cdot\left(\frac{2z_j-z_j^2}{(1-z_j)^2}+1\right)\right)-\lambda=$$

$$-T^{-2}+(m+k)^{-1}\cdot\left(\sum_{i\neq j}\left(\frac{z_i^2}{1-z_i}+T\right)^{-2}+\left(\frac{z_j^2}{1-z_j}+T\right)^{-2}\cdot\frac{1}{(1-z_j)^2}\right)-\lambda=$$

$$-\lambda-T^{-2}+(m+k)^{-1}\cdot\sum_{i=1}^{m}\left(\frac{z_i^2}{1-z_i}+T\right)^{-2}+$$

$$(m+k)^{-1}\cdot\left(\frac{z_j^2}{1-z_j}+T\right)^{-2}\left(\frac{1}{(1-z_j)^2}-1\right)=0,$$

thus $\left((1-z_j)^{-2}-1\right)\left(T+\frac{z_j^2}{1-z_j}\right)^{-2} = (m+k)(\lambda+T^{-2})-\sum_{i=1}^{m}\left(\frac{z_i^2}{1-z_i}+T\right)^{-2}$ is independent of j. By Lemma 3.1, all the z_j are equal to the same value z. Hence, the minimum is $(mz+k)^{-1}-(m+k)^{-1}m\cdot\left(\frac{z^2}{1-z}+mz+k\right)^{-1}$, which is at least $(m+k)^{-2}$ by Lemma 3.3. □

The following theorem is the first main result of the paper, providing an upper bound to the expected runtime of the gambler's ruin game on an arbitrary graph.

Theorem 3.4. *Let (V, E) be a graph with $|E| = m \in \mathbb{N}$ and let $\nu : V \to \mathbb{N}$ be a function representing the initial wealth of the vertices. In each round of a game, an edge is selected uniformly at random, and by a fair choice 1 unit wealth is transferred from an endpoint to the other. The process halts when a vertex has 0 wealth. Then the expected runtime is at most*

$$\frac{m^2}{\sum\limits_{uv \in E} (\nu(u)\nu(v))^{-1}}$$

.

Proof. Given a state by the function $\mu : V \to \mathbb{N}$, let the guesstimate of the expected runtime from this state be $m^2 \cdot \left(\sum\limits_{uv \in E} (\mu(u)\mu(v))^{-1} \right)^{-1}$. Note that the guesstimate is 0 for absorbing states. We show that this guesstimate satisfies the inequality that the difference of its value at the given state and the average value in all states we can reach in one step is at least 1. It is going to be convenient in the calculation that the formula returns 0 in the absorbing states of the Markov process: thus we can simply subtract from $m^2 \cdot \left(\sum\limits_{uv \in E} (\mu(u)\mu(v))^{-1} \right)^{-1}$ the average of the expressions we obtain by adding 1 and -1 to the endpoints of an edge in all the $2m$ possible ways.

Let us focus on a pair of such modified expressions corresponding to a given edge $uv \in E$. By introducing the notation $\mu(u) = a, \mu(v) = b$, $\sum\limits_{uw \in E} \mu(w)^{-1} = c^{-1}$, $\sum\limits_{vw \in E} \mu(w)^{-1} = d^{-1}$, and $\sum\limits_{st \in E, \{s,t\} \cap \{u,v\} = \emptyset} (\mu(s)\mu(t))^{-1}) = K$, we have $a, b \geq 1$, $c, d, K > 0$, and the sum of the two modified expressions is

$$m^2 \cdot \left(K + \frac{1}{(a+1)(b-1)} + \frac{1}{c(a+1)} + \frac{1}{d(b-1)} \right)^{-1} + \tag{1}$$

$$m^2 \cdot \left(K + \frac{1}{(a-1)(b+1)} + \frac{1}{c(a-1)} + \frac{1}{d(b+1)} \right)^{-1}. \tag{2}$$

The crucial estimation in the proof is to show that this sum is at most

$$2m^2 \cdot \left(K + \frac{1}{ab-1} + \frac{1}{ca} + \frac{1}{db}\right)^{-1}. \tag{3}$$

First note that if $a = b = 1$, then $(1) = (2) = (3) = 0$. If $a = 1$ and $b \geq 2$, then the claim simplifies to

$$\left(K + \frac{1}{2(b-1)} + \frac{1}{2c} + \frac{1}{d(b-1)}\right)^{-1} \leq 2\left(K + \frac{1}{b-1} + \frac{1}{c} + \frac{1}{db}\right)^{-1},$$

or equivalently,

$$2K + \frac{2}{2(b-1)} + \frac{2}{2c} + \frac{2}{d(b-1)} \geq K + \frac{1}{b-1} + \frac{1}{c} + \frac{1}{db},$$

which is trivial as $2K > K$, $\frac{2}{2(b-1)} = \frac{1}{b-1}$, $\frac{2}{2c} = \frac{1}{c}$ and $\frac{2}{d(b-1)} > \frac{1}{db}$. The case $a \geq 2$ and $b = 1$ is analogous, thus we may assume that $a, b \geq 2$. In particular,

$$P = \frac{1}{(a+1)(b-1)} + \frac{1}{c(a+1)} + \frac{1}{d(b-1)}$$

$$Q = \frac{1}{(a-1)(b+1)} + \frac{1}{c(a-1)} + \frac{1}{d(b+1)}$$

$$R = \frac{1}{ab-1} + \frac{1}{ca} + \frac{1}{db}$$

are all positive real numbers, and the estimation we need to show (after simplification by m^2) is $\frac{1}{K+P} + \frac{1}{K+Q} \leq \frac{2}{K+R}$. By multiplying the inequality with all three denominators, we obtain the equivalent inequality

$$2K^2 + (P+Q+2R)K + (P+Q)R \leq 2K^2 + 2(P+Q)K + 2PQ.$$

Hence, by comparing the coefficients of the powers of K, it is enough to show that $R \leq \frac{P+Q}{2}$ and $R \leq \frac{2PQ}{P+Q}$. As the harmonic mean is at most the arithmetic mean, we have $\frac{2PQ}{P+Q} \leq \frac{P+Q}{2}$, thus it suffices to show that $R \leq \frac{2PQ}{P+Q}$, i.e., $(P+Q)R \leq 2PQ$. We can cancel $c^2d^2(a^2-1)(b^2-1)$ from the common denominator, and then multiplying both sides by the denominator obtained yields the equivalent inequality that

$$2a^3bcd + 4a^2b^2cd + 2ab^3cd + 2ab^2c^2d + 2a^2bcd^2 - 2a^2cd - 4abcd - 2b^2cd - 2ac^2d - 2bcd^2$$

is non-negative. Pulling out the common factor $2cd$ leads to the quartic polynomial

$$a^3b + 2a^2b^2 + ab^3 + ab^2c + a^2bd - a^2 - 2ab - b^2 - ac - bd$$

that can be written as $(ab-1)(a+b)^2 + (b^2-1)ac + (a^2-1)bd$, hence it is non-negative. This concludes the argument showing that $(1) + (2) \leq (3)$.

By listing the edges and denoting the value $\mu(u)\mu(v)$ corresponding to the i-th edge $uv \in E$ by y_i, the inequality we showed provides the following lower estimation to the difference of the expression $m^2 \cdot \left(\sum_{uv\in E} (\mu(u)\mu(v))^{-1}\right)^{-1}$ and the average of this expression on the $2m$ states at distance one:

$$m^2 \cdot \Bigg(\left(y_1^{-1} + \cdots + y_m^{-1}\right)^{-1} - \\ - \frac{1}{m} \cdot \left(\sum_{i=1}^{m} y_1^{-1} + \cdots + y_{i-1}^{-1} + (y_i - 1)^{-1} + y_{i+1}^{-1} + \cdots + y_m^{-1}\right)\Bigg) = \\ m^2 \cdot \Bigg(\left(y_1^{-1} + \cdots + y_m^{-1}\right)^{-1} - \\ - \frac{1}{m} \cdot \left(\sum_{i=1}^{m} y_1^{-1} + \cdots + y_{i-1}^{-1} + \frac{y_i^{-1}}{1 - y_i^{-1}} + y_{i+1}^{-1} + \cdots + y_m^{-1}\right)\Bigg).$$

Putting $y_i^{-1} = z_i$, we have $z_i \in]0,1]$, and then

$$m^2 \cdot \Bigg((z_1 + \cdots + z_m)^{-1} - m^{-1} \cdot \left(\sum_{i=1}^{m} z_1 + \cdots + z_{i-1} + \frac{z_i}{1 - z_i} + z_{i+1} + \cdots + z_m\right)\Bigg) = \\ m^2 \cdot \Bigg((z_1 + \cdots + z_m)^{-1} - m^{-1} \cdot \left(\sum_{i=1}^{m} \frac{z_i^2}{1 - z_i} + z_1 + \cdots + z_m\right)\Bigg) \geq 1$$

according to Theorem 3.2 with the substitution $k = 0$. □

Corollary 3.5. *Under the assumptions of Theorem 3.4, if all vertices have the same initial wealth ν, then the expected runtime is at most $m\nu^2$.*

If the graph consists of one single edge, then the game reduces to the classical gambler's ruin problem, a.k.a. the drunkard's walk with two players u and v. It is well-known that the expected runtime of that process is $\nu(u)\nu(v)$, hence the formula in Theorem 3.4 is sharp in that special case.

The harmonic mean is in general much smaller than the geometric mean, which is smaller than the arithmetic mean. This is well-reflected in the fact that if one element in an m-tuple z_1 is fixed and the others $z_2, \dots, z_m$ tend to infinity, then the geometric and arithmetic means tend to infinity, while the harmonic mean tends to mz_1. Thus if every agent in a network has infinite wealth except for two agents u, v that are linked by an edge, the gambler's ruin game still has a finite expected runtime at most $m^2\nu(u)\nu(v)$ according to Theorem 3.4, something we could not conclude from estimations involving the geometric or arithmetic mean (which would in return be easier to verify).

4. Moments

It is well-known that there exists a degree k polynomial f with leading coefficient $k!$ such that the vector of k-th moments of the absorption time of an absorbing Markov chain with fundamental matrix N is of the form $f_k(N)\underline{c}_0$. This polynomial can be computed by finding the k-th derivative of the moment generating function. For example, $f_1(N) = N, f_2(N) = 2N^2 - N, f_3(N) = 6N^3 - 6N^2 + N$ and $f_4(N) = 24N^4 - 36N^3 + 14N^2 - N$. For the method and the precise computation of these polynomials, cf. [21, Theorem 3.2]; see also [22]. Thus our strategy is to provide estimations for $k!N^k\underline{c}_0$, as it is going to turn out that the smaller degree terms of $f_k(N)$ are negligible.

For all $k \geq 0$ let $\underline{c}_k \in \mathbb{R}^{\mathrm{Tran}}$ be the vector whose entry at a state with parameters (β, ϱ) is $(\beta\varrho)^k$. Let $\underline{x}_0 := \underline{c}_1$, and for all $k \geq 1$ let $\underline{x}_k = \frac{1}{(k+1)(2k+1)}\underline{c}_{k+1} + \frac{kn^2}{4k+2}\underline{x}_{k-1}$. Note that the x_k can be computed recursively from this definition.

Example 4.1.

- $\underline{x}_0 = \underline{c}_1$
- $\underline{x}_1 = \frac{1}{6}\underline{c}_2 + \frac{n^2}{6}\underline{x}_0 = \frac{1}{6}\underline{c}_2 + \frac{n^2}{6}\underline{c}_1$
- $\underline{x}_2 = \frac{1}{15}\underline{c}_3 + \frac{n^2}{5}\underline{x}_1 = \frac{1}{15}\underline{c}_3 + \frac{n^2}{30}\underline{c}_2 + \frac{n^4}{30}\underline{c}_1$
- $\underline{x}_3 = \frac{1}{28}\underline{c}_4 + \frac{3n^2}{14}\underline{x}_2 = \frac{1}{28}\underline{c}_4 + \frac{n^2}{70}\underline{c}_3 + \frac{n^4}{140}\underline{c}_2 + \frac{n^6}{140}\underline{c}_1$

For all $k \geq 0$ let $\underline{d}_k \in \mathbb{R}^{\mathrm{Tran}}$ be the vector whose entry at a state with parameters (β, ϱ) where spreading of the colors blue and red have probability p and q, respectively, is $(p-q)(k+1)(\beta\varrho - 1)^k(\beta - \varrho)$. As this expression has order of magnitude $O(n^{2k+1})$, we can copy the proof of [19, Lemma 4] to obtain the following estimation.

Lemma 4.2. *Let N be the fundamental matrix of the discordant oblivious, push, or pull protocol. Then $N\underline{d}_k = O(rn^{2k+1})$.*

Proof. By considering the same absorbing Markov chain as in the proof of [19, Lemma 4], we have that the expression $|(p-q)(k+1)(\beta\varrho - 1)^k(\beta - \varrho)|$ is zero on non-penultimate states (those before some runs are merged). Moreover, in penultimate states, the expression divided by the probability of absorption is at most Cn^{2k+1} with some absolute constant C (depending only on k). Thus during every phase of the process where the number of runs stagnate, the expected increment of the expression is at most Cn^{2k+1}, making the total sum during the process at most Crn^{2k+1}. □

Lemma 4.3. *Let N be the fundamental matrix of the discordant oblivious, push, or pull protocol. Then $N\underline{c}_k = \underline{x}_k + O(rn^{2k+1})$.*

Proof. Let the vector $\underline{\varepsilon}_k$ be defined by the equation $Q\underline{x}_k = \underline{x}_k - \underline{c}_k + \underline{\varepsilon}_k$. We use induction on k to show that $N\underline{\varepsilon}_k = O(rn^{2k+1})$, which is equivalent to the assertion. The initial case $k = 0$ follows from [19, Theorem 5], as $\underline{\varepsilon}_0 = \underline{c}_0 - (I-Q)\underline{x}_0 = \underline{c}_0 - (I-Q)\underline{c}_1$, making $N\underline{\varepsilon}_0 = N\underline{c}_0 - \underline{c}_1 = O(rn)$. We assume that the assertion holds for $k-1$ and show it for k.

The entry of $Q\underline{c}_{k+1}$ at a state with parameters (β, ϱ) where spreading of the colors blue and red have probability p and q, respectively, is

$$
\begin{aligned}
&p((\beta+1)(\varrho-1))^{k+1}+q((\beta-1)(\varrho+1))^{k+1}= \\
&\qquad p((\beta\varrho-1)-(\beta-\varrho))^{k+1}+q((\beta\varrho-1)+(\beta-\varrho))^{k+1}= \\
&\qquad (\beta\varrho-1)^{k+1}+\frac{(k+1)k}{2}(\beta\varrho-1)^{k-1}(\beta-\varrho)^2- \\
&\qquad -(p-q)(k+1)(\beta\varrho-1)^k(\beta-\varrho)+O(n^{2k-1})= \\
&\qquad (\beta\varrho)^{k+1}-(k+1)(\beta\varrho)^k+\frac{(k+1)k}{2}(\beta\varrho)^{k-1}(\beta-\varrho)^2- \\
&\qquad\qquad -(p-q)(k+1)(\beta\varrho-1)^k(\beta-\varrho)+O(n^{2k-1}).
\end{aligned}
$$

Because $(\beta-\varrho)^2=(\beta+\varrho)^2-4\beta\varrho=n^2-4\beta\varrho$, and since the last expression in the calculation is the entry of $-\underline{d}_k$ at the given state, we obtain

$$
\begin{aligned}
&Q\underline{c}_{k+1}=\underline{c}_{k+1}-(k+1)\underline{c}_k+\frac{(k+1)kn^2}{2}\underline{c}_{k-1}-2(k+1)k\underline{c}_k-\underline{d}_k+O(n^{2k-1})= \\
&\qquad \underline{c}_{k+1}-(k+1)(2k+1)\underline{c}_k+\frac{(k+1)kn^2}{2}\underline{c}_{k-1}+\underline{d}_k+O(n^{2k-1}).\ \text{Hence,}
\end{aligned}
$$

$$
\begin{aligned}
&Q\underline{x}_k=\frac{1}{(k+1)(2k+1)}Q\underline{c}_{k+1}+\frac{kn^2}{4k+2}Q\underline{x}_{k-1}= \\
&\quad \frac{1}{(k+1)(2k+1)}\left(\underline{c}_{k+1}-(k+1)(2k+1)\underline{c}_k+\frac{(k+1)kn^2}{2}\underline{c}_{k-1}-\underline{d}_k\right)+ \\
&\qquad +\frac{kn^2}{4k+2}(\underline{x}_{k-1}-\underline{c}_{k-1}+\underline{\varepsilon}_{k-1})+O(n^{2k-1})= \\
&\quad \frac{1}{(k+1)(2k+1)}\underline{c}_{k+1}-\underline{c}_k+\frac{kn^2}{4k+2}\underline{c}_{k-1}-\frac{1}{(k+1)(2k+1)}\underline{d}_k+ \\
&\qquad +\frac{kn^2}{4k+2}\underline{x}_{k-1}-\frac{kn^2}{4k+2}\underline{c}_{k-1}+\frac{kn^2}{4k+2}\underline{\varepsilon}_{k-1}+O(n^{2k-1})= \\
&\qquad\qquad \underline{x}_k-\underline{c}_k-\frac{1}{(k+1)(2k+1)}\underline{d}_k+\frac{kn^2}{4k+2}\underline{\varepsilon}_{k-1}+O(n^{2k-1}).
\end{aligned}
$$

This makes $\underline{\varepsilon}_k=-\frac{1}{(k+1)(2k+1)}\underline{d}_k+\frac{kn^2}{4k+2}\underline{\varepsilon}_{k-1}+O(n^{2k-1})$, and consequently, by the induction hypothesis and Lemma 4.2 we have

$$
N\underline{\varepsilon}_k=-\frac{1}{(k+1)(2k+1)}N\underline{d}_k+\frac{kn^2}{4k+2}N\underline{\varepsilon}_{k-1}+O(n^{2k+1})=O(rn^{2k+1}).
$$

□

Lemma 4.3 makes it possible to compute the leading terms of $k!N^k\underline{c}_0$ recursively. If $\beta\varrho$ is quadratic in n, as it happens to be in the extremal case when $\beta = n/2 + O(1)$ and $\varrho = n/2 + O(1)$, then these terms have degree n^{2k}. Similarly, $N^i\underline{c}_0 = O(n^{2i})$, thus the rest of the polynomial f_k (such that $f_k(N)\underline{c}_0$ is the k-th moment) is $O(n^{2k-2})$, hence negligible.

Example 4.4.

- $N\underline{c}_0 = \underline{x}_0 + O(rn) = \underline{c}_1 + O(rn)$
- $2N^2\underline{c}_0 = 2N\underline{x}_0 + O(rn^3) = 2N\underline{c}_1 + O(rn^3) = 2\underline{x}_1 + O(rn^3) = \frac{1}{3}\underline{c}_2 + \frac{n^2}{3}\underline{c}_1 + O(rn^3)$
- $6N^3\underline{c}_0 = 3N(\frac{1}{3}\underline{c}_2 + \frac{n^2}{3}\underline{c}_1) + O(rn^5) = \underline{x}_2 + n^2\underline{x}_1 + O(rn^5) = \frac{1}{15}\underline{c}_3 + \frac{n^2}{5}\underline{c}_2 + \frac{n^4}{5}\underline{c}_1 + O(rn^5)$
- $24N^4\underline{c}_0 = 4N(\frac{1}{15}\underline{c}_3 + \frac{n^2}{5}\underline{c}_2 + \frac{n^4}{5}\underline{c}_1) + O(rn^7) = \frac{4}{15}\underline{x}_3 + \frac{4n^2}{5}\underline{x}_2 + \frac{4n^4}{5}\underline{x}_1 + O(rn^7) = \frac{1}{105}\underline{c}_4 + \frac{2n^2}{35}\underline{c}_3 + \frac{17n^4}{105}\underline{c}_2 + \frac{17n^6}{105}\underline{c}_1 + O(rn^7)$

In general, let $k!N^k\underline{c}_0 = \sum_{i=0}^{k} a_{k,i}n^{2k-2i}\underline{c}_i + O(rn^{2k-1})$, where the coefficients $a_{k,i}$ can be computed recursively as illustrated in Example 4.4, and the constant factor in $O(rn^{2k-1})$ only depends on k. Then the k-th moment is also $f_k(N)\underline{c}_0 = \sum_{i=0}^{k} a_{k,i}n^{2k-2i}\underline{c}_i + O(rn^{2k-1})$. Note that $\beta\varrho \leq n^2/4$ and $r \leq n$ implies the upper bound $f_k(N)\underline{c}_0 \leq C(k)n^{2k}$ with a universal constant $C(k) > 0$. Let $m_k := \sum_{i=0}^{k} a_{k,i}4^{-i}$; e.g., $m_1 = \frac{1}{4}, m_2 = \frac{5}{48}, m_3 = \frac{61}{960}$ and $m_4 = \frac{277}{5376}$ according to Example 4.4. Let $M_k(\beta, \varrho)$ denote the k-th moment of the standard drunkard walk starting from the state (β, ϱ), i.e., the fair gambler's ruin problem on the graph with two vertices and an edge, and the vertices having initial wealth β and ϱ. Note that this process coincides with the oblivious push protocol with a starting state $r = 1$ and number of blue and red vertices β and ϱ, respectively. Thus according to Lemma 4.3, the k-th moment of the drunkard walk is the corresponding entry of $f_k(N)\underline{c}_0 = k!N^k\underline{c}_0 + O(n^{2k-1}) = \sum_{i=0}^{k} a_{k,i}n^{2k-2i}\underline{c}_i + O(n^{2k-1})$, that is, $M_k(\beta, \varrho) = \sum_{i=0}^{k} a_{k,i}n^{2k-2i}(\beta\varrho)^i + O(n^{2k-1})$. Hence, up to an error of order of magnitude $O(n^{2k-1})$, the $M_k(\beta, \varrho)$ are easy to determine recursively, cf. Examples 4.1 and 4.4. We are ready to summarize our results in the main theorem of the current section.

Theorem 4.5. *The k-th moment of the runtime of the discordant oblivious, push, and pull protocols from an initial state with parameters (β, ϱ) and r runs is $M_k(\beta, \varrho) + O(rn^{2k-1})$. In particular, if $r = o(n)$, then these k-th moments are asymptotically equal to the k-th moment of the drunkard walk. Moreover, the maximum of the k-th moments (ranging through all possible initial states) is $m_k n^{2k} + O(n^{2k-1/2} \log_2 n)$, which is asymptotically equal to the maximal k-th moment of the drunkard walk.*

Proof. The series of equations were already obtained as a corollary of Lemma 4.3. We prove the final assertion of the theorem by cutting the process into two parts. The first part consists of the first $80n^{3/2} \log_2 n$ moves (unless the process halts earlier). Starting from any initial state, the expected number of moves to reach a state with $r \leq n^{1/2}$ is at most $40n^2/n^{1/2} = 40n^{3/2}$ by [19, Proposition 3]. Thus by Markov's inequality, the probability that after $80n^{3/2}$ moves we have $r > n^{1/2}$ is at most $1/2$. The probability that this happens $\log_2 n$ times is at most $(1/2)^{\log_2 n} = 1/n$. That is, if A is the event that in the initial state of the second part we have $r > n^{1/2}$, then $\mathbb{P}(A) \leq 1/n$.

The total time T to absorption is at most $T_2 + 80n^{3/2} \log_2 n$, where T_2 is the number of steps in the second part. If in the initial state of the second part we have $r > n^{1/2}$, we can use the universal upper bound $E(T_2) \leq C(k)n^{2k}$. If in the initial state of the second part we have $r \leq n^{1/2}$, then the error term $O(rn^{2k-1})$ in the expression estimating $f_k(N)_{\underline{c}_0}$ is at most $O(n^{2k-1/2})$. Hence, by the law of total expectation we have

$$E(T^k) \le E((T_2+80n^{3/2}\log_2 n)^k) \le \frac{1}{n}C(k)n^{2k} + E((T_2+80n^{3/2}\log_2 n)^k|A) =$$

$$\sum_{i=0}^{k} E\left(\binom{k}{i} T_2^{k-i}(80n^{3/2}\log_2 n)^i|A\right) + C(k)n^{2k-1} =$$

$$E(T_2^k|A) + (80kn^{3/2}\log_2 n)E(T_2^{k-1}|A)+$$

$$+\sum_{i=2}^{k}\binom{k}{i} 80n^{3i/2}(\log_2 n)^i E(T_2^{k-i}|A) + C(k)n^{2k-1} =$$

$$E(T_2^k|A) + O(n^{2k-1/2}\log_2 n) \le m_k n^{2k} + O(n^{2k-1/2}\log_2 n),$$

where the last inequality follows from $\beta\varrho \le n^2/4$ yielding the entry of the vector $f_k(N)\underline{c}_0 = \sum_{i=0}^{k} a_{k,i} n^{2k-2i}\underline{c}_i + O(rn^{2k-1})$ at any state with $r \le n^{1/2}$ at most $\sum_{i=0}^{k} a_{k,i} n^{2k-2i} n^{2i}/4^i + O(n^{2k-1/2}) = \left(\sum_{i=0}^{k} a_{k,i}4^{-i}\right) n^{2k} + O(n^{2k-1/2})$.

The upper bound is attained when $\beta = n/2+O(1)$ and $\varrho = n/2+O(1)$. □

As $M_k(\beta,\varrho) = \sum_{i=0}^{k} a_{k,i} n^{2k-2i}\underline{c}_i + O(n^{2k-1})$, the maximum of the k-th moment of the drunkard walk is $m_k n^{2k} + O(n^{2k-1/2})$. Hence, Theorem 4.5 states that up to an error of order of magnitude $O(n^{2k-1/2}\log_2 n)$ the maximum of the k-th moment of all three discordant protocols discussed coincide with that of the drunkard walk, and the same holds for any initial state with parameters (β,ϱ) as long as r is not too close to n.

Corollary 4.6. *The variance of the runtime of the discordant oblivious, push, and pull protocols from an initial state with parameters* (β,ϱ) *and* r *runs is* $\frac{n^2}{3}\beta\varrho - \frac{2}{3}(\beta\varrho)^2 + O(rn^3)$. *The maximum variance (when ranging through all possible initial states) is* $\frac{1}{24}n^4 + O(n^{7/2}\log_2 n)$.

Proof. According to Theorem 4.5 and Example 4.4, we have $\mathrm{Var}(T) = E(T^2) - E(T)^2$, where $E(T^2)$ is the corresponding entry in $2N^2\underline{c}_0 + O(rn^3) = \frac{1}{3}\underline{c}_2 + \frac{n^2}{3}\underline{c}_1 + O(rn^3)$ and $E(T)$ is in $N\underline{c}_0 + O(rn) = \underline{c}_1 + O(rn)$. Hence, $E(T)^2$ is the corresponding entry in $\underline{c}_2 + O(rn^3)$, and consequently, $\mathrm{Var}(T)$ is the corresponding entry in $\frac{n^2}{3}\underline{c}_1 - \frac{2}{3}\underline{c}_2 + O(rn^3)$.

The maximum of the function $\frac{n^2}{3}\beta\varrho - \frac{2}{3}(\beta\varrho)^2 = \frac{1}{6}\cdot(2\beta\varrho)\cdot(n^2 - 2\beta\varrho)$ under the conditions that $0 \le \beta, \varrho$ and $\beta + \varrho = n$ is attained at $\beta = \varrho = n/2$, since $2\beta\varrho \le n^2/2$ where equality holds if and only if $\beta = \varrho = n/2$. Putting $\beta = n/2 + O(1)$ and $\varrho = n/2 + O(1)$ we have $\frac{n^2}{3}\beta\varrho - \frac{2}{3}(\beta\varrho)^2 = n^4/24 + O(n^3)$. This yields the estimate $n^4/24 + O(rn^3)$ to the maximum variance by the first assertion of the corollary. The desired upper bound $n^4/24 + O(n^{7/2}\log_2 n)$ can be obtained from this estimation by using the same technique of cutting the process into two parts as in the proof of Theorem 4.5. □

In [19, Theorem 7], it was shown that the probability for the blue consensus to win in the discordant push and pull protocols is $\beta/n + O(r/n)$. That is, if $r = o(n)$, then the probability is asymptotically β/n, making these games nearly fair. In contrast, it was also shown in [19, Theorem 10] that for n large enough, there is always an initial state such that the estimate β/n to the winning probability has an error at least 0.005. The proof hinges on the fact that if the initial state has alternating blue-red runs of lengths 1 and 2, respectively, then there exist numbers $M \in \mathbb{N}, 0 < a, b < 1$ such that after M steps the process reaches a state where the proportion of red vertices is below $1/3 - a$ with probability at least b. Then the law of total probability implies that the estimation β/n cannot be within a certain error range for all these states and the initial one at the same time. The same argument can be copied to show that without the assumption $r = o(1)$, the estimate $M_k(\beta, \varrho)$ to $E(T^k)$, where T is the runtime of the discordant push or pull protocol, cannot always be within a certain range of relative error.

5. An open problem

As it was mentioned in Section 3, the estimate provided in Theorem 3.4 is sharp for the two-vertex graph with one edge. However, if the graph is a 3-cycle with vertices u, v, w, then it is easy to see (e.g., by using Lemma 2.1) that the exact expected runtime of the gambler's ruin game is

$$3/((\nu(u)\nu(v))^{-1} + (\nu(v)\nu(w))^{-1} + (\nu(w)\nu(u))^{-1}),$$

rather than $9/((\nu(u)\nu(v))^{-1}+(\nu(v)\nu(w))^{-1}+(\nu(w)\nu(u))^{-1})$, which is the upper bound provided by Theorem 3.4. The proof of Theorem 3.2 suggests that the supremum of the function $\underline{x}-Q\underline{x}$ is obtained when all vertices has equal wealth tending to infinity. This limit of $\underline{x}-Q\underline{x}$ when the graph is an n-cycle is $3/n^2$, suggesting that the upper bound provided by Theorem 3.4 might be possible to improve for cycle graphs by a factor 3.

Question 5.1. *Let T be the runtime of the gambler's ruin game on an n-cycle. Is it true that $E(T) \leq \frac{n^2}{3\cdot \sum_{uv\in E}(\nu(u)\nu(v))^{-1}}$?*

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