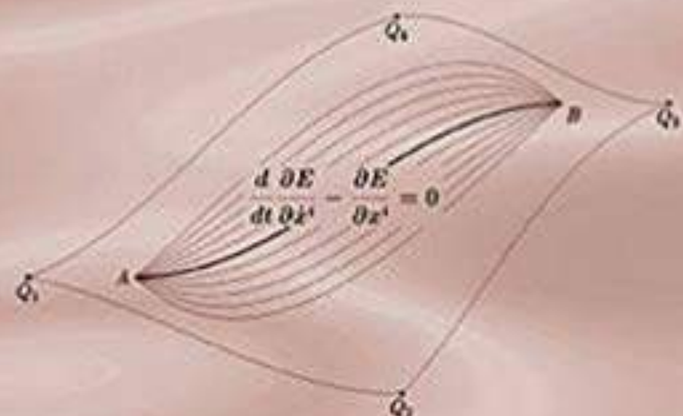
Variational Principles for Second-order Differential Equations

Application of the Spencer Theory to Characterize Variational Sprays



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VARIATIONAL PRINCIPLES FOR SECOND-ORDER DIFFERENTIAL EQUATIONS
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Preface

The objective of this book is to familiarize the reader with the basic tools of over-determined partial differential equations, namely the Spencer version of the Cartan-Kähler theorem, via the study of an old problem of differential geometry almost open until now, the characterization of second order ordinary equations, also called sprays, coming from a variational problem. Note that this problem is not the same as the characterisation of Euler-Lagrange operators, which is now well understood thanks to the variational bicomplex. Using the terminology of I.M. Anderson, we will study the *variational integrating factors problem*, since for any non singular matrix a_β^α the differential operators P_α and $a_\beta^\alpha P_\alpha$ define the same differential equation, we ask if for a quasi-linear differential operator P_α there exists a non-singular matrix a_β^α and a Lagrangian E such that

$$a_\beta^\alpha P_\alpha = \frac{d}{dt}\frac{\partial E}{\partial \dot{x}^\beta} - \frac{\partial E}{\partial x^\beta}$$

The solution to this problem requires the study of the integrability of a partial differential system called the *Euler-Lagrange system*.

The most significant contribution to this problem is a famous paper of J. Douglas where, using the Riquier's theory, the variational differential equation on 2-dimensional manifolds are classified. The generalization of its results in the higher dimensional case is a very hard problem because the Euler-Lagrange system is still extremely over-determined.

Using techniques from the Spencer theory of over-determined systems such as prolongation, Spencer cohomology, involutivity, 2-acyclicity, and the natural framework of the tangent bundle such as algebraic differential characterization of connections and derivations by Frölicher-Nijenhuis brackets, we can present the obstructions which appear in the 2-dimensional cases in an intrinsic and natural way. When the dimension of the manifold is n, we apply this technique to the study of a special class of sprays, which we call *isotropic*. Roughly speaking, a variational isotropic spray corresponds to the geodesic flow of a Finsler resp. Riemann manifold with a constant sectional curvature in the homogeneous resp. quadratic case. However it is more general because we will also consider the case of non-homogeneous second order equations. The main theorems provide complete working illustration of the techniques employed, such as prolongation, computation of torsion, involutivity, Spencer cohomology, 2-acyclicity etc.

We briefly describe the contents of each chapter.

Chapter I offers an elementary introduction to the formal integrability theory of partial differential systems. No proofs are given, but all the notions are illustrated with simple examples, so that the formalism of the theory, which usually disheartens the reader, can be easily absorbed.

Chapter II and Chapter III are devoted to the presentation of the connection theory based on the Frölicher-Nijenhuis graded Lie algebra. It provides an adapted formalism for our problem, which allows us to present all the differential relations and obstructions easily and intrinsically.

In Chapter IV we study variational sprays. We establish the necessary relations which they satisfy and introduce a natural graded Lie algebra associated to the spray which plays an important role in our study.

The application of formal integrability theory of partial differential equations to the the inverse problem begins in Chapter V. We study the problem in the general case, i.e. without any restriction on the dimension or on the curvature. We give the first obstructions so that a spray is variational. This chapter provides useful examples for the reader interested in the application of the technique. The complete classification of the variational sprays seems to be impossible in the general situation, because a lot of obstructions, determined by the elements of the graded Lie algebra introduced in Chapter IV, appear. However, it is instructive to see how they arise and this Chapter offers quite a clear idea of the methods employed. In order to obtain complete results we restrict ourselves to particular cases. That is what we do in the following Chapters.

In Chapter VI we treat the 2-dimensional case. Of course, this chapter, like the original paper of J. Douglas, is quite complicated, because many cases and sub-cases have to be considered and many obstructions arise. Nevertheless, it is the very chapter that the theory of the over-determined system is fully applied and all the possible situations appear: involutivity, 2-acyclicity, non-zero higher order cohomology groups, restriction of the system etc. As we will see, from this study a particular class of sprays emerges naturally which we call *typical*, because it also contains the quadratic and homogeneous second order equations, which are the more frequent in differential geometry. Although they require a special treatment, the necessary computations are easier.

In the last Chapter we return to the n-dimensional case, but we limit ourselves to the study of isotropic sprays. When the non-holonomy is weak, we obtain the necessary and sufficient conditions for the spray to be variational. Some of the results of this Chapter was published in *Annales de l'Institut Fourier* recently.

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Contents

Chapter 1

An Introduction to Formal Integrability Theory of Partial Differential Systems

In this chapter we give an elementary introduction to the Spencer-Goldschmidt version of the Cartan-Kähler Theorem. Our goal is to study the variational problem related to the integrability of the Euler-Lagrange differential operator. Since it is a second order linear partial differential operator, in this chapter we look at the theory of integrability of linear differential operators. In section 1.7 the non linear case is mentioned.

1.1 Introduction

The fundamental theorem about partial differential equations (PDE) is the well known Cauchy-Kowaleska Theorem:

Theorem 1.1 . (CAUCHY-KOWALESKA) *Consider the system*

$$\frac{\partial z^\mu}{\partial x^n} = \Phi^\mu\left(x, z, \frac{\partial z}{\partial x^i}\right) \tag{1.1}$$

where $i = 1, \ldots, n-1$, $\mu = 1, \ldots, m$, $x = (x_1, \ldots, x_n)$, $z = (z_1, \ldots, z_m)$ *and the functions* $\Phi^\mu(x, z, \frac{\partial z}{\partial x^i})$ *are analytic on a neighborhood of* $(0, A^\mu, A_i^\mu)$. *Given analytic functions* $f^\mu(x^1, \ldots, x^{n-1})$ *on a neighborhood of* 0 *such that*

$$f^\mu(0) = A^\mu, \qquad \frac{\partial f^\mu}{\partial x^i}(0) = A_i^\mu,$$

there exists a unique analytic solution $z = F(x^1, \ldots, x^n)$ *of the system*

(1.1) on a neighborhood of $0 \in \mathbb{R}^n$ *such that:*

$$z^\mu(x^1, \ldots x^{n-1}, 0) = f^\mu(x^1, \ldots, x^{n-1}).$$

Note that the system of PDE in the Cauchy-Kowaleska Theorem has two particularities:

(1) The number m of (first order) equations is equal to the number of unknown functions z^μ.
(2) One of the independent functions, x^n, plays a particular role.

The idea of the proof is as follows:

1. One begins to look at the formal integrability, i.e. one looks for formal power series in a neighborhood of $0 \in \mathbb{R}^n$, which satisfy the system and the initial condition. Taking into account the particular form of the system (the partial derivatives $\frac{\partial z^\mu}{\partial x^n}$ are expressed in terms of the other components of the 1-jet of z^μ) it is not difficult to prove that formal solutions exist.

2. By the technique of the "majorant series" one proves that the formal series converge.

The Cartan-Kähler Theorem generalizes the Cauchy-Kowaleska Theorem, in the sense that the number of equations is not necessarily equal to the number of unknown functions, and that none of the variables play a particular role.

As we have said, the particular form of the Cauchy-Kowaleska system implies that formal solutions always exist. This is not the case for a general system: obstructions can arise and be explicitly computed. However, if formal integrability of an analytic system is ensured, the formal solutions converge, as in the Cauchy-Kowaleska case.

The situation is similar to the one for systems of linear algebraic equations and the analogy is not only formal. If a linear system is in the Cramer form (i.e. the number of equations is the same as the number of unknown variables and the matrix is regular) then a solution exists and it is unique. For a general system, obstructions appear: they can be obtained by computing the "characteristic determinants" (which amounts to giving all the linear relations between the equations). When these compatibility conditions are satisfied, the system can be put in the Cramer form with some free parameters. Then a parametrized family of solutions can be obtained, and the number of the parameters depends on the rank of the system. For the systems of PDE the situation is similar. The obstructions arising from the Spencer cohomology correspond to the fact that the characteristic determinants have to vanish and the number of parameters (which in this case are arbitrary functions) can be explicitly computed.

Let us consider the system of partial differential equations

$$F^{\nu}(x^{\alpha}, z^{\mu}, z^{\mu}_{\alpha}, \dots, z^{\mu}_{\alpha_1 \dots \alpha_k}) = 0 \tag{1.2}$$

where $\nu = 1 \dots p$ and

$$z^{\mu}_{\alpha} = \frac{\partial z^{\mu}}{\partial x^{\alpha}}, \qquad \dots, \qquad z^{\mu}_{\alpha_1 \dots \alpha_k} = \frac{\partial^k z^{\mu}}{\partial x^{\mu}_1 \dots \partial x^{\mu}_k}.$$

Definition 1.1 The system 1.2 is *locally integrable* in a neighborhood of x_0, if for any real numbers A^{μ}, A^{μ}_{α}, ..., $A^{\mu}_{\alpha_1 \dots \alpha_k} \in \mathbf{R}$ verifying

$$F^{\nu}(x_0, A^{\mu}, \dots A^{\mu}_{\alpha_1 \dots \alpha_k}) = 0, \tag{1.3}$$

there exists a neighborhood U of x_0 and a solution $z^{\mu}(x)$ defined on U such that

$$\begin{cases} z^{\mu}(x_0) = A^{\mu}, \\ z^{\mu}_{\alpha}(x_0) = A^{\mu}_{\alpha}, \\ \quad \vdots \\ z^{\mu}_{\alpha_1 \dots \alpha_k}(x_0) = A^{\mu}_{\alpha_1 \dots \alpha_k}. \end{cases}$$

The set of the $(A^{\mu}, A^{\mu}_{\alpha}, \dots, A^{\mu}_{\alpha_1 \dots \alpha_k})$ satisfying (1.3) is called a *kth-order formal solution* (or *initial data* at x_0). The set of all the kth order formal solutions at x_0 is noted R_{k,x_0}.

In other words, the PDE (1.2) is (locally) integrable in a neighborhood of x_0 if

> *for every $F_0 \in R_{k,x_0}$ there exists a neighborhood U of x_0 and $f \in C^{\infty}(U, \mathbf{R}^p)$ verifying the differential equation (1.2), such that $(j_k f)_{x_0} = F_0$.*

To study the integrability Taylor series can be used. At first one looks for a formal solution $z = (z^1, \dots, z^m)$ i.e. a formal series satisfying the equation

$$z = \sum_{\alpha} \frac{A_{\alpha}}{\alpha!}(x - x_0)^{\alpha}$$

(where α is a multi-index and $A_\alpha = (D_\alpha z)(x_0)$). Putting z into the equation, we can compute A_α by solving algebraic systems. Then we can look at the convergence of the formal solution.

Example 1.1 Let us consider the equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

With obvious notations it can be written:

$$z_{11} + z_{22} = 0.$$

The coefficients of the formal series verifying the equation must satisfy:

$$A_{11} + A_{22} = 0$$

Taking for example $A_{11} = 1, A_{22} = -1$ and choosing $A_1 = A_2 = A_{12} = 0$, we have a 2^{nd}-order solution $(0, 0, 1, 0, -1)$.

If we want to compute the other coefficients of the formal expansion, we need to derive the equation and study the the *first prolongation* of the system

$$\begin{cases} z_{11} & + & z_{22} & = 0 \\ z_{111} & + & z_{122} & = 0 \\ z_{211} & + & z_{222} & = 0. \end{cases}$$

The numbers A_i, A_{ij}, A_{ijk} must satisfy the system:

$$\begin{cases} A_{11} & + & A_{22} & = 0 \\ A_{111} & + & A_{122} & = 0 \\ A_{211} & + & A_{222} & = 0. \end{cases}$$

If we take the A_i and A_{ij} as above, we can build a 3^{rd}-order solution which extends the 2nd order solution already found. For example

$$A_1 = 0, \; A_2 = 0, \; A_{11} = 1, \; A_{12} = 0, \; A_{22} = -1$$

and

$$A_{111} = 0, \; A_{112} = 0, \; A_{122} = 0, \; A_{222} = 0.$$

When we carry out this operation a second time, we obtain a system of five equations. If this system is consistent then we find the 2nd order solutions which are lifted to 4^{th}-order solutions, etc.

Formal integrability at x_0 means that any kth order formal solution at x_0 can be lifted into an infinite order solution.

As is shown by the above example, in order to prove that the kth order solutions can be lifted in infinite order solutions (i.e. there exists a formal solution), we need to study the consistency of an algebraic system containing an infinite number of unknowns and equations. Roughly speaking the Cartan-Kähler Theorem says that if the system is "involutive" and "regular" (these notions will be introduced in the next sections) one only needs to study the first prolongation:

> *Consider the kth order system*
>
> $$F^\nu(x, z^\alpha, \ldots, z^\alpha_{\alpha_1 \ldots \alpha_k}) = 0, \qquad \nu = 1, \ldots, p$$
>
> *which is supposed to be "regular". If the system is "involutive", and every k^{th}-order solution can be lifted in a $(k+1)^{th}$ order solution, then the system is formally integrable.*

1.2 Notations and definitions

Let M be an n-dimensional manifold. We shall denote by (x^μ) local coordinates on M. Where there is no possibility of confusion TM and T^*M will be noted as T and T^*. Moreover $\Lambda^p T^*$ and $S^p T^*$ will designate the vector bundles of the skew-symmetric and symmetric forms.

Let E be a fibred bundle over the manifold M with the projection π. One denotes by $Sec\,E$ the sheaf of the sections of E over M. Two sections of E determine the same kth order jet if in one, and hence in every, local coordinate system their Taylor series coincide up to order k. The class determined by the section $s \in Sec\,E$ at the point $x_0 \in M$ is denoted by $j_k(s)_{x_0}$, and the set of all k-jets is denoted by $J_k(E)$. With the projection $\pi_{k,0}$ defined by $\pi_{k,0}(j_k(s)_{x_0}) = x_0$, $J_k(E)$ is a fibred manifold over M which is called the bundle of k-jets of sections of E. If $l > k$, one defines the projection $\pi_{l,k}$ as follows: $\pi_{l,k}(j_l(s)_{x_0}) = j_k(s)_{x_0}$, and $J_l(E)$ is also a fibred manifold over $J_k(E)$, in the case where E is a vector bundle over M.

Let $(U, x^1, \ldots, x^n)$ be a local coordinate system on M such that E is trivialisable over $\pi^{-1}(U)$, and let $(x^1, \ldots, x^n, z^1, \ldots, z^m)$ be local coordinates on $\pi^{-1}(U)$. A standard local coordinate system $(x^i, z^\mu, z^\mu_\alpha)$ of $J_k(E)$ on $\pi_k^{-1}(U)$ is defined for a section s of E by

$$z^\mu(j_k(s)_{x_0}) = z^\mu(s(x_0)),$$
$$z^\mu_\alpha(j_k(s)_{x_0}) = D^\alpha z^\mu(s(x))(x_0),$$

where $i = 1, \ldots, n$, $j = 1, \ldots, m$, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ are multi-indices satisfying $1 \leq \alpha_1 + \ldots + \alpha_n \leq k$.

To simplify the notation we denote π_{k-1} for $\pi_{k,k-1}$:

$$\begin{array}{ccc} J_k E & \xrightarrow{\pi_{k-1}} & J_{k-1} E \\ (x^\alpha, z^\mu, \ldots, z^\mu_{\alpha_1 \ldots \alpha_k}) & & (x^\alpha, z^\mu, \ldots, z^\mu_{\alpha_1 \ldots \alpha_{k-1}}) \end{array}$$

It is easy to see that :*

$$\mathrm{Ker}\, \pi_{k-1} \simeq S^k T^* \otimes E.$$

Then we have the exact sequence.

$$0 \longrightarrow S^k T^* \otimes E \xrightarrow{\ \varepsilon\ } J_k E \xrightarrow{\pi_{k-1}} J_{k-1} E \longrightarrow 0,$$

where ε is defined at $x \in M$ in the following way: if $f_1, \ldots, f_k \in C^\infty(M)$ are functions vanishing in x_0 and $s \in Sec(E)$, then:

$$\varepsilon(df_1 \odot \ldots \odot df_k \otimes s)_{x_0} = j_k(f_1 \cdots f_k s)_{x_0},$$

where $\odot$ denotes the symmetric product.

Definition 1.2 Let E and F be two vector bundles over the same manifold M. A partial differential operator (PDO) is a map

$$P : Sec(E) \longrightarrow Sec(F).$$

*Using local coordinates, if $\xi \in \mathrm{Ker}\, \pi_{k-1}$, then $\xi = (x^\alpha, 0, \ldots, 0, z^\mu_{\alpha_1 \ldots \alpha_k})$ and the $z^\mu_{\alpha_1 \ldots \alpha_k}$ are identified with the components of a multi-linear symmetric map $\underbrace{T \times \cdots \times T}_{k \text{ times}} \to E$.

P is a 0th *order* operator if $P(fs) = fP(s)$ for every $f \in C^\infty(M)$ and $s \in Sec(E)$. The order is k if the map

$$\begin{array}{ccc} Sec(E) & \longrightarrow & Sec(F) \\ s & \longrightarrow & P(fs) - fP(s) \end{array}$$

is a $k-1$ order operator for any $f \in C^\infty(M)$.

It is easy to see that the order of P is k if and only if $P(s)$ can be expressed in terms of the k-jet of s; then P can be identified with a map

$$\mu_0(P) : J_k E \longrightarrow F.$$

P is called a *linear* if $\mu_0(P)$ is a morphism of vector bundles on M.

Example 1.2 *Exterior derivative*
The operator $d : Sec(T^*) \longrightarrow Sec(\Lambda^2 T^*)$ can be identified with the morphism:

$$\begin{array}{cccc} \mu_0(d) : & J_1 T^* & \longrightarrow & \Lambda^2 T^* \\ & (x^a, \omega_a, \omega_{a,b}) & \to & (x^a, \omega_{a,b} - \omega_{b,a}). \end{array}$$

Example 1.3 *Linear connection.*
Let ∇ be a linear connection on a vector bundle $F \longrightarrow M$ characterized by the differential operator:

$$\begin{array}{ccc} \nabla : Sec(F) & \longrightarrow & Sec(T^* \otimes F) \\ s & \longmapsto & \nabla s \end{array}$$

where

$$\begin{array}{ccc} \nabla s : Sec(TM) & \longrightarrow & Sec(F) \\ X & \longmapsto & \nabla_X s \end{array}$$

Then ∇ can be identified with the morphism:

$$\begin{array}{cccc} \mu_0(\nabla) : & J_1 F & \longrightarrow & T^* \otimes F \\ & (x^a, z^\mu, z^\mu_a) & \longmapsto & (x^a, z^\mu_a - \Gamma^\mu_{a\nu}(x) z^\nu) \end{array}$$

Definition 1.3 A *partial differential equation* R_k of order k on E is a fibred sub-manifold of $J_k(E) \xrightarrow{\pi_{k,0}} M$. A solution of R_k is a section s of E such that $j_k(s)$ is a section of R_k.

As we explained in the Introduction we need to derive the system to find the integrability conditions. This is the notion of prolongation.

Definition 1.4 • Let $P : Sec\,E \to Sec\,F$ be a kth order PDO. The map

$$p_l(P) : J_{k+l}E \longrightarrow J_lF$$
$$j_{k+l}(s) \longmapsto j_l(Ps)$$

is called the l^{th}*-order prolongation of* P.

It is easy to see that the lth prolongation of a kth order operator depends only on the $(k+l)$-jet of $s \in Sec\,E$.

Definition 1.5 A kth order jet $(j_k s)_x \in J_k(E)$ is called kth order solution of P in x, if $P(s)_x = 0$. More generally $(j_{k+l}s)_x \in J_{k+l}(E)$ is called $(k+l)$th order solution of P in x, if $p_l(P)(s)_x = 0$, $l \geq 0$. Let us set.

$$R_{k+l,x}(P) = \{\ (j_{k+l}s)_x \in J_{k+l}(E)_x \mid s \in Sec(E) \ \text{ and } \ p_l(P)(s)_x = 0\ \}.$$

$R_{k+l,x}$ is called the *space of* $(k+l)$*th order solutions* of the operator P at x.

From now on we will suppose that the differential operator P is such that $R_k(P)$ is regular, that is $R_k(P)$ is a fibred sub-manifold of $J_kE \longrightarrow M$. It can be proved that it is the case if $p_0(P)$ has a locally constant rank (cf. [BCG[3]] p. 396). If the PDO is written locally in the form

$$F^\nu(x^a, z^\mu, z^\mu_a, \ldots, z^\mu_{a_1,\ldots,a_k}) = 0, \qquad \nu = 1,\ldots,p$$

this amounts to the mapping F having locally constant rank.

Definition 1.6 The partial differential equation corresponding to a kth order partial differential operator P is the fibred manifold $R_k(P)$. A *solution* of the operator P on an open set $U \subset M$ is a section $s \in Sec(E)$ defined on U, such that $Ps = 0$, or equivalently:

$$p_0(P)(j_k f)_x = 0, \quad \forall x \in U.$$

Let

$$\pi_{k+l-1} : R_{k+l} \longrightarrow R_{k+l-1}$$

be the restriction of the projection $\pi_{k+l,-1} : J_{k+l}E \to J_{k+l-1}E$ to R_{k+l}. The surjectivity of the $\bar{\pi}_{k+l-1}$ for every $l \geq 1$ means that every k^{th}-order solution can be lifted to a ∞-order solution. Now we can formulate the following definition:

Definition 1.7 A PDO is called *formally integrable* at x_0 if

(1) R_{k+l} is a vector bundle for all $l \geq 0$,
(2) $\bar{\pi}_{k+l-1,x_0} : R_{k+l,x_0} \to R_{k+l-1,x_0}$ for every $l \geq 1$ is onto.

In the analytical context, formal integrability implies the existence of solutions for all the initial data:

Theorem 1.2 *(cf. [BCG³], p.397) – Let P be a regular analytical PDO. Suppose that P is formally integrable at x_0. Then for every $F_0 \in R_{k,x_0}$, there exists an analytical section s of E defined on a neighborhood U of x_0, such that $Pf = 0$, and $(j_k f)(x_0) = F_0$.*

The convergence of the power series was first established by Ehrenpreis, Guillemin and Sternberg in 1965 [EGS] and later by Sweeney [Swe] proving the so-called Poincaré δ-estimate formulated by Spencer [Spe]. In 1972, Malgrange gave direct proof of the theorem with the method of "majorants" [Ma]

1.3 Involutivity

The notion of involutivity has been introduced by E. Cartan in his theory of exterior differential systems [Ca]. It can be explained as follows. Let us consider the system

$$\begin{cases} \dfrac{\partial f}{\partial x} &= P(x,y), \\ \dfrac{\partial f}{\partial y} &= Q(x,y) \end{cases}$$

If we add to this system the obstruction arising from the first prolongation ($\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$), then it is easy to see that the system

$$\begin{cases} \dfrac{\partial f}{\partial x} &= P(x,y) \\ \dfrac{\partial f}{\partial y} &= Q(x,y) \\ \dfrac{\partial P}{\partial y} &- \dfrac{\partial Q}{\partial x} = 0 \end{cases}$$

is locally integrable. In other words, all the obstructions are contained in the first prolongation. Cartan stated that for a general PDO all the obstructions are contained in the system obtained after a *finite number* of prolongations. This theorem was later proved by Kuranishi [Ku] and Quillen [Qui].

In the Cartan theory, the involutivity is checked by the so-called *Cartan test*. It consists of computing the dimensions of a flag related to the prolongation of the system. This amounts to producing a certain basis of the tangent space, called a *quasi-regular bases*, satisfying some conditions. In 1963 Serre expressed the involutivity in terms of cohomological algebra [Se]. At present the involutivity is too strong a condition : the obstructions to the formal integrability belong to some cohomological groups of a complex called the "Spencer complex" (the involutivity is equivalent to the vanishing of all the cohomological groups of the Spencer complex). Finally Quillen proved that the cohomology of the Spencer complex vanishes from a certain order (i.e. there exists a prolongation of the system which is involutive).

Nota - From now on, the PDO will be considered as linear (the non-linear case will be considered at the end of this chapter).

Let $P : Sec\, E \longrightarrow Sec\, F$ be a linear differential operator and $p_0(P) : J_k E \longrightarrow F$ the corresponding morphism on the jet bundle.

Definition 1.8 The map $\sigma_k(P) : S^k T^* \otimes E \longrightarrow F$ defined by $\sigma_k(P) =$

$p_k(P) \circ \varepsilon$ is called the *symbol* of P:

$$\begin{array}{ccccc} S^kT^* \otimes E & \xrightarrow{\varepsilon} & J_kE & \xrightarrow{\pi_{k-1}} & J_{k-1}E \\ & \searrow^{\sigma_k(P)} & \downarrow{\scriptstyle p_k(P)} & & \\ & & F & & \end{array}$$

Where there is no possibility of confusion, we will simply denote it by σ_k. This amounts to restricting the PDO to its maximal order part.

The lth *prolongation* $\sigma_{k+l}(P)$ of the symbol is defined by the following diagram:

$$\begin{array}{ccc} S^{k+l}T^* \otimes E & \xrightarrow{\sigma_{k+l}} & S^lT^* \otimes F \\ \downarrow{\scriptstyle i} & \nearrow_{id \otimes \sigma_k} & \\ S^lT^* \otimes S^kT^* \otimes E & & \end{array}$$

It is easy to see that σ_{k+l} is naturally identified with the symbol of the lth order prolongation of P. In particular

$$\sigma_{k+1} : S^{k+1}T^* \otimes E \longrightarrow T^* \otimes F$$

is defined by

$$i_X\sigma_{k+1}(t) = \sigma_k(i_X t)$$

for $X \in T$ and $t \in S^{k+1}T^*$.

Let P be a kth order linear differential operator. We put

$$g_{k+l}(P) := \operatorname{Ker} \sigma_{k+l}(P), \qquad l \geq 0.$$

Example 1.4 Let us consider the exterior derivative :

$$d : Sec(T^*) \longrightarrow Sec(\Lambda^2T^*)$$

or, in terms of jets $p_1(d) : J_1T^* \longrightarrow \Lambda^2T^*$. We have the diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & \begin{matrix} T^* \otimes T^* \\ (x, A_{ij}) \end{matrix} & \xrightarrow{\varepsilon} & \begin{matrix} J_1T^* \\ (x, f, A_{ij}) \end{matrix} & \xrightarrow{\pi_0} & T^* & \longrightarrow 0 \\ & & \searrow^{\sigma_1(d)} & \downarrow{\scriptstyle p_1(d)} & & & \\ & & & \begin{matrix} \Lambda^2T^* \\ (x, A_{ij} - A_{ji}) \end{matrix} & & & \end{array}$$

The symbol is the map

$$\begin{aligned}\sigma_1(d) : T^* \otimes T^* &\longrightarrow \Lambda^2 T^* \\ A &\longmapsto \sigma_1 A\end{aligned}$$

defined by

$$[\sigma_1(d)A](X,Y) = A(X,Y) - A(Y,X),$$

$X, Y \in T$. Taking into account that $(i_X\sigma_2)(B) = \sigma_1(i_X B)$, the prolongation of the symbol is the map

$$\begin{aligned}\sigma_2(d) : S^2T^* \otimes T^* &\longrightarrow T^* \otimes \Lambda^2 T^* \\ B &\longmapsto \sigma_2 B\end{aligned}$$

given by

$$[\sigma_2(d)B](X,Y,Z) = B(X,Y,Z) - B(X,Z,Y).$$

Example 1.5 Let us consider the covariant derivative

$$\begin{aligned}p_{fd}(\nabla) : \quad J_1F &\longrightarrow T^* \otimes F \\ (x^\alpha, z^\mu, z^\mu_\alpha) &\longmapsto (x^\alpha, z^\mu_\alpha + \Gamma^\mu_{\alpha\lambda}(x)z^\lambda)\end{aligned}$$

The symbol is the map

$$\begin{aligned}\sigma_1(\nabla) : T^* \otimes F &\longrightarrow T^* \otimes F \\ A^\mu_\alpha &\longmapsto A^\mu_\alpha\end{aligned}$$

Then $\sigma_1(\nabla)(A)(X,\xi) = A(X,\xi)$ i.e. $\sigma_1(\nabla) = \mathrm{id}_{T^*\otimes F}$. Now

$$\sigma_2(\nabla) : S^2T^* \otimes F \longrightarrow T^* \otimes T^* \otimes F$$

is defined by $\sigma_2(\nabla)(B)(X,Y,\xi) = B(X,Y,\xi)$

Definition 1.9 Let $\{e_1, ..., e_n\}$ be an ordered basis of T_xM. We can write, for $j = 1, ..., n-1$,

$$g_k(F)_{x,e_1 ... e_j} = \{A \in g_k(F)_x \mid i_{e_1}A = ... = i_{e_j}A = 0\}.$$

A basis is called "*quasi-regular*" if:

$$\dim g_{k+1}(P)_x = \dim g_k(P)_x + \sum_{j=1}^{n-1} \dim g_k(P)_{x,e_1\ldots e_j}.$$

A symbol is called *involutive* at x if there exists a quasi-regular basis at $x \in M$.[1]

REMARKS

(1) For any basis we have

$$\dim(g_{k+1})_x \leq \dim(g_k)_x + \sum_{j=1}^{n-1} \dim(g_k)_{x,e_1\ldots e_j}. \tag{1.4}$$

(2) The *characters* defined by E. Cartan are related to the dimensions of the $(g_k)_{x,e_1\ldots e_j}$ by

$$\begin{aligned} s_1 &= \dim g_k - \dim (g_k)_{e_1} \\ s_2 &= \dim (g_k)_{e_1} - \dim (g_k)_{e_1 e_2} \\ &\vdots \\ s_j &= \dim (g_k)_{e_1\ldots e_{j-1}} - \dim (g_k)_{e_1\ldots e_j} \\ &\vdots \end{aligned}$$

With these notations the condition for a quasi-regular basis can be written:

$$\dim g_{k+1} = s_1 + 2s_2 + \cdots + ns_n.$$

This is the so-called *Cartan test*.

(3) The following property holds:
Let P be a formally integrable involutive PDO, $\{e_i\}$ a quasi-regular basis and s_j the Cartan characters. Let ℓ be the largest integer such that $s_\ell \neq 0$. Then the general solution depends on s_ℓ arbitrary functions of ℓ variables.

[1] There is a slight problem of language here. In the works of Cartan, and more generally in the theory of exterior differential systems, "involutivity" means more than the existence of a quasi-regular basis and it amounts to "integrability" (cf. [BCG[3]], p.107, 140). Here we are following the terminology of Goldschmidt (cf. [BCG[3]], p. 409).

Example 1.6 Let us consider the exterior derivative. Using the computation of the Example 1.4 one finds

$$g_1(d) = \{A \in T^* \otimes T^* \mid A(X,Y) = A(Y,X)\} = S^2T^*$$

and therefore

$$\dim g_1(d) = \frac{n(n+1)}{2}.$$

Taking into account the expression of $\sigma_2(d)$ one has

$$g_2(d) = \{B \in S^2T^* \otimes T^* \mid B(X,Y,Z) = B(X,Z,Y)\} = S^3T^*$$

thus

$$\dim g_2(d) = \frac{n(n+1)(n+2)}{6}$$

Let us consider an arbitrary basis $\{e_1, \ldots, e_n\}$ of TM. We have:

$$\begin{aligned}
(g_1)_{e_1} &= \{A \in S^2T^* \mid i_{e_1}A = 0\} && \simeq \mathbb{R}_2[x_2, \ldots, x_n],\\
(g_1)_{e_1 e_2} &= \{A \in S^2T^* \mid i_{e_1}A = i_{e_2}A = 0\} && \simeq \mathbb{R}_2[x_3, \ldots, x_n],\\
&\ \vdots\\
(g_1)_{e_1 \ldots e_{n-1}} &= \{A \in S^2T^* \mid i_{e_1}A = \ldots = i_{e_{n-1}} = 0\} && \simeq \mathbb{R}_2[x_n].
\end{aligned}$$

and therefore

$$\begin{aligned}
\dim (g_1)_{e_1} &= \tfrac{n(n-1)}{2}\\
\dim (g_1)_{e_1 e_2} &= \tfrac{(n-1)(n-2)}{2}\\
&\ \vdots\\
\dim (g_1)_{e_1 \ldots e_{n-1}} &= \tfrac{2 \cdot 1}{2}
\end{aligned}$$

Now

$$\sum_{k=1}^{n} (k+1)k = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k = \frac{n(n-1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

then the basis is quasi-regular and the partial differential operator d is involutive.

REMARK - In general, the quasi-regular basis arises naturally and depends on the geometrical objects given in the problem. For example, if a Riemannian metric is given, it is natural to look at the orthonormal basis; if an endomorphism occurs in the problem, it is natural to look at the Jordan basis, etc. In the case of the exterior derivative there is no particular basis so we have chosen one arbitrarily (in other words: either all the bases are quasi-regular, or none of them are).

In practice, in order to prove involutivity, one starts from a "natural basis" and, taking into account the inequality 1.4 which says that for a quasi-regular basis the dimensions of the $(g_k)_{e_1 \dots e_i}$ are minimal, one tries to minimize these dimensions by changing the basis. We will give a simple example here; more complicated ones will appear in the study of the inverse problem of the calculus of variations.

Example 1.7 Let h be a (1-1) tensor field on M such that $h^2 = h$, and consider the PDO

$$d_h : C^\infty(M) \to \Lambda^1(M)$$

defined by

$$d_h f(X) := df(hX)$$

for $X \in \mathcal{X}(M)$ or, in terms of jets,

$$\begin{aligned} p_0(d_h) : J_1\mathbb{R} &\longrightarrow T^* \\ (x, z, z_a) &\longmapsto (x, h^a_b z_a) \end{aligned}$$

Thus $\sigma_1(d_h) : T^* \longrightarrow T^*$ is defined by

$$[\sigma_1(d_h)\omega](X) = \omega(hX),$$

while the first prolongation of the symbol $\sigma_2(d_h) : S^2T^* \longrightarrow T^* \otimes T^*$ by

$$(\sigma_2 B)(X, Y) = B(X, hY).$$

Therefore

$$g_1(d_h) = \{\, \omega \in T^* \mid \omega|_{\operatorname{Im} h} = 0 \}$$

and

$$g_2(d_h) = \{ B \in S^2T^* \mid B(X, hY) = 0 \ \ \forall X, Y \in T \}$$

Then $\dim g_1 = n - r$, where $r = \operatorname{rank} h$. Since $T_x M = \operatorname{Ker} h \oplus \operatorname{Im} h$, we can take a basis $B = (e_1, \ldots, e_r, \varepsilon_1, \ldots, \varepsilon_{n-r})$ with the $e_i \in \operatorname{Im} h$ and $\varepsilon_\alpha \in \operatorname{Ker} h$ (this is a "natural basis"). We have,

$$B \in g_2 \iff \begin{cases} B(e_i, e_j) &= 0, \quad i, j = 1, \ldots, r, \\ B(\varepsilon_\alpha, e_i) &= 0, \quad \alpha = 1, \ldots, n-r, \quad i = 1, \ldots, r \end{cases}$$

Then B is determined by the $\frac{(n-r)(n-r+1)}{2}$ components $B(\varepsilon_\alpha, \varepsilon_\beta)$ and we get

$$\dim g_2 = \frac{(n-r)(n-r+1)}{2}.$$

On the other hand we have

$$(g_1)_{e_1} = \left\{\omega \,\middle|\, \omega\,|_{\operatorname{Im} h} = 0\,,\, \omega(e_1) = 0\right\} = g_1$$

since $e_1 \in \operatorname{Im} h$. In the same way

$$(g_1)_{e_1 \ldots e_j} = g_1, \quad \text{for} \quad j = 1, \ldots, r$$

and

$$\dim (g_1)_{e_1 \ldots e_j} = n - r, \qquad \text{for } j = 1, \ldots, r.$$

Now

$$\dim (g_1)_{e_1 \ldots e_r, \varepsilon_1} = \dim \left\{\omega \in T^* \mid \omega_{|\operatorname{Im} h} = 0,\ \omega(\varepsilon_1) = 0\right\} = n - r - 1$$

and more generally

$$\dim (g_1)_{e_1 \ldots e_r, \varepsilon_1 \ldots \varepsilon_\alpha} = n - r - \alpha.$$

So we have

$$\begin{aligned} \dim g_1 + \sum_{j=1}^{r} \dim (g_1)_{e_1 \ldots e_j} + \sum_{\alpha=1}^{n-r} \dim (g_1)_{e_1 \ldots e_r \varepsilon_1 \ldots \varepsilon_\alpha} &= \\ = (r+1)(n-r) + \frac{(n-r)(n-r-1)}{2} &= \frac{(n-r)(n+r+1)}{2} > \dim g_2 \end{aligned}$$

which shows that the basis B is not quasi-regular. The reason is that the dimensions of the $(g_1)_{e_1 \ldots e_j}$ do not decrease sufficiently quickly (at present do not decrease at all).

Let us consider now the basis $\mathcal{B}' = \{\varepsilon_1, \ldots, \varepsilon_{n-r}, e_1, \ldots, e_r\}$. We have

$$\dim(g_1)_{\varepsilon_1} = n - r - 1,$$

$$\dim(g_1)_{\varepsilon_1, \ldots, \varepsilon_a} = n - r - a,$$

$$\dim(g_1)_{\varepsilon_1, \ldots, \varepsilon_{n-r}, e_1, \ldots, e_r} = 0,$$

and an easy computation shows that $\mathcal{B}'$ is quasi-regular

1.4 First compatibility conditions for a PDO

In this section we will explain how to find integrability conditions or how to check the surjectivity of $\overline{\pi}_1$. Obstructions to the integrability, also called *torsion*, arise at this stage.

Let $P : Sec(E) \longrightarrow Sec(F)$ be a kth order linear PDO. We have the following diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & S^{k+1}T^* \otimes E & \xrightarrow{\sigma_{k+1}} & T^* \otimes F & \overset{\tau}{\dashrightarrow} & K \longrightarrow 0 \\
 & & \downarrow \varepsilon & & \nabla \uparrow \downarrow \varepsilon & & \\
R_{k+1} & \longrightarrow & J_{k+1}E & \overset{p_1(P)}{\dashrightarrow} & J_1F & & \\
\downarrow \overline{\pi}_k & & \uparrow \downarrow \pi_k & & \downarrow \pi_0 & & \\
R_k & \dashrightarrow & J_kE & \xrightarrow{p_k(P)} & F & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array} \tag{1.5}$$

where K denotes the cokernel of the morphism σ_{k+1}, $K := \dfrac{T^* \otimes F}{\operatorname{Im} \sigma_{k+1}}$ and ∇ is an arbitrary linear connection on F. A classical result in homological algebra gives the following

Proposition 1.1 *There exists a morphism $\varphi : R_k \longrightarrow K$ such that the sequence*

$$R_{k+1} \xrightarrow{\pi_k} R_k \xrightarrow{\varphi} K$$

is exact. [‡] *In particular, the morphism $\overline{\pi}_k$ is onto if and only if $\varphi = 0$*

Let us construct the map φ. Consider $z \in R_k$, take $z_1 \in J_{k+1}E$ such that $\pi_k z_1 = z$ and compute $p_1(P)z_1$. We have

$$\pi_0 p_1(P)z_1 = p_0(P)\pi_k z_1 = p_0(P)z = 0.$$

Since the sequence $T^* \otimes F \to J_1F \to F$ is exact, this proves that $p_1(P)z_1 \in \operatorname{Im}\varepsilon$. Consequently there exists $A \in T^* \otimes F$ such that $\varepsilon A = p_1(P)z_1$ (A is uniquely determined because ε is injective).

Let us put $\varphi(z) = \tau A$. We must prove that $\varphi(z)$ does not depend on the choice of z_1. Let z_1' also be an element of R_{k+1} such that $\pi_k z_1' = z$. Of course we find that $\pi_k(z_1' - z_1) = 0$, so $z_1' - z_1 \in \operatorname{Im}\epsilon$. Let A' be an element of $T^* \otimes F$ such that $\epsilon A' = p_1(P)z_1'$. We must check that $\tau A = \tau A'$, i.e. $A - A' \in \operatorname{Ker}\tau = \operatorname{Im}\sigma_{k+1}$. We have

$$\varepsilon(A - A') = p_1(P)z_1' - p_1(P)z_1 = p_1(P)(z_1' - z_1).$$

hence

$$\varepsilon(A - A') \in p_1(P)\epsilon(S^{k+1}T^* \otimes E) = \varepsilon\left(\sigma_{k+1}(S^{k+1}T^* \otimes E)\right) = \varepsilon(\operatorname{Im}\sigma_{k+1})$$

But ε is onto, so $A - A' \in \operatorname{Im}\sigma_{k+1}$. This proves that φ is well-defined.

Now let us check that

$$\varphi = 0 \Longleftrightarrow \overline{\pi}_k \text{ is onto.}$$

We just need to prove that $\operatorname{Ker}\varphi = \operatorname{Im}\overline{\pi}_k$. We have $\varphi(z) = \tau A$, with A such that $\varepsilon A = p_1(P)z_1$ and z_1 such that $\pi_k z_1 = z$. Now

$$\tau A = 0 \Longleftrightarrow A \in \operatorname{Im}\sigma_{k+1} \Longleftrightarrow \exists B \in S^{k+1}T^* \otimes E \text{ such that } A = \sigma_{k+1}B$$

Let us consider ϵB; we have

$$p_1(P)\epsilon B = \varepsilon\sigma_{k+1}B = \varepsilon A = p_1(P)z_1$$

[‡] This morphism is represented in the diagram by dashed arrows

and so

$$p_1(P)(z_1 - \varepsilon B) = 0 \quad \text{that is} \quad z_1 - \varepsilon B \in R_{k+1}.$$

Define $\bar{z} := z_1 - \varepsilon B$. We have

$$\bar{\pi}_k(\bar{z}) = z - \pi_k \varepsilon B = z$$

which proves that

$$\varphi(z) = 0 \iff \exists\, \bar{z} \in R_{k+1} \quad \text{such that} \quad \bar{\pi}_k(\bar{z}) = z,$$

i.e. $\mathrm{Ker}\,\varphi = \mathrm{Im}\,\bar{\pi}_k$. □

Thus the surjectivity of $\bar{\pi}_k$ can be checked by showing that $\varphi = 0$. We will now explain how this can be carried out.

First notice that if ∇ is a connection on the vector bundle F, we have $\sigma_1(\nabla) = Id$ (cf. Example 1.5), more precisely $p_0(\nabla) \circ \varepsilon = Id_{T^*\otimes F}$. So $p_0(\nabla)$ is a splitting of $\varepsilon : T^* \otimes F \longrightarrow J_1 F$ and therefore it can be used in the diagram to get φ. To construct φ, start from $s_k \in R_k$, consider a section $s \in Sec(E)$ such that $s_k = j_k(s)_x$ i.e. such that $P(s)_x = 0$; lift it to $j_{k+1}(s)_x$ and compute $p_1(P)j_{k+1}(s)_x = j_1(P(s))_x$. By mapping $p_0(\nabla)$ we obtain $p_0(\nabla)j_1(P(s))_x$ which amounts to $(\nabla P(s))_x$ in terms of sections. So we obtain the following statement:

> *Consider $s_k \in R_k$ and $s \in Sec(E)$ such that $s_k = (j_k s)_x$ (i.e. such that $P(s)_x = 0$). We have :*
>
> $$\varphi(s_k) = \left(\tau \nabla P(s)\right)_x$$
>
> *where ∇ is an arbitrary linear connection on the vector bundle $F \xrightarrow{\pi} M$. Moreover $\bar{\pi}_k$ is onto if and only if:*
>
> $$P(s)_x = 0 \implies \left(\tau \nabla P(s)\right)_x = 0.$$

Example 1.8 Let us consider the exterior derivative

$$d : Sec(T^*) \longrightarrow Sec(\Lambda^2 T^*)$$

We have the following diagram:

$$\begin{array}{ccccccc}
 & & S^2T^*\otimes T^* & \xrightarrow{\sigma_2} & T^*\otimes\Lambda^2T^* & \xrightarrow{\tau} & \dfrac{T^*\otimes\Lambda^2T^*}{\operatorname{Im}\sigma_2} \longrightarrow 0 \\
 & & \downarrow \epsilon & & \downarrow \epsilon & & \\
R_2 & \longrightarrow & J_2T^* & \xrightarrow{p_1(d)} & J_1\Lambda^2T^* & & \\
\downarrow \bar{\pi}_1 & & \downarrow \pi_1 & & \downarrow \pi_0 & & \\
R_1 & \longrightarrow & J_1T^* & \xrightarrow{p_0(d)} & \Lambda^2T^* & &
\end{array}$$

Let us show that $\bar{\pi}_1$ is onto. At first let us compute $K := \operatorname{Coker}\sigma_2$.

$$\begin{aligned}
\dim K &= \dim(T^*\otimes\Lambda^2T^*) - \operatorname{rank}\sigma_2 = \frac{n^2(n-1)}{2} - [\dim S^2T^*\otimes T^* - \dim \operatorname{Ker}\sigma_2] \\
&= \frac{n(n+1)(n+2)}{6} - n^2 = \frac{n(n-1)(n-2)}{6},
\end{aligned}$$

hence $K \simeq \Lambda^3T^*$. To compute the first compatibility conditions we need a morphism τ, such that the following sequence is exact:

$$S^2T^*\otimes T^* \xrightarrow{\sigma_2} T^*\otimes\Lambda^2T^* \xrightarrow{\tau} \Lambda^3T^* \longrightarrow 0 \tag{1.6}$$

Let consider the map: $\tau : T^*\otimes\Lambda^2T^* \longrightarrow \Lambda^3T^*$ defined by

$$\tau(C)(X,Y,Z) := C(X,Y,Z) + C(Y,Z,X) + C(Z,X,Y). \tag{1.7}$$

It is easy to check that the sequence (1.6) is exact. In fact, we have that $\tau\circ\sigma_2$ is zero, so $\operatorname{Im}\sigma_2 \subseteq \operatorname{Ker}\tau$. On the other hand, if $\Omega \in \Lambda^3T^*$, we can take $C = \frac{1}{3}\Omega$ and get $\tau(C) = \Omega$, so τ is onto.

Now we can compute the first compatibility condition of the operator d. Let us consider a linear connection ∇ on M. We also denote by ∇ its action on Λ^2T^*. It is well-known that d can be obtained from a linear connection ∇ with the help of the antisymmetrisation. More precisely, for every 2-form Ω we have:

$$d\Omega(X,Y,Z) = \sum_{cycl(X,Y,Z)} \left[(\nabla\Omega)(X,Y,Z) + \Omega(T(X,Y),Z)\right],$$

where T is the torsion of ∇. Let $\omega \in Sec(T^*)$ now be a 1-form, so that $(j_1\omega)_x \in J_1T^*$ is a first order solution of the operator d. This means that $d\omega$ vanishes at the point x. We obtain that

$$\varphi(\omega)_x = (\tau\nabla(d\omega))_x = (d^2\omega)_x = 0$$

and therefore $\varphi \equiv 0$. Therefore every first order solution of the operator d can be lifted in a second order solution.

Remark. Note that τ maps $T^* \otimes K$ on the Cokernel of σ_{k-1}. Then τ is just the map which gives *the linear relations relating the equations of the first prolongation.*

Example 1.9 Let us consider the exterior derivative and take the dimension of the manifold as 3. The equations of the symbol are (with obvious notations)

$$\begin{cases} A_{12} - A_{21} = 0, \\ A_{23} - A_{32} = 0, \\ A_{31} - A_{13} = 0, \end{cases}$$

and then the equations of the first prolongation are

$$\begin{cases} C_{112}: & B_{112} - B_{121} = 0, \\ C_{113}: & B_{113} - B_{131} = 0, \\ C_{212}: & B_{212} - B_{221} = 0, \\ C_{123}: & B_{123} - B_{132} = 0, \\ C_{223}: & B_{223} - B_{232} = 0, \\ C_{323}: & B_{323} - B_{332} = 0, \\ C_{131} & B_{131} - B_{113} = 0, \\ C_{231} & B_{231} - B_{213} = 0, \\ C_{331} & B_{331} - B_{313} = 0. \end{cases}$$

This is a system of 9 equations with 12 variables, the $B_{ij,k}$. One has $\sigma_1 : S^2T^* \otimes T^* \to T^* \otimes \Lambda^2 T^*$ and $\dim(T^* \otimes \Lambda^2 T^*) = 9$, $\dim(S^2 T^* \otimes T^*) = 12$. It is easy to check that these equations are related by one and only one linear relation:

$$C_{123} + C_{231} + C_{312} = 0,$$

which is just the above relation defining τ in (1.7). Thus the rank of the system (that is the rank of σ_1) is 8, or equivalently $\dim K = 1$.

This example enables us to understand how K and τ can be found :

One writes the system which defines $\operatorname{Im} \sigma_{k+1}$. *Then:*

- τ *is defined by the relations between the equations of the system;*
- $\dim K$ *is the number of these relations.*

Example 1.10 Let us consider a manifold M endowed with a linear connection ∇. We want to study the first lift of the first order formal solutions of the operator ∇ defined by

$$\begin{aligned} \nabla : \mathfrak{X}(M) &\longrightarrow Sec(T^* \otimes T) \\ X \quad &\longmapsto \quad \nabla X \end{aligned} \tag{1.8}$$

where

$$\begin{aligned} \nabla X : \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(M) \\ Y \quad &\longmapsto \nabla_Y X \end{aligned}$$

We have the following diagram:

$$\begin{array}{ccccccc}
 & & S^2T^*\otimes T & \xrightarrow{\sigma_2(\nabla)} & T^*\otimes(T^*\otimes T) & \longrightarrow & \mathrm{Coker}\,\sigma_2(\nabla) \longrightarrow 0 \\
 & & \downarrow \epsilon & & \downarrow \epsilon & & \\
R_2 & \longrightarrow & J_2T & \xrightarrow{p_1(\nabla)} & J_1(T^*\otimes T) & & \\
\downarrow \pi_1 & & \downarrow \pi_1 & & \downarrow \pi_0 & & \\
R_1 & \longrightarrow & J_1T & \xrightarrow{p_0(\nabla)} & (T^*\otimes T) & &
\end{array}$$

As we have already seen (cf. Example 1.5) the symbol σ_1 is the identity map of $T^* \otimes T$ and the first prolongation is the identity map of $S^2T^* \otimes T$. It follows that:

$$g_1(\nabla) = 0 \quad \text{and} \quad g_2(\nabla) = 0.$$

On the other hand,

$$\dim \mathrm{Coker}\,\sigma_2(\nabla) = \dim(T^* \otimes T^* \otimes T) - \mathrm{rank}\,\sigma_2 = n^3 - \frac{n^2(n+1)}{2} = \frac{n^2(n-1)}{2},$$

therefore $\mathrm{Coker}\,\sigma_2(\nabla)$ is isomorphic to $\Lambda^2T^* \otimes T$. Let us consider the morphism $\tau : T^* \otimes T^* \otimes T \longrightarrow \Lambda^2T^* \otimes T$ defined by

$$\tau(C)(X, Y) = C(X, Y) - C(Y, X).$$

For every $B \in S^2T^* \otimes T$, we have

$$\tau \circ \sigma_2(B)(X, Y) = \sigma_2(B)(X, Y) - \sigma_2(B)(Y, X) = B(X, Y) - B(Y, X) = 0.$$

Now τ is clearly onto, so the sequence

$$S^2T^* \otimes T \xrightarrow{\sigma_2} T^* \otimes (T^* \otimes T) \xrightarrow{\tau} \Lambda^2T^* \otimes T \longrightarrow 0$$

is exact. With the help of the morphism τ, one can compute the first compatibility condition corresponding to the operator (1.8). Let $X \in \mathfrak{X}(M)$ be a vector field which determines at x_0 a first order solution of our operator, i.e. $j_1(X) \in J_1(T)$ satisfies the equation $(\nabla X)_{x_0} = 0$, and let us compute $\varphi(j_2X)_{x_0} = \tau\left(\nabla(\nabla X)\right)_{x_0}$. If T denotes the torsion tensor of ∇, then at x_0 one has

$$\begin{aligned}
\tau(\nabla\nabla X)(Y, Z) &= (\nabla\nabla X)(Y, Z) - (\nabla\nabla X)(Z, Y) = \nabla_Y(\nabla X)(Z) - \nabla_Z(\nabla X)(Y) \\
&= \nabla_Y\nabla_Z X - \nabla_{\nabla_Y Z} X - \nabla_Z\nabla_Y X + \nabla_{\nabla_Z Y} X = R(Y, Z)X + \nabla_{T(Y,Z)}X \\
&- R(Y, Z)X
\end{aligned}$$

because $(\nabla_{T(Y,Z)}X)_{x_0} = 0$

Of course, the compatibility condition $\varphi = 0$ is not satisfied in the generic case. But in the case where the curvature of the connection ∇ vanishes, the compatibility condition is identically satisfied. In this situation every first order solution can be lifted in a second order solution: $\bar{\pi}_1$ is onto.

1.5 The Cartan-Kähler theorem

The Theorem 1.2 shows that the formal integrability guarantees the existence of analytical solutions for a *regular* analytical PDO. The notion of involutivity, which we shall study in the next section, allows us to check the formal integrability in quite a simple way : *if the PDO is "involutive" then the surjectivity of $\bar{\pi}_k$ implies the formal integrability* (no need to check that *all* the maps $\bar{\pi}_{k-1}$ are onto).

Theorem 1.3 - (CARTAN-KÄHLER) *Let P be a linear partial differential operator. Suppose that $g_{k+1}(P)$ is a vector bundle on R_k i.e. P is is regular. If*

a) $\bar{\pi}_k : R_{k+1} \longrightarrow R_k$ *is onto,*

b) *the symbol is involutive,*

then P is formally integrable.

Example 1.11 Let consider the example of the exterior derivative. The Example 1.4 shows that the symbol of the operator d is involutive and the Example 1.8 shows that $\pi_1 : R_2 \to R_1$ is onto. Using the above Theorem we find that the operator $d : Sec(T^*) \longrightarrow Sec(\Lambda^2 T^*)$ is *formally integrable*.

Therefore, in the analytic case[5], we have the following property: for every $x_0 \in M$ and $\omega_0 \in T^*_{x_0}M$ there exists a neighborhood U of x_0 and $\omega \in Sec(T^*U)$ such that

$$d\omega = 0 \quad \text{and} \quad \omega_{x_0} = \omega_0.$$

Example 1.12 Let us consider the differential operator defined in the Example 1.5:

$$\begin{aligned} \nabla : \mathfrak{X}(M) &\longrightarrow Sec(T^* \otimes T) \\ X &\longmapsto \nabla X \end{aligned}$$

As we remarked in the $g_1(\nabla) = 0$ and $g_2(\nabla) = 0$ (see page 19). So all the bases are quasi-regular and therefore the PDO is involutive. On the other hand, as we showed in the Example 1.8 in the case where the curvature of the connection ∇ vanishes, the compatibility condition is identically satisfied and therefore every first order solution can be lifted in a second order solution: π_1 is onto. Therefore we proved that if the curvature of ∇ vanishes, then the operator ∇ is formally integrable.

In the analytical case this means that if the curvature of the connection vanishes, then every $X_0 \in T_{x_0}$ can be lifted into a parallel vector field on a neighborhood of x_0 [6].

1.6 Spencer cohomology

It can be shown that the condition of the existence of a quasi-regular basis can be replaced by a weaker condition. The obstructions to the higher order successive lift of the kth order solution are contained in some of the cohomological groups of a certain complex called *Spencer complex*.

[5] It can be checked much more easily in the differentiable case using the Frobenius Theorem. Our proof is then nothing more than an example which clarifies the method.

[6] This result can be shown more easily in the differentiable case by using the Frobenius Theorem.

Let

$$\delta : S^{k+1}T^* \longrightarrow S^kT^* \otimes T^*$$

be the natural injection. δ can be lifted to a morphism of vector bundles:

$$S^{k+1}T^* \otimes \Lambda^jT^* \longrightarrow S^kT^* \otimes \Lambda^{j+1}T^*$$

noted again δ, and defined by $\delta(\varphi \otimes \omega) = (\delta\varphi)\wedge\omega$. More precisely, δ is defined by the following diagram:

$$\begin{array}{ccc} S^{k+1}T^* \otimes \Lambda^jT^* & \xrightarrow{\quad\delta\quad} & S^kT^* \otimes \Lambda^{j+1}T^* \\ & {\scriptstyle \delta\otimes id_{\Lambda^jT^*}} \searrow \qquad \nearrow {\scriptstyle id_{S^kT^*}\otimes\wedge} & \\ & S^kT^* \otimes T^* \otimes \Lambda^jT^* & \end{array}$$

It is not difficult to show that $\delta^2 = 0$ and that the sequence

$$0 \to S^kT^* \xrightarrow{\delta} S^{k-1}T^* \otimes T^* \xrightarrow{\delta} \cdots \to S^{k-n}T^* \otimes \Lambda^nT^* \to 0 \qquad (1.9)$$

is exact (By definition $S^lT^* = 0$ for $l < 0$).

Let P now be a kth order partial differential equation. We have the following commutative diagram:

$$\begin{array}{ccc} S^{k+l+1}T^* \otimes E \otimes \Lambda^jT^* & \xrightarrow{\sigma_{k+l+1}\otimes id} & S^{l+1}T^* \otimes F \otimes \Lambda^jT^* \\ \downarrow \delta & & \downarrow \delta \\ S^{k+l}T^* \otimes E \otimes \Lambda^{j+1}T^* & \xrightarrow{\sigma_{k+l}\otimes id} & S^lT^* \otimes F \otimes \Lambda^{j+1}T^* \end{array}$$

From the commutativity of the diagram, we can deduce that δ can be restricted to $g_{k+l+1} := \mathrm{Ker}\,\sigma_{k+l+1} \subset S^{k+l+1}T^* \otimes E$. So we have a map

$$\delta : g_{k+l+1} \otimes \Lambda^jT^* \longrightarrow g_{k+l} \otimes \Lambda^{j+1}T^*,$$

and therefore for every $l \geq k$ we obtain the complex

$$0 \to g_l \xrightarrow{\delta} g_{l-1} \otimes T^* \xrightarrow{\delta} g_{l-2} \otimes \Lambda^2T^* \to \cdots \to g_{l-n} \otimes \Lambda^nT^* \to 0 \quad (1.10)$$

with the convention that $g_m = S^mT^* \otimes E$ if $m < k$. This complex is called the *Spencer complex*.

Definition 1.10 The cohomology group of the Spencer complex at $g_m \otimes \Lambda^j T^*$ is noted H^j_m:

$$H^j_m := \frac{\operatorname{Ker}\left(g_m \otimes \Lambda^j T^* \xrightarrow{\delta} g_{m-1} \otimes \Lambda^{j+1} T^*\right)}{\operatorname{Im}\left(g_{m+1} \otimes \Lambda^{j-1} T^* \xrightarrow{\delta} g_m \otimes \Lambda^j T^*\right)}.$$

Theorem 1.4 - (J.-P. SERRE) (cf. [BCG[3]] page 410). *The following properties are equivalent:*

(1) The symbol is involutive (that is: there exists a quasi-regular basis).

(2) All the groups of the Spencer cohomology vanish.

Definition 1.11 The symbol is called "r-acyclic" if $H^j_m = 0$ for every $m \geq k$ and $0 \leq j \leq r$.

It is easy to see that the symbol is always 1−acyclic, i.e. $H^0_m = 0$ and $H^1_m = 0$ for $m \geq k$.

Goldschmidt proved that the 2-acyclicity is a sufficient condition to lift the $(k+1)$th order solutions in an infinite order solution. So one has the

Theorem 1.5 - (H. GOLDSCHMIDT) (cf. [BCG[3]] page 410). *Let P be a kth order regular linear partial differential operator. If*

a) $\bar{\pi}_k : R_{k+1} \longrightarrow R_k$ is onto,

b) the symbol is 2-acyclic,

then P is formally integrable.

Taking into account that the symbol is always 1-acyclic, we can replace the study of the involutivity by the study of the cohomology in the fourth terms of the Spencer complexes:

$$0 \longrightarrow g_{l+2} \longrightarrow g_{l+1} \otimes T^* \longrightarrow g_l \otimes \Lambda^2 T^* \longrightarrow g_{l-1} \otimes \Lambda^3 T^* \longrightarrow \cdots$$

for every $l \geq k$.

In practice, only a finite number of these cohomology groups do not vanish. In fact we have the

Theorem 1.6 – (D.G. QUILLEN) (cf. [BCG³], page 409) *If the dimension of the fibres of E is uniformly bounded on M by the same constant, then there exists an integer k_0 such that*

$$H^j_m = 0 \qquad \forall m \geq k_0 \ \text{ and } \ \forall j \geq 0$$

In our case the condition of the above theorem will be satisfied because we will suppose that E is a vector bundle and so its rank is constant. Notice however that there is no method allowing us to compute the order of this prolongation. From this we have the following version of the Cartan-Kuranishi "finiteness Theorem":

Theorem 1.7 *Let P be a kth order regular linear PDO. If there exists an integer $k_0 \geq k$ such that:*

1) $g_{k+1}, g_{k+2}, \ldots, g_{k_0}$ *are vector bundles over R_k,*
2) $\pi_l : R_{l+1} \to R_l$. *is onto, for all $k \leq l \leq k_0$,*

then P is formally integrable.

* * *

In short, in order to show the formal integrability of a linear differential operator $P : Sec\, E \longrightarrow Sec\, F$ we have to

(1) check the regularity hypothesis, i.e. that g_{k+1} is a vector bundle;
(2) show that there exists a quasi-regular basis or shows the 2-acyclicity;
(3) find the compatibility conditions. It requires:
   (a) "a good interpretation" of the obstruction space $K = (T^* \otimes F)/\operatorname{Im} \sigma_{k+1}$;
   (b) the definition of a morphism $\tau : T^* \otimes F \longrightarrow K$ such that the sequence

$$S^{k+1}T^* \otimes T^* \xrightarrow{\sigma_{k+1}} T^* \otimes F \xrightarrow{\tau} K \longrightarrow 0$$

   is exact;
   (c) to compute the morphism $\varphi : R_k \to K$ defined by $\varphi = \tau(\nabla P)$ where ∇ is an arbitrary linear connection on F. If one finds that $\varphi = 0$ then the operator P is formally integrable.

If in the step (c) one finds that $\varphi \neq 0$, then one obtains a compatibility condition for the operator.

1.7 The nonlinear case

Let us now consider the case of a nonlinear kth order PDO

$$P : Sec(E) \longrightarrow Sec(F)$$

where E and F are two vector bundles on M. The set

$$R_{k,x} = \left\{ (j_k s)_x \mid s \in Sec(E) \text{ and } P(s)_x = 0 \right\}$$

is called the *set of kth order solutions at x*.

Let $s \in Sec(E)$ be a section of E and let $P'_s : Sec(E) \longrightarrow Sec(F)$ be the *linearized* of P along s, defined by

$$P'_s(u) = \frac{d}{dt} P(s + tu) \mid_{t=0} .$$

It is a linear k-order PDO: $p_0(P'_s) : J_k E \longrightarrow F$.

It is easy to see that the symbol at x of the operator P'_s does not depend on s, but only on $\xi = j_k(s)_x \in R_{k,x}$. It is noted as $\sigma_{(\xi)k}$, and is called the *symbol of P at ξ*.

Definition 1.12 - A (non-linear) PDO is called involutive if for any kth order solution ξ the symbol of the corresponding linearised operator is involutive (i.e. for any $\xi \in R_{k,x}$ there exists a quasi regular basis of T_ξ).

By using the following Theorem, the study of the formal integrability of a non linear PDO is reduced to the study of a linear one:

Theorem 1.8 - (H. Goldschmidt) *Let P be a kth order PDO. If the linearised P'_s is formally integrable for any kth order solution s, then P is formally integrable.*

Chapter 2

Frölicher-Nijenhuis Theory of Derivations

In 1956 A. Frölicher and A. Nijenhuis developed an elegant theory permitting the classification of the derivations of the exterior algebra of the differential forms on a manifold ([FN]). The main result of this theory was the discovery of a natural structure of Lie-graded algebra on the module of the vector-valued differential forms, generalizing the Lie-algebra structure defined by the bracket on the vector fields.

Further papers of Frölicher and Nijenhuis deal with some applications of their theory, in particular with the Dolbeault cohomology in complex manifolds. Later J. Klein, in his papers devoted to the intrinsic presentation of Lagrangian mechanics (cf. [Kl]) demonstrated the relevance of applying this theory to the vector valued differential forms in the tangent bundle. In [Gr] the Frölicher-Nijenhuis theory is used to give an algebraic presentation of the theory of connections and Finsler geometry, which we will explain in the next chapter, because it plays a central role in our treatment of the inverse problem of the calculus of variations.

However, in spite of the interest this structure offers due to the fact that it arises naturally in the differential calculus on manifolds, the Frölicher-Nijenhuis bracket is not well-known, except for some particular cases (like the so-called *Nijenhuis torsion* which appears as an obstruction for the integrability of almost complex structures, or in the theory of the completely integrable systems, cf. for example [MM]). In this chapter we shall give a simple presentation of this theory and we shall note the most important formulas in the Appendix.

2.1 Derivations of the exterior algebra

In this chapter we note by $\Lambda(M) = \bigoplus_{p\in\mathbb{N}} \Lambda^p(M)$ the graded algebra of the exterior differential forms: $\Lambda^p(M) = Sec(\Lambda^p(TM))$. We also write $\Psi(M) = \bigoplus_{l\in\mathbb{N}} \Psi^l(M)$ for the graded algebra of the vector-valued exterior forms : $\Psi^l(M) = Sec(\Lambda^l(M) \otimes TM)$. In other words, an element $L \in \Psi^l(M)$ is a skew-symmetric $C^\infty(M)$-multi-linear map

$$L : \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{l \text{ times}} \longrightarrow \mathfrak{X}(M)$$

where $\mathfrak{X}(M)$ is the $C^\infty(M)$-module of the vector fields on M

Definition 2.1 *A derivation of degree r* of $\Lambda(M)$ is a map $D : \Lambda(M) \to \Lambda(M)$ such that

1) $D\,k = 0, \qquad k \in \mathbb{R}$,
2) $D\,\Lambda^p(M) \subset \Lambda^{p+r}(M)$,
3) $D\,(\varphi + \psi) = D\,\varphi + D\,\psi$,
4) $D\,(\pi\wedge\omega) = D\,\pi\wedge\omega + (-1)^{pr}\,\pi\wedge D\omega, \qquad \pi \in \Lambda^p(M)$.

It is easy to prove that the derivations of $\Lambda(M)$ are local operators. The proof of the following proposition is a straightforward verification.

Proposition 2.1 *The commutator of two derivations D_1 and D_2 defined by*

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{r_1 r_2} D_2 \circ D_1 \tag{2.1}$$

is a derivation of degree $r_1 + r_2$, r_1 and r_2 being the degrees of D_1 and D_2.

For example, the exterior differential d is a derivation of degree 1, and for $X \in \mathfrak{X}(M)$ the inner product i_X, the Lie derivative $\mathcal{L}_X$ are derivations of degrees $-1, 0$ respectively. $\mathcal{L}_X$ is the commutator of i_X and d . $\mathcal{L}_X = i_X d + d i_X = [i_X, d]$.

We can also verify the following properties which show that the set of the derivations is a graded Lie algebra:

Proposition 2.2

1) $[D_1, D_2] = (-1)^{r_1 r_2 + 1}[D_2, D_1]$.

2) $(-1)^{r_1 r_3}[D_1, [D_2, D_3]] + (-1)^{r_2 r_1}[D_2, [D_3, D_1]] + (-1)^{r_3 r_2}[D_3, [D_1, D_2]] = 0$

The Frölicher-Nijenhuis Theory is founded on the following property:

Proposition 2.3 *The derivations of $\Lambda(M)$ are determined by their action on $\Lambda^0(M) = C^\infty(M)$ and $\Lambda^1(M)$ alone.*

Proof. The derivations are local operators, so we can prove the Proposition in a local coordinate system (U, x^α). On the other hand the derivations are additive, therefore it is sufficient to show the Proposition for a p-form which has the local form $\omega = f\, dx^{i_1} \wedge \ldots \wedge dx^{i_p} \in \Lambda^p(U)$. Applying D, we find

$$D\omega = (Df) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} + \sum_{k=1}^{p} (-1)^{r(k-1)} f\, dx^{i_1} \wedge \ldots \wedge D dx^{i_k} \wedge \ldots \wedge dx^{i_p}$$

which proves that D is known as soon as its action is known on $\Lambda^0(M)$ and $\Lambda^1(M)$.

□

A straightforward application of the properties in the definition 2.1 shows that the above formula does not depend on local coordinates. This allows us to define D globally by its action on $\Lambda^0(M)$ and $\Lambda^1(M)$. We can deduce the following

Corollary 2.1 *Every map $D : \Lambda^0(M) \oplus \Lambda^1(M) \longrightarrow \Lambda(M)$ satisfying the properties in the definition 2.1 can be extended in a unique way to a derivation of $\Lambda(M)$.*

Taking into account that $\Lambda^p(M) = \{0\}$ if $p \leq -1$, we have

Corollary 2.2 *All the derivations of a degree less or equal to -2 are trivial.*

2.2 Derivations of type i_* and d_*

Definition 2.2 - A derivation is of type i_* if it is trivial on the functions (i.e. on $\Lambda^0(M)$).

For example the inner product i_X is an i_*-derivation. Note that the derivations of type i_* are determined by their action on $\Lambda^1(M)$.

Let us give the basic example of i_*-derivation:

Definition 2.3 To any $L \in \Psi^l(M)$ there is an associated derivation of type i_* of degree $l-1$, noted i_L and defined by

$$\begin{array}{lll} a) & i_X\omega = \omega(X) & \text{if } X \in \Psi(M)^0 = \mathfrak{X}(M), \\ b) & i_L\omega\,(X_1,\dots,X_l) = \omega(L(X_1,\dots,X_l)) & \text{if } L \in \Psi^l(M), \quad l \geq 1, \end{array}$$

where $\omega \in \Lambda^1(M)$ and $X_1,\dots,X_l \in \mathfrak{X}(M)$. *

Extending i_L to $\Lambda(M)$ the following formula can be proved

$$i_L\omega\,(X_1,\dots,X_{p+l-1}) = \tag{2.2}$$
$$= \frac{1}{l!(p-1)!}\sum_{s\in\mathfrak{S}_{p+l-1}} \varepsilon(s)\omega\,(L(X_{s(1)}\dots X_{s(l)}), X_{s(l+1)},\dots,X_{s(l+p-1)})$$

for $\omega \in \Lambda^p(M)$ and $L \in \Psi^l(M)$ ($\mathfrak{S}_{p+l-1}$ denotes the $(p+l-1)!$-order symmetric group and $\varepsilon(s)$ the signature of s). For example if $L \in \Psi^1(M)$, then one obtains that

$$i_L\omega(X_1,\dots,X_p) = \sum_{i=1}^{p}\omega(X_1,\dots,LX_i,\dots,X_p) \tag{2.3}$$

In particular, for the identity endomorphism I and $\omega \in \Lambda^p(M)$ one finds.

$$i_I\omega = p\omega.$$

As the following theorem shows, i_L is the only example of i_* derivation:

Theorem 2.1 *Let D be an i_*-derivation of a degree $l-1 \geq -1$. Then there exists a unique $L \in \Psi^l(M)$ such that*

$$D = i_L.$$

Proof. Since an i_*-derivation is determined by its action on $\Lambda^1(M)$, it is sufficient to construct $L \in \Psi^l(M)$ such that the equality $D\omega = i_L\omega$ holds

*In [FN] the i_*-derivation is noted also $\bar{\wedge}$: $i_L\omega = \omega\bar{\wedge}L$. It is also called "exterior-inner product" of ω by L.

for any $\omega \in \Lambda^1(M)$. Let us notice that the elements L of $\Psi^l(M)$ can be identified with the $C^\infty(M)$-multilinear maps

$$\tilde{L} : \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{l-times} \times \Lambda^1(M) \longrightarrow C^\infty(M)$$

which are skew-symmetric in the first l arguments by the following identification :

$$\tilde{L}(X_1, \ldots, X_l, \omega) = \omega(L(X_1, \ldots, X_l)).$$

Now if we define $\tilde{L}$ by

$$\tilde{L}(X_1, \ldots, X_l, \omega) = (D\omega)(X_1, \ldots, X_l),$$

then $\tilde{L}$ verifies the required conditions and we can define (via the above isomorphism) $L \in \Psi^l(M)$ such that $D\omega = i_L\omega$. L is unique because if $\bar{L}$ is another vector valued l-form such that $D\omega = i_{\bar{L}}\omega$, then one has $(i_L - i_{\bar{L}})\omega = 0$ for any $\omega \in \Lambda^1(M)$, that is

$$\omega((L - \bar{L})(X_1, \ldots, X_l)) = 0,$$

for any $\omega \in \Lambda^1(M)$ and $X_1, \ldots, X_l \in \mathfrak{X}(M)$. So $L = \bar{L}$. $\Box$

Definition 2.4 A derivation is of type d_* if it commutes (in the sense of (2.1)) with the exterior derivative d.

The exterior differential d and the Lie derivative $\mathcal{L}_X$ are d_* type derivations.

Remark. - The d_*-derivations are determined by their action on $\Lambda^0(M) = C^\infty(M)$. Indeed, let (U, x^α) be a local coordinate system and $\omega = \sum_{i=1}^{\nu} a_i dx^i \in \Lambda^1(U)$. If D is a d_*-derivation of degree r, then:

$$\begin{aligned}(D\omega) = D\Big(\sum_{i=1}^{\nu} a_i dx^i\Big) &= \sum_{i=1}^{\nu}(Da_i \wedge dx^i + a_i\, Ddx^i) \\ &= \sum_{i=1}^{\nu}(Da_i \wedge dx^i + (-1)^r a_i\, dDx^i)\end{aligned}$$

This proves that $D\omega$ is determined by the action of D on $\Lambda^0(M)$.

Proposition 2.4 *Let $L \in \Psi^l(M)$ be a vectorial l-form. If we consider*

$$d_L := [i_L, d] \tag{2.4}$$

then d_L is a d_*-derivation of degree l.

Indeed,

$$[i_L, d]d = (i_L d - (-1)^{l-1} d i_L)d = -(-1)^{l-1} d i_L d.$$

Besides

$$d[i_L, d] = d(i_L d - (-1)^{l-1} d i_L) = d i_L d$$

leads to $[d, d_L] = 0$ □

Example 2.1

(1) If $X \in \Psi^0(M) = \mathfrak{X}(M)$, one finds

$$d_X = i_X d + d i_X = \mathcal{L}_X \tag{2.5}$$

where $\mathcal{L}_X$ denotes the Lie-derivative with respect to X.

(2) If $L \in \Psi^1(M)$, then

$$d_L = i_L d - d i_L. \tag{2.6}$$

In particular, if L is the identity endomorphism I of $\mathfrak{X}(M)$, then we have

$$d_I \omega = i_I d\omega - d i_I \omega = (p+1) d\omega - d(p\omega) = d\omega$$

for every $\omega \in \Lambda^p(M)$, hence $d_I = d$.

The following Theorem states that all the d_*-derivations are of this type.

Theorem 2.2 *Let D be a d_*-derivation of degree l. Then there exists a unique $L \in \Psi^l(M)$ such that*

$$D = d_L$$

Proof. It is sufficient to construct $L \in \Psi^l(M)$ so that, for any functions f, one has $Df = d_L f$. For any given $X_1, \dots, X_l \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, let us define L by

$$L(X_1, \dots, X_l)f := (Df)(X_1, \dots, X_l).$$

It is easily shown that $L(X_1, \dots, X_l)$ is a derivation of $C^\infty(M)$, i.e. a vector field on M, and that the map

$$L : (X_1, \dots, X_l) \longmapsto L(X_1, \dots, X_l)$$

is $C^\infty(M)$-multilinear and skew-symmetric. Then $L \in \Psi^l(M)$. Moreover,

$$(Df)(X_1, \dots, X_l) = L(X_1, \dots, X_l) \cdot f = df(L(X_1, \dots, X_l)) = i_L df(X_1, \dots, X_l),$$

which yields $Df = i_L df = d_L f$. The uniqueness of L can be proved as in the Theorem 2.2. □

Theorem 2.3 *Every derivation of $\Lambda(M)$ can be decomposed uniquely into the sum of one i_*-type and one d_*-type derivation.*

Proof. Let D be a derivation of $\Lambda(M)$. The action of D on $\Lambda^0(M)$ defines a d_*-derivation d_K:

$$Df = d_K f.$$

Moreover $(D - d_K)$ acts trivially on $\Lambda^0(M)$ and so is an i_*-derivation. Therefore there exists $L \in \Psi(M)$ such that

$$D - d_K = i_L \quad \text{on } \Lambda^1(M).$$

Now

$$(i_L + d_K)f = d_K f = Df \qquad \text{for any } f \in \Lambda^0(M),$$
$$(i_L + d_K)\omega = (D - d_K)\omega + d_K\omega = D\omega \qquad \text{for any } \omega \in \Lambda^1(M).$$

Then by Corollary 2.1, $D = i_L + d_K$.

We shall frequently use the following

Proposition 2.5 *Let us consider $\omega \in \Lambda^p(M)$ such that $\omega_x = 0$ ($x \in M$), and $L \in \Psi^1(M)$. Then*

$$(d_L\omega)_x = (i_L\nabla\omega)_x,$$

where ∇ is an arbitrary linear connection on M and $\tau : T^*M \otimes \Lambda^p T^*M \longrightarrow \Lambda^{p+1} T^*M$ is defined by

$$(\tau\Omega)(X_1, \ldots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \Omega(LX_i, X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}),$$

where $\hat{}$ symbolizes the term which does not appear in the corresponding expression.

Proof. Since $\omega_x = 0$, the following formula holds:

$$(d\omega)_x(X_1, \ldots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} X_i \left(\omega_x(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})\right).$$

Let us compute the terms of $d_L\omega = i_L d\omega - d i_L \omega$. We have

$$\begin{aligned}(i_L d\omega)_x(X_1, \ldots X_{p+1}) &= \sum_{i=1}^{p+1} (d\omega)_x(X_1, \ldots, LX_i, \ldots, X_{p+1}) \\ &= \sum_{i=1}^{p+1} (-1)^{i+1} LX_i \left(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})\right) + \\ &\quad + \sum_{i=1}^{p+1} (-1)^{i+1} X_i \left((i_L\omega)_x(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})\right)\end{aligned}$$

Since $(i_L\omega)_x = 0$ because $\omega_x = 0$, we have

$$(di_L\omega)_x(X_1, \ldots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} X_i \left((i_L\omega)_x(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})\right).$$

Thus

$$(d_L\omega)_x(X_1, \ldots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} LX_i \cdot \left(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})\right).$$

On the other hand

$$
\begin{aligned}
(\tau\nabla\omega)_x(X_1,\dots,X_{p+1}) &= \sum_{i=1}^{p+1}(-1)^{i+1}(\nabla\omega)_x(LX_i; X_1,\dots,\hat{X}_i,\dots,X_{p+1}) \\
&= \sum_{i=1}^{p+1}(-1)^{i+1}(\nabla_{LX_i}\omega)_x(X_1,\dots,\hat{X}_i,\dots,X_{p+1}) \\
&= \sum_{i=1}^{p+1}(-1)^{i+1}LX_i(\omega(X_1\dots\hat{X}_i\dots X_{p+1})),
\end{aligned}
$$

because $\omega_2 = 0$. Thereby $(d_L\omega)_x = (\tau\nabla\omega)_x$. □

2.3 Graded Lie algebra structure on the module of vector-valued forms

The most important application of the theory developed in the previous sections is to define a bracket on the module of the vector-valued differential forms which arises naturally in differential geometry. A particular case of this bracket is the well-known *Nijenhuis torsion*. It appears for example on a manifold endowed with an almost complex structure and its vanishing characterizes the integrability of the structure.

The exterior inner product on the graded module of the vector valued differential forms $\Psi(M)$ can also be introduced by the same formula as (2.2):

Definition 2.5 If $K \in \Psi^k(M)$ ($k \geq 1$) and $L \in \Psi^l(M)$, then its exterior inner product $K\bar{\wedge}L \in \Psi^{k+l-1}(M)$ is given by

$$
(K\bar{\wedge}L)(X_1,\dots,X_{k+l-1}) = \\
= \frac{1}{(k-1)!l!}\sum_{s\in S^{k+l-1}}\varepsilon(s)\,K\left(L(X_{s(1)},\dots,X_{s(l)}),X_{s(l+1)},\dots,X_{s(l+k-1)}\right).
$$

and for $X \in \mathfrak{X}(M)$ one defines $X\bar{\wedge}M = 0$.

In particular we have

$$L\bar{\wedge}X = L(X) \quad \text{for} \quad L \in \Psi^1(M),\ X \in \mathfrak{X}(M)$$

and

$$L \bar{\wedge} K = L \circ K \quad \text{for} \quad L, K \in \Psi^1(M).$$

The following Proposition can be proved by a simple computation:

Proposition 2.6 *The commutator of two i_* (resp. d_*) -derivations is also an i_* (resp. d_*) -derivation.*

With the above notations, the bracket of two i_* type derivations can be given by the following formula (cf. [FN], (5.6)):

$$[i_L, i_K] = i_{K \barwedge L} - (-1)^{(l-1)(k-1)} i_{L \barwedge K}. \tag{2.7}$$

Using the Propositions 2.2 and 2.6 one can introduce the following:

Definition 2.6 For $K \in \Psi^k(M)$ and $L \in \Psi^l(M)$, the bracket $[L, K]$ is the vector valued (l+k)-form defined by the relation

$$[d_K, d_L] = d_{[K,L]}.$$

We have the following

Proposition 2.7

a) If $X, Y \in \mathfrak{X}(M)$, then $[X, Y]$ is the usual bracket on the vector fields;
b) $[L, K] = (-1)^{lk-1}[K, L]$;
c) $[id, L] = 0$;
d) $(-1)^{ln}[L, [K, N]] + (-1)^{kl}[K, [N, L]] + (-1)^{nk}[N, [L, K]] = 0$;

where l, k, n are the degrees of L, K, N respectively.

Proof *a)* can immediately be verified by checking the effect of its action on the functions. In fact,

$$[X, Y] f = d_{[X,Y]} f = [d_X, d_Y] f = [\mathcal{L}_X, \mathcal{L}_Y] f = X(Yf) - Y(Xf).$$

The properties *b)*, *c)* and *d)* follow from the analogous identities for the derivations. □

The following formula expresses the bracket of an i_*-derivation and a d_*-derivation :

$$[i_L, d_K] = d_{K \barwedge L} + (-1)^k i_{[L,K]} \tag{2.8}$$

(For the proof, cf. [FN], (5.9)).

Finally, the following formula (cf. [FN], (5.22)) allows us to compute the bracket of two vector forms recursively:

$$[L,K]\bar\wedge Y = [L\bar\wedge Y, K] - (-1)^{k(\ell-1)} L\bar\wedge[Y,K] + \\ + (-1)^{\ell k+1}\left([K\bar\wedge Y, L] - (-1)^{\ell(k-1)} K\bar\wedge[Y,L]\right)$$

For example, for $L \in \Psi^1(M)$, $K = X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(M)$:

$$[L,X]\bar\wedge Y = [L\bar\wedge Y, X] - L\bar\wedge[Y,X] - [X\bar\wedge Y, L] + X\bar\wedge[Y,L],$$

therefore

$$[X,L]Y = [X,LY] - L[X,Y] \tag{2.9}$$

so $[X,L]$ is the Lie derivative $\mathcal{L}_X L$ of L with respect to X.

For $L, K \in \Psi^1(M)$ and $X \in \mathfrak{X}(M)$, one has

$$[L,K]\bar\wedge X = [LX, K] - L[X,K] + [KX, L] - K[X,L],$$

since

$$[L,K](X,Y) = ([L,K]\bar\wedge X)\bar\wedge Y,$$

one arrives at.

$$\begin{aligned}[L,K](X,Y) = \; & [LX,KY] + [KX,LY] + LK[X,Y] + KL[X,Y] \\ & - L[KX,Y] - K[LX,Y] - L[X,KY] - K[X,LY].\end{aligned} \tag{2.10}$$

In particular, for $K = L$, we get the so-called *Nijenhuis torsion*:

$$\frac{1}{2}[L,L](X,Y) = [LX,LY] + L^2[X,Y] - L[LX,Y] - L[X,LY] \tag{2.11}$$

Local expression

Let (U, x^α) be a local system of coordinates, $L \in \Psi^1(M)$, and $X \in \mathfrak{X}(M)$, where the local expressions are $L = L^\alpha_\beta dx^\beta \otimes \frac{\partial}{\partial x^\alpha}$ and $X = X^\alpha \frac{\partial}{\partial x^\alpha}$. Let us compute the components of the tensor $[L, X]$. One has

$$[L,X]\left(\frac{\partial}{\partial x^\lambda}\right) = \left[L\left(\frac{\partial}{\partial x^\lambda}\right), X^\gamma \frac{\partial}{\partial x^\gamma}\right] - L\left[\frac{\partial}{\partial x^\lambda}, X^\gamma \frac{\partial}{\partial x^\gamma}\right]$$
$$= \left[L^\mu_\lambda \frac{\partial}{\partial x^\mu}, X^\gamma \frac{\partial}{\partial x^\gamma}\right] - L\left(\frac{\partial X^\gamma}{\partial x^\lambda}\frac{\partial}{\partial x^\gamma}\right) = L^\mu_\lambda \frac{\partial X^\gamma}{\partial x^\mu}\frac{\partial}{\partial x^\gamma} - X^\gamma \frac{\partial L^\mu_\lambda}{\partial x^\gamma}\frac{\partial}{\partial x^\mu}$$
$$- \frac{\partial X^\gamma}{\partial x^\lambda} L^\mu_\gamma \frac{\partial}{\partial x^\mu} = \left(L^\gamma_\lambda \frac{\partial X^\alpha}{\partial x^\gamma} - L^\alpha_\gamma \frac{\partial X^\gamma}{\partial x^\lambda} - X^\gamma \frac{\partial L^\alpha_\lambda}{\partial x^\gamma}\right)\frac{\partial}{\partial x^\alpha}$$

Therefore

$$[L,X] = \left(L^\gamma_\lambda \frac{\partial X^\alpha}{\partial x^\gamma} - L^\alpha_\gamma \frac{\partial X^\gamma}{\partial x^\lambda} - X^\gamma \frac{\partial L^\alpha_\lambda}{\partial x^\gamma}\right) dx^\lambda \otimes \frac{\partial}{\partial x^\alpha} . \qquad (2.12)$$

In the same way one can prove that the local expression of the Nijenhuis torsion of a vectorial 1-form L is

$$\frac{1}{2}[L,L] = \left(L^\mu_\alpha \frac{\partial L^\lambda_\beta}{\partial x^\mu} - L^\mu_\beta \frac{\partial L^\lambda_\alpha}{\partial x^\mu} + L^\lambda_\mu \frac{\partial L^\mu_\alpha}{\partial x^\beta} - L^\lambda_\mu \frac{\partial L^\mu_\beta}{\partial x^\alpha}\right) dx^\alpha \otimes dx^\beta \otimes \frac{\partial}{\partial x^\lambda} .$$

Chapter 3

Differential Algebraic Formalism of Connections

3.1 The tensor algebra of the tangent vector bundle

In this chapter we will explain the differential algebraic formalism of connection theory introduced by J. Grifone in [Gr] which will be specially adapted to the inverse problem of variational calculus.

Semi-basic forms

Let $\tau : TM \longrightarrow M$ be the tangent bundle and $\pi_{TM} : TTM \longrightarrow TM$ the second tangent bundle. We have the following diagram:

$$\begin{array}{ccc} TTM & \xrightarrow{\tau_*} & TM \\ {\scriptstyle \pi_{TM}}\downarrow & & \downarrow{\scriptstyle \tau} \\ TM & \xrightarrow{\tau} & M \end{array}$$

and the exact sequence

$$0 \longrightarrow TM \underset{M}{\times} TM \xrightarrow{i} TTM \xrightarrow{j} TM \underset{M}{\times} TM \longrightarrow 0 \qquad (3.1)$$

where $i(v,w) := \frac{d}{dt}(v + tw)\Big|_{t=0}$ is the natural injection and $j := (\pi_{TM}, \tau_*)$. Using an adapted coordinate system (x^α, y^α) on TM, where (x^α) are the coordinates on M and y^α are the components of a vector of TM on the basis $\left(\frac{\partial}{\partial y^\alpha}\right)$, we have

$$i\big((x^\alpha, y^\alpha), (x^\alpha, Z^\alpha)\big) = (x^\alpha, y^\alpha, 0, Z^\alpha)$$

and

$$j(x^\alpha, y^\alpha, X^\alpha, Y^\alpha) = \big((x^\alpha, y^\alpha), (x^\alpha, X^\alpha)\big).$$

If $T^vTM := \operatorname{Ker} \pi_*$ is the vertical bundle, then

$$T^vTM = \operatorname{Im} i = \operatorname{Ker} j.$$

NOTE. *From now on we will work on the manifold TM. Where there is no possibility of confusion, TTM, T^*TM and T^vTM will be noted as T, T^* and T^v respectively.*

For any $z \in TM$, there is a natural isomorphism

$$\begin{aligned} i_z : T_{\pi(z)}M &\longrightarrow T^v_z \\ u &\longmapsto i(z, u) \end{aligned}$$

We use the formula :

$$\xi_z = (i_z)^{-1}.$$

Locally, if $X = X^\alpha(x, y)\dfrac{\partial}{\partial y^\alpha}$ is a vertical field, then $\xi_z X = X^\alpha(x, z)\dfrac{\partial}{\partial x^\alpha}$.

Definition 3.1 A p-form $\omega \in \otimes^p T^*$ is semi-basic if $\omega(X_1, ..., X_p) = 0$ when one of the vectors X_i is vertical. A vector valued l-form $L \in \otimes^l T^* \otimes T$ is semi-basic if it takes its values in the vertical bundle and $L(X_1, ..., X_l) = 0$ when one of the vectors X_i is vertical.

The set of the skew-symmetric (resp: symmetric) semi-basic scalar forms will be noted $\Lambda^p T^*_v$ (resp: $S^p T^*_v$) (for $p = 1$, these sets will be noted T^*_v).

In an adapted coordinate system the semi-basic scalar and vector forms can be expressed as:

$$\begin{aligned} \omega &= \omega_{\alpha_1 \ldots \alpha_p}(x, y)\, dx^{\alpha_1} \otimes \ldots \otimes dx^{\alpha_p} \\ L &= L^\beta_{\alpha_1 \ldots \alpha_l}(x, y)\, dx^{\alpha_1} \otimes \ldots \otimes dx^{\alpha_l} \otimes \frac{\partial}{\partial y^\beta}. \end{aligned}$$

For any $z \in TM$, there is a natural isomorphism

$$\xi_z^* : \left(\Lambda^p T^*_v\right)_z \longrightarrow \Lambda^p T^*_{\pi(z)}.$$

defined in the following way: if $\omega \in \Lambda^p T^*_{v\,z}$ and $u_1, \ldots, u_p \in T_{\pi(z)}$, then $(\xi_z \omega)(u_1, \ldots, u_p) := \omega(U_1, \ldots, U_p)$, where $U_i \in T_z$ are such that $\pi_* U_i = u_i$. ξ_z is well-defined because the difference of two vectors which project on the same vector is a vertical vector.

In the same way, for any $z \in TM$, there is a natural isomorphism

$$(\Lambda^p T^*_v)_z \otimes T^v_z \xrightarrow{\ \xi'\ } \Lambda^p T^*_{\pi(z)} \otimes T_{\pi(z)}M$$

defined by $\xi''_z := \xi'_z \otimes \xi_z$. Later on, ξ, ξ' and ξ'' will be noted ξ. If $L \in \Phi^\ell(M)$ is a vectorial ℓ-form and its expression in a local coordinate system is $L = L^\alpha_{\beta_1 \ldots \beta_\ell}(x,y)\, dx^{\beta_1} \wedge \ldots \wedge dx^{\beta_\ell} \otimes \frac{\partial}{\partial y^\alpha}$, then we have:

$$\xi_z L = L^\alpha_{\beta_1 \ldots \beta_\ell}(x,z)\, dx^{\beta_1} \wedge \ldots \wedge dx^{\beta_\ell} \otimes \frac{\partial}{\partial x^\alpha}.$$

Vertical endomorphism

Definition 3.2 The tensor $J \in T^* \otimes T$ defined by

$$J := i \circ j$$

is called *vertical endomorphism*.

The following properties can immediately be verified:

Proposition 3.1

$$1)\quad J^2 = 0.$$
$$2)\quad \operatorname{Ker} J = \operatorname{Im} J = T^v.$$

Locally $J = dx^\alpha \otimes \frac{\partial}{\partial y^\alpha}$, or in other words

$$J\left(\frac{\partial}{\partial x^\alpha}\right) = \frac{\partial}{\partial y^\alpha} \qquad \text{and} \qquad J\left(\frac{\partial}{\partial y^\alpha}\right) = 0 \tag{3.2}$$

Using these formulae, it is easy to check that the Nijenhuis torsion of J vanishes

$$[J, J] = 0. \tag{3.3}$$

Canonical field and homogeneity

Definition 3.3 The *canonical vertical field* of TM (also called *Liouville field*) is the vector field $C = i \circ \delta$ on TM, where $\delta : TM \to TM \times_M TM$ is the diagonal map defined by $\delta(z) = (z, z)$.

Locally:

$$C = y^a \frac{\partial}{\partial y^a}. \tag{3.4}$$

Remark. C is the infinitesimal transformation associated with the group of the homotheties with a positive ratio.
Indeed, if we consider $\varphi_t : TM \longrightarrow TM$, the one-parameter group of infinitesimal transformations defined by $\varphi_t(v = (x^a, y^a)) = (x^a, e^t y^a)$ we obtain:

$$\frac{d}{dt}(\varphi_t v)\Big|_{t=t} = (x^a, e^t y^a, 0, e^t y^a)\Big|_{t=0} = (x^a, y^a, 0, y^a) = C_v$$

The relation

$$[C, J] = -J \tag{3.5}$$

can easily be checked in a coordinate system, taking into account (3.2) and (3.4).

A function $f \in C^\infty(TM \setminus \{0\})$ is *(positively) homogeneous of degree* r if

$$f(\lambda v) = \lambda^r f(v)$$

for any $\lambda > 0$. In this case we will say that f is $h(r)$. It is well known that this property is equivalent to the Euler identity, which in a local coordinate system can be written:

$$y^a \frac{\partial f}{\partial y^a}(x^i, y^i) = r f(x^i, y^i),$$

or using the canonical field

$$\mathcal{L}_C f = r f.$$

Remark. Consider $f : TM \to \mathbf{R}$ such that $f|_{TM\setminus\{0\}}$ is C^∞ and $h(r)$. If f is C^r on the 0-section then it is a polynomial of the degree r on the fibers. Indeed, let $x \in M$ be an arbitrary point. Notice that if f is $h(0)$ then it is constant on the straight lines starting from the origin of T_xM; so if it is C^0 in the origin, then it is constant on T_xM (i.e. f is basic). Now suppose that f is $h(1)$ and C^1; the partial derivatives $\frac{\partial f}{\partial y^\alpha}$ are $h(0)$ and C^0 and thus constant on the fibers. Hence, by the Euler identity, f is linear on the fibers. A recursion argument easily yields the general property.

Definition 3.4 A tensor t on $TM \setminus \{0\}$ is homogeneous of the degree r (t is $h(r)$) if

$$\mathcal{L}_C\, t = rt.$$

Notice that if $L \in \Psi^1(TM \setminus \{0\})$ is a skew-symmetric vector-valued l-form, this condition can be written: $[C, L] = rL$. In local coordinates, let us consider for example, $L \in \Psi^1(TM)$:

$$L = L^\beta_\alpha dx^\alpha \otimes \frac{\partial}{\partial x^\beta} + L^\beta_{\bar\alpha} dy^\alpha \otimes \frac{\partial}{\partial x^\beta} + L^{\bar\beta}_\alpha dx^\alpha \otimes \frac{\partial}{\partial y^\beta} + L^{\bar\beta}_{\bar\alpha} dy^\alpha \otimes \frac{\partial}{\partial y^\beta},$$

which means that the matrix of the endomorphism L in the basis $\{\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^\alpha}\}$ is

$$L = \begin{pmatrix} L^\beta_\alpha & L^\beta_{\bar\alpha} \\ L^{\bar\beta}_\alpha & L^{\bar\beta}_{\bar\alpha} \end{pmatrix}.$$

Thus L is $h(r)$ if and only if the functions L^β_α and $L^{\bar\beta}_{\bar\alpha}$ are $h(r)$, $L^\beta_{\bar\alpha}$ are $h(r-1)$, and $L^{\bar\beta}_\alpha$ are $h(r+1)$.

3.2 Sprays and connections

The notion of sprays has been introduced in [APS] to give an intrinsic presentation of ordinary second order differential equations.

Definition 3.5 A *spray* on M is a vector field on the tangent bundle $S \in \mathfrak{X}(TM)$ such that:

$$JS = C.$$

S is a spray if and only if, in a natural local coordinate system of TM, there are functions f^α, such that

$$S = y^\alpha \frac{\partial}{\partial x^\alpha} + f^\alpha(x,y)\frac{\partial}{\partial y^\alpha}. \tag{3.6}$$

The spray S is called *homogeneous* if $[C,S] = S$ and if it is C^∞ on $TM \setminus \{0\}$ and C^1 on the zero section. In this case the functions $f^\alpha(x,y)$ are homogeneous of degree 2 in the variables y^α. If, in addition, S is C^2 on the zero section then the $f^\alpha(x,y)$ are quadratic in the y^α. In that case, we will say that the spray is *quadratic**.

Definition 3.6 The vertical vector field

$$S^* = [C,S] - S$$

which measures the non homogeneity of S is called the *deflection* of S.

All sprays are associated to a second order system of ordinary differential equations, and reciprocally: a spray can be associated to any second order system of ordinary differential equations in the following way :

Definition 3.7 Let S be a spray. A *path* of S is a parametrized curve $\gamma : I \longrightarrow M$ such that γ' is an integral curve of S, that is:

$$S_{\gamma'} = \gamma''.$$

In a local coordinate system, if $(x^\alpha(t))$ is a path on M, and the spray S has the local form (3.6), then $S_{\gamma'} = (x^\alpha, \frac{dx^\alpha}{dt}, \frac{dx^\alpha}{dt}, f^\alpha(x(t), \frac{dx}{dt}))$ and $\gamma'' = (x^\alpha, \frac{dx^\alpha}{dt}, \frac{dx^\alpha}{dt}, \frac{d^2x^\alpha}{dt^2})$. So the paths of S are solution to the second order differential system

$$\frac{d^2x^\alpha}{dt^2} = f^\alpha\left(x, \frac{dx}{dt}\right), \tag{3.7}$$

$\alpha = 1,\dots,n$. Reciprocally, if a system (3.7) is given, one defines S in a local coordinates system by $S = y^\alpha \frac{\partial}{\partial x^\alpha} + f^\alpha(x,y)\frac{\partial}{\partial y^\alpha}$ and one verifies that the definition does not depend on the coordinates system.

Definition 3.8 Let L be a semi-basic (scalar or vector) l-form. The *potential* of L is the semibasic $(l-1)$-form L^0 defined by

$$L^0 = i_S L \tag{3.8}$$

*In [APS] the term "spray" is reserved to the quadratic sprays.

where S is an arbitrary spray.

L^0 is well defined: it does not depend on the choice of S. In fact let S' be another spray. Since $J(S - S') = C - C = 0$, we see that $S - S'$ is vertical. So $i_{S-S'}L = 0$ and $i_{S'}L = i_S L$.

Locally if $L = \frac{1}{l!} L^{\beta}_{\alpha_1 \ldots \alpha_l}(x,y) dx^{\alpha_1} \otimes \ldots \otimes dx^{\alpha_l} \otimes \frac{\partial}{\partial y^{\beta}}$, we have

$$L^0 = \frac{1}{(l-1)!} y^{\gamma} L^{\beta}_{\gamma\alpha_1 \ldots \alpha_{l-1}} dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{l-1}}.$$

Connections

In this section we will recall the differential algebraic presentation of the connections theory introduced in [Gr], which we shall constantly use later on

Definition 3.9 *A connection on M is a tensor field of type (1-1) Γ on TM (i.e. $\Gamma \in \Psi^1(TM)$) such that*

i) $J\Gamma = J$,

ii) $\Gamma J = -J$.

The connection is called *homogeneous* if $[C, \Gamma] = 0$, it is C^∞ on $TM \setminus \{0\}$ and C^0 on the 0 section. In addition, if Γ is C^1 on the 0 section, then it is called *linear*.

Proposition 3.2 *If Γ is a connection, then $\Gamma^2 = I$ and the eigenvector space corresponding to the eigenvalue -1 is the vertical space. Then, at any $z \in TM$, we have the splitting*

$$T_z TM = H_z \oplus T^v_z,$$

where H_z is the eigenspace corresponding to $+1$. H_z is called horizontal space.

Proof. From *i)* we have $J(\Gamma - I) = 0$ then $\mathrm{Im}(\Gamma - I) \subset \mathrm{Ker}\, J = T^v$. From *ii)* we have $(\Gamma - I)J = 0$, then $T^v \equiv \mathrm{Im} J \subset \mathrm{Ker}(\Gamma + I)$. Then $\mathrm{Im}(\Gamma - I) \subset \mathrm{Ker}(\Gamma + I)$ that is

$$(\Gamma + I)(\Gamma - I) = 0.$$

so $\Gamma^2 = I$. On the other hand, $T^v \subset \mathrm{Ker}(\Gamma + I)$ by *ii*). Reciprocally, if $\Gamma X = -X$, one has $J\Gamma X = -JX$, that is $JX = -JX$ and then $JX = 0$. So X is vertical and $\mathrm{Ker}(\Gamma + I) \subset T^v$. Finally: $T^v = \mathrm{Ker}(\Gamma + I)$. □

Expression of Γ *in local coordinates:*

The matrix of the vertical endomorphism J in the natural basis $\{ \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^\alpha} \}$ is

$$J = \begin{pmatrix} 0 & 0 \\ \delta^\beta_\alpha & 0 \end{pmatrix}.$$

The condition *i)* and *ii)* of the definition of the connection implies that the matrix of Γ is

$$\Gamma = \begin{pmatrix} \delta^\beta_\alpha & 0 \\ 2\Gamma^\beta_\alpha(x, y) & \delta^\beta_\alpha \end{pmatrix},$$

where Γ^β_α are functions called *coefficients of the connection*. If the connection is homogeneous (resp. linear) the coefficients $\Gamma^\beta_\alpha(x, y)$ are $h(1)$ (resp. linear in y). In the linear case, one states.

$$\Gamma^\beta_\alpha(x, y) = y^\gamma \Gamma^\beta_{\alpha\gamma}(x).$$

Definition 3.10 The semi-basic tensor $H = \frac{1}{2}[C, \Gamma]$, which measures the non homogeneity of the connection will be called the *tension*.

We denote

$$h := \frac{1}{2}(I + \Gamma), \qquad v := \frac{1}{2}(I - \Gamma),$$

the *horizontal* and *vertical* projectors. They verify :

$$\begin{cases} Jh = J, & hJ = 0, \\ Jv = 0, & vJ = J. \end{cases}$$

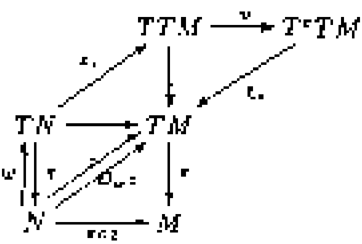
Locally we have,

$$\begin{cases} h\left(\dfrac{\partial}{\partial x^\alpha}\right) = \dfrac{\partial}{\partial x^\alpha} - \Gamma^\beta_\alpha(x, y)\dfrac{\partial}{\partial y^\beta}, \\ h\left(\dfrac{\partial}{\partial y^\alpha}\right) = 0. \end{cases}$$

Definition 3.11 Let N and M be two manifolds, $w \in \mathfrak{X}(N)$ and $\xi_z : T_z^v \to T_{\pi(z)}M$ the natural injection. The *covariant derivative* of z with respect to w is defined by:

$$D_{w(x)}z = \xi_{z(x)}(v \circ z_* \circ w). \tag{3.9}$$

We have the following diagram.



In particular, for $N = M$ and $w, z \in \mathfrak{X}(M)$, we have

$$D_w z = w^\alpha \left(\frac{\partial z^\beta}{\partial x^\alpha} + \Gamma^\beta_\alpha(x, z(x)) \right) \frac{\partial}{\partial x^\beta} \tag{3.10}$$

If $N = [a, b]$ is an interval of $\mathbf{R}$, $w = \frac{d}{dt}$ and $z : [a, b] \to TM$, $z(t) = (x(t), y(t))$ is a vector field along a curve $\gamma : [a, b] \to M$ (i.e. $\gamma = \pi \circ z$), we arrive at .

$$D_{\frac{d}{dt}} z = \left(\frac{dy^\beta}{dt} + \Gamma^\beta_\alpha(x(t), y(t)) \frac{dx^\alpha}{dt} \right) \frac{\partial}{\partial x^\beta}. \tag{3.11}$$

Definition 3.12 A vector field $z \in TM$ along a curve γ is called *parallel* if $D_{\frac{d}{dt}} z = 0$, that is $v(z') = 0$. A *geodesic* is a curve $\gamma : [a, b] \to M$ such that

$$D_{\frac{d}{dt}} \gamma' = 0$$

In others word, γ is a geodesic if and only if

$$v \circ \gamma'' = 0.$$

Canonical decomposition of a connection

In this paragraph we explain the relations between sprays and connections.

Definition 3.13 Let Γ be a connection, h the corresponding horizontal projector. The spray S associated to the connection is defined by

$$S = h\dot{S},$$

where $\dot{S}$ is an arbitrary spray.

Indeed, S is a spray because $JS = Jh\dot{S} = J\dot{S} = C$. On the other hand S does not depend on the choice of $\dot{S}$, because if S' is another spray $J(\dot{S} - S') = C - C = 0$; then $\dot{S} - S'$ is vertical so $h(\dot{S} - S') = 0$.

Locally:

$$S = y^\alpha \frac{\partial}{\partial x^\alpha} - y^\beta \Gamma^\alpha_\beta(x, y) \frac{\partial}{\partial y^\alpha}$$

where the $\Gamma^\alpha_\beta(x, y)$ are the coefficients of the connection.

It is easy to verify the following

Proposition 3.3 *The paths of the spray associated to the connection Γ are the geodesics of Γ.*

Indeed, for any curve γ on M, one has : $J_{\gamma'} = C_{\gamma'}$. Then, for any spray $\bar{S}$, $\bar{S} - \gamma''$ is vertical, so $h\bar{S}_{\gamma'} = h\gamma''$, i.e. $S_{\gamma'} = h\gamma'' = \gamma'' - v\gamma''$. The property follows from this equality. □

Reciprocally, a connection can be associated to any spray in the following way:

Proposition 3.4 (cf. [Gr]). *Let S be a spray on M. Then $\Gamma := [J, S]$ is a connection. The spray associated to $[J, S]$ is $S + \frac{1}{2}S^*$, where S^* is the deflection of S. If S is homogeneous (resp. quadratic), then $[J, S]$ is homogeneous (resp. linear).*

Proof. Using the equation (3.3) we have

$$0 = \frac{1}{2}[J, J](S, X) = [C, JX] - J[C, X] - J[S, JX] = [C, J]X - J[S, JX],$$

and taking into account (3.5) we find

$$J[JX, S] = JX. \tag{3.12}$$

Now

$$J[J,S]X = J[JX,S] - J^2[X,S] = JX,$$

and

$$[J,S]JX = [J^2X,S] - J[JX,S] = -JX,$$

which proves that $[J,S]$ is a connection. The other properties can be easily verified. □

Locally, if $S = y^\alpha \frac{\partial}{\partial x^\alpha} + f^\alpha(x,y)\frac{\partial}{\partial y^\alpha}$, then the coefficients of the connection associated are:

$$\Gamma^\alpha_\beta = -\frac{1}{2}\frac{\partial f^\alpha}{\partial y^\beta}.$$

Note that as the spray of the connection $[J,S]$ is not S (except in the case where S is homogeneous), then in general the geodesics of $[J,S]$ are not the paths of S. To avoid this difficulty let us introduce the notion of torsion.

Definition 3.14 The weak *torsion* is the vector-valued semi-basic 2-form defined by $t := \frac{1}{2}[J,\Gamma]$. The *strong* torsion is the vector-valued semi-basic 1-form $T := t^\circ - H$, where H is the tension of the connection.

As we will see, if the strong torsion vanishes, then the weak torsion is zero, but the converse is not generally true.

Locally

$$t(X,Y) = X^\alpha Y^\beta\left(\frac{\partial\Gamma^\lambda_\alpha}{\partial y^\beta} - \frac{\partial\Gamma^\lambda_\beta}{\partial y^\alpha}\right)\frac{\partial}{\partial y^\lambda},$$

$$T(X) = X^\beta\left(y^\alpha\frac{\partial\Gamma^\lambda_\beta}{\partial y^\alpha} - \Gamma^\lambda_\beta\right)\frac{\partial}{\partial y^\lambda}.$$

Then for a linear connection:

$$t(X,Y) = X^\alpha Y^\beta(\Gamma^\lambda_{\alpha\beta} - \Gamma^\lambda_{\beta\alpha})\frac{\partial}{\partial y^\lambda},$$

$$T(X) = y^\alpha X^\beta(\Gamma^\lambda_{\alpha\beta} - \Gamma^\lambda_{\beta\alpha})\frac{\partial}{\partial y^\lambda}.$$

so:

$$\xi_z t(Z,W) = \xi_z T(W) = D_z w - D_w z - [z,w]$$

where $Z, W \in TTM$ and $z = \pi_* Z$, $w = \pi_* W$. Therefore in the linear case, t and T coincide with the classical torsion up the natural identification of the tensor algebra of TM with the tensor algebra of the vertical bundle.

An easy computation enables us to check that the strong torsion "counter-balances" the spray, in the sense that its potential makes up the deflection of S, that is:

$$T^0 + S^* = 0 \tag{3.13}$$

The following Theorem shows that a connection is determined by its sprays and its strong torsion:

Theorem 3.1 - *(Canonical decomposition) ([Gr] (I.55)) - Let S be a spray and T a semi-basic vector valued 1-form counterbalancing S. Then there exists one and only one connection Γ whose spray is S and whose strong torsion is T. Γ is given by*

$$\Gamma = [J,S] + T.$$

PROOF Consider S and T as in the Theorem and put $\Gamma := [J,S] + T$. It is easy to verify that Γ is a connection whose spray is S and the strong torsion is T. Reciprocally, let us consider a connection Γ whose spray is S. We have to prove that:

$$\Gamma = [J,S] + \frac{1}{2}([J,\Gamma]\bar{\wedge}S - [C,\Gamma])$$

Taking into account that the spray associated to Γ is $\frac{1}{2}(I+\Gamma)S$, where S is an arbitrary spray, for any connection Γ and for any spray S we have

$$2\Gamma = [J, S + \Gamma S] + [J,\Gamma]\bar{\wedge}S - [C,\Gamma].$$

Now, we have:

$$[J,\Gamma]\bar{\wedge}S = [C,S] - J[S,\Gamma] + [\Gamma S, J] - \Gamma[S,J]$$

But we also have:

$$-J[S,\Gamma] = J[\Gamma,S] = J[I - 2v, S] = -2v = 2\Gamma - I$$

and thus

$$-\Gamma[S,J] = \Gamma[J,S] = \Gamma(I - 2\bar{v}) = \Gamma + 2\bar{v} = \Gamma + I + [J,S]$$

where $\bar{v}$ is the vertical projector of the connection $[J,S]$. We immediately find the Theorem. □

Notice that if $T = 0$, then $\Gamma = [J,S]$, and $[J,\Gamma] = [J,[J,S]] = \frac{1}{2}[[J,J],S] = 0$. Therefore $t = 0$ and so $H = 0$. Reciprocally if H and t vanish, then $T = 0$. So $T = 0$ if and only if $t = 0$, and the connection is homogeneous

The almost complex structure associated to a connection

Definition 3.16 Let Γ be a connection on M, h the corresponding horizontal projector. The *almost complex structure* associated to Γ is the unique vector valued 1-form F on TM such that:

$$FJ = h \quad \text{and} \quad Fh = -J.$$

F is well defined. Indeed, let V be a vertical field and $Y, Y' \in \mathcal{X}TM$ two vector fields such that $JY = JY' = V$. Since $J(Y - Y') = 0$, $Y - Y'$ is vertical, then $h(Y - Y') = 0$ and thus $FV = hY = hY'$. In the same way one proves that the action of F on horizontal vectors is well defined. On the other hand, F is unique because it is determined on the horizontal and vertical vectors. Obviously we have $F^2 = -I$.

It is easy to prove the following properties (for example by computing the two members on the horizontal and vertical vectors):

$$F = h[S,h] - J, \tag{3.14}$$

$$JF = v. \tag{3.15}$$

Berwald connection

A linear connection on TM is associated to any connection Γ. This linear connection, given by the covariant derivative D, called the *Berwald* connection, can be characterized by the following system of axioms:

$$\begin{aligned} DF &= 0, \\ D_{hX}JY &= [h, JY]X, \\ D_{JX}JY &= [J, JY]X. \end{aligned}$$

Indeed if D exists it is unique because one has:

$$D_X JY = D_{hX}JY + D_{JFX}JY = [h, JY]X + [J, JY]FX$$

and

$$D_X hY = D_X FJY = FD_X JY = F\{[h, JY]X + [J, JY]FX\}.$$

Then:

$$\begin{aligned} D_X Y &= D_X hY + D_X JFY \\ &= F\{[h, JY]X + [J, JY]FX\} + [h, JFY]X + [J, JFY]FX \end{aligned}$$

so we find

$$D_X Y = F\{[h, JY]X + [J, JY]FX\} + [h, vY]X + [J, vY]FX.$$

Reciprocally it is easy to verify that this formula defines a connection on TM. □

An easy computation shows that

$$DJ = 0, \tag{3.16}$$

$$D\Gamma = 0. \tag{3.17}$$

In a local coordinates system the Berwald connection is determined by

$$\begin{cases} D_{\frac{\partial}{\partial x^\alpha}} \dfrac{\partial}{\partial x^\beta} = \dfrac{\partial \Gamma^\lambda_\alpha}{\partial y^\beta} \dfrac{\partial}{\partial x^\lambda} + \left(\dfrac{\partial \Gamma^\mu_\beta}{\partial x^\alpha} + \Gamma^\lambda_\alpha \dfrac{\partial \Gamma^\mu_\beta}{\partial y^\lambda} - \Gamma^\mu_\lambda \dfrac{\partial \Gamma^\lambda_\alpha}{\partial y^\beta} \right) \dfrac{\partial}{\partial y^\mu}, \\ D_{\frac{\partial}{\partial x^\alpha}} \dfrac{\partial}{\partial y^\beta} = \dfrac{\partial \Gamma^\lambda_\alpha}{\partial y^\beta} \dfrac{\partial}{\partial y^\lambda}, \\ D_{\frac{\partial}{\partial y^\alpha}} \dfrac{\partial}{\partial x^\beta} = \dfrac{\partial \Gamma^\lambda_\beta}{\partial y^\alpha} \dfrac{\partial}{\partial y^\lambda}, \\ D_{\frac{\partial}{\partial y^\alpha}} \dfrac{\partial}{\partial y^\beta} = 0. \end{cases}$$

3.3 Curvature and Douglas tensor

The notions of this section will often be used in the following chapters.

Curvature

Definition 3.16 The *curvature* of the connection Γ is the vector-valued 2-form R defined by

$$R = -\frac{1}{2}[h, h],$$

where h is the horizontal projection defined by Γ.

Remark. Taking into account that $h[vX, vY] = 0$, because the vertical distribution is integrable, an easy computation gives

$$R(X, Y) = -v[hX, hY], \tag{3.18}$$

which proves that R is semi-basic and that: *$R = 0$ if and only if the horizontal distribution is integrable.*

Locally one has

$$R = \left(\frac{\partial \Gamma^\lambda_\beta}{\partial x^\alpha} - \frac{\partial \Gamma^\lambda_\alpha}{\partial x^\beta} + \Gamma^\mu_\beta \frac{\partial \Gamma^\lambda_\alpha}{\partial y^\mu} - \Gamma^\mu_\alpha \frac{\partial \Gamma^\lambda_\beta}{\partial y^\mu} \right) dx^\alpha \otimes dx^\beta \otimes \frac{\partial}{\partial y^\lambda}$$

Thus for a *linear connection* we have

$$\xi_u R(Z, W) = D_z D_w u - D_w D_z u - D_{[z,w]} u$$

where $Z, W \in T_u TM$, and $z = \pi_* Z$, $w = \pi_* W$, and D is the covariant derivative associated to Γ.

The following properties are the *Bianchi identities* for a nonlinear connection:

$$[J, R] = [h, t], \tag{3.19}$$

$$[h, R] = 0 \tag{3.20}$$

They can easily be proved by using the Jacobi identity

The almost complex structure and the curvature are related to each other by the following property:

$$\frac{1}{2}h^*[F,F] = Ft + R \tag{3.21}$$

Indeed, $h^*[F,F](X,Y) = [F,F](hX,hY)$ and

$$\frac{1}{2}h^*[F,F](X,Y) = [JX,JY] - [hX,hY] + F[JX,hY] + F[hX,JY].$$

On the other hand we have

$$Ft(X,Y) = F[JX,hY] + F[hX,JY] - h[hX,hY] + [JX,JY] \tag{3.22}$$

and therefore

$$\begin{aligned}\frac{1}{2}h^*[F,F](X,Y) &= Ft(X,Y) - [hX,hY] + h[hX,hY]\\ &= Ft(X,Y) - v[hX,hY] = Ft(X,Y) + R(X,Y).\end{aligned}$$

From this property one deduces

Corollary 3.1 *The almost complex structure associated with a connection is integrable if and only if the connection is "weakly flat", i.e.* $t = 0$ *and* $R = 0$.

Indeed, $[F,F] = 0$ implies $t = 0$ and $R = 0$ because $Ft(X,Y)$ is horizontal and $R(X,Y)$ is vertical. The converse follows from the fact that $[F,F] = 0$ is equivalent to $h^*[F,F] = 0$. By a simple computation one can prove that

$$\begin{aligned}[F,F](hX,JY) &= F \circ h^*[F,F](X,Y),\\ [F,F](JX,JY) &= -h^*[F,F](X,Y).\end{aligned}$$

□

We also have the following identities:

$$[J,F] = t \barwedge F - Ft - R \tag{3.23}$$

$$[h,F] = -R \barwedge F - t \tag{3.24}$$

which can be proved by a straightforward verification. The following Proposition can easily be checked with the help of the above formulae.

Proposition 3.5 *The following properties are equivalent:*

$$1)\quad [F, F] = 0,$$
$$2)\quad [J, F] = 0,$$
$$3)\quad [h, F] = 0,$$
$$4)\quad R = 0,\ \ t = 0.$$

Indeed from (3.23) we have $J[J, F] = -t$. Therefore $[J, F] = 0$ implies $t = 0$ and then $R = 0$. The converse is trivial. Moreover we get $[h, F](hX, JY) = -R(X, Y)$. Therefore $[h, F] = 0$ implies $R = 0$ and then $t = 0$.

Douglas tensor

In his classical work on the inverse problem of the calculus of variations ([Dou]) J. Douglas introduced a tensor which plays an essential role in the theory [1]. His coordinate-free presentation is the following (cf. [Kl]):

Definition 3.17 The *Douglas tensor* is the (1-1) tensor A on TM defined by

$$A := v[h, S],$$

where h and v are the horizontal and vertical projectors of the connection $\Gamma = [J, S]$.

It is easy to see that A is semi-basic and

$$A = [h, S] - F + J, \tag{3.25}$$

where F is the almost complex structure associated to Γ. A is related to the curvature by the formula

$$R = \frac{1}{3}[J, A]. \tag{3.26}$$

We can prove this by the following :

$$[J, A] = [J, [h, S]] + [J, F] = [h, [J, S]] - R$$
$$= -[h, \Gamma] - R = -2[h, h] - R = 3R.$$

[1] This tensor is called Jacobi endomorphism in [Sa], [CSMBP]

Typical and atypical sprays

In this paragraph we give some definitions and describe properties related to the Douglas tensor.

Proposition 3.6 *Let $\mathcal{D}$ be the distribution spanned by the spray S and the canonical vertical vector field C. Then $\mathcal{D}$ is an integrable distribution if and only if there exists a function μ such that $vS = \mu C$. In particular, if S is homogeneous, then $\mathcal{D}$ is integrable.*

Indeed, $\Gamma S = [J,S]S = [C,S]$, so

$$vS = \frac{1}{2}(I - \Gamma)S = \frac{1}{2}(S - [C,S]).$$

Consequently if $vS = \mu C$, then $[C,S] = S - 2\mu C$, so $\mathcal{D}$ is integrable. Conversely, if $\mathcal{D}$ is an integrable distribution, then there exist functions a, b such that $[C,S] = aS + bC$. Therefore we have $J[C,S] = C = aC$. Hence $a = 1$. Since $S - 2vS = aS + bC$, we find that $vS = -\frac{b}{2}C$.

◇

Definition 3.16 Let L be a semi-basic vector-valued 1-form on TM. We use

$$\tilde{L} = LF + FL,$$

where F is the almost complex structure associated with the connection $[J,S]$.

Locally, if $L = L^{\beta}_{\alpha}(x,y)\, dx^{\alpha} \otimes \frac{\partial}{\partial y^{\beta}}$, i.e. in the matrix form

$$L = \begin{pmatrix} 0 & 0 \\ L^{\beta}_{\alpha} & 0 \end{pmatrix},$$

then we have:

$$\tilde{L} = \begin{pmatrix} L^{\beta}_{\alpha} & 0 \\ L^{\beta}_{\gamma}\Gamma^{\gamma}_{\alpha} - \Gamma^{\beta}_{\gamma}L^{\gamma}_{\alpha} & L^{\beta}_{\alpha} \end{pmatrix}$$

It follows that the eigenvalues of $\tilde{L}$ are the eigenvalues of the matrix (L^{β}_{α}) with double multiplicity, and $\tilde{L}$ is diagonalizable if and only if the matrix (L^{β}_{α}) is diagonalizable. More precisely, we have

Proposition 3.7 *The following properties are equivalent.*

1) $\bar{L}X = \lambda X$,
2) $\bar{L}FX = \lambda FX$,
3) $LX = \lambda JX$ and $LFX = \lambda vX$,
   i.e. if $X = X^\alpha \frac{\partial}{\partial x^\alpha} + \overline{X}^\alpha \frac{\partial}{\partial y^\alpha}$, then
   $$L^\beta_\alpha X^\alpha = \lambda X^\beta \quad \text{and} \quad L^\beta_\alpha(\overline{X}^\alpha + X^\gamma\Gamma^\alpha_\gamma) = \lambda(\overline{X}^\beta + X^\gamma\Gamma^\beta_\gamma).$$

Corollary 3.2 *If Γ is a connection and h denotes the horizontal projection associated to Γ, then if X is an eigenvector of $\bar{L}$ with eigenvalue λ and $hX \neq 0$, then hX and JX are also eigenvectors of $\bar{L}$ with eigenvalue λ.*

In order to present all computations in a coordinate free way, we will present it in terms of local bases, chosen in relation to the natural geometrical structures which come with the given system. We introduce the

Definition 3.19 Let L be a semi-basic vector-valued 1-form on TM and Γ a connection on TM. The basis $B := \{h_i, v_i\}_{i=1,2}$ of $T_v(TM)$ is called an *adapted basis* of $\bar{L}$ if B is a Jordan basis of $\bar{L}$ such that the vectors h_i are horizontal, and $v_i = Jh_i$.

In the next chapters we will study the cases where the spray S is or is not an eigenvector of the Douglas tensor. We will consider the following

Definition 3.20 The spray S is called *typical*, if it is an eigenvector of the tensor $\bar{A}$.

The terminology is justified by the fact that the class of typical sprays contains the quadratic and the homogeneous sprays, and also the spray of the geodesics of linear connections. More generally we have the

Proposition 3.8 *If the distribution D spanned by S and C is integrable, then S is typical. In particular the homogeneous (quadratic) sprays are typical.*

Indeed, if D is integrable, then by the above proposition there exists a function μ such that $vS = \mu C$. Therefore

$$\begin{aligned} AS &= v[h,S]S = v[hS,S] = -v[vS,S] = -v[\mu C,S] \\ &= \mathcal{L}_S\mu C - v[C,S] = \mathcal{L}_S\mu C + vS = (\mathcal{L}_S\mu + \mu)C \end{aligned}$$

Since $FC = hS$, one finds that $\bar{A}(hS) = \lambda(hS)$ where $\lambda := \mathcal{L}_S\mu - \mu$, so hS is a horizontal eigenvector of $\bar{A}$. Therefore $vS = 0$, and the spray is typical.

If $vS \neq 0$ we can see, using the Proposition 3.5, that $C = J(hS)$ is a vertical eigenvector of $\bar{A}$ corresponding to the eigenvalue λ. Then $vS = \mu C$ is also an eigenvector of $\bar{A}$ corresponding to the eigenvalue λ. Therefore $S = hS + vS$ is an eigenvector of $\bar{A}$, i.e. S is typical.

◇

3.4 The Lagrangian

Definition 3.21 A *Lagrangian* is a map $E : TM \to \mathbf{R}$ smooth on $TM \setminus \{0\}$ and C^1 on the 0-section. E is called *regular* if the 2-form $\Omega_E := dd_J E$ has maximal rank.

If $\mathcal{L}_C E = 2E$ and E is C^2 on the 0-section, then E is quadratic and it defines a *(pseudo)-Riemannian* metric on M by

$$g(v, v) = 2E(v),$$

$v \in TM$. If $\mathcal{L}_C E = 2E$ and E is C^1 on the null-section, then E defines a *Finsler* structure.

Note that from the equation $[J, J] = 0$ we find that $i_J dd_J = d_J^2 = d_{[J,J]} = 0$, so for every Lagrangian E we have

$$i_J \Omega_E = 0. \tag{3.27}$$

Proposition 3.9 *A regular Lagrangian E allows us to define a (pseudo)-Riemannian metric on the vertical bundles, by putting*

$$g_E(JX, JY) = \Omega_E(JX, Y) \tag{3.28}$$

Indeed, $g_E(JX, JY)$ is well-defined because if Y' is another vector on TM such that $JY = JY'$, so $Y - Y'$ is vertical, then $\Omega_E(JX, Y - Y') = 0$ (because $i_J\Omega = 0$) and therefore $\Omega_E(JX, Y) = \Omega_E(JX, Y')$. On the other hand, $g_E(JX, JY) = \Omega_E(JX, Y) = -\Omega_E(X, JY) = g_E(JY, JX)$, because $i_J\Omega_E = 0$.

Moreover g_E is not degenerated because if $g_E(JX, JY) = 0$ for any JY, then $\Omega_E(JX, Y) = 0$ for any Y, which is impossible, because Ω_E is not degenerated.

The local expression of the 2-form Ω_E is

$$\Omega_E = \frac{1}{2}\left(\frac{\partial^2 E}{\partial x^\alpha \partial y^\beta} - \frac{\partial^2 E}{\partial x^\beta \partial y^\alpha}\right) dx^\alpha \wedge dx^\beta - \frac{\partial^2 E}{\partial y^\alpha \partial y^\beta}\, dx^\alpha \wedge dy^\beta. \qquad (3.29)$$

and the Lagrangian E is regular if and only if

$$\det\left(\frac{\partial^2 E}{\partial y^\alpha \partial y^\beta}\right) \neq 0.$$

Using

$$g_{\alpha\beta} = g\left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\right)$$

we obtain

$$g_{\alpha\beta} = \frac{\partial^2 E}{\partial y^\alpha \partial y^\beta}.$$

Proposition 3.10 *[Go] - Let $E: TM \to R$ be a regular Lagrangian. The vector field S on TM defined by*

$$i_S\Omega_E = d(E - \mathcal{L}_C E) \qquad (3.30)$$

is a spray and the paths of S are the solutions to the Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial E}{\partial y^\alpha} - \frac{\partial E}{\partial x^\alpha} = 0, \qquad \alpha = 1, \ldots, n.$$

Indeed, if S is defined by this expression, one has

$$\begin{aligned} i_{JS}\Omega_E &= i_S i_J \Omega_E - i_J i_S \Omega_E = -i_J i_S \Omega_E = i_J(\mathcal{L}_C E - E) = d_J(\mathcal{L}_C E - E) \\ &= d_J \mathcal{L}_C E - d_J E = \mathcal{L}_C d_J E = i_C dd_J E = i_C \Omega_E, \end{aligned}$$

and then $JS = C$ because Ω_E is not degenerated. Locally, if f^α are the components of S (see 3.6), we have:

$$i_S\Omega_E = y^\alpha\left(\frac{\partial^2 E}{\partial x^\alpha \partial y^\beta} - \frac{\partial^2 E}{\partial x^\beta \partial y^\alpha}\right)dx^\beta - f^\alpha \frac{\partial^2 E}{\partial y^\alpha \partial y^\beta} dx^\beta - y^\alpha \frac{\partial^2 E}{\partial y^\alpha \partial y^\beta} dy^\beta$$

On the other hand we have

$$dE - \mathcal{L}_C dE = \left(\frac{\partial E}{\partial x^\alpha} - y^\beta \frac{\partial^2 E}{\partial x^\alpha \partial y^\beta}\right)dx^\alpha - y^\beta \frac{\partial^2 E}{\partial y^\alpha \partial y^\beta} dy^\alpha;$$

then $i_S\Omega_E = d(E - \mathcal{L}_C E)$ if and only if

$$f_\beta = \frac{\partial E}{\partial x_\beta} - y^\alpha \frac{\partial^2 E}{\partial x^\alpha \partial y^\beta},$$

where $f_\beta = g_{\alpha\beta} f^\alpha$. Now the paths of S are the solutions of the differential system

$$\frac{d^2 x^\alpha}{dt^2} - f^\alpha(x, \dot{x}) = 0.$$

which, taking the above relations into account, can be written as

$$\frac{\partial^2 E}{\partial \dot{x}^\alpha \partial \dot{x}^\beta}\ddot{x}^\alpha + \frac{\partial^2 E}{\partial \dot{x}^\alpha \partial x^\beta}\dot{x}^\alpha - \frac{\partial E}{\partial x^\beta} = 0$$

which are the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial E}{\partial \dot{x}^\alpha} - \frac{\partial E}{\partial x^\alpha} = 0, \qquad \alpha = 1, \ldots, n.$$

□

The connection $\Gamma = [J, S]$ is called the *natural connection associated* to E. In particular, if E is the quadratic form associated to a Riemannian metric, we can obtain the Levi-Civita connection. If E defines a Finsler structure, $\Gamma = [J, S]$ is the canonical connection defined by [Gr]

Definition 3.22 Let E be a Lagrangian; a vector $v \in TM$ has a *null length*, if $\Omega_E(C, S)_v = 0$, where S is an arbitrary spray.

This condition does not depend on the choice of S. In fact, in standard local coordinates (x, y) of TM, a vector $v \in TM$, with local expression $v = (x^\alpha, \dot{x}^\beta)$ has null length if and only if $g_{\alpha\beta}(x, \dot{x})\dot{x}^\alpha \dot{x}^\beta = 0$, where $g_{\alpha\beta} = \frac{\partial^2 E}{\partial y^\alpha \partial y^\beta}$.

Lemma 3.1 *If E is a regular Lagrangian, the interior of the set of the null length vectors is empty.*

Indeed $i_C\Omega_E = d_J(\mathcal{L}_C E - E)$, hence $\Omega_E(C, S) = \mathcal{L}_C(\mathcal{L}_C E - E)$. If $\Omega_E(C, S)$ vanishes on a open set U, then $\mathcal{L}_C E - E$ should be homogeneous of degree 0 on U and therefore, since it is C^0 on the zero section, it should be constant on the fibers of TM. Therefore $d_J(\mathcal{L}_C E - E) = 0$, that is $i_C\Omega_E = 0$, which is excluded because Ω_E has maximal rank.

◇

Definition 3.23 A spray S is called *variational* if there exists a smooth regular Lagrangian E which satisfies (3.30), the Euler-Lagrange equation.

Taking into account the local property of a Lagrangian associated to a variational spray given by the Lemma 3.1 we propose the following

Definition 3.24 A spray S is called *locally variational* in a neighborhood of $x \in TM$ if there exists an open neighborhood of x and a smooth regular Lagrangian E on U such that the interior of the set of the null length vectors is empty, and which satisfies (3.30), the Euler-Lagrange equation.

The aim of the following chapters is the local characterization of the second order ordinary differential equations which come from a variational principle, i.e. the local characterization of variational sprays.

Let us now introduce the following

Definition 3.25 Let E be a Lagrangian and S a spray on the manifold M, then the *Euler-Lagrange* form associated with E and S is

$$\omega_E := i_S\Omega_E + d\mathcal{L}_C E - dE. \tag{3.31}$$

It is easy to see that ω_E is semi-basic, and the local expression in the standard coordinate system on TM is

$$\omega_E = \sum_{i=1}^{n} \left[S\left(\frac{\partial E}{\partial y^i}\right) - \frac{\partial E}{\partial x^i}\right] dx^i.$$

Therefore along a curve $\gamma = (x(t))$ associated with S we have

$$\omega_E\big|_\gamma = \sum_{i=1}^{n} \left(\frac{d}{dt}\frac{\partial E}{\partial \dot{x}^i} - \frac{\partial E}{\partial x^i}\right) dx^i,$$

where d/dt denotes the derivation along γ. So, in order to find the solution to the inverse problem for a given second order differential system, we have to look for a regular Lagrangian such that $\omega_E \equiv 0$. Of course, for this purpose we must study the local integrability of the second order partial differential operator

$$P_1 : C^\infty(TM) \longrightarrow Sec\, T_1^*$$

called the *Euler-Lagrange operator* defined by

$$P_1 := i_S dd_J + d\mathcal{L}_C - d \tag{3.32}$$

Remark. To solve the inverse problem of the calculus of variations we will look for a *regular* Lagrangian associated with the spray. Supposing that the manifold M and the operator P_1 are analytical, we now only need to prove that

(1) the Euler-Lagrange operator (3.32), possibly enlarged with some compatibility conditions, is formally integrable, and
(2) there exists a second order regular formal solution.

To show 1) we use the theory of formal integrability of partial differential systems whose basic notions are given in Chapter 1, while the proof of 2) remains a simple linear algebraic computation in the space of the initial conditions.

3.5 Sectional curvature associated with a convex Lagrangian

Sectional curvature

Let E be a regular Lagrangian, $\Omega_E = dd_J E$, and g the associated metric defined by the equation (3.38) on the vertical bundle. If the Lagrangian E is convex, then the matrix $\left(\frac{\partial^2 E}{\partial y^i \partial y^j}\right)$ is positive definite, and g is a Riemannian metric. In this section we suppose that E is convex.

Lemma 3.2 *Let $L \in \Psi(TM)$ be an arbitrary (1-1) semi-basic tensor on TM. We define the function $\tilde{k}_L : T^v \setminus \{\lambda C\} \longrightarrow \mathbb{R}$ by:*

$$\tilde{k}_L(JX) = \frac{2g(L^o, JX)g(JX, C) - g(LX, JX)g(C, C)}{g(C, C)[g(JX, JX)g(C, C) - g(C, JX)^2]} .$$

Since $JX \neq \lambda C$, the denominator is not zero, according to the Cauchy-Schwartz inequality. If the equations $i_L \Omega_E = 0$ and $g(L^o, C) = 0$ hold, then

$$k_L(JX) = \tilde{k}_L(aJX + bC)$$

for any $a, b \in C^\infty(TM)$, $a \neq 0$.

This is borne out by

$$\dot{k}_L(aJX+bC) = \frac{2a^2 g(L^\circ, JX)g(JX,C) + 2ab\, g(L^\circ, JX)g(C,C)}{a^2 g(C,C)[g(JX,JX)g(C,C) - g(JX,C)^2]}$$
$$- \frac{[a^2 g(LX, JX) + ab\, g(L^\circ, JX) + ab\, g(LX, C)]g(C,C)}{a^2 g(C,C)[g(JX,JX)g(C,C) - g(JX,C)^2]} = \dot{k}_L(JX)$$

Geometrical Interpretation

Let $i : TM \times_M TM \to T^v$ be the natural isomorphism and $JX_{z_1} \in T^v_{z_1}$. Since $i^{-1}(JX_{z_1}) = (z_1, z_2)$ and $i^{-1}(C_{z_1}) = (z_1, z_1)$, JX and C are independent at $z_1 \in TM$ if and only if the vectors z_1 and z_2 are independent. Let $\mathcal{P}_{JX}$ be the plan spanned by (z_1, z_2). Since $i^{-1}(aJX+bC) = (z_1, az_2 + bz_1)$ we obtain:

$$\mathcal{P}_{JX} = \mathcal{P}_{aJX+bC}.$$

Thus the Lemma expresses the property that $\dot{k}_L$ depends only on the point $z_1 \in TM$ and on a 2-plan tangent to $\pi(z_1)$ containing z_1

Remark. Let L be a semi-basic vector-valued 1-form, such that $i_L \Omega_E = 0$, and put $\overline{L} := L - \frac{g(L^\circ, C)}{g(C,C)} J$. Then $i_{\overline{L}} \Omega_E = 0$ and $g(\overline{L}^\circ, C) = 0$. Therefore we can offer the following

Definition 3.26 Let E be a Lagrangian and L a semi-basic vector valued 1-form such that $i_L \Omega_E = 0$. The *sectional function* associated with E and L is the function defined by

$$k_L := \dot{k}_{\overline{L}}$$

In particular we will call the *sectional curvature* of E, denoted by k_A or more simply by k, the sectional function associated to the Douglas tensor A.

As we have seen, the sectional curvature depends only on a point $z \in TM$ and on a 2-plan tangent to $\pi(z)$ and containing z. A simple computation shows that

$$k_0(JX) = \frac{2g(L^v, C)}{g(C,C)^2} +$$

$$- \frac{2g(L^v, JX)g(C, JX) - g(LX, JX)g(C, C)] - g(JX, JX)g(L^v, C)}{g(C, C)[g(JX, JX)g(C, C) - g(JX, C)^2]}$$

Example 3.1 *The sectional curvature of Finsler manifolds.*

Let $E \in C^\infty(TM \setminus \{0\}$ be a homogeneous regular Lagrangian of degree 2 (i.e. E defines a Finsler structure on the manifold M) and $R = -\frac{1}{2}[h, h]$ the curvature of the canonical connection associated to E (cf. Paragraph 3.3). We have

$$\begin{aligned} R^0 &= -\frac{1}{2}[h, h]^v = -[hS, h] + [h[S, h] = [h, S] - [h, vS] - h[h, S] \\ &= v[h, S] - [h, vS] = A - [h, vS]. \end{aligned}$$

Moreover, from the homogeneity of E we have $[C, S] = S$ and therefore $vS = \frac{1}{2}(I - [J, S]S) = \frac{1}{2}(S - [C, S]) = 0$. Then:

$$k(JX) = \frac{g(R^0X, JX)}{g(C, JX)^2 - g(C, C)g(JX, JX)}.$$

So we find the sectional curvature usually introduced for a Finsler structure (cf. [Ru], page 117).

Example 3.2 *Sectional curvature of Riemann manifolds.*

In this case the Lagrangian E is quadratic. Let $\langle\ ,\ \rangle$ be the scalar product on the manifold M defined by $\langle v, v\rangle := E(v)$ and let $\mathcal{R}$ be the curvature tensor of the Levi-Civita connection associated to the scalar product:

$$\mathcal{R}(u, v)w = D_u D_v w - D_v D_u w - D_{[u,v]} w.$$

If $JX = i(u, v)$, we have

$$g(C, C) = \langle u, u\rangle, \quad g(C, JX) = \langle u, v\rangle, \quad \text{and} \quad g(JX, JX) = \langle v, v\rangle.$$

On the other hand $g(R^0X, JX) = \langle \mathcal{R}(u, v)u, v\rangle$, and so

$$k(u, v) = \frac{\langle \mathcal{R}(u, v)u, v\rangle}{\langle u, v\rangle^2 - \langle u, u\rangle\langle v, v\rangle}.$$

Therefore k is the usual sectional curvature of the Riemannian space.

Isotropy

Definition 3.27 We will say that the Lagrangian has *isotropic curvature* at $z \in TM$, $z \neq 0$, if the sectional curvature at z does not depend on the 2-plan containing the vector z.

Example 3.3 The case $A = \lambda J$

We find:

$$k(JX) = \frac{2\lambda}{g(C,C)} - \frac{-\lambda g(JX,JX)g(C,C) + 2\lambda g(C,JX)^2 - \lambda g(JX,JX)g(C,C)}{g(C,C)[g(JX,JX)g(C,C) - g(C,JX)^2]}$$

and therefore

$$k_{\lambda J} = 0$$

Example 3.4 The case $A = \mu i_C \Omega_E \otimes C$.

We have:

$$k(JX) = 2\frac{\mu g(C,C)^2}{g(C,C)^2} + \\ + \frac{\mu g(JX,C)^2 g(C,C) + 2\mu g(C,C)g(C,JX)^2 - \mu g(JX,JX)g(C,C)^2}{g(C,C)[g(JX,JX)g(C,C) - g(C,JX)^2]}$$

and therefore

$$k_{\mu i_C \Omega_E \otimes C} = \mu.$$

Remark. Since k_L is a C^∞-linear function on L, we have

$$k_{\lambda J + \mu i_C \Omega_E \otimes C} = \mu.$$

Note that in this case k does not depend on the 2-plan.

Definition 3.28 A spray S is called *flat*, if the associated Douglas tensor has the form $A = \lambda J$ for some function $\lambda \in C^\infty(TM)$.

Remark. The Example 3.3 shows us that if a flat spray is variational, then every associated Lagrangian has vanishing sectional curvature. For a 2-dimensional manifold, they correspond to the Case 1 in Douglas' terminology [Dou]

Proposition 3.11 *A Lagrangian has an isotropic sectional curvature k if and only if the Douglas tensor A has the form*

$$A = \lambda J + \alpha \otimes C + \beta \otimes A^0$$

where,

$$\begin{aligned}
\lambda &= \frac{g(A^0, C)}{g(C,C)} - k\, g(C,C),\\
\alpha &= \left(k - \frac{2g(A^0,C)}{g(C,C)^2}\right) i_C\Omega_E + \frac{1}{g(C,C)} i_{A^0}\Omega_E,\\
\beta &= \frac{1}{g(C,C)}\, i_C\Omega_E.
\end{aligned}$$

Indeed, the sectional curvature is isotropic if and only if for every vertical vector $JX \in T_z^v$ the quadratic form

$$\begin{aligned}
q(JX) &= \{kg(C,C) - 2g(A^0,C)\}\{g(JX,JX)g(C,C) - g(JX,C)^2\}\\
&\quad + g(AX,JX)g(C,C) - 2g(A^0,JX)g(C,JX) + g(JX,JX)g(A^0,C)
\end{aligned}$$

vanishes identically. By polarising the quadratic form q, this condition can be expressed by the following equation:

$$\begin{aligned}
0 =& \{kg(C,C) - 2g(A^0,C)\}\{g(JX,JY)g(C,C) - g(JX,C)g(JY,C)\}\\
&+ \frac{1}{2}\{g(AX,JY) + g(AY,JX)\}g(C,C) - g(A^0,JX)g(C,JY)\\
&- g(A^0,JY)g(C,JX) + g(JX,JY)g(A^0,C).
\end{aligned}$$

Taking into account that $g_E(AX,JY) = g_E(AY,JX)$ which follows from the equation $i_A\Omega_E = 0$ (see paragraph 4.2), we can easily obtain the expression of A.

Remark. Note that if the Douglas tensor of a variational spray has the form

$$(*) \qquad A = \lambda J + \alpha \otimes C + \beta \otimes A^0,$$

with some function λ and semi-basic 1-forms α, β, then the associated Lagrangian has not necessarily isutropic curvature.

Indeed, the equation $i_A\Omega_E = 0$ gives $\alpha \wedge i_C\Omega_E + \beta \wedge i_{A^0}\Omega_E = 0$. If A^0 and C are independent, then by Cartan's lemma we find $\alpha = a\, i_C\Omega_E + b\, i_{A^0}\Omega_E$ and $\beta = b\, i_C\Omega_E + d\, i_{A^0}\Omega_E$.

On the other hand, taking the potential of (*) we get $A^0 = \lambda C + \alpha^0 C + \beta^0 A^0$. Therefore $\beta^0 = 1$ and $\alpha^0 = -\lambda$. Since C and A^0 are independent, the conditions of Proposition (3.11) hold if and only if $d = 0$ (we can take $b = \frac{1}{g(C,C)}$, $\beta' = \frac{1}{g(C,C)} i_C\Omega_E$ and $k = a + \frac{2g(A^0,C)}{g(C,C)}$ for the sectional curvature)

Nevertheless, if the vectors C and $A^0 = A(S)$ are proportional (i.e. the Douglas tensor has the form $A = \lambda J + \alpha \otimes C$ where α is semi-basic 1-form), then the Lagrangian associated to the spray has an isotropic curvature. Indeed, if E is the Lagrangian associated to the spray S, $i_A\Omega_E = 0$, then $\alpha \wedge i_C\Omega_E = 0$, and therefore $\alpha = \mu\, i_C\Omega_E$. Thereby the conditions of the above Proposition hold, and the sectional curvature is isotropic (it is equal to μ as we have already computed).

Taking into consideration the preceding remark, we propose the following

Definition 3.29 A spray is *isotropic* if its Douglas tensor has the form

$$A = \lambda J + \alpha \otimes C, \tag{3.33}$$

where $\lambda \in C^\infty(TM)$ and α is a semi-basic 1-form on TM.

From the preceding remark we know that if an isotropic spray is variational, then every associated Lagrangian has isotropic curvature. Our goal in Chapter 7 is to examine the conditions for the existence of a Lagrangian (and therefore the existence of a Lagrangian with isotropic curvature) associated to a spray

Chapter 4

Necessary Conditions for Variational Sprays

In this chapter we define a graded Lie algebra associated to a second order differential equation. By using this notion in Theorem 4.1 we find a large system of differential equations on the Lagrangian (and algebraic conditions on the so-called "variational multiplier"). This gives more effective conditions to the existence of a solution to the inverse problem of the calculus of variations (Theorems 4.3 and 4.4).

4.1 Identities satisfied by variational sprays

Proposition 4.1 *Let E be a Lagrangian, Γ a connection on M, h the associated horizontal projection, and F the associated almost-complex structure. The following properties are equivalent:*

a) $i_\Gamma \Omega_E = 0$,

b) $i_F \Omega_E = 0$,

c) $\Omega_E(hX, hY) = 0 \quad \forall\, X, Y \in TTM$ *i.e. the horizontal distribution is Lagrangian.*

Indeed,

$$\begin{aligned} i_\Gamma \Omega_E(hX, hY) &= 2\Omega_E(hX, hY), \\ i_\Gamma \Omega_E(hX, JY) &= \Omega_E(hX, JY) - \Omega_E(hX, JY) = 0, \\ i_\Gamma \Omega_E(JX, JY) &= -2\Omega_E(JX, JY) = 0. \end{aligned}$$

so a) is equivalent to c). On the other hand we have

$$
\begin{aligned}
i_\Gamma\Omega_E(hX, hY) &= -\Omega_E(JX, hY) - \Omega_E(hX, JY) = -i_J\Omega(hX, hY) = 0,\\
i_\Gamma\Omega_E(hX, JY) &= -\Omega_E(JX, JY) + \Omega_E(hX, hY) = \Omega_E(hX, hY),\\
i_\Gamma\Omega_E(JX, JY) &= \Omega_E(hX, JY) + \Omega_E(JX, hY) = 0,
\end{aligned}
$$

so b) is equivalent to c).

□

Definition 4.1 A connection is called *Lagrangian* with respect to E, if it satisfies the above conditions.

Proposition 4.2 *Let E be a Lagrangian on M, and $\Gamma = [J, S]$ the connection associated to S. Then*

$$d_J\omega_E = i_\Gamma\Omega_E. \tag{4.1}$$

In particular, if the spray S is variational and E is a Lagrangian associated to S, then Γ is Lagrangian with respect to E.

Proof. The Euler-Lagrange form can be written as follows:

$$
\begin{aligned}
\omega_E &= i_S dd_J E + d\mathcal{L}_C E - \mathcal{L}_S d_J E - dE = d_J\mathcal{L}_S E - i_{[J,S]}dE\\
&= d_J\mathcal{L}_S E - 2d_h E.
\end{aligned}
$$

Since $[J, J] = 0$, we have $d_J^2 = d_J \circ d_J = d_{[J,J]} = 0$, so

$$
\begin{aligned}
d_J\omega_E &= -2d_J d_h E = 2d_h d_J E = 2(i_h dd_J E - di_h d_J E)\\
&= 2i_h\Omega_E - 2\Omega_E = i_\Gamma\Omega_E
\end{aligned}
$$

If the spray is variational and E is a Lagrangian associated with S, we have $\omega_E = 0$, then $i_\Gamma\Omega_E = 0$, so the connection associated to the spray is Lagrangian.

□

Proposition 4.3 *Let S be a spray, E a Lagrangian on M. Then*

$$i_A\Omega_E = d_h\omega_E - \frac{1}{2}\mathcal{L}_S d_J\omega_E + i_R\Omega_E, \tag{4.2}$$

where A is the Douglas tensor of S. In particular if S is variational and E is a Lagrangian associated to S, then

$$i_A\Omega_E = 0. \tag{4.3}$$

Proof Since $i_J\Omega_E = 0$, one has

$$\begin{aligned} i_A\Omega_E &= i_{[h,S]}\Omega_E + i_F\Omega_E = i_h\mathcal{L}_S\Omega_E - \mathcal{L}_S i_h\Omega_E + i_F\Omega_E \\ &= i_h d\omega_E - \mathcal{L}_S(\Omega_E + \frac{1}{2}d_J\omega_E) + i_F\Omega_E \\ &= i_h d\omega_E - d\omega_E - \frac{1}{2}\mathcal{L}_S d_J\omega_E + i_F\Omega_E = d_h\omega_E - \frac{1}{2}\mathcal{L}_S d_J\omega_E + i_F\Omega_E \end{aligned}$$

which shows (4.2).

Moreover, if E is a Lagrangian associated to S, then $\omega_E = 0$ and the connection Γ is Lagrangian. Therefore every term on the right side of the equation (4.2) vanishes, so $i_A\Omega_E = 0$.

□

Definition 4.2 Let S be a spray on M, and $L \in \Psi(TM)$ semi-basic. Then we put

$$L' := h^*v[S, L]. \tag{4.4}$$

where $h^*L(X_1, ..., X_l) := L(hX_1, ..., hX_l)$. The tensor L' is called the *semi-basic derivation* of L with respect to the spray S.

It is clear from the definition that L' is semi-basic.

Proposition 4.4 *Let S be a spray on M and $L \in \Psi(TM)$ semi-basic. We have the formula*

$$L' = [S, L] + FL - L\barwedge F. \tag{4.5}$$

In particular, suppose that S is variational, E being a Lagrangian associated to S. If the equation $i_L\Omega_E = 0$ holds, then the equations

$$i_{L'}\Omega_E = 0, \quad i_{L''}\Omega_E = 0, \quad i_{L'''}\Omega_E = 0, \quad etc. \tag{4.6}$$

hold too.

Proof. From the definition we find that

$$L'(X_1,\ldots,X_l) = v[S,L](hX_1,\ldots,hX_l) = v[S,L(X_1,\ldots,X_l)] - \sum_{i=1}^{l} L(X_1,\ldots,[S,hX_i],\ldots,X_l)$$

$$= [S,L(X_1,\ldots,X_l)] - h[S,L(X_1,\ldots,X_l)] - \sum_{i=1}^{l} L(X_1,\ldots,[S,h]X_i,\ldots,X_l)$$

$$- \sum_{i=1}^{l} L(X_1,\ldots,[S,X_i],\ldots,X_l) =$$

$$= [S,L](X_1,\ldots,X_l) + FL(X_1,\ldots,X_l) - \sum_{i=1}^{l} L(X_1,\ldots,h[S,h]X_i,\ldots,X_l).$$

Using the identity $h[S,h] = F + J$ and the hypothesis that L is semi-basic, we obtain the equation (4.5).

On the other hand, from (4.5) one has

$$i_{L'}\Omega_E = i_{[S,L]}\Omega_E + i_{FL}\Omega_E - i_{LF}\Omega_E = i_{[S,L]}\Omega_E + i_F i_L\Omega_E - i_L i_F\Omega_E$$
$$= \mathcal{L}_S i_L\Omega_E - d_L\Omega_E + i_F i_L\Omega_E - i_L i_F\Omega_E.$$

If S is variational and E is a Lagrangian associated to S, then $\omega_E = 0$, and the connection Γ is Lagrangian. Thus we have $i_F\Omega_E = 0$ (Propositions 4.1 and 4.2). Therefore if $i_L\Omega_E = 0$ holds, then $i_{L'}\Omega_E = 0$ holds too. Recursively one finds (4.6).

□

Definition 4.3 Let h be the horizontal projector associated to the connection $\Gamma = [J,S]$, and $L \in \Psi^1(TM)$ be semi-basic. We propose

$$d^h L := [h, L]. \tag{4.7}$$

Proposition 4.5 *If L is a semi-basic vector-valued form, then $d^h L$ is also semi-basic. Moreover, assume that S is variational and E is a Lagrangian associated to S. If the equation $i_L\Omega_E = 0$ holds, then the equation $i_{d^h L}\Omega_E = 0$ holds too.*

Proof. First we will show that $d^h L$ is semi-basic, that is $h^*\{v[h, L]\} = [h, L]$. Since L is semi-basic, i.e. $h \circ L = 0$ and $h^*(L) = L$, we have

$$[h, L](X_1, \dots, X_{l+1}) = \frac{-1}{(l-1)!} \sum_\sigma \varepsilon(\sigma) L([hX_{\sigma 1}, X_{\sigma 2}], X_{\sigma 3}, \dots, X_{\sigma(l+1)})$$

$$+ \frac{1}{l!} \sum_\sigma \varepsilon(\sigma) [hX_{\sigma 1}, L(X_{\sigma 2}, \dots, X_{\sigma(l+1)})] + \frac{(-1)^l}{l!} \sum_\sigma \varepsilon(\sigma) h[L(X_{\sigma 1}, \dots, X_{\sigma l}), X_{\sigma(l+1)}]$$

$$+ \frac{1}{(l-1)!\,2} \sum_\sigma \varepsilon(\sigma) L(h[X_{\sigma 1}, X_{\sigma 2}], X_{\sigma 3}, \dots, X_{\sigma(l+1)})$$

$$= \sum_{i=1}^{n} (-1)^{i+1} \left\{ [hX_i, L(X_1, \dots, \hat{X}_i, \dots, X_{l+1})] - h[X_i, L(X_1, \dots, \hat{X}_i, \dots, X_{l+1})] \right\}$$

$$- \sum_{i<j} (-1)^{i+j} L([hX_i, X_j] + [X_i, hX_j] - h[X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{l+1})$$

$$= \sum_{i=1}^{n} (-1)^{i+1} [h, L(X_1, \dots, \hat{X}_i, \dots, X_{l+1})](X_i)$$

$$+ \sum_{i<j} (-1)^{i+j} L\left(h[hX_i, X_j] + h[X_i, hX_j] - h^2[X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{l+1}\right)$$

$$= \sum_{i=1}^{n} (-1)^{i+1} [h, L(X_1, \dots, \hat{X}_i, \dots, X_{l+1})](X_i)$$

$$+ \sum_{i<j} (-1)^{i+j} L\left(R(X_i, X_j) + [hX_i, hX_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{l+1}\right)$$

$$= \sum_{i=1}^{n} (-1)^{i+1} [h, L(X_1, \dots, \hat{X}_i, \dots, X_{l+1})](X_i)$$

$$+ \sum_{i<j} (-1)^{i+j} L\left(h[hX_i, hX_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{l+1}\right)$$

where $\hat{}$ symbolizes the term which does not appear in the corresponding expression, and $\varepsilon(\sigma)$ denotes the sign of the permutation σ. It is clear that the value of the second term is vertical, and it vanishes when one of the arguments is vertical. On the other hand:

$$[h, L(X_1, \dots, \hat{X}_i, \dots, X_{l+1})](X_i) = [h, L(hX_1, \dots, \widehat{hX_i}, \dots, hX_{l+1})](hX_i + vX_i)$$

$$= [h^2 X_i, L(hX_1, \dots, \widehat{hX_i}, \dots, hX_{l+1})] - h[hX_i, L(hX_1, \dots, \widehat{hX_i}, \dots, hX_{l+1})]$$

$$+ [h(vX_i), L(hX_1, \dots, \widehat{hX_i}, \dots, hX_{l+1})] - h[vX_i, L(hX_1, \dots, \widehat{hX_i}, \dots, hX_{l+1})]$$

$$= v[hX_i, L(hX_1, \dots, \widehat{hX_i}, \dots, hX_{l+1})]$$

where we used the fact that the vertical distribution is integrable. So we realize that the value of $d^h L$ is vertical, and vanishes when one of the arguments is vertical. Thus $d^h L$ is semi-basic.

Now assume that S is variational, E is a Lagrangian associated to S, and L is a vector-valued semi-basic l-form. By the relation

$$(-1)^l i_{[h,L]} = i_h d_L - d_L i_h - d_{L \bar{\wedge} h}$$

and taking into account that $L \bar{\wedge} h = l\, L$, because L is semi-basic, we have

$$(-1)^l i_{d^h L} \Omega_E = (-1)^l i_{[h,L]} dd_J E = i_h d_L dd_J E - d_L i_h dd_J E - l\, d_L dd_J E.$$

If the equation $i_L \Omega_E = 0$ holds, then

$$\begin{aligned}(-1)^l i_{d^h L} \Omega_E &= i_h d i_L dd_J E - d_L i_{\frac{1}{2}(I-\Gamma)} dd_J E - l\, d i_L dd_J E \\ &= -l\, d i_L dd_J E - \frac{1}{2} d_L i_\Gamma dd_J E = 0.\end{aligned}$$

□

4.2 Graded Lie algebra associated to a second order ODE

Definition 4.4 The graded Lie algebra $(\mathcal{A}_S, [\ ,\ ])$ associated to the spray S is the graded Lie sub-algebra of the vector-valued forms spanned by the vertical endomorphism J, the Douglas tensor A, and generated by the action of the semi-basic derivation defined in (4.4), the derivation d^h, and the Frölicher-Nijenhuis bracket $[\ ,\ ]$. The graduation of $\mathcal{A}_S$ is given by

$$\mathcal{A}_S = \oplus_{k=1}^{n} \mathcal{A}_S^k \qquad (4.8)$$

where $\mathcal{A}_S^k := \mathcal{A}_S \cap \Psi^k(TM)$.

Remark. Note that J and A are semi-basic and that, as we showed in the preceding paragraph, the space of semi-basic forms is stable by semi-basic derivation defined in (4.4), by the derivation d^h, and by the Frölicher-Nijenhuis bracket. It follows that $\mathcal{A}_S$ is a graded Lie sub-algebra of the vector-valued *semi-basic* forms.

The importance of the graded Lie algebra associated to a spray is given by the following

Theorem 4.1 *Let S be a variational spray and E a Lagrangian associated to S. Then for every element L of $\mathcal{A}_S$ the equation*

$$i_L\Omega_E = 0 \tag{4.9}$$

holds. Therefore every element of $\mathcal{A}_S$ gives a (necessary) algebraic condition on $g_{\alpha\beta} = \frac{\partial^2 E}{\partial y^\alpha \partial y^\beta}$.

Remark. The regular matrix $g_{\alpha\beta} = \frac{\partial^2 E}{\partial y^\alpha \partial y^\beta}$ is called a variational multiplier.

To prove Theorem 4.1 we will first show that J and A satisfy the equation (4.9). Then we will prove that all the vector-valued forms obtained from J and A by a finite number of successive operations which define $\mathcal{A}_S$, also satisfy the equation (4.9).

(1) From $[J, J] = 0$ we can easily obtain :

$$i_J\Omega_E = i_J dd_J E = d_J^2 E = d_{[J,J]}E = 0, \tag{4.10}$$

so the equation (4.9) holds for J.

(2) The Proposition 4.3 shows that the equation (4.9) also holds for $L = A$, where A is the Douglas tensor.

(3) from Propositions 4.4, and 4.5 respectively we know that if $i_L\Omega_E = 0$ holds for $L \in \mathcal{A}_S$ then

$$i_{L'}\Omega_E = 0,$$
$$i_{d_\Gamma L}\Omega_E = 0,$$

hold too.

(4) Let $K \in A^k_S(TM)$, $L \in A^l_S(TM)$ be semi-basic vector-valued forms, such that $i_K\Omega_E = 0$ and $i_L\Omega_E = 0$. Since K and L are semi-basic, we have $L\bar{\wedge}K \equiv 0$ and hence

$$\begin{aligned}(-1)^l i_{[K,L]}\Omega_E &= \left(i_K d_L - (-1)^{l(m-1)} d_L i_K - d_{L\bar{\wedge}K}\right)\Omega_E \\ &= i_K(i_L d - d i_L)\, dd_J E - (-1)^{l(m-1)} d_L i_K dd_J E - d_{L\bar{\wedge}K} dd_J E \\ &= i_K d i_L \Omega_E - (-1)^{l(m-1)} d_L i_K \Omega_E = 0.\end{aligned}$$

Therefore $i_J\Omega_E = 0$ and $i_A\Omega_E = 0$ hold, and the operations which generate $\mathcal{A}_S$ "preserve" the equation (4.9). Consequently for every $L \in \mathcal{A}_S$ the equation $i_L\Omega_E = 0$ holds.

□

We can see that the above Theorem gives *second order differential conditions* on the Lagrangian. Indeed, if the spray is variational and E is a regular Lagrangian associated with S, then locally one has

$$\Omega_E = \frac{1}{2}\left(\frac{\partial^2 E}{\partial x^\alpha \partial y^\beta} - \frac{\partial^2 E}{\partial y^\alpha \partial x^\beta}\right) dx^\alpha \wedge dx^\beta - \frac{\partial^2 E}{\partial y^\alpha \partial y^\beta} dx^\alpha \wedge dy^\beta. \qquad (4.11)$$

If $L \in \Psi^l(TM)$ is semi-basic, then

$$i_L \Omega_E = \frac{1}{l!} \sum_{\alpha \in \mathfrak{S}_{l+1}} \epsilon(\alpha) L^\beta_{\alpha_1 \dots \alpha_l} \frac{\partial^2 E}{\partial y^\beta \partial y^{\alpha_{l+1}}} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_{l+1}}$$

So if $L \in A^l_S$, then the equation $i_L \Omega_E = 0$ gives a 2nd order partial differential equation in E. ($\mathfrak{S}_{p+l-1}$ denotes the $(p+l-1)!$-order symmetric group and $\epsilon(\alpha)$ is the sign of α.) Thereby these 2nd order equations give the algebraic equation

$$\sum_{\alpha \in \mathfrak{S}_{l+1}} \epsilon(\alpha) L^\beta_{\alpha_1 \dots \alpha_l} g_{\beta \alpha_{l+1}} = 0 \qquad (4.12)$$

in $g_{\alpha\beta} = \frac{\partial^2 E}{\partial y^\alpha \partial y^\beta}$.

The graded Lie algebra associated with the spray appears in a natural way on studying the integrability of the Euler-Lagrange equation. As we will see in the following chapters, the elements of A_S, more precisely the equations (4.9) and the equations (4.12) on the variational multiplier appear in the compatibility conditions of the Euler-Lagrange equation.

4.3 The rank of sprays

Using Theorem 4.1 we found a large set of tensors (the elements of A_S) which result in linear constraints on the variational multipliers. Using these equations we can formulate necessary conditions for the spray to be variational. The first one is the generalization of Douglas' VIII Theorem to the n-dimensional case (see also [AT], [Sa]).

Theorem 4.2 *If at $x \in TM$ one has*

$$\operatorname{rank}\{J, A, A', \ldots, A^{(k)}, \ldots\}_{k \in \mathbb{N}} > \frac{n(n+1)}{2},$$

then S is not variational in the neighborhood of x.

Proof. Let us suppose that S is variational with an associated regular Lagrangian E, and g is the bilinear symmetric 2-form (non-degenerate since E is regular) on T^v defined by the formula (3.28). If $L \in T^* \otimes T^v$, the condition $i_L \Omega_E = 0$ gives

$$g(LX, JY) = g(JX, LY), \qquad \forall X, Y \in T_x TM,$$

i.e. locally

$$g_{ik} L^k_j = g_{jk} L^k_i$$

where $g_{ij} := \dfrac{\partial^2 E}{\partial y^i \partial y^j}$. This means that L is symmetric with respect to g. Since the tensors $J, A, A', A'', \ldots, A^{(\frac{n(n+1)}{2}-1)}$ are elements of $\mathcal{A}_S$, we have $i_{A^{(k)}} \Omega_E = 0$. Therefore, if the spray is variational, then the tensors $J, A, A', A'', \ldots, A^{(\frac{n(n+1)}{2}-1)}$ are self-adjoint with respect to g. But the space of the $(1-1)$ tensors which are self-adjoint with respect to a regular matrix is $\dfrac{n(n+1)}{2}$-dimensional. Consequently, if the spray is variational, then J, A, A', A'', $\ldots$, $A^{(\frac{n(n+1)}{2}-1)}$ are linearly dependent.

□

If $\dim M = 2$, then $\mathcal{A}_S$ only contains the hierarchy given by the Douglas tensor and its semi-basic derivatives A', A'', etc. However, if $\dim M > 2$, then we find other hierarchies in $\mathcal{A}_S$ which give, in the generic case, new necessary conditions for the variational multipliers. For example, the curvature tensor R of the connection associated to S belongs to $\mathcal{A}_S$, because $J, A \in \mathcal{A}_S$ and $R = \frac{1}{3}[J, A] \in \mathcal{A}_S$ by (3.26). Therefore its semi-basic derivatives R', R'', etc. are also elements of $\mathcal{A}_S$ – more precisely elements of $\mathcal{A}^2_S$ (see also [SCM], [GM]).

Moreover, in $\mathcal{A}^2_S$ we also have the tensors $[A^{(k)}, A^{(l)}]$ where $k, l \geq 1$ which are generally linearly independent of the curvature's hierarchy.

Theorem 4.3 *Let S be a spray and $x \in TM$. If there exists an integer $k < n$ for which*

$$\dim A_S^k(x) > k\binom{n+1}{k+1}, \tag{4.13}$$

then the spray is not variational in a neighborhood of x.

Note that for $k = 1$ we obtain the Theorem 4.2.

Proof Let S be a spray and E a regular Lagrangian. We consider for every $k = 1, \ldots, n$ the morphism

$$\begin{array}{ccc} \Lambda^k T_v^* \otimes T^v & \xrightarrow{\psi_k} & \Lambda^{k+1} T_v^* \\ L & \longrightarrow & i_L \Omega_E \end{array}$$

By the regularity of E the 2-form Ω_E is symplectic, and the morphism ψ_k is onto. Indeed, it is easy to see that if $\{X_1, \ldots, X_n\}$ is a basis of T^v, then $\alpha_1, \ldots, \alpha_n \in T^*$ defined by $\alpha_i = i_{X_i}\Omega_E$, gives a basis of T_v^*. Consequently

$$\{\alpha_{i_1} \wedge \ldots \wedge \alpha_{i_k} \wedge \alpha_{i_{k+1}}\}_{1 \le i_1 < \ldots < i_{k+1} \le n} \tag{4.14}$$

is a basis of $\Lambda^{k+1} T_v^*$ and

$$\{\alpha_{i_1} \wedge \ldots \wedge \alpha_{i_k} \otimes X_{i_{k+1}}\}_{1 \le i_1 < \ldots < i_k \le n,\ 1 \le i_{k+1} \le n} \tag{4.15}$$

gives a basis of $\Lambda^k T_v^* \otimes T^v$. Moreover, if the components of $\Lambda \in \Lambda^{k+1} T_v^*$ with respect to the basis (4.14) are $\Lambda^{i_1 \ldots i_{k+1}}$, then $\Lambda = \psi_k(L)$ where $L = \Lambda_{i_1 \ldots i_{k+1}} \alpha_{i_1} \wedge \ldots \wedge \alpha_{i_k} \otimes X_{i_{k+1}}$. This proves that ψ_k is onto.

Theorem (4.1) shows us that if the spray is variational and E is a regular Lagrangian associated with S, then for every $x \in TM$ we have

$$A_S^k(x) \subset \operatorname{Ker} \psi_k(x).$$

Consequently if $\dim A_S^k(x) > \dim \operatorname{Ker} \psi_k$, then the spray is not variational. On the other hand ψ_k is onto, so

$$\operatorname{rank} \psi_k = \dim \Lambda^{k+1} T_v^* = \binom{n}{k+1} = \frac{n!}{(k+1)!(n+1-k)!}.$$

Thus

$$\dim \operatorname{Ker} \psi_k = n\binom{n}{k} - \binom{n}{k+1} = \frac{k(n+1)n!}{(k+1)!(n-k)!} = k\binom{n+1}{k+1} \tag{4.16}$$

and therefore we obtain Theorem 4.3.

□

Remark. In the 2-dimensional case, this condition means that the spray is not variational in the neighborhood of $x \in TM$ if the dimension of $A_S^1(x)$ is greater than 3, so we find the criterion given by Douglas' theorem VIII.

Definition 4.5 Let S be a spray, $x \in TM$, and let us consider the system of linear equations

$$\left\{ \sum_{\sigma \in \mathfrak{S}_{l+1}} \epsilon(\sigma) L^{j}_{i_1 \dots i_l} x_{j i_{l+1}} = 0 \;\middle|\; L \in A_S(x) \right\} \tag{4.17}$$

in the symmetric variables x_{ij} $(x_{ij} = x_{ji})$, where $L^j_{i_1 \dots i_l}$ are the components of $L \in A_S(x)$. The rank of the linear equations (4.17) is called the *rank of the spray at x*.

The flat sprays (cf. Definition 3.29) are obviously the rank null sprays.

Remark. As equation (4.12) shows, the rank of a spray gives the number of independent equations satisfied by the variational multipliers. Consequently, if the system (4.17) does not have a solution with $\det(x_{ij}) \neq 0$, then there is no variational multiplier for S, and therefore the spray is non-variational. Thus we can easily obtain the following

Theorem 4.4 *If at $x \in TM$ we have*

$$\operatorname{rank} S(x) \geq \frac{n(n+1)}{2},$$

then S is non-variational in a neighborhood of x.

□

Chapter 5

Obstructions to the Integrability of the Euler-Lagrange System

In this section we will consider the inverse problem of the calculus of variations in the case of n-dimensional manifolds and we will examine the integrability of the Euler-Lagrange equation. As far as possible we will carry out the study without restrictions either on the dimension or on the curvatures.

5.1 First obstructions for the integrability of the Euler-Lagrange operator

We denote by $J_k(\mathbb{R})$ the space of k-jets of the real-valued functions on TM, and let $R_2(P_1) \subset J_2(\mathbb{R})$ be the differential equation of the second order formal solutions of the Euler-Lagrange operator

$$P_1 := i_S dd_J + d\mathcal{L}_C - d.$$

We have the following

Proposition 5.1 *Let $p = j_2(E)_x$ be a second order formal solution in $x \in TM \setminus \{0\}$ of $R_2(P_1)$. Then p can be lifted into a 3rd order solution if and only if*

$$(i_\Gamma \Omega_E)_x = 0. \tag{5.1}$$

Remark. By Proposition 4.1 we know that this condition means that the connection associated to the spray must be Lagrangian with respect to the solution E in x.

Proof. Since the Euler-Lagrange operator is of a second order to find the first compatibility conditions, i.e. to examine if a given 2nd order formal solution can be lifted into a 3rd order solution, we will consider the following diagram:

$$\begin{array}{ccccccc}
 & & S^3T^* & \xrightarrow{\sigma_3(P_1)} & T^*\otimes T_x^* & \xrightarrow{\tau_C} & K_1 \dashrightarrow 0 \\
 & & \downarrow \iota & & \downarrow \iota & & \\
R_3 & \xrightarrow{\iota} & J_3(\mathbb{R}) & \xrightarrow{p_1(P_1)} & J_1(T_v^*) & & \\
\downarrow \pi_2 & & \downarrow \pi_2 & & \downarrow \pi_C & & \\
R_2 & \xrightarrow{\iota} & J_2(\mathbb{R}) & \xrightarrow{p_0(P_1)} & T_{v,x}^* & &
\end{array}$$

where $K_1 := \operatorname{Coker}\sigma_3(P_1)$. A simple calculation shows that the symbol of the Euler-Lagrange operator $\sigma_2(P_1)$ is

$$\sigma_2(P_1) \;:\; S^2T^* \longrightarrow T_v^*, \qquad [\sigma_2(P_1)\alpha](X) = \alpha(S, JX), \tag{5.2}$$

and the symbol of the prolonged system is given by

$$\sigma_3(P_1) \;:\; S^3T^* \longrightarrow T^*\otimes T_v^*, \qquad [\sigma_3(P_1)\beta](X,Y) = \beta(X,S,JY),$$

where $\alpha \in S^2T_x^*$, $\beta \in S^3T_x^*$, and $X, Y \in T_x$. Indeed, let $f, g \in C^\infty(TM)$ be two functions vanishing at x. We have

$$\begin{aligned}
\sigma_2(P_1)_x(df\odot dg)(X) &= [i_S dd_J - d\mathcal{L}_C - d]_x(fg)(X) = \\
&= \big\{i_S[dd_J f\, g - d_J f\wedge dg + df\wedge d_J g - f\, dd_J g] + dd\mathcal{L}_C f\, g + \\
&\quad + d\mathcal{L}_C f\, g + df\, d\mathcal{L}_C g + f\, dd\mathcal{L}_C g\big\}_x(X) = df(X)\, dg(S) + dg(X)\, df(S) \\
&= (df\odot dg)_x(S, JX).
\end{aligned}$$

and we obtain the expression (5.2) of the symbol of P_1.

Let us note that the operator P_1 is regular on $TM\setminus\{0\}$, because S only vanishes on the null-section.

In order to interpret the obstruction space $K_1 := \operatorname{Coker}\sigma_3(P_1)$, we first compute the dimension of $g_3(P_1) = \operatorname{Ker}\sigma_3(P_1)$. A symmetric tensor $B \in S^3T^*$ is an element of $g_3(P_1)$ if and only if

$$B(X, S, JY) = 0 \tag{5.3}$$

for every pair of vectors $X, Y \in T$. If $B = \{h_1 \dots h_n, v_1 \dots v_n\}$ is a basis adapted to the horizontal distribution determined by Γ, i.e. $h_i \in T^h$ and $v_i := Jh_i$ for $i = 1, \dots, n$, then the equation (5.3) gives

$$\begin{aligned} a) \quad & B(h_i, S, v_j) = 0, \\ b) \quad & B(v_i, S, v_j) = 0, \end{aligned} \tag{5.4}$$

$i, j = 1, \dots, n$. Using the symmetry of B, we find that (5.4a) produces n^2 independent equations, and (5.4b) produces $\frac{n(n+1)}{2}$ independent equations. So

$$\operatorname{rank} \sigma_3(P_1) = n^2 + \frac{n(n+1)}{2},$$

and the obstruction space K_1 is isomorphic to the space of semi-basic 2-forms: $K_1 \cong \Lambda^2 T_v^*$. So $\Lambda^2 T_v^*$ can be seen as the obstruction space. With the help of this interpretation we can compute the first integrability - or compatibility conditions for the Euler-Lagrange operator. Let $\tau_\Gamma : T^* \otimes T_v^* \to \Lambda^2 T_v^*$ be the morphism defined by

$$(\tau_\Gamma B)(X, Y) := B(JX, Y) - B(JY, X).$$

It is easy to see that $\tau_\Gamma \circ \sigma_3 = 0$ and $\dim(\operatorname{Ker} \tau_\Gamma) = n^2 + \frac{n(n+1)}{2}$. So we have the exact sequence

$$S^2 T^* \xrightarrow{\sigma_3(P_1)} T^* \otimes T_v^* \xrightarrow{\tau_\Gamma} \Lambda^2 T_v^* \longrightarrow 0. \tag{5.5}$$

Let ∇ be a linear connection on the tangent manifold of M, and let E be a second order solution at $x \in TM$. Using the results of the Paragraph 1.4 we know that $(j_2 E)_x$ can be lifted into a third order formal solution if and only if $\tau_\Gamma[\nabla(P_1 E)]_x = 0$. The Euler-Lagrange 1-form $\omega_E = P_1 E$ is semi-basic and vanishes at x so, using the Proposition 4.2, we arrive at

$$\tau_\Gamma[\nabla(P_1 E)]_x = d_J(P_1 E)_x = (d_J(i_S dd_J E + dL_C E - JE))_x = (i_\Gamma \Omega_E)_x$$

which proves the Proposition 5.1.

∎

If $n (= \dim M) = 1$, the above computation shows that every second order solution can be lifted into a third order solution. Indeed, in this case $\Lambda^2 T_v^* = 0$, every semi-basic 2-form vanishes, and therefore we find that $\tau_\Gamma \circ \nabla = 0$. Moreover, it is easy to show that the Euler-Lagrange operator is involutive and then it is formally integrable. Therefore every spray on a 1-dimensional manifold is variational.

The situation is different if the dimension of the manifold M is greater than one. The above computation shows that in higher dimensional cases there exists a compatibility condition for the Euler-Lagrange operator which is not identically satisfied for all the second order solutions. Therefore the Euler-Lagrange operator is not formally integrable: the space of second order solutions – or initial conditions – is too large; some of them cannot be lifted into a higher order. In order to eliminate the ones which cannot give a solution, we have introduced the compatibility conditions laid down in Proposition 5.1 into the Euler-Lagrange system. So we can consider the operator

$$P_2 := (P_1, P_\Gamma) \;:\quad C^\infty(TM) \longrightarrow Sec(T_v^* \oplus \Lambda^2 T_v^*), \tag{5.6}$$

where

$$P_\Gamma := i_\Gamma dd_J \;:\quad C^\infty(TM) \longrightarrow Sec(\Lambda^2 T_v^*).$$

5.2 Second obstructions for the Euler-Lagrange operator

Proposition 5.2 *A second order formal solution $p = j_2(E)_x$ of the operator P_2 at $x \in TM \setminus \{0\}$ can be lifted into a third order solution if and only if the equations*

$$\begin{cases} (i_A \Omega_E)_x = 0, \\ (i_R \Omega_E)_x = 0 \end{cases} \tag{5.7}$$

hold, where one denotes $\Omega_E := dd_J E$.

Proof. It is easy to show that the symbol of P_Γ is the morphism $\sigma_2(P_\Gamma)$: $S^2T^* \longrightarrow \Lambda^2 T_v^*$ given by

$$[\sigma_2(P_\Gamma)\alpha](X, Y) = 2\big[\alpha(hX, JY) - \alpha(hY, JX)\big] \tag{5.8}$$

$\alpha \in S^2T^*$, $X, Y \in T_x$. In fact, if $f, g \in C^\infty(TM)$ are two functions vanishing in $x \in TM$, we find

$$\begin{aligned}
\sigma_2(P_2)_x(df \odot dg)(X,Y) &= [i_\Gamma dd_J]_x(fg)(X,Y)\\
&= [i_{2h-I} dd_J]_x(df \odot dg)(X,Y) = 2(i_h dd_J - dd_J)_x(fg)(X,Y)\\
&= \left[i_h(-d_J \wedge dg + df \wedge d_J) + d_J \wedge dg - df \wedge d_J\right]_x(X,Y)\\
&= 2\left[df(hX)dg(JY) - df(hY)dg(JX) - df(JX)dg(hY) - df(JY)dg(hX)\right]_x\\
&= 2\left[(df \odot dg)_x(hX, JY) - (df \odot dg)(hY, JX)\right]_x .
\end{aligned}$$

so we obtain (5.8).

Since $P_2 = (P_1, P_\Gamma)$, where both P_1 and P_Γ are of second order, we have $\sigma_2(P_2) = (\sigma_2(P_1), \sigma_2(P_\Gamma))$, where $\sigma_2(P_2) : S^2T^* \longrightarrow T^* \oplus \Lambda^2T_1^*$. Of course, we also have $\sigma_3(P_2) = (\sigma_3(P_1), \sigma_3(P_\Gamma))$.

It is easy to see that P_2 is a regular differential operator on $TM \setminus \{0\}$. Now let us consider $v \in TM \setminus \{0\}$ and let $R_{2,v}(P_2) \subset J_2(\mathbb{R})$ be the space of the second order formal solution of P_2 in v. Then $R_{2,v}(P_2)$ will contain second order regular formal solutions.

Indeed, let (x^i) be a local coordinate system on M, (x^i, y^i) the associated coordinate system on TM in the neighborhood of v. If $p = j_k(E)_v \in J_k(TM, \mathbb{R})$ is a kth order jet of a real valued function E on TM we set

$$s_{a_1\ldots a_k \underline{b_{k+1}\ldots b_l}}(p) := \frac{\partial^l E}{\partial x^{a_1}\ldots\partial x^{a_k}\partial y^{b_{k+1}}\ldots\partial y^{b_l}}(v), \qquad 1 \leq l \leq k. \tag{5.9}$$

Then $(x^i, y^i, s, s_i, s_{\underline{i}})$, and $(x^i, y^i, s, s_i, s_{\underline{i}}, s_{ik}, s_{i\underline{k}}, s_{\underline{ik}})$ give a coordinate system on $J_1(\mathbb{R})$ and $J_2(\mathbb{R})$ respectively. A second order jet $(j_2E)_v = (x^i, y^i, p, p_i, p_{\underline{i}}, p_{ik}, p_{i\underline{k}}, p_{\underline{ik}})$ is a second order regular solution of P_2 in $v = (x^i, v^i)$ if and only if

$$\det(p_{\underline{ij}}) \neq 0, \tag{5.10}$$

and $(P_1E)_v = 0$, and $(P_\Gamma E)_v = 0$ are satisfied, i.e. if we have (5.10) and the *linear system*:

$$v^\alpha p_{\alpha\underline{i}} + f^\alpha p_{\underline{\alpha i}} - p_i = 0, \tag{5.11}$$

$$p_{ji} - p_{i\underline{j}} + \Gamma^\alpha_i p_{\underline{\alpha j}} - \Gamma^\alpha_j p_{\underline{\alpha i}} = 0 \tag{5.12}$$

for $i, j = 1, \ldots, n$, where f^α are the components of the spray and Γ^α_j are the coefficients of the connection $\Gamma = [J, S]$.

Choosing p_{j_1} such that the inequality (5.10) is realized, we can solve the system (5.11) and (5.12) for the pivot terms p_i and p_{j_1}. Therefore we can find a regular second order formal solution to P_2 in $u \in TM$.

In order to find the dimension of the obstruction space, we have to compute the kernel of the symbol of the prolonged operator. A symmetric tensor $B \in S^3T_x^*$ is an element of $g_3(P_2)_x$ if and only if the equations (5.3) and the equations

$$B(X, hY, JZ) - B(X, hZ, JY) = 0 \qquad (5.13)$$

hold for every $X, Y, Z \in T_x$. Using the basis $\{h_i, v_j\}_{\{i,j=1,\dots,n\}}$ of T_x adapted to the connection Γ, that is h_i is horizontal and $v_i = Jh_i$ for every $i = 1, \dots, n$, we find that $B \in S^3T^*$ is found in $g_3(P_2)$ if and only if (5.4) and the equations

$$\begin{aligned} &a) \quad B(h_i, h_j, v_k) = B(h_i, h_k, v_j), \\ &b) \quad B(v_i, h_j, v_k) = B(v_i, h_k, v_j), \end{aligned} \qquad (5.14)$$

hold for $i, j = 1, \dots, n$. Let us introduce the notation $B_{ij\underline{k}} := B(h_i, h_j, v_k)$ and $B_{i\underline{jk}} := B(h_i, v_j, v_k)$. The equations (5.14) show that a symmetric tensor B is an element of $g_3(P_2)$ if and only if the $B_{ij\underline{k}}$ and the $B_{i\underline{jk}}$ are symmetric in i, j, k. Since there is no other condition imposed on the symmetric components $B_{ijk} := B(h_i, h_j, h_k)$ and $B_{\underline{ijk}} := B(v_i, v_j, v_k)$, we can deduce that

$$\dim g_3(P_2) = 4\binom{n+2}{3}.$$

On the other hand, an element B of the space $g_3(P_2)$ is contained in $g_3(P_1)$ if and only if the equations (5.4) hold. Using the symmetry of B (5.4.b) we obtain $\frac{n(n+1)}{2}$ independent equations. Employing the equation of $g_3(P_2)$, the components which appear in the equations (5.4.a) are also completely symmetric, so in the system (5.4.a) there are $\frac{n(n-1)}{2}$ relations. Therefore we arrive at

$$\dim g_3(P_1) = \frac{4n(n+1)(n+2)}{6} - \frac{2n(n+1)}{2} = \frac{n(n+1)(2n+1)}{3} \qquad (5.15)$$

and

$$\operatorname{rank} \sigma_3(P_2) = \dim S^3T^* - \dim g_3(P_2) = \frac{n(n+1)(2n+1)}{3}. \qquad (5.16)$$

Let

$$\tau_2 : (T^* \otimes T_v^*) \oplus (T^* \otimes \Lambda^2 T_v^*) \longrightarrow \Lambda^2 T_v^* \oplus \Lambda^2 T_v^* \oplus \Lambda^3 T_v^* \oplus \Lambda^3 T_v^*$$

be the morphism defined by $\tau_2 = (\tau_F, \tau_A, \tau_H, \tau_{(J,J)})$, where

$$\begin{aligned}
\tau_F(B,C)(X,Y) &:= B(JX,Y) - B(JY,X) \\
\tau_A(B,C)(X,Y) &:= B(hX,Y) - B(hY,X) - \frac{1}{2}C(S,X,Y) \\
\tau_H(B,C)(X,Y,Z) &:= C(hX,Y,Z) + C(hY,Z,X) + C(hZ,X,Y) \\
\tau_{(J,J)}(B,C)(X,Y,Z) &:= C(JX,Y,Z) + C(JY,Z,X) + C(JZ,X,Y)
\end{aligned}$$

for $B \in S^2T^*$, $C \in S^3T^*$, and $X,Y,Z \in T_v$. If $K_2 = \operatorname{Im} \tau_2$, then the sequence

$$S^3T^* \xrightarrow{\sigma_3(P_2)} (T^* \otimes T_v^*) \oplus (T^* \otimes \Lambda^2 T_v^*) \xrightarrow{\tau_2} K_2 \longrightarrow 0 \qquad (5.17)$$

is exact.

Indeed, it is easy to check that $\tau_2 \circ \sigma_3(P_2) = 0$. On the other hand, $\operatorname{Ker} \tau_2$ is defined by the systems $\tau_F = 0$, $\tau_A = 0$, $\tau_H = 0$ and $\tau_{(J,J)} = 0$. These systems are independent, so that the rank of τ_2 is the sum of the ranks of τ_F, τ_A, τ_H and $\tau_{(J,J)}$. It is easy to see that $\operatorname{rank} \tau_F = \operatorname{rank} \tau_A = \frac{1}{2}n(n-1)$, $\operatorname{rank} \tau_H = \operatorname{rank} \tau_{(J,J)} = \frac{1}{6}n(n-1)(n-2)$. Then $\operatorname{rank} \tau_2 = \frac{n(n-1)(n+1)}{3}$, and

$$\begin{aligned}
\dim \operatorname{Ker} \tau_2 &= \dim((T^* \otimes T_v^*) \oplus (T^* \otimes \Lambda^2 T_v^*)) - \operatorname{rank} \tau_2 = \frac{n(n+1)(2n+1)}{3} \\
&= \operatorname{rank} \sigma_3(P_2).
\end{aligned}$$

Consequently $\operatorname{Im} \sigma_3(P_2) = \operatorname{Ker} \tau_2$, and the sequence (5.17) is exact.

Let us compute the compatibility condition of the operator P_2. We have the diagram

$$\begin{array}{ccccccc}
 & & S^3T^* & \xrightarrow{\sigma_3(P_2)} & (T^*\otimes T_v^*)\oplus(T^*\otimes\Lambda^2T_v^*) & \xrightarrow{\tau_2} & K_2 \longrightarrow 0 \\
 & & \downarrow \varepsilon & & \downarrow \varepsilon & & \\
R_3 & \longrightarrow & J_3R & \xrightarrow{p_1(P_2)} & J_1(T_v^*\oplus\Lambda^2T_v^*) & & \\
\downarrow \pi_2 & & \downarrow \pi_2 & & \downarrow \pi_0 & & \\
R_2 & \longrightarrow & J_2R & \xrightarrow{p_0(P_2)} & T_v^*\oplus\Lambda^2T_v^* & &
\end{array}$$

Let ∇ be an arbitrary linear connection on TM, and $p = (j_2E)_x$ a second order solution at the point $x \in TM \setminus \{0\}$. Then $(j_2E)_x$ can be lifted into a 3rd order solution if and only if $\tau_2\nabla(P_2E)|_x = 0$. Since $(\omega_E)_x = 0$ and $(i_\Gamma\Omega_E)_x = 0$, we find that

$$\begin{aligned}
\tau_{VV}[\nabla(P_2E)]_x &= (d_JP_1E)_x = (d_J\omega_E)_x = (i_J\Omega_E)_x = 0 \\
\tau_{VH}[\nabla(P_2E)]_x &= (d_h\omega_E - \tfrac{1}{2}\mathcal{L}_S i_J\Omega)_x = (i_A dd_J E)_x \\
\tau_{H}[\nabla(P_2E)]_x &= (d_hP_\Gamma E)_x = (d_h(i_\Gamma\Omega_E))_x = \tfrac{1}{2}(i_{[h,h]}\Omega_E)_x - (i_R\Omega_E)_x \\
\tau_{J,J}[\nabla(P_2E)]_x &= (d_JP_\Gamma E)_x = d_J(d_J\omega_E)_x = \tfrac{1}{2}(d_{[J,J]}\omega_E)_x = 0
\end{aligned}$$

Therefore

$$\tau_2[\nabla(P_2E)]_x = \left(0,\, i_A\Omega_x,\, i_R\Omega_x,\, 0\right), \tag{5.18}$$

which shows the proposition.

□

The equations (5.7) are not satisfied in the generic case. However, we can find a certain special class of sprays, for which these obstructions are identically satisfied. We will consider this class of sprays in the Paragraph 7.1

Chapter 6

The Classification of Locally Variational Sprays on Two-dimensional Manifolds

In his paper [Dou], using Riquier's theory, Douglas classifies second order variational differential equations, also called variational sprays, with two degrees of freedom. In this chapter we will reconsider this problem i.e. the classifications of variational sprays on 2-dimensional manifolds. However we will use a different approach to Douglas': instead of working with a differential system on the variational multiplier, we will study directly the integrability of the *Euler-Lagrange* system, as it is more natural. This approach allows us to present all the obstructions in a natural and intrinsic way.

As we saw in the previous chapter, the first non-trivial case is when the dimension of the manifold is two. Its study is interesting, because all kinds of obstructions to solving an over-determined partial differential system arise (problems with the first and higher order compatibility, involutivity, 2-acyclicity etc). In this chapter we give the complete, explicit and coordinate-free classification of the variational second order differential equations.

We note that the analysis is much more complicated on higher dimensional manifolds because we have to take the equation involving the curvature tensor into consideration. However, if the dimension is fixed, the study is analogous to the 2-dimensional case.

We assume that the manifolds and the other objects (tensors, functions etc.) are analytic. If an object is defined on the tangent bundle, then it is assumed to be analytic away from the zero section.

We have shown in Section 5.1 that the Euler-Lagrange differential operator is not formally integrable. Introducing its compatibility conditions in the system we defined the differential operator P_1. The compatibility conditions of this second operator are given by the equations (5.7). Both forms $i_A \Omega_E$ and $i_R \Omega_E$ are semi-basic. If $\dim M = 2$, then the space of the semi-basic 3-forms is the trivial null-space, so the equation $i_R \Omega_E = 0$ holds for every Lagrangian E.

On the other hand, as we saw in Chapter 4, using the graded Lie algebra associated with the spray and the notion of the rank of sprays, we can give simple criteria for the existence of a solution of the inverse problem. In particular, on a 2-dimensional manifold we have $A_S^k = 0$ for every $k > 1$. Therefore the rank of the spray is determined only by the dimension of A_S^1, i.e. by the dimension of the space of vector-valued 1-forms spanned by the vertical endomorphism J, the Douglas tensor A, and its semi-basic derivations: A', A'' etc. Since the equation (4.10) holds identically, the vertical endomorphism J does not give any restriction on the variational multipliers. Therefore on a 2-dimensional manifold the rank of the spray is given by the rank of the (1-1) semi-basic tensor field $(J, A, A', \ldots A^{(n)}, \ldots)_{n \in N}$:

$$\operatorname{rank} S + 1 = \operatorname{rank} \{J, A, A', \ldots A^{(n)}, \ldots\}_{n \in N}.$$

Proposition 6.1 *We have*

$$(fL)' = f'L + fL'$$

for any $L \in \Lambda^1 T_v^* \otimes T^v$ *and* $f \in C^\infty(TM)$, *where* $f' := \mathcal{L}_S f$. *So, if* $A^{(p+1)} = f_0 J + f_1 A + \ldots + f_p A^{(p)}$ *with* $f_0, \ldots, f_p \in C^\infty(TM)$, *then for every* $r > p+1$ *there exist* $g_0, g_1, \ldots, g_p \in C^\infty(TM)$ *such that*

$$A^{(r)} = g_0 J + g_1 A + \ldots + g_p A^{(p)}.$$

In particular, the rank of the spray is r *if and only if* $\{J, A, A', \ldots, A^{r-1}\}$ *is a basis of the* $C^\infty(TM)$*-module spanned by the tensors* $(J, A, A', A'', \ldots, A^{(k)}, \ldots)_{k \in N}$.

The rank of the spray only offers the first constraints on the second order solutions. However, it is natural to organize the study of the inverse problem depending on the rank of the spray, as Douglas does in his paper [Dou].

6.1 Flat sprays

We consider in this section the case when the spray is flat. This means that the Douglas tensor is proportional to the vertical endomorphism, thus there exists a function λ such that $A = \lambda J$. In this case the spray has rank 0, and it is isotropic (see Definition 3.29). Moreover, the sectional curvature of a Lagrangian associated with S vanishes (see Example 3.3).

Theorem 6.1 *Every flat spray is locally variational in a neighborhood of a point $x \in TM \setminus \{0\}$.*

Proof. Let us consider the second order differential operator P_2 defined in (5.6). It is regular and, as we showed in the Remark on page 87, at every point $x \in TM \setminus \{0\}$ there exists a regular second order formal solution of P_2.

Moreover, as we have shown (cf. Proposition 5.2), a second order formal solution $(j_2E)_x$ of P_2 can be lifted into a 3rd order solution if and only if $(i_A\Omega_E)_x = 0$. Now

$$(i_A\Omega_E)_x = (i_{\lambda J}\Omega_E)_x = \lambda(d_J d_J E)_x = \lambda(d_{[J,J]}E)_x = 0,$$

so every second order solution can be lifted into a third order solution.

The theorem is proved if we show that P_2 is also involutive. The construction of a quasi-regular basis is slightly different according to whether S is horizontal or not.

a) The spray is horizontal

Assume that the spray is horizontal, and let $\mathcal{B} = \{h_1, h_2 := S, v_1, v_2 := C\}$ be a basis, with h_1 horizontal and $v_1 := Jh_1$. Let $B \in S^2T^*$ be a symmetric tensor, and set $a_{ij} := B(h_i, h_j)$, $b_{ij} := B(h_i, v_j)$ and $c_{ij} := B(v_i, v_j)$. From (5.2) and (5.8) we find that if $B \in g_2(P_2) = \operatorname{Ker}\sigma_2(P_2)$, then

$$\begin{aligned} b_{12} - b_{21} &= B(h_1, C) - B(S, v_1) = 1/2\,[\sigma_2(P_2)B](h_1, S) = 0, \\ b_{21} &= B(S, v_1) = [\sigma_2(P_2)B](h_1) = 0, \\ b_{22} &= B(S, C) = [\sigma_2(P_2)B](C) = 0 \end{aligned}$$

Therefore

$$g_2(P_2) = \left\{ \begin{bmatrix} a_{11} & a_{12} & b_{11} & 0 \\ a_{12} & a_{22} & 0 & 0 \\ b_{11} & 0 & c_{11} & c_{12} \\ 0 & 0 & c_{12} & c_{22} \end{bmatrix} \right\},$$

where the components a_{11}, a_{12}, a_{22}, b_{11}, c_{11}, c_{12} and c_{22} can be chosen arbitrarily. Let us consider the basis $\bar{B} := \{e_1, e_2, v_1, v_2\}$ where $e_1 := h_1 + v_1$ and $e_2 := S + v_1 + v_2$. We shall prove that this basis is quasi-regular. Denoting the new components of B by $\bar{a}_{ij}$, $\bar{b}_{ij}$ and $\bar{c}_{ij}$ respectively, we find that $\bar{c}_{ij} = c_{ij}$ and that the block $\bar{b}_{ij}$ is given by

$$\bar{b} = \begin{bmatrix} b_{11} + c_{11} & c_{12} \\ c_{11} + c_{12} & c_{12} + c_{22} \end{bmatrix}.$$

Therefore the components $\bar{c}_{ij}$ are determined by the components $\bar{b}_{ij}$ by the following relations:

$$\begin{aligned} \bar{c}_{12} &= \bar{b}_{12}, \\ \bar{c}_{11} &= \bar{b}_{21} - c_{12} = \bar{b}_{21} - \bar{b}_{12}, \\ \bar{c}_{22} &= \bar{b}_{22} - c_{12} = \bar{b}_{22} - \bar{b}_{12}. \end{aligned}$$

Thus, in the basis $\bar{B}$ an element $B \in g_2(P_2)$ is determined by the components $\bar{a}_{11}$, $\bar{a}_{12}$, $\bar{a}_{22}$, $\bar{b}_{11}$, $\bar{b}_{12}$, $\bar{b}_{21}$ and $\bar{b}_{22}$. Now

$$g_2(P_2)_{e_1} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \bar{a}_{22} & \bar{b}_{21} & \bar{b}_{22} \\ 0 & \bar{b}_{21} & c_{11} & c_{12} \\ 0 & \bar{b}_{22} & c_{12} & c_{22} \end{bmatrix} \right\}$$

Then $\dim g_2(P_2)_{e_1} = 3$ because there are only 3 free parameters: $\bar{a}_{22}$, $\bar{b}_{21}$ and $\bar{b}_{22}$. Moreover

$$g_2(P_2)_{e_1,e_2} = g_2(P_2)_{e_1,e_2,v_1} = g_2(P_2)_{e_1,e_2,v_1,v_2} = \{0\}.$$

So we get

$$\dim g_2(P_2) + \sum_{i=1,2} \dim \left(g_2(P_2)\right)_{e_1,\dots,e_i} + \sum_{i=1,2} \dim \left(g_2(P_2)\right)_{e_1,e_2,v_1,\dots,v_i} = \dim g_3(P_2)$$

which shows that the basis $\tilde{B}$ is quasi-regular.

b) The spray is not horizontal

Let S not be horizontal, and consider the basis $B' = \{h_1, h_2, v_1, v_2\}$, where $h_2 := hS$, $v_2 := C$, $Jh_1 := v_1$ and $vS := v_1 + C$. We have

$$g_2(P_2) = \left\{ \begin{bmatrix} a_{11} & a_{12} & b_{12} & -(c_{11}+c_{12}) \\ a_{12} & a_{22} & -(c_{11}-c_{12}) & -(c_{12}+c_{22}) \\ b_{11} & -(c_{11}+c_{12}) & c_{11} & c_{12} \\ -(c_{11}+c_{12}) & -(c_{12}+c_{22}) & c_{12} & c_{22} \end{bmatrix} \right\}$$

where the components a_{11}, a_{12}, a_{22}, b_{11}, c_{11}, c_{12} and c_{22} can be chosen arbitrarily. Let us consider the basis $\tilde{B}' := \{e_1, e_2, v_1, v_2\}$, where $e_1 := h_1 + v_1$ and $e_2 := h_2 + v_2(-hS + C)$. In this basis we find that the new components are $\tilde{c}_{ij} = c_{ij}$ and

$$\tilde{b} = \begin{bmatrix} b_{11} + c_{11} & (c_{11} + c_{12}) + c_{12} \\ -(c_{11} + c_{12}) + 2c_{12} & -(c_{12} + c_{22}) + 2c_{22} \end{bmatrix}.$$

As in the previous case, the block $\tilde{c}_{ij}$ can be expressed with the help of the block $\tilde{b}_{ij}$:

$$\begin{aligned} \tilde{c}_{11} &= \tilde{b}_{12} \\ \tilde{c}_{12} &= \tilde{b}_{21} - \tilde{b}_{12} \\ \tilde{c}_{22} &= \tilde{b}_{22} + \tilde{b}_{21} - \tilde{b}_{12}. \end{aligned}$$

Consequently $B \in g_2(P_2)$ is determined by its components: a_{11}, $\tilde{a}_{12}$, $\tilde{a}_{22}$, $\tilde{b}_{11}$, $\tilde{b}_{12}$, $\tilde{b}_{21}$ and $\tilde{b}_{22}$. So, $\dim g_2(P_2)_{e_1} = 3$ and

$$g_2(P_2)_{e_1,e_2} = g_2(P_2)_{e_1,e_2,v_1} = g_2(P_2)_{e_1,e_2,v_1,v_2} = \{0\}.$$

Thus we obtain

$$\dim g_2(P_2) + \sum_{i=1,2} \dim \left(g_2(P_2)\right)_{e_1,e_i} + \sum_{i=1,2} \dim \left(g_2(P_2)\right)_{e_1,e_2,v_i} = \dim g_3(P_2)$$

which shows that $\tilde{B}'$ is a quasi-regular basis

□

Example 6.1 The simplest example of flat sprays is given by the following system [*]

$$\begin{cases} \ddot{x}_1 = 0, \\ \ddot{x}_2 = 0. \end{cases} \tag{6.1}$$

Of course, we have $\Gamma^i_j = 0$, $A = 0$, so the rank of the spray is 0. Therefore this spray is locally variational.

Example 6.2 Another example of flat sprays is given by the system

$$\begin{cases} \ddot{x}_1 = -\dfrac{1 + \dot{x}_1^2 + \dot{x}_2^2}{x_1}, \\ \ddot{x}_2 = y_2^2. \end{cases} \tag{6.2}$$

An easy computation gives: $\Gamma^1_1 = \frac{y_1}{x_1}$, $\Gamma^1_2 = \frac{y_2}{x_1}$, $\Gamma^2_1 = 0$ and $\Gamma^2_2 = -y_2$. Moreover, $A = 0$ and therefore the system (6.2) is locally variational.

6.2 Rank $S = 1$: Typical sprays.

As we have seen in the previous section, a second order formal solution $j_2(E)_x$ of the operator P_2 can be lifted into a third order solution if and only if $(i_A \Omega_E)_x = 0$. Note that this obstruction is expressed in terms of the unknown function E. In order to obtain a condition in terms only of S, we must introduce it into the system and study

$$\begin{cases} \omega = 0, \\ i_J \Omega = 0, \\ i_A \Omega = 0. \end{cases} \tag{6.3}$$

In other words, we have to study the differential operator

$$P_3 : C^\infty(TM) \longrightarrow Sec\,(T_v^* \oplus \Lambda^2 T_v^* \oplus \Lambda^2 T_v^*)$$

[*] In order to simplify the notation, in the sequel for the examples we denote by (x_i) (respectively (x_i, y_i)) the standard coordinate system on the manifold M (respectively on TM).

defined by

$$P_1 := \{P_2, P_A\}, \qquad \text{where} \qquad P_A = i_A dd_J. \tag{6.4}$$

The problem is completely different according to whether S is typical or not. If S is typical, then P_1 is involutive and the Cartan-Kähler Theorem leads to a simple result. Note that the class of typical sprays contains the homogeneous and the quadratic sprays which are the most important examples in differential geometry. In the non-typical case, the Spencer cohomology is not trivial and the results are much more complicated.

NOTATION. In the sequel, if $\{e_i\}$ is a base, then we will denote by $\xi^X_{e_i}$ the component of the vector X on e_i:

$$X := \sum \xi^X_{e_i} e_i. \tag{6.5}$$

In this section we will prove the following

Theorem 6.2 *Let S be a rank one typical spray and $x \in TM \setminus \{0\}$.*

(1) If $\dot{A}$ is diagonalizable, let $\{h_1, hS, Jh_1, C\}$ be an adapted Jordan base of $\bar{A}$, and α the semi-basic 1-form defined by $i_{hS}\alpha = 1$ and $i_{h_1}\alpha = 0$. Then S is variational on a neighborhood of $x \neq 0$ if and only if

$$D_{hX}\alpha \wedge \alpha = 0, \qquad \forall X \in \operatorname{Ker} \alpha, \tag{6.6}$$

where D is the Berwald connection associated with the spray (cf Paragraph 3.2).

(2) If $\dot{A}$ is non-diagonalizable, then S is non-variational. However, there exists a regular Lagrangian associated to S if and only if the function

$$\zeta := \xi^S_C - \mathcal{L}_C \xi^S_C + \xi^{[h_2,C]}_{h_2}$$

vanishes in a neighborhood of x, where h_2 is a horizontal vector field such that $\{hS, h_2, C, Jh_1\}$ is an adapted base of $\bar{A}$ in a neighborhood of x. In particular, if S is quadratic or homogeneous, then the condition $\zeta \equiv 0$ is identically satisfied.

Before proving the theorem we will show the following

Lemma 6.1 *Let S be a variational spray such that* rank $S \geq 1$, *E an associated Lagrangian, and $\{h_i, v_i\}_{i=1,2}$ an adapted Jordan base of $\bar{A}$.*

- *If $\bar{A}$ is diagonalizable, then*

$$\Omega_E(v_1, h_1) \neq 0 \quad \text{and} \quad \Omega_E(v_2, h_2) \neq 0, \tag{6.7}$$

- *if $\bar{A}$ is non-diagonalizable, then*

$$\Omega_E(v_1, h_2) \neq 0 \tag{6.8}$$

Indeed, using the adapted base $\{h_i, v_i\}_{i=1,2}$ the equation $i_A\Omega_E = 0$ gives $\Omega_E(v_1, h_2) = 0$ in the diagonalizable case, and $\Omega_E(v_1, h_1) = 0$ in the non-diagonalizable case. But E being regular, we have locally $\det\left(\frac{\partial^2 E}{\partial y^i \partial y^j}\right) \neq 0$ i.e.

$$\det\begin{pmatrix} \Omega_E(v_1, h_1), & \Omega_E(v_1, h_2) \\ \Omega_E(v_2, h_1), & \Omega_E(v_2, h_2) \end{pmatrix} \neq 0$$

Therefore we have the inequality (6.8) in the non-diagonalizable case and (6.7) in the diagonalizable case $\quad \diamond$

Remark. Taking the Remark of page 64 into account, Lemma 6.1 yields that in the case, where the Euler-Lagrange system with its compatibility conditions has no second order formal solutions so that the inequality (6.8) or (6.7) is satisfied, S is non-variational.

Let us return to the Theorem 6.2. The proof involves two steps. The first step is to show that P_1 is involutive (Lemma 6.2) and the 2nd step is to show that if the hypotheses of the theorem hold, then every second order solution can be lifted into a 3rd order solution (Lemmas 6.3 and 6.4).

Lemma 6.2 *The operator P_1 is involutive at $x \in TM \setminus \{0\}$.*

It is clear that P_1 is a second order differential operator. A simple computation shows that the symbol $\sigma_2(P_A) : S^2T^* \longrightarrow \Lambda^2 T_v^*$ of P_A is

$$[\sigma_2(P_A)\alpha](X, Y) = \alpha(AX, JY) - \alpha(AY, JX), \tag{6.9}$$

and the symbol $\sigma_3(P_A) : S^3T^* \longrightarrow T^* \otimes \Lambda^2 T_v^*$ of the prolonged system is

$$[\sigma_3(P_A)\beta](X, Y) = \beta(X, AY, JZ) - \beta(X, AZ, JY)$$

where $\alpha \in S^2T^*$, $\beta \in S^3T^*$ and $X, Y \in T$.

Indeed, let f, g be C^∞ functions both vanishing at $z \in T$; we have.

$$
\begin{aligned}
\sigma_2(P_A)_z(df \odot dg)(X,Y) &= \{\iota_A(-d_J f \wedge dg + df \wedge d_J g)\}(X,Y) \\
&= df(AX)dg(JY) - df(AY)dg(JX) + dg(AX)df(JY) - dg(AY)df(JX) \\
&= (df \odot dg)(AX, JY) - (df \odot dg)(AY, JX),
\end{aligned}
$$

which gives the formula (6.9) of the symbol of P_3. Since $g_2(P_3) = g_2(P_1) \cap g_2(P_F) \cap g_2(P_A)$, $B \in S^2T^*$ lies in $g_2(P_3)$ if and only if the equations

$$
\left\{
\begin{aligned}
B(S, JX) &= 0, \\
B(hX, JY) - B(hY, JX) &= 0, \\
B(AX, JY) - B(AY, JX) &= 0
\end{aligned}
\right.
\qquad (6.10)
$$

hold.

If we suppose that S is typical, then it belongs to an eigenspace of $\tilde{A}$. On the other hand, this 2-dimensional eigenspace is spanned by hS and C (see Corollary 3.19), so from Proposition 3.6, vS and C are proportional. Taking into account that $C \neq 0$ at $z \in TM \setminus \{0\}$, we find that $vS = \zeta^2 C$.

The proof of the involutivity is slightly different in the diagonalizable and in the non-diagonalizable case, so we will treat them separately.

a) $\tilde{A}$ diagonalizable

Let us denote the eigenspace generated by S and C by Δ_1 and the corresponding eigenvalue by λ_1. Let us consider the base $\mathcal{B} = (h_i, v_i)_{i=1,2}$ of T_z, where

a) $h_1 \in T_z^h \cap \Delta_2$,
b) $h_2 = hS$,
c) $Jh_i = v_i$, $i = 1, 2$

Writing the equations (6.10) which express that $B \in g_2(P_3)$ in this basis, we get

$$\begin{cases} B(v_2, v_1) = 0, \\ B(h_2, v_1) = 0, \\ B(h_2, v_2) + \xi^S_C\, B(v_2, v_2) = 0, \\ B(h_1, v_2) - B(h_2, v_1) = 0. \end{cases} \tag{6.11}$$

Indeed

$B(Ah_2, v_1) - B(Ah_1, v_2) = 0$ by (6.10c) then $\lambda_2 B(v_2, v_1) - \lambda_1 B(v_1, v_2) = 0$ so we have $B(v_1, v_2) = 0$,

$B(h_2, v_1) = B(S, v_1) = B(vS, v_1) - B(S, v_1) = \xi^S_C B(v_2, v_1) - B(S, v_1) \overset{(6.10a)}{=} 0$,

$B(h_2, v_2) = B(S, C) = B(vS, C) \overset{(6.10a)}{=} -\xi^S_C\, B(v_2, v_2)$,

$B(h_1, v_2) - B(h_2, v_1) \overset{(6.10e)}{=} 0$.

Hence

$$g_2(P_3) = \left\{ \begin{pmatrix} a_{ij} & b_{ij} \\ b_{ij} & c_{ij} \end{pmatrix}_{i,j=1,2} \;\middle|\; \begin{matrix} a_{12} = a_{21},\ b_{12} = b_{21},\ c_{12} = c_{21}, \\ b_{12} = 0,\ c_{12} = 0,\ b_{22} = \xi^S_C \cdot c_{22} \end{matrix} \right\},$$

so $\dim g_2(P_3) = 6$.

In the same way, $g_3(P_3) = g_3(P_1) \cap g_3(P_2) \cap g_3(P_4)$, and a tensor $B \in S^3T^*$ is found in $g_3(P_3)$ if and only if the equations

$$\begin{cases} B(X, S, JY) = 0, \\ B(X, hY, JZ) - B(X, hZ, JY) = 0, \\ B(X, AY, JZ) - B(X, AZ, JY) = 0 \end{cases} \tag{6.12}$$

hold. We note that P_3 is a regular operator in a neighborhood of $x \in TM \setminus \{0\}$. It is easy to check that this system contains 12 independent equations; so $\dim g_3(P_3) = \dim S^3T^* - 12 = 8$.

Let us consider the basis $\bar{\mathcal{B}} = \{e_1, e_2, v_1, v_2\}$, where

$$e_1 := h_1 + v_1, \qquad \text{and} \qquad e_2 := h_2 + v_1 + v_2$$

and denote by $\bar{a}_{ij}$, $\bar{b}_{ij}$, $\bar{c}_{ij}$ the coefficients of B in this basis. It is easy to see that the elements of $g_2(P_3)$ are determined by the components $\bar{a}_{11}$, $\bar{a}_{12}$,

$\bar a_{22}$, $\bar b_{11}$, $\bar b_{12}$, and $\bar b_{22}$. Now

$$\dim (g_2)_{e_1} = 2,$$
$$\dim (g_2)_{e_1,e_2} = \dim (g_2)_{e_1,e_2,v_1} = \dim (g_2)_{e_1,e_2,v_1,v_2} = 0,$$

and so

$$\dim g_2(P_3) + \sum_{i=1}^{2} \dim(g_2(P_3))_{e_1 \cdots e_i} + \sum_{k=1}^{2} \dim(g_2(P_3))_{e_1,e_2,v_1 \cdots v_k} = \dim g_3(P_3),$$

which shows that the base $\bar{\mathcal{B}}$ is quasi-regular, and so P_3 is involutive.

b) $\dot A$ non-diagonalizable

In this case the two eigenvalues are equal. We set down $\lambda := \lambda_1 = \lambda_2$. Now S lies in the eigenspace Δ because it is typical, and by Proposition 3.8 the vectors hS and C span Δ. Let $h_1 \in T^h$ and $v_1 := Jh_1$ be vector fields in a neighborhood of $z \in TM$ so that $\{h_1, hS, v_1, C\}$ gives an adapted Jordan base of $\dot A$.

Let us consider the base $\{h_1, S, v_1, C\}$. We have $B \in g_2(P_3)$ if and only if the equations (6.10) hold, i.e.

$$\begin{cases} B(S, v_1) = B(S, C) = 0, \\ B(h_1, C) + \xi^2_2 B(v_1, C) = 0, \\ B(C, C) = 0. \end{cases}$$

Hence

$$g_2(P_3) = \left\{ \begin{pmatrix} a_{ij} & b_{ij} \\ b_{ji} & c_{ij} \end{pmatrix}_{ij=1,2} \;\middle|\; \begin{array}{c} a_{12} = a_{21},\ c_{12} = c_{21}, \\ b_{21} = b_{22} = 0,\ c_{22} = 0 \end{array} \right\}.$$

where the parameters a_{11}, a_{12}, a_{22}, b_{11}, b_{12}, and c_{11} are arbitrary. Therefore $\dim g_2(P_3) = 6$.

Let us now consider the basis $\bar{\mathcal{B}} = \{e_1, e_2, v_1, v_2\}$, where $e_1 := h_1$, and $e_2 := S + v_1$, and let $\bar a_{ij}$, $\bar b_{ij}$, $\bar c_{ij}$ be the coefficients of B in this basis. It is easy to see that the elements of $g_2(P_3)$ are determined by the components $\bar a_{11}$, $\bar a_{12}$, $\bar a_{22}$, $\bar b_{11}$, $\bar b_{12}$ and $\bar b_{21}$. Consequently $g_2(P_3)_{e_1}$ is determined by only two parameters: $\bar a_{22}$, and $\bar b_{21}$, so $\dim g_2(P_3)_{e_1} = 2$. Moreover,

$$\dim g_2(P_3)_{e_1,e_2} = \dim g_2(P_3)_{e_1,e_2,v_1} = \dim g_2(P_3)_{e_1,e_2,v_1,v_2} = 0.$$

and so

$$\dim g_2(P_3) + \sum_{k=1}^{2} \dim (g_2(P_3))_{e_k e_k} + \sum_{k=1}^{2} \dim (g_2(P_3))_{e_1 e_2\, v_1 v_k} = 9.$$

It is easy to see that, as in the diagonalizable case, $\dim g_3(P_3) = 8$, so the base $\bar{B}$ is quasi-regular and P_3 is involutive. This proves the Lemma $\diamond$

Lemma 6.3 *Let S be a rank 1 spray so that $\bar{A}$ is diagonalizable. Then every regular 2nd order solution of P_1 can be lifted into a 3rd order solution if and only if the condition (6.6) is satisfied.*

Indeed, let us consider the map

$$(T^* \otimes T_0^*) \oplus (T^* \otimes \Lambda^2 T_0^*) \oplus (T^* \otimes \Lambda^2 T_0^*) \xrightarrow{\ \tau_3\ } \Lambda^2 T_0^* \oplus \Lambda^2 T_1^* \oplus \Lambda^2 T_0^*$$

defined by $\tau_3 = (\tau_\Gamma, \tau_A, \tau_{A'})$, where

$$\begin{aligned}
\tau_\Gamma(B, C_\Gamma, C_A)(X, Y) &= B(JX, Y) - B(JY, X),\\
\tau_A(B, C_\Gamma, C_A)(X, Y) &= B(hX, Y) - B(hY, X) - \tfrac{1}{2} C_\Gamma(S, X, Y),\\
\tau_{A'}(B, C_\Gamma, C_A)(X, Y) &= C_A(S, X, Y) - B(AX, Y) + B(AY, X),
\end{aligned}$$

and, considering an adapted base $\{hS, h_2, C, v_2\}$ of $\bar{A}$, let τ_c be the function

$$\tau_c \;:\; (T^* \otimes T_0^*) \oplus (T^* \otimes \Lambda^2 T_0^*) \oplus (T^* \otimes \Lambda^2 T_1^*) \longrightarrow \mathbb{R}$$

defined by

$$\begin{aligned}
\tau_c(B, C_\Gamma, C_A) &:= B(v_2, h_2)\\
&\quad - \tfrac{1}{2} C_\Gamma(v_2, S, h_2) + \frac{\xi\xi^2}{\lambda_2 - \lambda_1} C_A(v_2, S, h_2) + \frac{1}{\lambda_2 - \lambda_1} C_A(h_2, S, h_2).
\end{aligned}$$

for $B \in T^* \otimes T_0^*$, $C_\Gamma \in T^* \otimes \Lambda^2 T_0^*$, $C_A \in T^* \otimes \Lambda^2 T_1^*$. Let us prove that if one puts $K_3^c := \operatorname{Im}(\tau_3 \oplus \tau_c)$, then the sequence

$$S^3 T^* \xrightarrow{\ \sigma_3(P_3)\ } T^* \otimes (T_0^* \oplus \Lambda^2 T_0^* \oplus \Lambda^2 T_0^*) \xrightarrow{\ \tau_3 \oplus \tau_c\ } K_3^c \longrightarrow 0$$

is exact.

Indeed, it is easy to check that rank $g_3(P_3) = 12$. On the other hand, it is not difficult to see that the four equations defining $\operatorname{Ker}(\tau_3 \oplus \tau_c)$ are independent (using an adapted base, the terms $B(v_2, h_1)$, $B(v_2, h_2)$, $C_\Gamma(S, h_1, h_2)$,

$C_A(S, h_1, h_2)$ are pivots for the equations $\tau_\Gamma = 0$, $\tau^\Sigma = 0$, $\tau_A = 0$ and $\tau_{A'} = 0$ respectively). Therefore

$$\dim \mathrm{Ker}(\tau_3 \oplus \tau_c) = \dim\left[T^* \otimes (T_v^* \oplus \Lambda^2 T_v^* \oplus \Lambda^2 T_v^*)\right] - 4 = \mathrm{rank}\, \sigma_3(P_3),$$

which proves that the sequence is exact

Thereby a 2nd order solution $j_2(E)_x$ of P_3 can be lifted into a 3rd order solution if and only if $(\tau_3 \oplus \tau_c)\nabla(P_3 E)_x = 0$. Now

$$((\tau_3 \oplus \tau_c)\nabla(P_3 E))_x = \left(\tau_2\nabla(P_2 E), \tau_{A'}\nabla(P_3 E), \tau_c\nabla(P_3 E)\right)_x.$$

The computation on page 90 shows that $(\tau_2\nabla P_2 E)_x = (i_\Gamma\Omega_E, i_A\Omega_E)_x$ and so $(\tau_2\nabla P_2 E)_x = 0$. On the other hand

$$\tau_{A'}\left(\nabla(P_3 E)\right)_x = (\mathcal{L}_S i_A dd_J E)_x - (d_A P_1 E)_x = (i_{A'} dd_J E)_x \qquad (6.13)$$

Since rank $S = 1$, there exist μ_1 and μ_2 such that $A' = \mu_1 J + \mu_2 A$. Thus

$$(\tau_{A'}\nabla(P_3 E))_x = (i_{(\mu_1 J + \mu_2 A)} dd_J E)_x \overset{(3.27)}{=} \mu_2 (i_A dd_J E)_x = 0 \qquad (6.14)$$

and

$$\left(\tau_2\nabla(P_3 E)\right)_x = 0. \qquad (6.15)$$

Moreover,

$$\begin{aligned} \tau_c\nabla(P_3 E)_x &= \left(v_2\omega_E(h_2)\right)_x - v_2\left(\tfrac{1}{2}\, i_\Gamma\Omega_E(S, h_2) + \tfrac{\zeta\Sigma^S}{\lambda_2 - \lambda_1}\, i_A\Omega_E(S, h_2)\right)_x \\ &\quad + h_2\left(\tfrac{1}{\lambda_2 - \lambda_1}\, i_A\Omega_E(S, h_2)\right)_x \\ &= \left(d\omega_E(v_2, h_2) - v_2\Omega_E(hS, h_2) - v_2\Omega_E(vS, h_2) + h_2\Omega(S, v_2)\right)_x, \end{aligned}$$

and $d\omega_E = \mathcal{L}_S\Omega_E$, so

$$d\omega_E(v_2, h_2) = v_2\Omega_E(S, h_2) - h_2\Omega_E(S, v_2) + \Omega_E(S, [v_2, h_2])$$

Thus

$$d\omega_E(v_2, h_2) - v_2\Omega_E(S, h_2) + h_2\Omega_E(S, v_2) = -\Omega_E(S, [v_2, h_2]),$$

and therefore

$$\tau_c(\nabla(P_3 E))_x = i_S\Omega_E([v_2, h_2])_x = i_C\Omega_E(F[v_2, h_2])_x. \qquad (6.16)$$

Let $\alpha \in T_v^*$ be the semi-basic 1-form defined by $\alpha(hS) = 1$ and $\alpha(h_2) = 0$. We have

$$A = \lambda_2 J + (\lambda\alpha) \otimes C,$$

where $\lambda := \lambda_1 - \lambda_2$. The equation $i_A\Omega_E(x) = 0$ gives $(\lambda\alpha)_x \wedge i_C\Omega_E(x) = 0$. This implies $\lambda i_C\Omega = g(C, C)\alpha$. Thus, by Proposition 6.1 and equation (6.16), the condition $\tau_r(\nabla(P_3 E))_x = 0$ is identically satisfied by the regular second order solution $(j_2 E)_x$ if and only if

$$\alpha_x(F[v_2, h_2]) = 0. \tag{6.17}$$

Since $h_2 \in \alpha^\perp \cap T^h$ and $\alpha(F[h_2, Jh_2])$ depends only on the value of h_2 at x, we obtain

$$\alpha_x(F[h_2, Jh_2]) = \alpha_x(FD_{h_2}Jh_2) = \alpha_x(FJD_{h_2}h_2)_x = (D_{h_2}\alpha)(h_2)_x,$$

where D is the Berwald connection. Thus

$$\tau^r[\nabla(P_3 E)]_x = 0 \quad \text{if and only if} \quad (D_{hX}\vartheta \wedge \vartheta)_x = 0 \quad \forall X \in \operatorname{Ker}\vartheta.$$

Lemma 6.2 and 6.3 prove the Theorem in the case when $\hat{A}$ is diagonalizable. ◇

Let us suppose now that A is non-diagonalizable. Firstly we have the following

Remark. If E is a regular Lagrangian associated to S on a neighborhood U of $x \in TM \setminus \{0\}$, then every vector $v \in U$ has null length.

Indeed, E has to satisfy the compatibility condition $P_A E = 0$, i.e. the equation $i_A\Omega_E = 0$, on U. Computing it on the vectors S and h_2 we find

$$i_A\Omega_E(S, h_2) = \Omega_E(AS, h_2) - \Omega_E(Ah_2, S) = -\lambda\Omega_E(C, S).$$

Consequently $\Omega_E(C, S)\,|_U = 0$, and therefore S is non-variational.

In order to prove the second part of 2) of Theorem 6.2 we show the following

Lemma 6.4 *Let S be a rank 1 spray with $\hat{A}$ non-diagonalizable, and $x \in TM \setminus \{0\}$. Then every regular second order formal solution of P_1 can be lifted into a third order solution in x if and only if $\zeta(x) = 0$. In* *particular if S is quadratic or homogeneous, then every second order formal solution can be lifted into a 3rd order solution.*

Let us consider the map τ_c

$$\tau_c(B, C_\Gamma, C_A) := B(C, h_2) + C_A(h_2, S, h_2) + \xi_C^S C_A(v_2, S, h_2) - \tfrac{1}{2} C_\Gamma(C, hS, h_2),$$

where $\{hS, h_2, C, v_2\}$ is an adapted base of $\bar{A}$. If we set $K_3^c := \operatorname{Im}(\tau_3 \oplus \tau_c)$, then the sequence

$$S^3T^* \xrightarrow{\sigma_3(P_3)} T^* \otimes (T_v^* \oplus \Lambda^2 T_v^* \oplus \Lambda^2 T_v^*) \xrightarrow{\tau_3 \oplus \tau_c} K_3^c \to 0$$

is exact

Indeed, the four equations defining $\operatorname{Ker}(\tau_3 \oplus \tau_c)$ are independent, since $B(u_1, h_2)$, $C_\Gamma(S, h_1, h_2)$, $C_A(S, h_1, h_2)$ and $B(C, h_2)$ are pivots for the equations $\tau_\Gamma = 0$, $\tau_A = 0$, $\tau_{A'} = 0$ and $\tau^c = 0$. Therefore

$$\dim \operatorname{Ker}(\tau_3 \oplus \tau_c) = \dim \left[T^* \otimes (T_v^* \oplus \Lambda^2 T_v^* \oplus \Lambda^2 T_v^*)\right] - 4 = \operatorname{rank} \sigma_3(P_3),$$

which proves that the sequence is exact.

To compute the compatibility conditions of P_3, let us consider $j_2(E)_x$ a second order formal solution of P_3. It can be lifted into a third order solution if and only if $(\tau_3 \oplus \tau_c)\nabla(P_3E)_x = 0$. The same computation as in the diagonalizable case – actually only the definition of the function τ_c is different – shows that (6.15) holds, thus

$$\big((\tau_3 \oplus \tau_c)\nabla(P_3E)\big)_x = \big(\tau_3\nabla(P_3E), \tau_c\nabla(P_3E)\big)_x = \big(0, \tau_c\nabla(P_3E)\big)_x.$$

Moreover, we have

$$\begin{aligned}\tau_c\nabla(P_3E)_x &= \\ &- \Big(C(w_E(h_2)) + \bar{n}_3(i_A\Omega_E(S, h_2)) + \xi_C^S n_3(i_A\Omega_E(S, h_2)) - \tfrac{1}{2} C(i_\Gamma\Omega_E(hS, h_2))\Big)_x \\ &= (\xi_C^S - \mathcal{L}_C\xi_C^S + \xi_{h_2}^{[v_2, C]})_x \Omega_E(v_2, S)_x = \zeta(x)\,\Omega_E(v_2, S)_x.\end{aligned}$$

Using Proposition 6.1 we find that if $\zeta(x) \neq 0$, then there is no regular formal solution E of P_3 satisfying the compatibility conditions $\tau_c\nabla(P_3E) = 0$. In the case $\zeta \equiv 0$ we find that every second order formal solution can be lifted into a 3rd order solution.

If the spray S is quadratic or homogeneous, then $vS = 0$, so $\xi_C^S \equiv 0$. On the other hand, the horizontal projection h is also homogeneous: $[h, C] = 0$

and therefore $v_1[h_2, C] = [h, C](h_2) = 0$ i.e. $\xi^{[h_2,C]}_{v_2} = 0$. Thus $\zeta = 0$, and every second order formal solution can be lifted into a 3rd order solution.

Lemmas 6.2 and 6.4 prove the Theorem in the case when $\bar{A}$ is non-diagonalizable.

□

Example 6.3 [CPST]. Let us consider the system

$$\begin{cases} \ddot{x}_1 = -2\dot{x}_1\dot{x}_2 \\ \ddot{x}_2 = \dot{x}_2^2. \end{cases} \tag{6.18}$$

Note that as the system (6.18) is homogeneous, the corresponding spray is typical. We have $hS = S$, and S is an eigenvector of the Douglas tensor $AS = v[h, S]hS = v[S, S] = 0$. An easy computation gives

$$\begin{array}{llll} \Gamma^1_1 = y_2, & \Gamma^1_2 = y_1, & \Gamma^2_1 = 0, & \Gamma^2_2 = -y_2, \\ A^1_1 = 2y_2^2, & A^1_2 = -2y_1y_2, & A^2_1 = 0, & A^2_2 = 0, \\ A'^1_1 = 4y_2^3, & A'^1_2 = 8y_1y_2^2, & A'^2_1 = 0, & A'^2_2 = 0. \end{array}$$

Thus rank $S = 1$ and $\bar{A}$ is diagonalizable. An adapted base is offered by the eigenvectors

$$\begin{aligned} h_1 &= S = y^1\frac{\partial}{\partial x_1} + y^2\frac{\partial}{\partial x_2} - 2y_1y_2\frac{\partial}{\partial y_1} + y_2^2\frac{\partial}{\partial y_2}, \\ v_1 &= C = y^1\frac{\partial}{\partial y_1} + y^2\frac{\partial}{\partial y_2}, \\ h_2 &= \frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial y_1}, \\ v_2 &= \frac{\partial}{\partial y_1}. \end{aligned}$$

Since $[h_2, v_2] = 0$, the equation (6.17) holds, so the condition (6.6) is identically satisfied for every second order solution $j_2(E)_x$ of Γ_1. It follows from Theorem 6.2 that the spray S is variational.

6.3 Rank $S = 1$: Atypical sprays.

Firstly, let us note that, in contrast to the typical case, there are no obstructions to lifting the second order formal solutions of P_3 into a third order solutions, when the spray is atypical.

Indeed, as in the typical case, the space $g_2(P_3)$ is determined by the equations (6.10) and $\dim g_2(P_3) = 6$. But if the spray is non-typical, then $\operatorname{rank}\sigma_3(P_3) = 13$. Defining τ_3 by $\tau_3 = (\tau_{P'}, \tau_A, \tau_{A'})$, as we did in the typical case and $K_3 = \operatorname{Im}\tau_3$, we obtain the exact sequence

$$S^3T^* \xrightarrow{\sigma_3(P_3)} T^*\otimes(T_v^*\oplus\Lambda^2T_v^*\oplus\Lambda^2T_v^*) \xrightarrow{\tau_3} K_3 \longrightarrow 0. \qquad (6.19)$$

Since $[\tau_3\nabla(P_3E)]|_z = 0$, for a second order formal solution $j_2(E)_z$, any 2nd order formal solution can be lifted to a 3rd order one.

Despite this fact, the inverse problem for atypical sprays is much more complicated, because the symbol of P_3 is not involutive. Indeed, in this case we have $\dim g_2(P_3) = 6$ and so for any $v \in T$, $\dim(g_2(P_3))_v \geq 2$, (because $i_vB = 0$ yields a maximum of four equations on $B \in g_2(P_3)$). On the other hand, $\dim g_3(P_3) = \dim S^3T^* - \operatorname{rank}\sigma_3(P_3) = 7$, and so for any basis $B = \{e_i\}_{i=1}^4$, we have

$$\dim g_3(P_3) < \sum_{i=0}^{4}\dim(g_2(P_3))_{e_1\ldots e_i}.$$

Therefore a quasi-regular basis does not exist, and P_3 is not involutive.

Note that involutivity is not necessary for the formal integrability: 2-acyclicity suffices. Unfortunately P_3 is not 2-acyclic: there are non-trivial higher order cohomological groups in the Spencer complex. This means that obstructions for integrability arise in the higher order prolongations. This is the reason why in the study of atypical sprays we need to prolong the system.

Although the study presented in this section may seem too complicated, we will expose it in detail because it may be instructive to see how all the difficulties of formal integrability appear and can be solved.

6.3.1 *Non-triviality of the Spencer cohomology.*

The aim of this section is to compute the Spencer cohomology groups of the operator P_3 and in particular to prove the following

Proposition 6.2 *In the atypical case, the operator P_3 is not 2-acyclic. The first non-trivial Spencer cohomological group is $H_2^2(P_3)$.*

Proof. We will begin by showing the following formula:

$$\dim g_m(P_3) = m + 4 \tag{6.20}$$

for any $m \geq 0$.

To prove it, we introduce the following notation. let P be a differential operator and put

$$G_m(P) := \{B \in g_m(P) \mid h^* B = 0\}. \tag{6.21}$$

Of course $G_m(P) \cong g_m(P)/S^m T_v^*$.

In our case the elements of $g_m(P_3)$ are evaluated on at last one vertical vector and so

$$g_m(P_3) = S^m T_h^* \oplus G_m(P_3). \tag{6.22}$$

Since $\dim S^m T_v^* = m + 1$, we have just to prove that

$$\dim G_m(P_3) = 3. \tag{6.23}$$

The proof is slightly different according to whether $\bar{A}$ is diagonalizable or not.

a) $\bar{A}$ is diagonalizable

Property 1.

> *Let S be a rank 1 atypical spray and suppose that $\bar{A}$ is diagonalizable. Then:*
>
> *(1) the eigenspaces Δ_i of $\bar{A}$ are invariants with respect to S, i.e. $[\Delta_i, S] \subset \Delta_i$ for i=1,2.*
>
> *(2) Let pr_i be the projection on the eigenspace Δ_i. Then $pr_i(hS) \neq 0$.*

Let us prove *(1)*. Consider an adapted basis $\{h_i, v_i\}_{i=1,2}$. We have to show that $[h_i, S] \in \Delta_i$ and $[v_i, S] \in \Delta_i$, i=1,2. If h_i is an eigenvector of $\bar{A}$ we have $Ah_i = v([h_i, S]) = \lambda_i v_i$, and so

$$pr_j(v[h_i, S]) = 0 \quad \text{for} \quad i \neq j \tag{6.24}$$

Since rank $S = 1$, A' is a linear combination of J and A. Hence the eigenspaces are invariant by A'. On the other hand

$$A'h_i = v[A, S](h_i) = v[Ah_i, S] - A[h_i, S] = v[\lambda_i v_i, S] - A[h_i, S] = -(S\lambda_i)v_i$$
$$+ \lambda_i v(\Gamma h_i + J[h_i, S]) - A[h_i, S] = -(S\lambda_i)v_i + (\lambda_i J - A)([h_i, S])$$

and $A - \lambda_i J = \lambda_j pr_j$, so $pr_j(h[h_i, S]) = 0$ for $j \neq i$. This equation and (6.24) show that

$$[h_i, S] \in \Delta_i. \tag{6.25}$$

Moreover $[v_i, S] = h_i + J[h_i, S]$, so, by (6.25),

$$[v_i, S] \in \Delta_i \tag{6.26}$$

$i = 1, 2$, which proves *(1)*.

Now, we will prove *(2)*. Let $\{h_i, v_i\}_{i=1,2}$ be an adapted basis to $\bar{A}$. Using the notation (6.5) the spray is

$$S := \xi^S_{h_1} h_1 + \xi^S_{h_2} h_2 + \xi^S_{v_1} v_1 + \xi^S_{v_2} v_2$$

We have to show that $\xi^S_{h_1} \neq 0$ and $\xi^S_{h_2} \neq 0$. We have $(hS)_x \neq 0$ because $J(hS) = C$ and C does not vanish at $x \neq 0$. Thus one of the coefficients $\xi^S_{h_i}$ is different from 0, say $\xi^S_{h_1} \neq 0$. Let us suppose that $\xi^S_{h_2} = 0$. Since the spray is atypical, $S \notin \Delta_1$, so $\xi^S_{v_2} \neq 0$. Now $hS = \xi^S_{h_1} h_1$ is an eigenvector of $\bar{A}$, hence we can take $h_1 = hS$ and therefore $v_1 = C$. Thus we have $vS = \xi^S_{v_1} C + \xi^S_{v_2} v_2$. Since $\xi^S_{v_2} \neq 0$, $vS \notin \Delta_1$. Now $vS = v(\Gamma S) = v[J, S]S - v[C, S]$ and $C \in \Delta_1$. Since Δ_1 is invariant by S, as we have just seen, $[C, S]$ should be in Δ_1, that is $[C, S] = ah_1 + bv_1$. Hence $vS = bv_1$, that is $vS \in \Delta_1$, which leads a contradiction. Thus Property 2. is proved.

Let us now return to the formula (6.23).

NOTATION: *In order to simplify the notation, we will use the symbol* v^k, *if the vector* v *is repeated* k *times in the argument of a symmetric* *tensor* i.e.

$$B(\dots, v^k, \dots) := B(\dots \underbrace{v, \dots, v}_{k-\text{times}} \dots). \tag{6.27}$$

To prove the formula we will show by induction that every element B_m of $G_m(P_3)$ $m > 0$ is determined by the three independent parameters

$$B_m(h_2^{m-1}, v_2), \quad B_m(v_1^{m-1}, v_1), \quad B_m(v_2^{m-1}, v_2). \tag{6.28}$$

Let $m = 2$ and $B_2 \in g_2(P_3)$. B is symmetric and satisfies the equations (6.10). Using an adapted basis, these equations become

$$\begin{cases} B_2(S, v_1) = B_2(S, v_2) = 0, \\ B_2(h_1, v_2) - B_2(h_2, v_1) = 0, \\ B_2(v_1, v_2) = 0 \end{cases} \tag{6.29}$$

A direct computation shows that $\dim g_2(P_3) = 3$ and $B_2 \in g_2(P_3)$ is determined by its components $B_2(h_2, v_2)$, $B_2(v_1, v_1)$ and $B_2(v_2, v_2)$.

For $m = 3$ the explicit computation shows that $B_3 \in G_3(P_3)$ is determined by the components $B_3(h_2, h_2, v_2)$, $B_3(v_1, v_1, v_1)$, $B_3(v_2, v_2, v_2)$. These parameters are independent, so $\dim G_3(P_3) = 3$.

Let us assume that for any $m \geq 3$ we have $\dim G_{m-1}(P_3) = 3$, and that an element $B_{m-1} \in G_{m-1}(P_3)$ is determined by the components $B_{m-1}(h_2^{m-2}, v_2)$, $B_{m-1}(v_1^{m-1})$ and $B_{m-1}(v_2^{m-1})$. Since $G_m(P_3)$ is a linear subspace of $T^* \otimes G_{m-1}(P_3)$, by the recursion hypothesis every element B_m in $G_m(P_3)$ is determined by a maximum of 12 components:

$$B_m(e_i, h_2^{m-2}, v_2), \qquad B_m(e_i, v_1^{m-1}), \qquad B_m(e_i, v_2^{m-1}) \tag{6.30}$$

$i = \{1, \dots, 4\}$, where $\{e_i\}_{i=1,\dots,4} = \{h_j, v_j\}_{j=1,2}$. From (6.29) it follows that $B_m \in g_m(P_3)$ is characterised by the equations

$$\begin{cases} B_m(\dots S, \dots, v_i, \dots) = 0, \\ B_m(\dots h_1, \dots, v_2 \dots) - B_m(\dots h_2, \dots, v_1, \dots) = 0, \\ B_m(\dots v_1, \dots, v_2, \dots) = 0. \end{cases} \tag{6.31}$$

$i = 1, 2$. Thus for the components (6.30) we have : $B_m(h_2, v_1^{m-1}) = 0$, $B_m(v_2, v_1^{m-1}) = 0$, $B_m(h_1, v_2^{m-1}) = 0$, $B_m(v_1, v_2^{m-1}) = 0$, $B_m(v_1, h_2^{m-2}, v_2) =$

0, and the following relations hold:

$$\begin{aligned}
B_m(h_1, v_1^{m-1}) &= -\frac{\xi^S_{v_1}}{\xi^S_{h_1}} B_m(v_1^m),\\
B_m(h_2, v_2^{m-1}) &= -\frac{\xi^S_{v_2}}{\xi^S_{h_2}} B_m(v_2^m),\\
B_m(v_2, h_2^{m-2}, v_2) &= \left(\frac{\xi^S_{v_2}}{\xi^S_{h_2}}\right)^{m-2} B_m(v_2^m),\\
B_m(h_1, h_2^{m-2}, v_2) &= -\frac{\xi^S_{h_2}}{\xi^S_{h_1}} B_m(h_2^{m-1}, v_2) - \frac{\xi^S_{v_1}}{\xi^S_{h_1}} B_m(v_2, h_2^{m-2}, v_2).
\end{aligned}$$

Therefore B_m is determined by its components (6.28). This proves the formula (6.23) when $\bar{A}$ is diagonalisable.

b) $\bar{A}$ is non-diagonalizable

To prove the formula (6.28) in this case we need the following

Property 2.

> *Let S be a variational spray and suppose that $\bar{A}$ is not diagonalizable. Then hS is not an eigenvector of $\bar{A}$. In particular, if $\{h_i, v_i\}_{i=1,2}$ is a Jordan basis adapted to $\bar{A}$, with $h_1, v_1 = Jh_1$ spanning the eigen-distribution Δ and $h_2, v_2 = Jh_2$ spanning the other characteristic space, then $\xi^S_{h_2} \neq 0$.*

Indeed, $S \notin \Delta$, so either $\xi^S_{h_2} \neq 0$ or $\xi^S_{v_2} \neq 0$. Let us suppose that $\xi^S_{h_2} = 0$; we can choose $hS = h_1$, so that $v_1 = C$, and we have $Ah_2 = \lambda v_2 + C$. Thus $0 = i_A\Omega(hS, h_2) = \Omega(AhS, h_2) + \Omega(hS, Ah_2) = \lambda\Omega(v_1, h_2) + \lambda\Omega(h_1, v_2) + \Omega(S, C) = \Omega(S, C)$, which is excluded by Lemma 6.1.

Let us now prove the formula (6.23). We will show that, in an adapted Jordan basis for $\bar{A}$, an element $B_m \in C_m(F_3)$ is determined by the components

$$B_m(h_2^{m-1}, v_2), \qquad B_m(v_1, v_2^{m-1}), \qquad B_m(v_2^m). \qquad (6.32)$$

Indeed, in an adapted Jordan basis the equations (6.10) for an element B_2 are:

$$\begin{cases} B_2(S, v_i) = 0, \qquad i = 1, 2, \\ B_2(h_1, v_2) - B_2(h_2, v_1) = 0, \\ B_2(v_1, v_1) = 0. \end{cases} \tag{6.33}$$

A direct computation shows that $\dim G_2(P_3) = 3$, and $B_2 \in G_2(P_3)$ is determined by the components $B_2(h_2, v_2)$, $B_2(v_1, v_2)$ and $B_2(v_2, v_2)$. Likewise $B_3 \in G_3(P_3)$ is also determined by the three components $B_3(h_2, h_2, v_2)$, $B_3(v_1, v_2, v_2)$ and $B_3(v_2, v_2, v_2)$, so $\dim G_3(P_3) = 3$.

To show that $\dim G_m(P_3) = 3$ for $m > 3$, we suppose by induction that $\dim G_{m-1}(P_3) = 3$, and that $B_{m-1} \in G_{m-1}(P_3)$ is determined by the components $B_{m-1}(h_2^{m-2}, v_2)$, $B_{m-1}(v_1, v_2^{m-2})$, $B_{m-1}(v_2^{m-1})$.
Since $G_m(P_3)$ is a linear subspace of $T^* \otimes G_{m-1}(P_3)$, an element B_m of $G_m(P_3)$ is determined by the components

$$B_m(e_i, h_2^{m-2}, v_2), \quad B_m(e_i, v_1, v_2^{m-2}), \quad B_m(e_i, v_2^{m-1}), \tag{6.34}$$

$i = \{1, \dots, 4\}$, where $\{e_i\}_{i=1..4} = \{h_j, v_j\}_{j=1,2}$. But $B_m \in g_m(P_3)$, so the equations

$$\begin{cases} B_m(\dots S, \dots, v_i, \dots) = 0, \qquad i = 1, 2, \\ B_m(\dots h_1, \dots, v_2, \dots) - B_m(\dots h_2, \dots, v_1, \dots) = 0, \\ B_m(\dots v_1, \dots, v_1, \dots) = 0 \end{cases} \tag{6.35}$$

hold. Using these equations and the symmetry of B_m, we find the following relations between the components (6.34):

$$\begin{aligned} B_m(v_1, v_1, v_2^{m-2}) &\overset{(6.35c)}{=} 0, \\ B_m(h_2, v_1, v_2^{m-2}) &\overset{(6.35a)}{=} -\frac{\zeta^2}{\zeta_1^2} B_m(v_1, v_2^{m-1}), \\ B_m(h_1, v_2^{m-1}) &\overset{(6.35b)}{=} B_m(h_2, v_1, v_2^{m-2}) = -\frac{\zeta^2}{\zeta_2^2} B_m(v_1, v_2^{m-1}) \end{aligned}$$

To find the relations between the other components, we notice that $B_m(h_2^{l+1}, v_2^{m-1-l})$, and $B_m(h_2^{l+1}, v_2^{m-2-l}, v_1)$ can be determined with the help of $B_m(h_2^l, v_2^{m-l})$, $B_m(h_2^l, v_2^{m-1-l}, v_1)$, for any l, m such that $0 \le l < m - 2$.

Indeed, we have

$$
\begin{aligned}
B(h_2^{j+1}, v_2^{m-j-1}, v_1) &\overset{(6.35a)}{=} \\
&= -\frac{1}{\xi^S_{h_2}}\left(\xi^S_{h_1} B(h_1, h_2^j, v_2^{m-j-1}, v_1) + \xi^S_{v_1} B(v_1, h_2^j, v_2^{m-j-1}, v_1)\right) \\
&= -\frac{1}{\xi^S_{h_2}}\left(\xi^S_{h_1} B(v_2, h_2^j, v_2^{m-j-1}, v_1)\right) \overset{(6.31)}{=} -\frac{\xi^S_{h_1}}{\xi^S_{h_2}} B(h_2^j, v_2^{m-j-1}, v_1),
\end{aligned}
$$

and so

$$
\begin{aligned}
&B(h_2^{j+1}, v_2^{m+1-j}) \overset{(6.35a)}{=} \\
&= -\frac{1}{\xi^S_{h_2}}\left(\xi^S_{h_1} B(h_1, h_2^j, v_2^{m+1-j}) + \xi^S_{v_1} B(v_1, h_2^j, v_2^{m+1-j}) + \xi^S_{v_2} B(v_2, h_2^j, v_2^{m+1-j})\right) \\
&= -\frac{1}{\xi^S_{h_2}}\left(\xi^S_{h_1} B(h_2^{j+1}, v_2^{m-j}, v_1) + \xi^S_{v_1} B(h_2^j, v_2^{m+1-j}, v_1) + \xi^S_{v_2} B(v_2, h_2^j, v_2^{m+1-j})\right) \\
&= -\frac{1}{\xi^S_{h_2}}\left(\left[\xi^S_{h_1}\left(-\frac{\xi^S_{h_1}}{\xi^S_{h_2}}\right)^{j+1} + \xi^S_{v_1}\left(-\frac{\xi^S_{h_1}}{\xi^S_{h_2}}\right)^{j}\right] B(v_2^{m+1}, v_1) + \xi^S_{v_2} B(v_2, h_2^j, v_2^{m+1-j})\right).
\end{aligned}
$$

It follows that $B_m(h_2, v_2^{m+1})$, $B_m(v_2, h_2^m, v_2)$, $B_m(v_1, h_2^m, v_2)$ are determined by $B_m(v_2^{m+1}, v_1)$ and $B_m(v_1, h_2^{m+1})$. On the other hand

$$
B_m(h_1, h_2^m, v_2) = \frac{-1}{\xi^S_{h_1}}\left(\xi^S_{h_2} B(h_2, h_2^m, v_2) + \xi^S_{v_2} B(v_2, h_2^m, v_2) + \xi^S_{v_1} B(v_1, h_2^m, v_2)\right),
$$

hence it is also determined by the components (6.32). Since the components of (6.32) are independent, it follows that $\dim G_m(P_S) = 3$ for any $m \geq 2$ and this proves the formula (6.23). ◇

Let us now prove Proposition 6.2.

NOTATION. *To distinguish the maps δ appearing in the Spencer sequence associated with a differential operator P, we denote them by*

$$
\delta_{1,m}(P) : T^* \otimes g_{m+1}(P) \longrightarrow \Lambda^2 T^* \otimes g_m(P), \tag{6.36}
$$

and

$$
\delta_{2,m}(P) : \Lambda^2 T^* \otimes g_m(P) \longrightarrow \Lambda^3 T^* \otimes g_{m-1}(P). \tag{6.37}
$$

Firstly let us compute the rank of $\delta_{1,m}(P_S)$. Since the sequences

$$
0 \to g_{m+2} \xrightarrow{\delta} T^* \otimes g_{m+1} \xrightarrow{\delta_{1,m}} \Lambda^2 T^* \otimes g_m \xrightarrow{\delta_{2,m}} \Lambda^3 T^* \otimes g_{m-1} \to \dots \tag{6.38}
$$

are exact in the terms g_{m+2} and $T^* \otimes g_{m+1}$ for any $m \geq 0$, according to (6.20) we have

$$\operatorname{rank} \delta_{1,m}(P_3) = \dim (T^* \otimes g_{m+1}(P_3)) - \dim g_{m+2}(P_3) = 3m + 14 \quad (6.39)$$

for $m \geq 2$, and so $\operatorname{rank} \delta_{1,2}(P_3) = 20$.

Consider now the sequence

$$0 \to S^4T^* \xrightarrow{\delta} T^* \otimes S^3T^* \xrightarrow{\delta_1} \Lambda^2T^* \otimes S^2T^* \xrightarrow{\delta_2} \Lambda^3T^* \otimes T^* \xrightarrow{\delta} \Lambda^4 \to 0 \quad (6.40)$$

Since it is exact, $\operatorname{Ker} \delta_2$ is determined by 15 independent equations. But the morphism $\delta_{2,2}(P_3)$ is the restriction to δ_2 on the subspace $\Lambda^2T^* \otimes g_2(P_3)$, so the system $\delta_{2,2}(P_3) = 0$ contains a maximum of 15 independent equations, and thus

$$\dim \operatorname{Ker} \delta_{2,2}(P_3) \geq \dim \Lambda^2T^* \otimes g_2(P_3) - 15 = 21,$$

and so

$$\dim H_2^2(P_3) \geq 1,$$

which proves that P_3 is not 2-acyclic. A direct computation shows that $\dim H_2^2(P_3) = 1$.

□

The obstruction to the second lift

Since $H_2^2(P_3) \neq 0$, some obstructions arise to lifting the second order formal solutions of P_3 twice. In this section we will compute them.

Proposition 6.3 *Let $p = j_3(E)_x$ be a third order solution of P_3 at x. The semi-basic tensor $\varphi_E \in \Lambda^2T_v^* \otimes \Lambda^2T_v^*$ defined by*

$$\varphi_E(X,Y,Z,U) = \frac{1}{2}\big((\nabla\nabla i_\Gamma \Omega_E)(hX,JY,Z,U) - (\nabla\nabla i_\Gamma \Omega_E)(hY,JX,Z,U)$$
$$- (\nabla\nabla i_\Lambda \Omega_E)(hZ,JU,X,Y) + (\nabla\nabla i_\Lambda \Omega_E)(hU,JZ,X,Y)\big),$$

where ∇ is an arbitrary connection and $\Omega_E = dd_J E$, depends at x only on $j_3(E)_x$ (and not on the fourth jet of E at x). Moreover $j_3(E)_x$ can be lifted to a 4^{th} order solution if and only if $(\varphi_E)_x = 0$.

Proof Let us denote by $P_3^1 := (P_3, \nabla P_3)$ the first prolongation of P_3. To simplify notations we will use

$$F_3 := T_v^* \oplus \Lambda^2 T_v^* \oplus \Lambda^2 T_v^*.$$

Then $P_3^1 : C^\infty(TM) \longrightarrow F_3 \oplus (T^* \otimes F_3)$ is defined by

$$P_3^1(E) := (\omega_E, i_\Gamma \Omega_E, i_A \Omega_E, \nabla \omega_E, \nabla i_\Gamma \Omega_E, \nabla i_A \Omega_E). \tag{6.41}$$

A standard computation which takes into account that $P_3^1(E)_x = 0$, shows that

$$\begin{aligned}(\varphi_E)_x(X, Y, Z, U) &:= [hU, AY]\Omega_E(X, JZ) - [hU, AX]\Omega_E(Y, JZ) \\ &+ AX\Big\{\sum_{cycl}\Big(\Omega_E([hZ, U], JY) + \Omega_E([Z, hU], JY) - \Omega_E([Z, U], JY)\Big)\Big\} \\ &- AY\Big\{\sum_{cycl}\Big(\Omega_E([hZ, U], JX) + \Omega_E([Z, hU], JX) - \Omega_E([Z, U], JX)\Big)\Big\} \\ &- hZ\Big\{\sum_{cycl}\Big(\Omega_E([JU, AX], Y) - \Omega_E([JU, AY], X)\Big)\Big\} + [hZ, AX]\Omega_E(Y, JU) \\ &+ hU\Big\{\sum_{cycl}\Big(\Omega_E([JZ, AX], Y) - \Omega_E([JZ, AX], Y)\Big)\Big\} - [hZ, AY]\Omega_E(X, JU)\end{aligned}$$

This formula proves that $(\varphi_E)_x$ is semi-basic and depends only on the third order jet of E at x.

Let us now compute the obstruction to the second lift. Note that P_3^1 is a third order operator. Of course its second order part does not appear in the symbol, so $\sigma_3(P_3^1) = 0 \times \sigma_3(P_3)$. Thus to compute the space of the obstruction we only need to construct an exact sequence

$$S^4T^* \xrightarrow{\sigma_4(P_3^1)} S^2T^* \otimes F_3 \xrightarrow{\tilde{\tau}} \tilde{K} \longrightarrow 0, \tag{6.42}$$

because $\operatorname{Coker} \sigma_4(P_3^1) \cong \operatorname{Im}(id_{T^* \otimes F_3} \times \tilde{\tau})$ where $id_{T^* \otimes F_3}$ denotes the identity map of $T^* \otimes F_3$.

Let us propose $\tau_3^1 := id_{T^*} \otimes \tau_3$. The morphism τ_3^1 is the first prolongation of the morphism τ_3 defined in Lemma 6.3. We have the following diagram:

$$\begin{array}{ccc} S^2T^* \otimes F_3 & \xrightarrow{\quad \tau_3^1 \quad} & T^* \otimes K_3 \\ & {\scriptstyle i} \searrow \qquad \nearrow {\scriptstyle id_{T^*} \otimes \tau_3} & \\ & T^* \otimes (T^* \otimes F_3) & \end{array}$$

Now $\dim H_3^2(P_3) = 1$, hence $\dim\left(\operatorname{Im}\sigma_3(P_3)/\operatorname{Ker}\tau_3^1\right) = 1$. Therefore, in order to find the exact sequence (6.42) we have to complete the morphism τ_3^1 with a new one, which gives exactly one new independent relation with respect to the system defined by $\operatorname{Ker}\tau_3^1$. Let

$$S^2T^* \otimes F_3 \xrightarrow{\tau_{(h,A)}} \Lambda^2T_v^* \otimes \Lambda^2T_v^* \tag{6.43}$$

be the morphism defined by the formula

$$\begin{aligned}\tau_{(h,A)}(B, B_\Gamma, B_A)(X,Y,Z,U) := B_A(hU, JZ, X, Y) - B_A(hZ, JU, X, Y) \\ + \tfrac{1}{2}\left(B_\Gamma(AX, JY, Z, U) - B_\Gamma(AY, JX, Z, U)\right).\end{aligned}$$

where $(B_S, B_\Gamma, B_A) \in S^2T^* \otimes F_3$ and $X, Y, Z, U \in T$. It is easy to check that the equation $\tau_{(h,A)} = 0$ is independent of the equation $\tau_3^1 = 0$. Therefore the sequence

$$S^4T^* \xrightarrow{\sigma_4(P_3^1)} S^2T^* \otimes F_3 \xrightarrow{\tilde{\tau}_3^1} \tilde{K}_3^1 \longrightarrow 0 \tag{6.44}$$

is exact, where $\tilde{\tau}_3^1 := \tau_3^1 \times \tau_{(h,A)}$, and $\tilde{K}_3^1 = \operatorname{Im}\tilde{\tau}_3^1$.

The diagram corresponding to the prolonged operator is

$$\begin{array}{ccccccc}
 & & S^4T^* & \xrightarrow{\sigma_4(P_3^1)} & (T^* \otimes F_3) \oplus (S^2T^* \otimes F_3) & \xrightarrow{\tau_3 \times \tilde{\tau}_3^1} & K_3 \oplus \tilde{K}_3^1 \to 0 \\
 & & \downarrow \varepsilon & & \downarrow \varepsilon & & \\
R_4 & \longrightarrow & J_4R & \xrightarrow{p_1(P_3^1)} & J_1(F_3 \oplus (T^* \otimes F_3)) & & \\
\downarrow \bar{\pi}_3 & & \downarrow \pi_3 & & \downarrow \pi_0 & & \\
R_3 & \longrightarrow & J_3R & \xrightarrow{p_0(P_3^1)} & F_3 \oplus (T^* \otimes F_3) & &
\end{array}$$

Thus the compatibility condition for the first prolongation of P_3 is

$$\left(\tau_3^1 \times \tau_{(h,A)}\right)\left[\nabla(P_3^1 E)\right]_x = \left(0, \nabla(i_{A'}\Omega_E)_x, (\varphi_E)_x\right).$$

Now $i_J\Omega_E$ vanishes identically and therefore $\nabla i_J\Omega_E = 0$. On the other hand a 3rd order formal solution $j_3(E)_x$ of P_3^1 satisfies the equations $(i_A\Omega_E)_x = 0$ and $(\nabla i_A\Omega_E)_x = 0$. So, since the rank of the spray is one, A' is a linear combination of J and A and thus $(\nabla i_{A'}\Omega_E)_x = 0$. Therefore the compatibility condition for P_3^1 is given by the equation $\varphi_E = 0$ only

□

6.3.2 The inverse problem when $\bar{A}$ is diagonalizable

As we have seen in the preceding sections, if we study the differential operator P_3^1 i.e. the Euler-Lagrange system with the first compatibility conditions

$$\begin{cases} \omega_E = (i_S dd_J + d\mathcal{L}_C - d)E = 0, \\ i_\Gamma dd_J E = 0, \\ i_A \Omega_E = 0, \end{cases}$$

a higher order compatibility condition appears. This obstruction, noted as φ_E, can be second or third order PDE. This leads us to define three different kinds of sprays: *reducible* sprays if φ_E is of the second order, and *semi-reducible* and *irreducible* sprays if it is of the third order, according to its complexity. Note that this classification is very close to Douglas' classification of separable, semi-separable and non-separable sprays, but it is not exactly the same.

6.3.2.1 Reducibility of sprays

Lemma 6.5 *Let S be a rank 1 atypical spray and suppose that A is diagonalizable. If $j_3(E)_x$ is a 3rd order formal solution of P_3^1 at $x \in TM \setminus \{0\}$ and $(h_i, v_i)_{(i=1,2)}$ is an adapted local basis on a neighborhood U of x, then*

$$(\varphi_E)_x(h_1, h_2, h_1, h_2) = \sum_{i=1}^{2} \chi_i (v_i \Omega_E(v_i, h_i))_x + \sum_{i=1}^{2} k_i \, \Omega_E(h_i, v_i)_x, \quad (6.45)$$

where χ_i and k_i are functions on U, depending on the h_i, v_i, completely determined by the spray (their definition is given by (6.53) resp. by (6.54)).

Proof. Let E be a Lagrangian on M such that $j_3(E)_x$ is a 3rd order solution of P_3^1. Note that Ω_E depends on the second order jet of E and so the terms

$$v_1 \Omega_E(v_2, h_2), \quad v_2 \Omega_E(v_1, h_1), \quad h_1 \Omega_E(v_2, h_2), \quad h_2 \Omega_E(v_1, h_1), \quad (6.46)$$

contain the third derivatives of E. We prove first that they can actually be expressed at x in terms of the coefficients of Ω_E, without its derivatives.

Since the 2-form Ω_E vanishes identically on the vertical bundle, we have $h_2\Omega_E(v_1, v_2) = 0$. On the other hand $(\nabla P_3 E)_x = 0$, so

$$(\nabla P_T)_x(v_j, h_i, h_j) = (v_j\,\Omega_E(h_j, v_i))_x = 0,$$
$$(\nabla P_A)_x(v_j, h_i, h_j) = (\lambda_j - \lambda_i)_x\,(v_j\,\Omega_E(h_j, v_i))_x = 0.$$

Since $\lambda_1 - \lambda_2 \neq 0$, we get $(v_j\,\Omega_E(h_j, v_i))_x = 0$, for $i \neq j$. Moreover, $\Omega_E = dd_J E$ is an exact 2-form, hence

$$\begin{aligned} d\Omega_E(v_i, v_j, h_j) &= v_i\Omega_E(v_j, h_j) + v_j\Omega_E(h_j, v_i) + h_j\Omega_E(v_i, v_j) \\ &\quad - \Omega_E([v_i, v_j], h_j) - \Omega_E([v_j, h_j], v_i) - \Omega_E([h_j, v_i], v_j) = 0, \\ d\Omega_E(h_i, v_j, h_j) &= h_i\Omega_E(v_j, h_j) + v_j\Omega_E(h_j, h_i) + h_j\Omega_E(h_i, v_j) \\ &\quad - \Omega_E([h_i, v_j], h_j) - \Omega_E([v_j, h_j], h_i) - \Omega_E([h_j, h_i], v_j) = 0, \end{aligned}$$

so

$$v_i\Omega_E(v_j, h_j) = \Omega_E([v_i, v_j], h_j) + \Omega_E([v_j, h_j], v_i) + \Omega_E([h_j, v_i], v_j),$$

and

$$h_i\Omega_E(v_j, h_j) = \Omega_E([h_i, v_j], h_j) + \Omega_E([v_j, h_j], h_i) + \Omega_E([h_j, h_i], v_j),$$

for $i \neq j$, $i, j = 1, 2$. Hence these terms can be expressed with the help of Ω_E without its derivatives. On the other hand, $(i_\Gamma\Omega_E)_x = 0$ and $(i_A\Omega_E)_x = 0$. Thus we have at x [1]:

$$\begin{aligned} v_i\Omega_E(v_j, h_j) &= \chi^j_{v_i}\,\Omega_E(v_j, h_j) + \chi^i_{v_i}\,\Omega_E(v_i, h_i), \\ h_i\Omega_E(v_j, h_j) &= \chi^j_{h_i}\,\Omega_E(v_j, h_j) + \chi^i_{h_i}\,\Omega_E(v_i, h_i), \end{aligned} \tag{6.47}$$

where

$$\begin{aligned} \chi^i_{v_i} &= -\xi_{h_i}^{[v_j, h_j]}, & \chi^j_{v_i} &= \xi_{v_j}^{[v_i, v_j]} - \xi_{h_j}^{[h_j, v_i]}, \\ \chi^i_{h_i} &= \xi_{v_i}^{[h_j, h_j]}, & \chi^j_{h_i} &= \xi_{v_j}^{[h_i, v_j]} - \xi_{h_j}^{[h_j, h_i]}, \end{aligned} \tag{6.48}$$

for $i \neq j$. Using these formulas, we find at x:

$$\begin{aligned} \varphi_E(h_1, h_2, h_1, h_2) &= l_1\,\Omega_E(v_1, h_1) + l_2\,\Omega_E(v_2, h_2) \\ &\quad + (\lambda_2 - \lambda_1)\Big([h_2, v_2](\Omega_E(h_1, v_1)) - [h_1, v_1](\Omega_E(h_2, v_2))\Big). \end{aligned}$$

[1]We are not using the summation convention in this paragraph.

where the f_i are functions on U depending on h_i, v_i:

$$
\begin{aligned}
f_i = {} & \lambda_j v_j \xi_{h_i}^{[h_i,h_j]} - \lambda_j v_j \xi_{v_i}^{[h_j,v_i]} - \lambda_j h_j \xi_{h_i}^{[v_i,v_j]} + \lambda_j h_j \xi_{h_i}^{[v_i,h_j]} - \lambda_i \xi_{h_i}^{[h_i,[v_i,h_i]]} \\
& - \lambda_i \xi_{v_j}^{[v_i,v_j]} \xi_{v_i}^{[v_i,h_j]} + \lambda_i \xi_{h_j}^{[v_i,h_j]} \xi_{v_i}^{[v_j,h_i]} + \lambda_i \xi_{h_i}^{[h_i,v_j]} \xi_{h_i}^{[v_i,h_i]} + \lambda_i \xi_{v_i}^{[v_i,[h_i,v_i]]} \\
& + (\lambda_i - \lambda_j) \left(\xi_{v_i}^{[h_i,v_i]} \xi_{h_i}^{[h_i,v_i]} - \xi_{h_i}^{[h_i,v_i]} \xi_{v_i}^{[h_i,v_i]} \right) - \lambda_i \xi_{v_j}^{[v_j,h_i]} \xi_{h_i}^{[v_i,h_j]}.
\end{aligned}
$$

for $i \neq j$, $i, j = 1, 2$. Since $pr_1(hS) \neq 0$ (cf. Property 6.3.1) we can choose $h_1 := pr_1(hS)$. We have at x:

$$
\begin{aligned}
\Omega_E([S, v_i], h_i) + \Omega_E(v_i, [S, h_i]) &= -\Omega_E([J,S]h_i, h_i) - 2\Omega_E([v_i, S], h_i) \\
&= -2\xi_{h_i}^{[v_i, S]} \Omega_E(v_i, h_i)
\end{aligned}
\tag{6.49}
$$

But using once the exactness of Ω_E we arrive at $\mathcal{L}_S \Omega_E = (di_S + i_S d) dd_J E = di_S dd_J E = d\omega_E$, and

$$
\begin{aligned}
S(\Omega_E(v_i, h_i)) &= (\mathcal{L}_S \Omega_E)(v_i, h_i) + \Omega_E([S, v_i], h_i) + \Omega_E(v_i, [S, h_i]) \\
&= 2\xi_{h_i}^{[S, v_i]} \Omega_E(v_i, h_i)
\end{aligned}
\tag{6.50}
$$

at x, for $i = 1, 2$. Thus for any 3rd order formal solution and for any adapted basis, we get

$$
h_i \Omega_E(v_i, h_i) = \mu_{h_i} v_i \Omega_E(v_i, h_i) + \mu^i_{h_i} \Omega_E(v_i, h_i) + \mu^j_{h_i} \Omega_E(v_j, h_j), \tag{6.51}
$$

where

$$
\begin{aligned}
\mu_{h_i} &= -\frac{\xi^S_{v_i}}{\xi^S_{h_i}}, \\
\mu^i_{h_i} &= -\frac{1}{\xi^S_{h_i}} \left(\xi^S_{h_i} \chi^i_{h_i} + \xi^S_{v_i} \chi^i_{v_i} - 2\xi_{v_i}^{[S, v_i]} \right) \\
\mu^j_{h_i} &= -\frac{1}{\xi^S_{h_i}} \left(\xi^S_{h_i} \chi^j_{h_i} + \xi^S_{v_i} \chi^j_{v_i} \right).
\end{aligned}
\tag{6.52}
$$

for $i \neq j$, $i, j = 1, 2$. Of course, $\xi^S_{h_1} = 1$ if we take $h_1 = S$. But

$$
\begin{aligned}
[h_i, v_i] \Omega_E(h_j, v_j) = {} & 2\xi_{h_j}^{[h_i, v_i]} \xi_{h_i}^{[S, v_i]} \Omega_E(v_j, h_j) + \left(\xi_{v_i}^{[h_i, v_i]} - \xi^S_{v_i} \xi_{h_j}^{[h_i, v_i]} \right) v_j \Omega_E(v_j, h_j) \\
& + \left(\left(\xi_{h_i}^{[h_i, v_i]} - \xi^S_{h_i} \xi_{h_j}^{[h_i, v_i]} \right) \chi^i_{h_i} + \left(\xi_{v_i}^{[h_i, v_i]} - \xi^S_{v_i} \xi_{h_j}^{[h_i, v_i]} \right) \chi^i_{v_i} \right) \Omega_E(v_i, h_i) \\
& + \left(\left(\xi_{h_i}^{[h_i, v_i]} - \xi^S_{h_i} \xi_{h_j}^{[h_i, v_i]} \right) \chi^j_{h_i} + \left(\xi_{v_i}^{[h_i, v_i]} - \xi^S_{v_i} \xi_{h_j}^{[h_i, v_i]} \right) \chi^j_{v_i} \right) \Omega_E(v_j, h_j).
\end{aligned}
$$

so $(\varphi_E)_x$ has the expression (6.45), where the coefficients of the third order terms are

$$\chi_2 = \lambda(\xi_{v_2}^{[h_1,v_1]} - \xi_{v_1}^{S}\xi_{h_2}^{[h_1,v_1]}), \tag{6.53}$$

and the coefficients of the second order terms are

$$\begin{aligned} k_i = {} & l_i + 2\xi_{h_i}^{[h_i,v_i]}\xi_{v_j}^{[S,v_i]} \\ & + (\xi_{h_i}^{[h_i,v_i]} - \xi_{h_i}^{S}\xi_{h_j}^{[h_i,v_i]})\chi^1_{h_i} + (\xi_{v_i}^{[h_i,v_i]} - \xi_{v_i}^{S}\xi_{h_i}^{[h_i,v_i]})\chi^1_{v_i} \\ & + (\xi_{h_j}^{[h_j,v_j]} - \xi_{h_j}^{S}\xi_{h_i}^{[h_j,v_j]})\chi^1_{h_j} + (\xi_{v_j}^{[h_j,v_j]} - \xi_{v_j}^{S}\xi_{h_i}^{[h_j,v_j]})\chi^1_{v_j}. \end{aligned} \tag{6.54}$$

where $i \neq j$.

◇

Note that the obstruction $\varphi_E = 0$ is of second order if and only if $\chi_1 = 0$ and $\chi_2 = 0$. It follows that the analysis of the problem is different according whether to the χ_i vanish or not. Thus we propose the following definition:

Definition 6.1 The spray S is called
- *reducible* if $\chi_1 = 0$, and $\chi_2 = 0$,
- *semi-reducible* if $\chi_1 = 0$, and $\chi_2 \neq 0$, (or $\chi_2 = 0$ and $\chi_1 \neq 0$),
- *irreducible* if $\chi_1 \neq 0$, and $\chi_2 \neq 0$.

Despite the fact that the functions χ_i depend on the choice of the basis $\{h_i, v_i\}$, these notions are intrinsic. In order to see this, we will adopt the following.

Definition 6.2 Let Δ be a distribution on TM and consider $\Delta^1 = \Delta$ and $\Delta^{r+1} = [\Delta^r, \Delta^r]$. The distribution Δ^r $(r \geq 2)$ is called *reducible at* x, if

$$\text{either} \quad \Delta^r_x = \Delta^{r-1}_x \quad \text{or} \quad S_x \in \Delta^r_x.$$

Δ^r is called reducible if it is reducible at any x.

Note that if the spray is found in Δ^r, then either Δ^r is involutive, or S is a characteristic field. This terminology is justified by the following

Proposition 6.4 *Let $x \in TM \setminus \{0\}$. Then $(\chi_i)_x = 0$ if and only if Δ^2_j is reducible at x for $i \neq j$.*

Indeed, we have

$$\chi_i(x) = \lambda(x)\det\begin{pmatrix}\xi^S_{h_i} & \xi^{[h_i,v_i]}_{h_j}\\ \xi^S_{h_j} & \xi^{[h_i,v_i]}_{h_j}\end{pmatrix}(x),$$

where $\xi^S_{h_i} = 1$ and $h_i = pr_i S$. Thus we obtain $\chi_j(x) = 0$ if and only if for $i \neq j$ the vectors $pr_j S_x$ and $pr_j[h_i, v_i]_x$ are linearly dependent, i.e. if and only if there exists $\mu \in \mathbb{R}$, such that

$$pr_j([h_i, v_i] - \mu S)_x = 0$$

But the spray is non-typical, ($S \notin \Delta_1$), hence $\chi_j(x) = 0$ if and only if $\mu = 0$, that is $\Delta^2_i = \Delta_i$ at x or $S_x \in \Delta^2_{i,x}$ at x, i.e. Δ^2_i is reducible at x.

◊

With this terminology we can state the

Corollary 6.1 *Let S be an atypical spray of rank 1 and suppose that $\bar{A}$ is diagonalizable. Then every 3rd order solution of the operator P^1_3 can be lifted into a 4th order solution if and only if the spray is reducible and $k_i = 0$ for $i = 1, 2$.*

6.3.2.2 *Completion Lemma*

Recall that we are studying the prolongation P^1_3 of the operator P_3 defined by the equations

$$\begin{cases} \omega_E = 0, \\ i_1\Omega_E = 0, \\ i_A\Omega_E = 0, \end{cases}$$

where $\omega_E = i_S dd_J E + d\mathcal{L}_C E - dE$ is the Euler-Lagrange form and $\Omega_E = dd_J E$. As we computed in the previous paragraph, an obstruction denoted by φ appears. Namely, if E is a Lagrangian associated to S, then φ_E must vanish. The complexity of the equation $\varphi = 0$ is expressed by the notion of the reducibility of the spray. Denoting by $P_{(k,\chi)} : C^\infty(TM) \to C^\infty(TM)$ the corresponding differential operator defined by

$$P_{(k,\chi)}E = \sum_{i=1,2} \chi_i \mathcal{L}_{h_i}(\Omega_E(v_i, h_i)) + \sum_{i=1,2} k_i \Omega_E(h_i, v_i).$$

we have to study the system defined by

$$(P^1_2, P_{(h,\lambda)}).$$

As we expected, obstructions to the integrability appear several times in this system's analysis, but they can be treated in a similar way using a general Lemma, which we will state in this subsection. To formulate this Lemma, we introduce the following

Notations

1. Consider an adapted local basis $B = \{h_i, v_i\}_{i=1,2}$ on a neighborhood U of $z \in TM \setminus \{0\}$ and two functions ϑ_1 and ϑ_2 on U, with $\vartheta_2 \neq 0$. We put forward

$$\begin{aligned}
\Theta^1_{\vartheta_1,\vartheta_2} &:= \left(\vartheta_1 \xi^S_{h_1} \xi^{[h_1,v_1]}_{v_1} - \vartheta_1 \xi^S_{v_1} \xi^{[h_1,v_1]}_{h_1} + 2\vartheta_1 \xi^{[v_1,S]}_{h_1} - (S\vartheta_1)\right) - \\
&\quad - \frac{\vartheta_1}{\vartheta_2}\left(\vartheta_2 \xi^S_{h_2} \xi^{[h_2,v_2]}_{v_2} - \vartheta_2 \xi^S_{v_2} \xi^{[h_2,v_2]}_{h_2} + 2\vartheta_2 \xi^{[v_2,S]}_{h_2} - (S\vartheta_2)\right), \\
\Theta^2_{\vartheta_1,\vartheta_2} &= \left(2\vartheta_1 \xi^{[v_1,S]}_{h_1} + \vartheta_1 \xi^{[S_1,h_1]}_{h_1} + \vartheta_1 \xi^{[S_1,v_1]}_{v_1} - (S_1\vartheta_1)\right) \\
&\quad - \frac{\vartheta_1}{\vartheta_2}\left(\vartheta_2 \xi^{[S_1,h_2]}_{h_2} + \vartheta_2 \xi^{[S_1,v_2]}_{v_2} - (S_1\vartheta_2)\right),
\end{aligned} \tag{5.55}$$

where S_i is the projection of S on the eigenspace Δ_i, $i = 1,2$.

2. Consider two functions g_1 and g_2 on U. We will denote by P_g the second order differential operator $P_g : C^\infty(TM) \longrightarrow C^\infty(TM)$ defined by

$$P_g E = g_1 \Omega_E(v_1, h_1) + g_2 \Omega_E(v_2, h_2)$$

and by M_{g_1,g_2} the matrix whose rows are the coefficients of

$$\nabla_{v_1}(\Omega_E(v_1,h_1)), \quad \nabla_{v_2}(\Omega_E(v_2,h_2)), \quad \Omega_E(v_1,h_1), \quad \Omega_E(v_2,h_2)$$

in the equations defined by the operators

$$P_{(h,\lambda)}, \ \nabla_{v_1}P_g, \ \nabla_{v_2}P_g, \ P_g.$$

Namely,

$$M_{g_1,g_2} := \begin{pmatrix} v_1 & v_2 & k_1 & k_2 \\ g_1 & 0 & g_1^1 & g_1^2 \\ 0 & g_2 & g_2^1 & g_2^2 \\ 0 & 0 & g_1 & g_2 \end{pmatrix} \tag{5.56}$$

where

$$g_1^1 = \mathcal{L}_{v_1} g_1 + g_2 \chi^1_{v_1}, \qquad g_1^2 = \mathcal{L}_{h_1} g_2 + g_1 \chi^2_{h_1},$$
$$g_2^1 = \mathcal{L}_{v_2} g_1 + g_1 \chi^1_{v_2}, \qquad g_2^2 = \mathcal{L}_{h_2} g_2 + g_1 \chi^2_{h_2}.$$

Lemma 6.6 (Completion Lemma)
Let $P_4 := (P_3 = i_S dd_J + d\mathcal{L}_C - d, i_{[S} dd_J, i_{[J} dd_J)$ and $\bar{P}_4 := (P_3, P_4)$. Consider $x \in TM \setminus \{0\}$.

(1) if $g_1(x) = 0$ or $g_2(x) = 0$, then there are no second order regular formal solutions of $\bar{P}_4$ at x;

(2) If $g_1(x) \neq 0$ and $g_2(x) \neq 0$, then

(a) there are regular formal solutions of the differential system $P_4(E) = 0$ on a neighborhood U of x if and only if

$$\Theta^1_{g_1, g_2} = 0, \quad \Theta^2_{g_1, g_2} = 0, \quad \text{and} \quad \det(M_{g_1, g_2}) = 0$$

on U.

(b) Moreover $\bar{P}_4$ is "complete", in the sense that if we add to $\bar{P}_4(E) = 0$ a new differential equation of type

$$a v_1 \Omega_E(v_1, h_1) + b v_2 \Omega_E(v_2, h_2) + c\, \Omega_E(v_1, h_1) + e\, \Omega_E(v_2, h_2) = 0$$

which is independent of the others at x, then the new system has no regular second order formal solutions at x.

Proof (1) is obvious: if, for example, $g_1(x) = 0$, then $P_3 E$ vanishes at x if and only if $\Omega_E(v_2, h_2)_x = 0$, so there are no regular solutions of second order at x.

(2 b) is almost evident. Indeed,

$$\det \begin{pmatrix} \chi_1 & \chi_2 & k_1 & k_2 \\ g_1 & 0 & g_1^1 & g_1^2 \\ 0 & g_1 & g_2^1 & g_2^2 \\ 0 & 0 & g_1 & g_1 \end{pmatrix} \neq 0$$

at x, so if $p = j_3(E)_x$ is a third order formal solution at x of the new operator, then $v_1 \Omega_E(v_1, h_1)$, $v_2 \Omega_E(v_2, h_2)$, $\Omega_E(v_1, h_1)$, $\Omega_E(v_2, h_2)$ vanish at x and p is not regular.

The Proof of (2.a) is more complicated. It will be carried out in three steps: in the first two steps we will find the obstructions for the first two lifts of the 2nd order formal solutions and in the third step we will prove that, after prolongation, $\dot{P}_g$ becomes 2-acyclic.

STEP 1 - *First lift of the 2nd order formal solutions:*

> *Every second order formal solution of $\dot{P}_g$ at x can be lifted into a third order formal solution if and only if $\Theta^1_{g_1,g_2}(x) = 0$ and $\Theta^2_{g_1,g_2}(x) = 0$.*

Proof. The symbol $\sigma_2(P_{g_1,g_2}) : S^2T^* \longrightarrow \mathbb{R}$, of P_{g_1,g_2} is defined by

$$\sigma_2(P_{g_1,g_2})(\alpha) := g_1\, \alpha(v_1, v_1) + g_2\, \alpha(v_2, v_2), \tag{6.57}$$

and its prolongation $\sigma_3(P_{g_1,g_2}) : S^3T^* \longrightarrow T^*$ is given by

$$[\sigma_3(P_{g_1,g_2})(\beta)](X) := g_1\, \beta(X, v_1, v_1) + g_2\, \beta(X, v_2, v_2), \tag{6.58}$$

$\alpha \in S^2T^*$, $\beta \in S^3T^*$, $X \in T_x$. With the notations of the preceding section, $F_3 := T^* \oplus \Lambda^2 T^*_v \oplus \Lambda^2 T^*_v$, we define

$$\tilde{\tau} : (T^* \otimes F_3) \oplus T^* \longrightarrow K_3 \oplus \mathbb{R} \oplus \mathbb{R}$$

by $\tilde{\tau} := (\tilde{\tau}_3, \rho_1, \rho_2)$, where:

$$\begin{aligned}
&\tilde{\tau}_3(B, C_\Gamma, C_A, C_g) = \tau_3(B, C_\Gamma, C_A),\\
&\rho_1(B, C_\Gamma, C_A, C_g) = g_1\, B(v_1, h_1) + g_2\, B(v_2, h_2) - C_g(S),\\
&\rho_2(B, C_\Gamma, C_A, C_g) = g_1\, B(v_1, h_1) + \xi^S_{\lambda_2}\left(\frac{g_1}{2} C_\Gamma(v_1, h_1, h_2) - \frac{g_1}{\lambda} C_A(h_1, h_1, h_2)\right)\\
&\quad - \xi^S_{h_1}\left(C_g(h_1) - \frac{g_2}{2} C_\Gamma(v_2, h_1, h_2) - \frac{g_2}{\lambda} C_A(h_2, h_1, h_2)\right)\\
&\quad - \xi^S_{v_2}\left(\frac{g_1}{\lambda} C_A(v_1, h_1, h_2)\right) - \xi^S_{v_1}\left(C_g(v_1) - \frac{g_2}{\lambda} C_A(v_2, h_1, h_2)\right),
\end{aligned}$$

and $\lambda = \lambda_1 - \lambda_2$. Let us put forward $\operatorname{Im} \dot{K} = \tilde{\tau}$, and show that the sequence

$$S^3T^* \xrightarrow{\sigma_3(\dot{P}_g)} (T^* \otimes F_3) \oplus T^* \xrightarrow{\tilde{\tau}} \dot{K} \longrightarrow 0 \tag{6.59}$$

is exact.

It is easy to check that $\operatorname{Im} \sigma_3(\dot{P}_g) \subset \operatorname{Ker} \tilde{\tau}$. On the other hand, $g_3(\dot{P}_g) = g_3(P_1) \cap g_3(P_{g_1,g_2})$ where $g_3(P_{g_1,g_2})$ is defined by the equation

$$g_1\, B(X, v_1, v_1) + g_2\, B(X, v_2, v_2) = 0, \tag{6.60}$$

where $X \in T_x$. If we replace X by the four elements of a basis of T_x, we obtain four equations, two of which are independent and two of which are related to the system which defines $g_3(\dot P_g)$. Indeed, we get

$$\begin{aligned}
[\sigma_3(P_{g_1,g_2})B](S) &= g_1\,[\sigma_3(P_1)B](v_1,h_2) + g_2\,[\sigma_3(P_1)B](v_2,h_2),\\
[\sigma_3(P_{g_1,g_2})B](pr_1 S) &= g_1[\sigma_3(P_1)B](v_1,h_1)\\
&\quad - \xi^S_{h_1}\left(\frac{g_2}{2}[\sigma_3(P_T)B](v_2,h_1,h_1) + \frac{g_2}{\lambda}[\sigma_3(P_A)B](h_2,h_1,h_1)\right)\\
&\quad - \xi^S_{v_2}\left(\frac{g_1}{\lambda}[\sigma_3(P_A)B](v_1,h_1,h_2)\right) - \xi^S_{v_1}\left(\frac{g_2}{\lambda}[\sigma_3(P_A)B](v_2,h_1,h_2)\right)\\
&\quad + \xi^S_{h_2}\left(\frac{g_2}{2}[\sigma_3(P_T)B](v_2,h_1,h_2) - \frac{g_1}{\lambda}[\sigma_3(P_A)B](h_1,h_1,h_2)\right),\\
[\sigma_3(P_{g_1,g_2})B](pr_2 S) &= g_2[\sigma_3(P_1)B](v_2,h_2)\\
&\quad - \xi^S_{h_1}\left(\frac{g_2}{2}[\sigma_3(P_T)B](v_2,h_2,h_1) + \frac{g_1}{\lambda}[\sigma_3(P_A)B](h_1,h_2,h_1)\right)\\
&\quad - \xi^S_{v_2}\left(\frac{g_2}{\lambda}[\sigma_3(P_A)B](v_2,h_2,h_1)\right) + \xi^S_{v_1}\left(\frac{g_1}{\lambda}[\sigma_3(P_A)B](v_1,h_2,h_1)\right)\\
&\quad + \xi^S_{h_2}\left(\frac{g_2}{2}[\sigma_3(P_T)B](v_2,h_2,h_1) - \frac{g_2}{\lambda}[\sigma_3(P_A)B](h_2,h_2,h_1)\right),
\end{aligned}$$

where $pr_i S$ is the projection of S on the eigenspace Δ_i, $i = 1,2$. Of course these equations are dependent, because $S = pr_1 S + pr_2 S$. On the other hand the equations $\rho_1 = 0$ and $\rho_2 = 0$ are independent of the equation $\tau_1 = 0$ (the corresponding pivot terms are $\omega(S)$ and $\omega(S_1)$). Thus $\dim g_3(\dot P_g) = \dim g_3(P_3) - 2$ and therefore

$$\operatorname{rank}\sigma_3(\dot P_g) = \operatorname{rank}\sigma_3(P_3) + 2.$$

Taking into account that the sequence (6.39) is exact, we find

$$\dim \operatorname{Ker}\dot\tau = \dim \operatorname{Ker}\tau_1 + \dim T^* - 2 = \dim \operatorname{Ker}\tau_1 + 2,$$

so $\dim \operatorname{Ker}\dot\tau = \operatorname{rank}\sigma_3(\dot P_g)$ and therefore the sequence (6.59) is exact.

Let us now compute the compatibility conditions. Taking a second order formal solution $j_2(E)_x$ of $\dot P_g$, the new obstructions are

$$\begin{aligned}
\rho_1(\nabla\dot P_g E) &= \sum_{i=1,2}\left(\sum_{cycl} g_i\Omega_E([h_i,v_i],S) - (Sg_i)\Omega_E(v_i,h_i)\right),\\
\rho_2(\nabla\dot P_g E) &= g_1\left(\Omega_E([S,h_1],v_1) - \Omega_E([S,v_1],h_1)\right)\\
&\quad - \xi^S_{h_1}\left((h_1 g_1)\Omega_E(v_1,h_1) + (h_1 g_2)\Omega_E(v_2,h_1) - \sum_{cycl} g_2\Omega_E([h_1,v_2],h_2)\right)\\
&\quad - \xi^S_{v_1}\left((v_1 g_1)\Omega_E(v_1,h_2) + (v_1 g_2)\Omega_E(v_2,h_2) + \sum_{cycl} g_2\Omega_E([v_1,v_2],h_2)\right)\\
&\quad + \xi^S_{h_2}\left(\sum_{cycl} g_1\Omega_E([v_1,h_1],h_2)\right) - \xi^S_{v_2}\left(\sum_{cycl} g_1\Omega_E([v_1,v_2],h_1)\right)
\end{aligned}$$

Since the spray is reducible, the vectors $pr_1[h_2, v_2]$ and $pr_1 S$ (resp. $pr_2[h_1, v_1]$, and $pr_2 S$) are linearly dependent and so

$$\begin{aligned}\Omega_E([h_i, v_i], S) &= \Omega_E([h_i, v_i], pr_i S) + \Omega_E([h_i, v_i], pr_j S)\\ &= \left(\xi^S_{h_i}\xi^{[h_i,v_i]}_{v_i} - \xi^S_{v_i}\xi^{[h_i,v_i]}_{h_i}\right)\Omega_E(v_i, h_i),\end{aligned}$$

where $i \neq j$. Using (6.49) we obtain:

$$\begin{aligned}p_1(\nabla \hat{P}_g E) &= \sum_{i=1,2}\left(k_i\xi^S_{v_i}\xi^{[h_i,v_i]}_{h_i} - k_i\xi^S_{h_i}\xi^{[h_i,v_i]}_{v_i} + 2k_i\xi^{[v_i,S]}_{h_i} - (Sk_i)\right)\Omega_E(v_i, h_i),\\ p_2(\nabla \hat{P}_g E) &= \left(2g_1\xi^{[v_1,S]}_{h_1} + g_1\xi^{[S_2,h_1]}_{h_1} + g_1\xi^{[S_2,v_1]}_{v_1} - (S_1 g_1)\right)\Omega_E(v_1, h_1)\\ &\quad + \left(g_2\xi^{[S_2,h_2]}_{h_2} + g_2\xi^{[S_1,v_2]}_{v_2} - (S_1 g_2)\right)\Omega_E(v_2, h_2).\end{aligned} \tag{6.61}$$

Since $j_2(E)_x$ is a second order formal solution also of P_{g_1, g_2}, the equation

$$g_1(x)\Omega_E(v_1, h_1)_x + g_2(x)\Omega_E(v_2, h_2)_x = 0$$

holds, and so

$$p_i(\nabla \hat{P}_g E) = \Theta^i_{g_1, g_2}\,\Omega_E(v_1, h_1).$$

Therefore $p_i(\nabla \hat{P}_g E)(x) = 0$ if and only if $\Theta^i_{g_1, g_2}(x) = 0$, $i = 1, 2$, which proves STEP 1.

Like P_3, $\hat{P}_g$ is not 2-acyclic and an explicit computation shows that $H^2_2(\hat{P}_g)$, the second Spencer cohomology group, is non-vanishing. Hence some obstructions arise for successive lifts. ◇

STEP 2 - *Second lift* :

Suppose that every regular second order formal solution at x of $\hat{P}_g$ can be lifted into a 3^{rd} order one. Then every 3rd order regular formal solution can be lifted at x into a 4th order formal solution if and only if $j_1(\Theta^i_{g_1, g_2})_x = 0$, $i = 1, 2$, and $\det(M_{g_1, g_2})(x) = 0$.

Proof. We begin by showing that the sequence

$$S^4T^* \xrightarrow{\sigma_4(\hat{P}_g)} S^2T^* \otimes F \xrightarrow{\;\tau^1\;} \hat{K}^1 \longrightarrow 0 \tag{6.62}$$

is exact, where $\tau^1 := (\mathrm{id} \otimes \tau) \circ \tau_{[h,4]}$ and $\tau_{[h,4]}$ is defined in (6.43).

The equations defining $g_4(\hat{P}_g) = \operatorname{Ker}\sigma_4(\hat{P}_g)$ give some restrictions on the components containing at least one vertical vector. Therefore we have the splitting $g_4(\hat{P}_g) = S^4T_u^* \oplus G_4(\hat{P}_g)$, where $G_m(\hat{P}_g)$ is defined by (6.21). Now $\dim G_4(\hat{P}_g) = 1$ and

$$\dim g_4(\hat{P}_g) = \dim S^4T_v^* + 1 = 6,$$

i.e. $\dim g_4(\hat{P}_g) = \dim g_4(P_g^1) - 2$. Using the exactness of the sequence (6.44) and the fact that the equations $id \otimes \rho_i = 0$, $i = 1, 2$ give 8 independent equations with respect to the equations $\hat{\tau}_3^1 = 0$, we have

$$\operatorname{rank}\sigma_4(\hat{P}_g) = \operatorname{rank}\sigma_4(P_g^1) + 2 = \dim\operatorname{Ker}\hat{\tau}_3^1 + \dim S^2T^* = 8 - \dim\operatorname{Ker}\hat{\tau}^1.$$

This proves that the sequence (6.62) is exact.

Let us compute the obstructions to lifting the 3rd order solutions of $\hat{P}_g$. A 3rd order formal solution $j_3(E)_x$ can be lifted into a 4th order solution if and only if $\hat{\tau}^1(\nabla\nabla\hat{P}_g E)_x = 0$. Now

$$\begin{aligned}\hat{\tau}^1(\nabla(\nabla\hat{P}_g E))_x &= \left(\nabla(\hat{\tau}\nabla(\hat{P}_g E)),\ \tau_{(h,A)}(\nabla\hat{P}_g E)\right)_x \\ &= \left(\nabla(\tau_1(\nabla\hat{P}_g E)),\ \nabla(\rho_i\nabla\hat{P}_g E))_{i=1,2},\ \tau_{(h,A)}(\nabla\hat{P}_g E)\right)_x,\end{aligned}$$

and

$$\begin{aligned}\tau_{(h,A)}(\nabla\hat{P}_g E)_x &= (P_{(h,A)}E)_x, \\ \nabla\left(\tau_1(\nabla\hat{P}_g E)\right)_x &= 0 \times \nabla(i_A\Omega_E)_x = 0, \\ \nabla\left(\rho_i(\nabla\hat{P}_g E)\right)_x &= \nabla\left(\Theta^i_{g_1,g_2}\Omega(v_i,h_i)\right)_x, \qquad i = 1, 2\end{aligned}$$

As we have just seen, every second order formal solution at x can be lifted into a 3rd order solution if and only if $\Theta^i_{g_1,g_2}(x) = 0$, $i = 1, 2$. Let us suppose that this condition is satisfied. Then

$$\hat{\tau}^1(\nabla\nabla\hat{P}_g E)_x = \left((P_{(h,A)}E)_x,\ (\nabla\Theta^i_{g_1,g_2})_x\Omega_E(v_i,h_i)_x\right)$$

Thus, if $j_3(E)_x$ is a 3rd order regular formal solution, which can be lifted into a 4th order solution, we have $\nabla\Theta^i_{g_1,g_2}(x) = 0$ for $i = 1, 2$, that is $j_1(\Theta^i_{g_1,g_2})_x = 0$, because we suppose that $(\Theta^i_{g_1,g_2})_x = 0$

On the other hand, note that $\det(M_{g_1,g_2})(x) = 0$ if and only if the first row of the matrix is a linear combination of the others, because $g_1(x) \neq 0$ and $g_2(x) \neq 0$. This means that the differential operator $P_{(h,A)}$ can be linearly expressed at x with the help of the operators $\nabla P_g(v_1)$, $\nabla P_g(v_2)$

and P_g. Hence, since $\nabla(P_3 E)(v_1)$, $(\nabla P_3 E)(v_2)$ and $P_3 E$ vanish at x if $j_3(E)_x$ is a third order formal solution, then $\det(M_{g_1,g_2})(x) = 0$ if and only if $(P_{(h,4)} E)_x = 0$ for a third order solution $j_3(E)_x$. This proves STEP 2.

STEP 3 - *Higher order lifts* :

The first prolongation $\dot{P}_g^1$ of $\dot{P}_g$ is 2-acyclic.

Proof. We have $g_m(\dot{P}_g^1) = g_m(\dot{P}_g)$ and $g_m(\dot{P}_g) = g_m(P_3) \cap g_m(P_g)$ for $m \geq 3$. An element $B \in g_2(P_g)$ is characterized by the equation

$$g_1\, B(v_1, v_1) + g_2\, B(v_2, v_2) = 0. \tag{6.63}$$

First we will compute $\dim g_m(\dot{P}_g)$. Note that

$$g_m(\dot{P}_g) = S^m T_V^* \oplus G_m(\dot{P}_g), \tag{6.64}$$

where $G_m(\dot{P}_g) = G_m(P_3) \cap G_m(P_g)$. Now, $\dim G_m(P_3) = 3$ and an element B_m in $G_m(P_3)$ is determined by the components (6.28). Moreover, from the equations (6.63), we find

$$B(v_i^m) \overset{(6.63)}{=} \frac{g_j}{g_i} B(v_i^{m-2}, v_j, v_j) \overset{(6.31)}{=} 0$$

for $i \neq j$, $(i, j = 1, 2)$, $m \geq 1$. Thus in $G_m(\dot{P}_g)$ there is only one free component: the $B(h_2^{m-1}, v_2)$. So we obtain

$$\dim g_m(\dot{P}_g) = \dim S^m T^* + 1 = m + 2 \tag{6.65}$$

for any $m \geq 3$. Hence

$$\operatorname{rank} \delta_{1,m}(\dot{P}_g) = \dim\left(T^* \otimes g_{m+1}(\dot{P}_g)\right) - \dim\left(g_{m+2}(\dot{P}_g)\right) = 3m + 8 \tag{6.66}$$

for any $m \geq 2$.

Remark. Using (6.66) we can note that $\operatorname{rank} \delta_{1,2}(\dot{P}_g) = 14$. On the other hand, from the exactness of the sequence (6.40) follows that rank $\delta_{2,2}(\dot{P}_g) \leq 15$. So $\dim \operatorname{Ker} \delta_{2,2}(\dot{P}_g) \geq \dim \Lambda^2 T^* \otimes g_2(\dot{P}_g) - 15 = 15$, and therefore

$$H_2^2(\dot{P}_g) = \frac{\operatorname{Ker} \delta_{2,2}(\dot{P}_g)}{\operatorname{Im} \delta_{1,3}(\dot{P}_g)} \neq 0,$$

and $\dot{P}_g$ is not 2-acyclic.

In order to show that $H^2_m(\dot{P}_g) = 0$ for any $m \geq 3$, we only need to compute $\dim \operatorname{Ker} \delta_{2,m}(\dot{P}_g)$. By the splitting (6.64) we have

$$\Lambda^3 T^* \otimes g_{m-1}(\dot{P}_g) = (\Lambda^3 T^* \otimes S^{m-1} T^*_u) \oplus (\Lambda^3 T^* \otimes G_{m-1}(\dot{P}_g)).$$

Thus for $B \in \Lambda^2 T^* \otimes S^m T^*$ we have $\delta_{2,m}(\dot{P}_g) B = 0$ if and only if the equations

$$\sum_{cycl(i,j,k)} B(e_i, e_j, e_k, h_2^{m-2}; v_2) = 0 \tag{6.67}$$

and the equations

$$\sum_{cycl(i,j,k)} B(e_i, e_j, e_k, h_1^{m-l}, h_2^{l-1}) = 0, \quad 1 \leq l \leq m \tag{6.68}$$

hold, where $(e_i)_{i=1,\dots,4}$ denotes the vectors of the adapted basis $\{h_i, v_i\}_{i=1,2}$. Thus $\dim \operatorname{Ker} \delta_{2,m}(\dot{P}_g) = \operatorname{rank} \{(6.67), (6.68)\}$. The rank of this system can be found by a completely analogous computation as the rank of the system $\{(6.70), (6.71)\}$ in the proof of the Theorem 6.3. This second one being slightly more complex, we would prefer to explore it in detail later on. However we can see, that the system (6.67) corresponds to the equations *a)* of (6.70); (6.68) is the same as (6.71), while *b)* and *c)* of (6.70) now hold identically. We find

$$\operatorname{rang} \delta_{2,m}(\dot{P}_g) = 3m + 4$$

and therefore

$$\dim \operatorname{Ker} \delta_{2,m}(\dot{P}_g) = \dim(\Lambda^2 T^* \otimes g_m(\dot{P}_g)) - (3m + 10) = 3m + 8$$

for every $m \geq 3$. So, by (6.66),

$$\operatorname{rank} \delta_{1,m}(\dot{P}_g) = \dim \operatorname{Ker} \delta_{2,m}(\dot{P}_g)$$

for $m \geq 3$ and then $H^2_m(\dot{P}_g) = 0$ for $m \geq 3$. Since $H^2_m(\dot{P}_g) = H^2_m(\dot{P}^1_g)$ for $m \geq 3$, we obtain

$$H^2_m(\dot{P}^1_g) = 0$$

for any $m \geq 3$, i.e. $\dot{P}^1_g$ is 2-acyclic. This proves STEP 3.

The point (2 a) follows from these three steps and then the Completion Lemma is proved.

□

6.3.2.3 *Reducible case*

We will now study the reducible case for the atypical sprays of rank 1 and we will prove the following

Theorem 6.3 *Let S be an atypical spray of rank 1 with $\tilde{\Lambda}$ diagonalizable and suppose that S is reducible.*

a) If $k_1 = k_2 = 0$, then S is locally variational;

b) if $k_1 \neq 0$ and $k_2 = 0$ (or $k_1 = 0$ and $k_2 \neq 0$), then S is not locally variational;

c) if $k_1 \neq 0$ and $k_2 \neq 0$, then S is locally variational if and only if

$$\Theta^1_{k_1, k_2} = 0, \qquad \text{and} \qquad \Theta^2_{k_1, k_2} = 0.$$

To prove the Theorem we just have to check that the differential operator P^1_3 is 2-acyclic. Indeed, if P^1_3 is 2-acyclic, then the proof of a) follows immediately from the Corollary 6.1. Note that we have here an example of a differentiable operator which is formally integrable though it is not 2-acyclic.

The statement of b) is obvious because the compatibility condition (6.45), i.e. $k_1 \Omega(v_1, h_1) = 0$, cannot be satisfied by a regular solution (cf. Lemma 6.1).

c) follows from the Completion Lemma with $g_i = k_i$ for $i = 1, 2$. Indeed, in the reducible case we have $\chi_1 = \chi_2 = 0$ and so $\det(M_{k_1, k_2}) = 0$.

To prove the 2-acyclicity of P^1_3, let us check first, with the notation of page 113, that

$$\dim \operatorname{Ker} \delta_{2,m}(P^1_3) = 3m + 14 \tag{6.69}$$

for any $m \geq 2$.

First we see that $\delta_{2,m}(P^1_3) = \delta_{2,m}(P_3)$. Consider now an element B of $\Lambda^2 T^* \otimes g_m(P_3)$, and let $\{e_i\}_{i=1,\dots,4} := \{h_1, h_2, v_1, v_2\}$ denote an adapted basis. Since

$$\Lambda^3T^*\otimes g_{m-1}(P_3^1) = (\Lambda^3T^*\otimes S^{m-1}T_h^*)\oplus(\Lambda^3T^*\otimes G_{m-1}(P_3^1)),$$

we have $\delta_2 B = 0$ if and only if the equations

$$\begin{aligned} &a) \quad \sum_{\text{cycl}\{ijk\}} B(e_i,e_j,e_k,h_2,\dots,h_2,v_2) = 0,\\ &b) \quad \sum_{\text{cycl}\{ijk\}} B(e_i,e_j,e_k,v_1,\dots,v_1,v_1) = 0, \qquad (6.70)\\ &c) \quad \sum_{\text{cycl}\{ijk\}} B(e_i,e_j,e_k,\underbrace{v_2,\dots,v_2,v_2}_{m-1}) = 0, \end{aligned}$$

and the equation

$$f) \quad \sum_{\text{cycl}\{ijk\}} B(e_i,e_j,e_k,\underbrace{h_1,\dots,h_1}_{m-l},\underbrace{h_2,\dots,h_2}_{l-1}) = 0, \qquad (6.71)$$

hold for $1 \le l \le m$, where $i,j,k = 1,\dots,4$ are all different. The system (6.70) gives 12 equations, and (6.71) gives $4m$ equations. The system (6.70) means that the components of $\delta_2 B$ corresponding to $\Lambda^3T^*\otimes G_{m-1}$ vanish, while the system (6.71) means that the components corresponding to $\Lambda^3T^*\otimes S^{m-1}T_h^*$ vanish.

The system (6.70) is composed of equations for which, among the last $m-1$ vectors on which B is computed, at least one is vertical. We will call these components "first type components" of B, and the others, for which the last $m-1$ vectors are horizontal, "second kind components".

Since the elements of $G_m(P_3)$, for $m \ge 2$, are determined by three parameters (cf. page 110), the elements of $\Lambda^3T^*\otimes G_{m-1}(P_3)$ are determined by a maximum of 18 parameters.

Let us now compute the rank of the system (6.70). In (6.70.b) the equations corresponding to the indices $(ijk) = (123), (134), (234)$ are independent (the corresponding pivots are $B(h_1,h_2,v_2^m)$, $B(h_1,v_2,v_2^m)$ and $B(h_2,v_2,v_2^m)$ respectively), whereas the equation corresponding to the index $(ijk) = (124)$ depends on the former, because we have the following relation:

$$B(h_1, h_2, v_1, v_1^{m-1}) + B(h_2, v_1, h_1, v_1^{m-1}) + B(v_1, h_1, h_2, v_1^{m-1}) \overset{(6.31)}{=} -\frac{\xi^S_{v_1}}{\xi^S_{h_1}} B(h_2, v_1, v_1^{m}) \overset{(ijk)=(234)}{=} 0 \tag{6.72}$$

In (6.70 c) the equations corresponding to $(ijk) = (124), (134), (234)$ are independent (the pivots are $B(h_1, h_2, v_2^m)$, $B(h_1, v_1, v_2^m)$ and $B(h_2, v_1, v_2^m)$), and the equation corresponding to the index $(ijk) = (123)$ is related to the others by the following :

$$B(h_1, h_2, v_2, v_2^{m-1}) + B(h_2, v_2, h_1, v_2^{m-1}) + B(v_2, h_1, h_2, v_2^{m-1}) \overset{(6.31)}{=} \frac{\xi^S_{v_2}}{\xi^S_{h_2}} B(h_1, v_2, v_2^{m}) \overset{(ijk)=(134)}{=} 0. \tag{6.73}$$

In (6.70 a) the equations corresponding to the indices $(ijk) = (123), (124), (234)$ are independent (the corresponding pivots are $B(v_1, h_1, h_2^{m-1}, v_2)$, $B(v_2, h_1, h_2^{m-1}, v_2)$ and $B(v_1, v_2, h_2^{m-1}, v_2)$), while the equation corresponding to $(ijk) = (134)$ depends on them. Indeed, using the notation (6.27), for every $X, Y, Z \in T_z$ and $0 \leq k \leq m-3$ we have

$$B(X, Y, Z, h_2^{k-1}, v_2^{(m+2)-(k+2)}) \overset{(6.31)}{=} -\frac{\xi^S_{v_2}}{\xi^S_{h_2}} B(X, Y, Z, h_2^{k}, v_2^{(m+2)-(k+1)}). \tag{6.74}$$

Thus if we denote the equation (6.70 a) corresponding to the index (ijk) by $\mathcal{E}_{ijk}$, then equation $\mathcal{E}_{234}$ can be expressed in terms of the other equations:

$$\begin{aligned}
&\xi^S_{v_2}\mathcal{E}_{134} + \xi^S_{h_2}\mathcal{E}_{234} \overset{(6.31c)}{=} \\
&\quad = B(v_1, v_2, hS, h_2^{m-2}, v_2) + \xi^S_{h_1} B(h_1, v_1, h_2^{m-1}, v_2) + \xi^S_{h_2} B(h_2, v_1, h_2^{m-2}, v_2, v_2) \\
&\quad = -\xi^S_{v_2} B(v_1, v_2, v_2, h_2^{m-2}, v_2) + \xi^S_{h_1} B(h_1, v_1, h_2^{m-1}, v_2) + \xi^S_{h_2} B(h_2, v_1, h_2^{m-1}, v_2) \\
&\quad \overset{(6.74)}{=} \xi^S_{h_2} B(v_2, v_1, h_2^{m-2}, v_2, v_2) + \xi^S_{v_2} B(h_2, v_1, h_2^{m-2}, v_2, v_2) \\
&\quad \overset{(6.31)}{=} \sum^{cycl}_{pr_2 S, v_1, h_2} B(pr_2 S, v_1, h_2, h_2^{m-3}, v_2^2) \\
&\quad \overset{(6.74)}{=} \left(-\frac{\xi^S_{v_2}}{\xi^S_{h_2}}\right)^{m-1} \sum^{cycl}_{pr_2 S, v_1, h_2} B(pr_2 S, v_1, h_2, v_2^{m-1}) \overset{(6.70b)}{=} 0,
\end{aligned}$$

where pr_2 denotes the projection on the eigenspace Δ_2. Thus the equation $\mathcal{E}_{234} = 0$ is a linear combination of the other equations and the rank of the system (6.70) is 9.

Let us now consider the system (6.71). In the $(l+1)$th block of (6.71) the equations are,

$$\sum_{cycl(ijk)} B(e_i, e_j, e_k, h_1^{m-l-1}, h_2^l) = 0,$$

where the $i, j, k = 1, \dots, 4$ are all different. In each equation of (6.71) we have two "first type" components (namely for $(ijk) = (134)$ and $(ijk) = (134)$), and one "second type" component $(ijk = 234)$. The equation

$$\sum_{cycl(h_2, v_1, v_2)} B(h_2, v_1, v_2, h_1^{m-l}, h_2^{l-1}) = 0 \tag{6.75}$$

in the l-th block contains the same "second type" component as the equation

$$\sum_{cycl(h_1, v_1, v_2)} B(h_1, v_1, v_2, h_1^{m-(l+1)}, h_2^l) = 0 \tag{6.76}$$

of the $(l+1)$-th block corresponding to the index $(ijk) = (134)$. Using the relations which determine the space $\Lambda^2 T^* \otimes \mathfrak{g}_\infty(P_3^1)$, it is easy to prove that these equations are linearly dependent. Indeed,

$$\begin{aligned}
(6.75) &\overset{(6.70)}{=} B(v_1, v_2, h_2, h_1^{m-l+1}, h_2^{l-2}) \\
&- (B(v_1, h_1, h_2, h_1^{m-l-1}, h_2^{l-1}, v_2) + B(h_1, h_2, v_1, h_1^{m-l+1}, h_2^{l-1}, v_2)) \\
&- (B(h_2, h_1, v_2, h_1^{m-l-1}, h_2^{l-1}, v_1) + B(h_1, v_2, h_2, h_1^{m-l+1}, h_2^{l-1}, v_1)) \overset{(6.70)}{=} (6.76).
\end{aligned}$$

Therefore the rank of the system does not change if we remove the equations of the l-th block corresponding to the index $(ijk) = (134)$ for $1 \le l \le m$. It is not difficult to check that the remaining equations are independent. Indeed, the pivots in the first block are,

$$B(h_2, v_1, h_1^m), \; B(h_2, v_2, h_1^m), \; B(v_1, v_2, h_1^m), \; B(v_1, v_2, h_1^{m-1}, h_2),$$

and in the l-th block, with $1 \le l \le m$:

$$B(h_2, v_1, h_1^{m-l}, h_2^{l-1}), \; B(h_2, v_2, h_1^{m-l}, h_2^{l-1}), \; B(v_1, v_2, h_1^{m-(l+1)}, h_2^l).$$

Thus there are $4 \dim S^{m-1} T_t^* - \dim S^{m-2} T_t^* = 3m + 1$ independent equations in the system (6.71) which are also independent of the system (6.70).

So, as the rank of the system (6.70) is 9, there are $3m + 10$ independent equations which determine $\mathrm{Ker}\ \delta_{2,m}(P_3)$. Thereby

$$\dim \mathrm{Ker}\ \delta_{2,m}(P_3) = \dim(\Lambda^2 T^* \otimes g_m(P_3)) - (3m+10) = 3m + 14,$$

which proves the formula (6.69).

◊

Taking into account (6.39) now we obtain:

$$\mathrm{rang}\, \delta_{1,m}(P_3) = \dim \mathrm{Ker}\ \delta_{2,m}(P_3)$$

for $m > 2$. Thus $H^2_m(P_3) = 0$ for $m > 2$. Since $g_m(P_3) = g_m(P_3^1)$ and so $H^2_m(P_3) = H^2_m(P_3^1)$ for $m \geq 3$, we get

$$H^2_m(P_3^1) = 0$$

for every $m \geq 3$. Therefore the operator P_3^1 is 2-acyclic.

□

Example 6.4 Let us consider the system[1]

$$\begin{cases} \ddot{x}_1 = F(x_1, \dot{x}_1), \\ \ddot{x}_2 = G(x_2, \dot{x}_2). \end{cases} \tag{6.77}$$

We have $\Gamma^1_2 = \Gamma^2_1 = 0$ and $A^1_2 = A^2_1 = A'^1_2 = A'^2_1 = 0$. For F and G generic $A^1_1 - A^2_2 \neq 0$ so the rank of S is 1. The eigenspaces Δ_i, $i = 1, 2$, are spanned by $\left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial \dot{x}_i} \right\}$. Thus the Δ_i are integrable and hence reducible. At a generic point S is not in Δ_i, so it is atypical. On the other hand, it is easy to check that the k_i vanish, thereby $\varphi_E = 0$ for every third order solution. According to Theorem 6.3 the spray is variational.

6.3.2.4 *Semi-reducible case*

In this section we will study the semi-reducible case for atypical sprays of rank 1. We recall that this means that in the obstruction $P_{(h,A)} = 0$, where

$$(P_{(h,A)}E)_x(h_1, h_2, h_3, h_4) = \sum_{i=1}^{2} (\chi_i)_x \mathcal{L}_{v_i}[\Omega_E(v_i, h_i)]_x + \sum_{i=1}^{2} (k_i)_x \Omega_E(h_i, v_i)_x$$

[1] It corresponds to Douglas' separated case IIa1.

has to be put in the prolongation P_3^1 of the system

$$\begin{cases} \omega = 0, \\ i_\Gamma \Omega = 0, \\ i_A \Omega = 0, \end{cases}$$

where $\omega = 0$ is the Euler-Lagrange equation. In the semi-reducible case one and only one of the coefficients χ_i, $(i = 1, 2)$ vanishes. In particular, the operator $P_{[h,A]}$ is of third order. We assume for example that $\chi_2 = 0$. To express the theorem in this case we need to introduce some matrices which naturally arise in the study. The theorem containing the results is given at the end of this section.

The computation is carried out in two steps: first we study the obstructions to lifting the third order solutions to a fourth order, then in the second step we check that the system is 2-acyclic. In the first step two cases have to be distinguished: $k_2 = 0$ and $k_2 \neq 0$.

STEP 1. *First lift of third order solutions.*

We will begin by computing the symbol of the operator $P_4^1 = (P_3^1, P_{[h,A]})$ and prove that

$$\operatorname{rank} \sigma_4(P_4^1) = \operatorname{rank} \sigma_4(P_3) + 1. \tag{6.78}$$

We have

$$[\sigma_3(P_{[h,A]})] B_3 = \chi_1 B_3(v_1, v_1, v_1) \tag{6.79}$$

and thus

$$[\sigma_4(P_{[h,A]}) B_4](X) = \chi_1 B_4(X, v_1, v_1, v_1), \tag{6.80}$$

where $B_3 \in S^3 T^*$, $B_4 \in S^4 T^*$ and $X \in T_x$. Since $g_m(P_4^1) = g_m(P_3) \cap g_m(P_{[h,A]})$ for $m \geq 3$, we have

$$g_m(P_4) = S^m T_h^* \oplus G_m(P_4), \tag{6.81}$$

where $G_m(P_4) = G_m(P_3) \cap G_m(P_{[h,A]})$.

As we have already shown, an element B_3 of $G_3(P_3)$ is determined by three free components: $B_3(h_1^1, v_2)$, $B_3(v_1^3)$ and $B_3(v_2^3)$ (see page 110). Therefore the equation which defines $g_3(P_{[h,A]})$, that is $\chi_1 B_3(v_1^3) = 0$, is linearly independent of the equations which determine $g_3(P_3)$ since $B_3(v_1^3)$ is a pivot. Thus

$$\dim g_3(P_4^1) = \dim g_3(P_3) - 1.$$

To compute $\dim g_4(P_4^1)$ we will replace X in the equation (6.80) by the vectors of the adapted basis h_i and v_i, $i = 1, 2$. We obtain four equations in addition to the equations which characterize the space $G_4(P_3)$. We know that $\dim G_4(P_3) = 3$ (see equation (6.23)) and an element B_4 in $G_4(P_3)$ is determined by its components (6.28). Therefore the new equation $\sigma_4(P_{(\lambda,A_1)})(v_1) = 0$ is linearly independent of the equations defining $g_4(P_3)$.

The other equations are related to the equations of $g_4(P_3)$: between the symbol of $P_{(\lambda,A_1)}$ and the symbol of the other operators of P_4^1 there exist the following relations:

$$(\sigma_4(P_{(\lambda,A_1)})B_4)(S) = \chi_1(\sigma_4(P_\Gamma)B_4)(v_1, v_1, h_1),$$
$$(\sigma_4(P_{(\lambda,A_1)})B_4)(h_2) = \frac{\lambda_1}{2}(\sigma_4(P_\Gamma)B_4)(v_1, v_1, h_1, h_2) + \frac{\chi_1}{\lambda}(\sigma_4(P_A)B_4)(v_1, h_1, h_1, h_2),$$
$$(\sigma_4(P_{(\lambda,A_1)})B_4)(v_2) = \frac{\chi_1}{\lambda}(\sigma_4(P_A)B_4)(v_1, v_1, h_1, h_1)$$

Thus $\dim g_4(P_4^1) = \dim g_4(P_3) - 1$, which proves (6.78).

Let $\tau_4^1 : (S^2T^* \otimes F_3) \oplus T^* \longrightarrow K_3^1 \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ be the morphism defined by

$$\tau_4^1 := (\tilde{\tau}_3^1; \tau_{(\lambda,A)}; \rho_4^1; \rho_4^2, \rho_4^3)$$

with

$$\tilde{\tau}_3^1(B, C_\Gamma, C_A, C_{(\lambda,A_1)}) = \tilde{\tau}_3^1(B, C_\Gamma, C_A),$$
$$\rho_4^1(B, C_\Gamma, C_A, C_{(\lambda,A_1)}) = C_{(\lambda,A_1)}(S) - \chi_1 \cdot B(v_1, v_1, h_1),$$
$$\rho_4^2(B, C_\Gamma, C_A, C_{(\lambda,A_1)}) = C_{(\lambda,A_1)}(h_2) + \frac{\chi_1}{2}C_\Gamma(v_1, v_1, h_1, h_2) - \frac{\chi_1}{\lambda}C_A(v_1, h_1, h_1, h_2),$$
$$\rho_4^3(B, C_\Gamma, C_A, C_{(\lambda,A_1)}) = C_{(\lambda,A_1)}(v_2) - \frac{\chi_1}{\lambda}C_A(v_1, v_1, h_1, h_1).$$

We shall prove that the sequence

$$S^4T^* \xrightarrow{\sigma_4(P_4^1)} (S^2T^* \otimes F_3) \oplus T^* \xrightarrow{\tau_4^1} K_4^1 \longrightarrow 0 \qquad (6.82)$$

is exact, where $K_4^1 = \operatorname{Im} \tau_4^1$.

It is easy to see that $\operatorname{Im} \sigma_4(P_4^1) \subset \operatorname{Ker} \tau_4^1$. On the other hand, let us consider the system $\tau_4^1 = 0$. The terms $C_{(\lambda,A_1)}(h_1)$, $C_{(\lambda,A_1)}(h_2)$, $C_{(\lambda,A_1)}(v_2)$

are pivots, so the equations $\rho_4^i = 0$, $i = 1,2,3$, are independent of the system defined by $\tau_3^1 = 0$. Since the sequence (6.44) is exact, we have

$$\operatorname{rank}\sigma_4(P_4^1) = \operatorname{rank}\tau_3^1 + 1 = \dim\operatorname{Ker}\tau_3^1 + \dim T^* - 3 = \dim\operatorname{Ker}\tau_4^1,$$

hence the sequence (6.82) is exact.

Let us now compute the obstructions. The new compatibility conditions for P_4^1 are given by the equations $\rho_4^i(\nabla P_4^1) = 0$, $i = 1,2,3$. Let $j_3(E)_x$ be a 3rd order solution of P_3^1 at $x \in TM \setminus \{0\}$. We have:

$$\begin{aligned}
\rho_4^1(\nabla P_4^1 E) &= S(F_{(h_1 h_2)}E) - \chi_1 v_1(\nabla_{v}\Omega_E(v_1, h_1)) - \chi_1[S, v_1]\Omega_E(v_1, h_1) - \\
&\quad - 2\chi_1(v_1\xi_{h_1}^{v_1 S})\Omega_E(v_1, h_1) - 2\chi_1\xi_{h_1}^{v_1 S} v_1\Omega_E(v_1, h_1) + (S\chi_1)v_1\Omega_E(v_1, h_1) + \\
&\quad + (Sk_1)\Omega_E(v_1, h_1) + (Sk_2)\Omega_E(v_2, h_2) + k_1 S\Omega_E(v_1, h_1) + k_2 S\Omega_E(v_2, h_2);
\end{aligned}$$

$$\begin{aligned}
\rho_4^2(\nabla P_4^1 E) &= h_2(F_{(h_1 h_2)}E) + \chi_1 v_1(\nabla_{h_2}\Omega_E(v_1, h_1, h_2)) - \frac{\chi_1}{\lambda} v_1(\nabla_{h_2}\Omega_E(h_1, h_1, h_2)) \\
&= \chi_1 v_1\Big(\sum_{cycl}\Omega_E([v_1, h_1], h_2)\Big) + (h_2\chi_1)v_1\Omega_E(v_1, h_1) + \chi_1[h_2, v_1]\Omega_E(v_1, h_1) + \\
&\quad + k_1 h_2\Omega_E(v_1, h_1) + k_2 h_2\Omega_E(v_2, h_2) + (h_2 k_1)\Omega_E(v_1, h_1) + (h_2 k_2)\Omega_E(v_2, h_2);
\end{aligned}$$

$$\begin{aligned}
\rho_4^3(\nabla P_4^1 E) &= v_2(F_{(h_1 h_2)}E) - \frac{\chi_1}{\lambda} v_1(\nabla_{v_2}\Omega_E(v_1, h_1, h_2)) \\
&\quad - \chi_1 v_1\Big(\sum_{cycl}\Omega_E([v_1, h_1], v_2)\Big) + (v_2\chi_1)v_1\Omega_E(v_1, h_1) + \chi_1[v_2, v_1]\Omega_E(v_1, h_1) + \\
&\quad + (v_2 k_1)\Omega_E(v_1, h_1) + (v_2 k_2)\Omega_E(v_2, h_2) + k_1 v_2\Omega_E(v_1, h_1) + k_2 v_2\Omega_E(v_2, h_2).
\end{aligned}$$

Since $\xi_{h_1}^{v_1 S} \neq 0$ (see the Property on page 108), the system $\rho_4^i = 0$, $i = 1,2,3$ is equivalent to the system $\rho_4^1 = 0$, $\bar\rho_4^2 = 0$, $\rho_4^3 = 0$, where

$$\bar\rho_4^2 := \xi_{h_1}^{v_1 S}\rho_4^2 + \xi_{h_2}^{v_1 S}\rho_4^3.$$

Taking into account the relations in $R_3(P_3^1)$ we have:

$$\begin{aligned}
\rho_4^1(\nabla P_4^1 E) &= \chi_1^1 v_1\Omega_E(v_1, h_1) + k_1^1\Omega_E(v_1, h_1) + k_2^1\Omega_E(v_2, h_2), \\
\bar\rho_4^2(\nabla P_4^1 E) &= \chi_1^2 v_1\Omega_E(v_1, h_1) + k_1^2\Omega_E(v_1, h_1) + k_2^2\Omega_E(v_2, h_2), \qquad (6.83)\\
\rho_4^3(\nabla P_4^1 E) &= \chi_1^3 v_1\Omega_E(v_1, h_1) + k_1^3\Omega_E(v_1, h_1) + k_2^3\Omega_E(v_2, h_2) + k_3 v_1\Omega_E(v_2, h_2),
\end{aligned}$$

where the coefficients χ_1^j and k_i^j can be expressed uniquely in terms of the spray S (their explicit formulae are given in the Appendix by (A.1) and (A.2)). We have proved that every 3rd order solution of P_3^1 can be lifted into a 4th order solution if and only if the equations $\rho_4^i(\nabla P_4^1 E) = 0$, $i = 1,3$ and $\bar\rho_4^2(\nabla P_4^1 E) = 0$ hold.

To discuss the different possibilities, we have to consider the cases $k_2 = 0$ and $k_2 \neq 0$ separately.

i) The case $k_2 = 0$.

Let us denote by M_1 the matrix formed by the coefficients of the operator $P_{[h,A]}$ and the coefficients of the equations of (6.83):

$$M_1 := \begin{pmatrix} \chi_1 & k_1 & k_2 \\ \chi_1^1 & k_1^1 & k_2^1 \\ \chi_1^2 & k_1^2 & k_2^2 \\ \chi_1^3 & k_1^3 & k_2^3 \end{pmatrix} \tag{6.84}$$

Firstly we prove the following

Lemma 6.7

(1) If rank $M_1 = 1$, *then every third order solution of P_4^1 can be lifted into a fourth order solution.*

(2) If rank $M_1 = 2$, *then a new second order condition has to be verified in order to lift the third order solutions and the Completion Lemma gives the explicit conditions for the spray to be variational.*

(3) If rank $M_1 > 2$, *then S is not variational.*

Proof. Let us suppose that rank $M_1 = 1$. Since $\chi_1 \neq 0$, there exist $a^i \in \mathbb{R}$, $i = 1,2,3$, such that $(\chi_1^i, k_1^i, k_2^i) = a^i(\chi_1, k_1, k_2)$. Thus if $(j_3E)_x$ is a third order solution of P_4^1 at $x \neq 0$, then we have

$$\rho_4^i(\nabla P_4^1 E)_x = a^i P_{[h,A]}(E)_x = 0,$$

$i = 1,2,3$. Therefore $\tau_4^1(\nabla P_4^1 E)_x = 0$, and every third order solution of P_4^1 can be lifted into a fourth order solution.

If rank $M_1 > 2$, then it is easy to see that there is no regular second order solution satisfying the compatibility conditions $\rho_4^i(\nabla P_4^1 E)_x = 0$, $i = 1,2,3$, because these equations imply $\Omega_E(v_1, h_1)_x = 0$ and $\Omega_E(v_2, h_2)_x = 0$. Hence in this case S is not variational.

If rank $M_1 = 2$, then the system $\rho_4^i(\nabla P_4^1 E)_x = 0$ gives exactly one new relation between the terms $\Omega_E(v_1, h_1)$ and $\Omega_E(v_2, h_2)$, i.e. a new second order compatibility condition, which we shall denote as

$$a_1\Omega(v_1, h_1)_x + a_2\Omega(v_2, h_2)_x = 0, \tag{6.85}$$

where a_1 and a_2 can easily be computed from (6.83). This relation yields a new differential operator

$$P_5 : C^\infty(TM) \longrightarrow C^\infty(TM)$$

defined by

$$P_5(E) := a_1\Omega_E(v_1, h_1) + a_2\Omega_E(v_2, h_2), \tag{6.86}$$

which has to be introduced into the system. So we have to study the differential operator

$$(P_4^1, P_5, \nabla P_5) = (P_3^1, P_{(h,A)}, P_5, \nabla P_5). \tag{6.87}$$

Considering the second order part of the system (6.87), (P_3, P_5), the Completion Lemma (Lemma 6.6) with $g_i := a_i$ gives the necessary and sufficient conditions for the spray to be variational.

ii) The case $k_2 \neq 0$

If $k_2 \neq 0$, the equation $\rho_4^1 = 0$, that is

$$k_2 v_2 \Omega_E(v_2, h_2) + b_1\Omega_E(v_1, h_1) + b_2\Omega_E(v_2, h_2) = 0,$$

with $b_i = (k_i^2 - \frac{\mu_i}{v_i})$, $i = 1, 2$, is of the third order. So, in order to apply the Completion Lemma, we need some supplementary computations. Let $P_6 : C^\infty(TM) \longrightarrow C^\infty(TM)$ be the differential operator defined by

$$P_6 E = k_2 v_2 \Omega_E(v_2, h_2) + b_1\Omega_E(v_1, h_1) + b_2\Omega_E(v_2, h_2). \tag{6.88}$$

Introducing it into the system we obtain the operator

$$P_5^1(E) := (P_4^1, P_6).$$

The symbol of P_6 is

$$\sigma_3(P_6) : S^3T^* \longrightarrow \mathbb{R}, \qquad [\sigma_3(P_6)B_3] := k_2 B_3(v_2, v_2, v_2),$$

and the symbol of the first prolongation is

$$\sigma_4(P_6) : S^4T^* \longrightarrow T^*, \quad [\sigma_4(P_6)B_4](X) := k_2 B_4(X, v_2, v_2, v_2),$$

where $B_3 \in S^3T^*$, $B_4 \in S^4T^*$, $X \in T_x$. The equation $[\sigma_3(P_6)B_3] = 0$ is independent of the equations of $g_3(P_4^1)$ because $B_3(v_2^3)$ is a pivot. Thus

$$\dim g_3(P_5^1) = \dim g_3(P_4^1) - 1.$$

On the other hand, $g_4(P_5)$ is characterized by the equations $[\sigma_4(P_5)B_4](e_i) = 0$, $i = 1, \ldots, 4$, where $\{e_i\}$ is a basis of T_x. From the preceding computations we know that $\dim G_4(P_4^1) = 2$, and an element B_4 of $G_4(P_4^1)$ is determined by its components $B_1(h_2^3, v_2)$ and $B_1(v_2^4)$. Hence the equation $[\sigma_4(P_5)B_4](v_2) = 0$ contains a pivot term. The other equations are related to the equations of $g_4(P_4^1)$. Therefore $\dim g_4(P_5^1) = \dim g_4(P_4^1) - 1$ and

$$\operatorname{rank} \sigma_4(P_5^1) = \operatorname{rank} \sigma_4(P_4^1) + 1.$$

Let us consider the morphism

$$\tau_5^1 : S^2T^* \otimes F_3 \oplus T^* \oplus T^* \longrightarrow K_4^1 \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}, \qquad (6.89)$$

defined by $\tau_5^1 := (\bar{\tau}_4^1, \rho_5^1, \rho_5^2, \rho_5^3)$, where

$$\begin{aligned}
\bar{\tau}_4^1(B, C_\Gamma, C_A, C_{(h,A)}, C_b) &= \tau_4^1(B, C_\Gamma, C_A, C_{(h,A)}),\\
\rho_5^1(B, C_\Gamma, C_A, C_{(h,A)}, C_b) &= C_b(S) - \chi_1 B(v_2, v_2, h_2),\\
\rho_5^2(B, C_\Gamma, C_A, C_{(h,A)}, C_b) &= C_b(h_1) - \frac{k_1}{2} C_\Gamma(v_2, v_2, h_1, h_2) - \frac{k_2}{\lambda} C_A(v_2, h_2, h_1, h_2),\\
\rho_5^3(B, C_\Gamma, C_A, C_{(h,A)}, C_b) &= C_b(v_1) - \frac{k_2}{\lambda} C_A(v_2, v_2, h_1, h_2)
\end{aligned}$$

We shall prove that the sequence

$$S^4T^* \xrightarrow{\sigma_4(P_5^1)} (S^2T^* \otimes F_3 \oplus T^*) \oplus T^* \xrightarrow{\tau_5^1} K_5^1 \longrightarrow 0 \qquad (6.90)$$

is exact, where $K_5^1 := \operatorname{Im} \tau_5^1$.

Indeed, $\operatorname{Im} \sigma_4(P_4^1) \subset \operatorname{Ker} \tau_4^1$. On the other hand, $C_b(h_1)$, $C_b(h_2)$ and $C_b(v_1)$ are pivots, thus the equations $\rho_5^i = 0$, $i = 1, 2, 3$ are independent of the equations $\operatorname{Ker} \bar{\tau}_4^1$. Therefore

$$\operatorname{rank} \sigma_4(P_5^1) = \operatorname{rank} \tau_4^1 + 1 = \dim \operatorname{Ker} \tau_4^1 + \dim T^* - 3 = \dim \operatorname{Ker} \tau_5^1,$$

which proves that the sequence is exact.

To compute the new compatibility conditions, we will consider a 3rd order formal solution $j_3(E)_x$ of P_5^1 at $x \in TM \setminus \{0\}$. We have:

$$\rho_5^1(\nabla P_5^1 E) = k_3[S, v_1]\Omega_E(v_2, h_1) + 2k_3 v_2\Omega_E([S, v_2], h_2) - (Sk_3)v_2\Omega_E(v_2, h_1)$$
$$+ (Sb_1)\Omega_E(v_1, h_1) - b_1 S\Omega_E(v_1, h_1) + (Sb_2)\Omega_E(v_2, h_2) + b_2 S\Omega_E(v_2, h_2).$$
$$\rho_5^2(\nabla P_5^1 E) = k_2 v_2\Big(\sum_{cyc}\Omega_E([h_1, v_2], h_2)\Big) + (h_1 k_2)v_2\Omega_E(v_2, h_1) + k_3[h_1, v_2]\Omega_E(v_2, h_2)$$
$$+ (h_1 b_1)\Omega_E(v_1, h_1) + b_1 h_1\Omega_E(v_1, h_1) + (h_1 b_2)\Omega_E(v_2, h_2) + b_2 h_1\Omega_E(v_2, h_2),$$
$$\rho_5^3(\nabla P_5^1 E) = k_3 v_2\Big(\sum_{cyc}\Omega_E([v_2, h_2], v_1)\Big) + (v_1 k_2)v_2\Omega_E(v_2, h_2) + k_3[v_1, v_2]\Omega_E(v_2, h_2)$$
$$+ (v_1 b_1)\Omega_E(v_1, h_1) + b_1 v_1\Omega_E(v_1, h_1) + (v_1 b_2)\Omega_E(v_2, h_2) + b_2 v_1\Omega_E(v_2, h_2).$$

Note that the system $(\rho_5^1, \tilde{\rho}_5^2, \rho_5^3)$ is equivalent to the system $(\rho_5^1, \rho_5^2, \rho_5^3)$, where

$$\tilde{\rho}_5^2 := \xi_{h_1}^S \rho_5^2 + \xi_{v_1}^S \rho_5^3,$$

because $\xi_{h_1}^S \neq 0$. In the space $R_{3,x}(P_5^1)$ of the third order solutions of P_5^1 at x, the 3rd order terms can be expressed with the help of second order terms using the equation (6.47) and the equations $P_{(h_1 h_2)}(E) = 0$ and $P_h(E) = 0$. Thus the equations (6.83 a,b) and the obstructions of P_5^1 can be written in the form

$$\begin{aligned} \rho_4^i(\nabla P_4^1 E) &= c_1^i\Omega_E(v_1, h_1) + c_2^i\Omega_E(v_2, h_2), & i = 1, 2 \\ \rho_5^i(\nabla P_5^1 E) &= c_1^i\Omega_E(v_1, h_1) + c_2^i\Omega_E(v_2, h_2), & i = 3, 4, 5 \end{aligned} \tag{6.91}$$

where the coefficients can be expressed in terms of the spray S (they are given explicitly in the Appendix, see (A.3)). Let

$$M_2 = \begin{pmatrix} c_1^1, & c_2^1 \\ \vdots & \vdots \\ c_1^5, & c_2^5 \end{pmatrix} \tag{6.92}$$

be the matrix whose rows are the coefficients of the equations (6.91), then we have the following

Lemma 6.8

(1) If $M_2 = 0$, then any 3rd order formal solution of P_5^1 can be lifted into a 4th order solution;

(2) If rank $M_2 = 1$, *then the equations (6.91) give a new second order compatibility condition which can be studied by the Completion Lemma.*

(3) *If* $\operatorname{rank} M_2 = 2$, *then* S *is not variational.*

Indeed, if $M_2 = 0$, then $\rho_4^i(\nabla P_4^1 E) = 0$ for $i = 1, 2$, and $\rho_5^i(\nabla P_5^1 E) = 0$ for $i = 3, 4, 5$. So $\tau_5^1(\nabla P_5^1 E) = 0$ and therefore the compatibility conditions of the operator P_5^1 are satisfied.

If $\operatorname{rank} M_2 = 2$, then $\Omega_E(v_2, h_1)_x = 0$ for any 3rd order solution $j_3(E)_x$ which satisfies the compatibility conditions, so E cannot be regular.

If $\operatorname{rank} M_2 = 1$, then we get a new second order compatibility condition for P_5^1:

$$c_1\Omega_E(v_1, h_1) + c_2\Omega_E(v_2, h_2) = 0. \tag{6.93}$$

Denoting by P_c the differential operator corresponding to equation (6.93),

$$P_c(E) := c_1\Omega_E(v_1, h_1) + c_2\Omega_E(v_2, h_2), \tag{6.94}$$

we must study the differential operator

$$(P_5^1, P_c, \nabla P_c) \tag{6.95}$$

Using the Completion Lemma with $g_i := c_i$, $i = 1, 2$, we have the following possibilities:

(1) If c_1 or c_2 vanishes at x, or the matrix

$$M_3 := \begin{pmatrix} \chi_1 & 0 & k_1 & k_2 \\ 0 & k_2 & h_1 & h_2 \\ c_1 & 0 & e_1^1 & e_2^1 \\ 0 & c_2 & e_1^2 & e_2^2 \\ 0 & 0 & c_1 & c_2 \end{pmatrix}, \tag{6.96}$$

where $e_1^1 := c_2\chi^2_{h_1} + (v_1 c_1)$ and $e_2^1 := c_1\chi^1_{h_2} + (v_2 c_1)$, is non-singular at x, then the spray S is not variational on a neighborhood of $x \neq 0$.

(2) If $c_1(x) \neq 0$, $c_2(x) \neq 0$ and $\det(M_3) = 0$, then S is variational on a neighborhood U of x if and only if $\Theta^1_{c_1, c_2} = 0$ and $\Theta^2_{c_1, c_2} = 0$ on U.

In order to complete the semi-reducible case, we only need to study the higher order lifts in the case when $M_2 = 0$.

STEP 2. *The higher order prolongations*

$P_4^1 = (P_3^1, P_{(h,\chi)})$ *and* $P_5^1 = (P_4^1, P_5)$ *are 2-acyclic*

Let us first consider the operator P_4^1. With the usual notations we have

$$g_m(P_4^1) = S^m T_h^* \oplus G_m(P_4^1) \tag{6.97}$$

Now $G_m(P_4^1) = G_m(P_3) \cap G_m(P_{(h,A)})$. An element B_m in $G_m(P_3)$ is determined by the three components (6.28), that is $B_m(h^{m-1}, v_2)$, $B_m(v_1^m)$ and $B_m(v_2^m)$. Now, if $B_m \in G_m(P_{(h,A)})$, then $\chi_1 B_m(v_1^m) = 0$, that is $B_m(v_1^m) = 0$. Thus

$$\dim G_m(P_4^1) = 2$$

and so

$$\dim g_m(P_4^1) = \dim S^m T_h^* + \dim G_m(P_4^1) = m + 3 \tag{6.98}$$

for every $m \geq 3$. Now the Spencer complex (1.10) corresponding to the operator P_4^1 is exact in the first two terms, that is in $g_{m+2}(P_4^1)$ and in $T^* \otimes g_{m+1}(P_4^1)$. So

$$\operatorname{rank} \delta_{1,m}(P_4^1) = \dim\left(T^* \otimes g_{m+1}(P_4^1)\right) - \dim\left(g_{m+2}(P_4^1)\right) = 3m + 11 \tag{6.99}$$

for every $m \geq 3$. On the other hand, from (6.97) we have

$$\Lambda^2 T^* \otimes g_{m-1}(P_4^1) = (\Lambda^2 T^* \otimes S^{m-1} T_h^*) \oplus (\Lambda^2 T^* \otimes G_{m-1}(P_4^1)).$$

Let B be an element of $\Lambda^2 T^* \otimes g_m(P_4^1)$. We have $\delta_{2,m}(P_4)B = 0$ if and only if the system consisting of the equations (6.71) and of the eight equations

$$\left\{ \begin{array}{l} \sum\limits_{cycl(ijk)} B(e_i, e_j, e_k, h_2^{m-2}, v_2) = 0, \\ \sum\limits_{cycl(ijk)} B(e_i, e_j, e_k, v_2^{m-1}) = 0, \end{array} \right. \tag{6.100}$$

holds, where $\{e_s\}_{s=1,\dots,4}$ are the vectors of the adapted basis $\{h_i, v_i\}_{i=1,2}$ and $i, j, k = 1, \dots, 4$ are all different. We can see that the equations (6.100) are the same as the equations of the system (6.70) without b). The analysis of (6.70) has shown that (6.70a) and (6.70c) both contain 3 independent equations (for the computation see page 133). Therefore the rank of (6.100) is 6.

Moreover, we also showed there that the system (6.71) contains $3m + 1$ independent equations with respect to (6.70) and therefore also with respect to (6.100). Hence Ker $\delta_{2,m}(P_4^1)$ is determined by $3m+7$ independent equations, and

$$\dim \operatorname{Ker} \delta_{2,m}(P_4^1) = \dim\left(\Lambda^2 T^* \otimes g_m(P_4^1)\right) - (3m+7) = 3m+11.$$

So rank $\delta_{1,m}(P_4^1) = \dim \operatorname{Ker} \delta_{2,m}(P_4^1)$ and thus $H_m^2(P_4^1) = 0$ for $m \geq 3$, which proves that P_4^1 is 2-acyclic.

Let us now consider the operator $P_5^1 = (P_4^1, P_6)$, with $k_2 \neq 0$. We have

$$g_m(P_5^1) = S^m T_h^* \oplus G_m(P_5^1), \tag{6.101}$$

where $G_m(P_5^1) = G_m(P_4^1) \cap G_m(P_6)$. Let $B_m \in G_m(P_5^1)$. If $B_m \in G_m(P_6)$, then one also has $B_m(v_2^m) = 0$. Thus

$$\dim G_m(P_5^1) = 1,$$

and so

$$\dim g_m(P_5^1) = \dim S^m T_h^* + \dim G_m(P_5^1) = m+2 \tag{6.102}$$

for every $m \geq 3$. From the 1-acyclicity of the Spencer complex we have rank $\delta_{1,m}(P_5^1) = \dim\left(T^* \otimes g_{m+1}(P_5^1)\right) - \dim\left(g_{m+2}(P_5^1)\right)$ and so, by (6.98), we get

$$\operatorname{rank} \delta_{1,m}(P_5^1) = 3m+8 \tag{6.103}$$

for every $m \geq 3$. On the other hand, using the decomposition (6.101) once again, we have

$$\Lambda^2 T^* \otimes g_{m-1}(P_5^1) = \left(\Lambda^2 T^* \otimes S^{m-1} T_h^*\right) \oplus \left(\Lambda^2 T^* \otimes G_{m-1}(P_5^1)\right).$$

If B is an element of $\Lambda^2 T^* \otimes g_m(P_5^1)$, then we have $\delta_{2,m}(P_5)B = 0$ if and only if the system consisting of the equations (6.71) and of the four equations

$$\sum_{\text{cycl}(i,j,k)} B(e_i, e_j, e_k, h_2^{m-2}, v_2) = 0, \tag{6.104}$$

with $i, j, k = 1, \ldots, 4$ different, holds.

The equations (6.104) are the same as the equations of the system *a)* of (6.70), which is, as we have already shown, composed of 3 independent equations. (For the computation see page 133.) Again using the fact that the system (6.71) contains $3m+1$ independent equations with respect to

(6.70) and therefore also with respect to (6.104) we find that $\operatorname{Ker} \delta_{2,m}(P_S^1)$ is determined by $3m + 4$ independent equations, and therefore

$$\dim \operatorname{Ker} \delta_{2,m}(P_S^1) = \dim(\Lambda^2 T^* \otimes g_m(P_S^1)) - (3m+4) = 3m + 8.$$

Thus $\operatorname{rank} \delta_{1,m}(P_S^1) = \dim \operatorname{Ker} \delta_{2,m}(P_S^1)$, for $m \geq 3$, which proves that $H_m^2(P_S^1) = 0$ for $m \geq 3$, that is P_S^1 is 2-acyclic.

□

Using the results of this section, we can state

Theorem 6.4 *Let S be a rank one atypical spray. Assume, that $\dot{A}$ is diagonalizable and S is semi-reducible.*

(1) If $k_2 = 0$, then

- *(a) if $\operatorname{rank} M_1 = 1$, then S is locally variational;*
- *(b) if $\operatorname{rank} M_1 = 2$, then S is locally variational if and only if $a_1 \neq 0$, $a_2 \neq 0$, $\det M_{a_1,a_2} = 0$ and $\Theta^1_{a_1,a_2} = 0$, $\Theta^2_{a_1,a_2} = 0$.*
- *(c) if $\operatorname{rank} M_1 \geq 2$, then S is non-variational.*

(2) If $k_2 \neq 0$, then

- *(a) if $M_2 = 0$, then S is locally variational.*
- *(b) if $\operatorname{rank} M_2 = 1$, then S is locally variational if and only if $c_1 \neq 0$, $c_2 \neq 0$, $\operatorname{rank} M_3 = 3$ and $\Theta^1_{c_1,c_2} = 0$, $\Theta^2_{c_1,c_2} = 0$;*
- *(c) if $\operatorname{rank} M_2 > 1$, then S is non-variational.*

□

6.3.2.5 *Irreducible case*

In this paragraph we consider the case where the spray is irreducible, that is the condition of the compatibility of P_1 in an adapted basis $\{h_i, v_i\}_{i=1,2}$ is

$$\sum_{i=1,2} \chi_i v_i \Omega_E(v_i, h_i) + \sum_{i=1,2} k_i \Omega_E(v_i, h_i) = 0. \tag{6.105}$$

with $\chi_1 \neq 0$, and $\chi_2 \neq 0$. That gives a 3rd order operator $P_{(h_i,h_i)}$ which has to be introduced into the system. The symbol $\sigma_3(P_{(h_i,h_i)}) : S^3T^* \longrightarrow \mathbb{R}$ of

$P_{(\lambda,A)}$ is

$$[\sigma_3(P_{(\lambda,A)})B_3] = \chi_1\, B_3(v_1,v_1,v_1) + \chi_2\, B_3(v_2,v_2,v_2), \tag{6.106}$$

and the symbol of the first prolongation $\sigma_4(P_{(\lambda,A)}) : S^4T^* \longrightarrow T^*$ is

$$[\sigma_4(P_{(\lambda,A)})B_4](X) = \chi_1\, B_4(X,v_1,v_1,v_1) + \chi_2\, B_4(X,v_2,v_2,v_2), \tag{6.107}$$

$B_3 \in S^3T^*$, $B_4 \in S^4T^*$ and $X \in T$. The equation (6.106) is independent of the equations which determine $g_3(P_3)$, therefore $\dim g_3(P_4^1) = \dim g_3(P_3^1) - 1$.

Let us now consider the prolonged system. Replacing X in the equation (6.107) of $g_4(P_{(\lambda,A)})$ by the four vectors of the adapted basis, we find that $[\sigma_4(P_{(\lambda,A)})B_4](v_i) = 0$, $i = 1,2$ are linearly independent of the equations of $g_4(P_3)$ for $i = 1,2$, (the pivot terms are $B_4(v_1^4)$ and $B_4(v_2^4)$), while the equations $[\sigma_4(P_{(\lambda,A)})B_4](h_i) = 0$, $i = 1,2$ are linearly related to the equations of $g_4(P_3)$. Now $\dim g_4(P_4^1) = \dim g_4(P_3) - 2$, that is

$$\operatorname{rank} \sigma_4(P_4^1) = \operatorname{rank} \sigma_4(P_3) + 2.$$

Let τ_4^1 be the map

$$\tau_4^1 : (S^2T^* \otimes F_3) \oplus T^* \longrightarrow K_3^1 \oplus \mathbb{R} \oplus \mathbb{R}, \tag{6.108}$$

defined by $\tau_4^1 := \{\tilde{\tau}_3^1, \tau_{(\lambda,A)}, \rho_1, \rho_2\}$, where

$$\tilde{\tau}_3^1(B, C_\Gamma, C_A, C_{(\lambda,A)}) := \tilde{\tau}_3^1(B, C_\Gamma, C_A)$$

as defined on page 116 and

$$\begin{aligned}
\rho_1(B, C_\Gamma, C_A, C_{(\lambda,A)}) &= \chi_1\, B(v_1,v_1,h_1) + \chi_2\, B(v_2,v_2,h_2) - C_{(\lambda,A)}(S),\\
\rho_2(B, C_\Gamma, C_A, C_{(\lambda,A)}) &= C_{(\lambda,A)}(pr_1 S) - \chi_1 \Big(B_S(v_1,v_1,h_1) + \\
&\quad + \frac{\xi^S_{h_2}}{2} C_\Gamma(v_1,v_1,h_1,h_2) - \frac{\xi^S_{h_2}}{\lambda} C_A(v_1,h_1,h_1,h_2) - \frac{\xi^S_{v_2}}{\lambda} C_A(v_1,v_1,h_1,h_2) \Big) - \\
&\quad - \chi_2 \Big(\frac{\xi^S_{h_1}}{2} C_\Gamma(v_2,v_2,h_1,h_2) + \frac{\xi^S_{h_1}}{\lambda}(v_2,h_2,h_1,h_2) + \frac{\xi^S_{v_1}}{\lambda} B_A(v_2,v_2,h_1,h_2) \Big).
\end{aligned}$$

$\lambda := \lambda_1 - \lambda_2$ and pr_i is the projection onto the eigenspace corresponding to the eigenvalue λ_i.

A simple computation shows that the sequence

$$S^4T^* \xrightarrow{\sigma_4(P_4^1)} (S^2T^* \otimes F_3) \oplus T^* \xrightarrow{\ \tau_4^1\ } K_4^1 \longrightarrow 0 \tag{6.109}$$

with $K_4^1 := \operatorname{Im} \tau_4^1$ is exact.

Indeed, $\operatorname{Im} \sigma_1(P_4^1) \subset \operatorname{Ker} \tau_4^1$. On the other side $C_{(h,A_1)}(S)$ and $C_{(h,A_2)}(pr, S)$ are pivot terms in the equations $\rho_4^1 = 0$ and $\rho_4^1 = 0$, and therefore they are independent of the equation $\tilde{\tau}_3^1 = 0$. Taking into account that the sequence (6.44) is exact, we have

$$\operatorname{rank} \sigma_1(P_4^1) = \operatorname{rank} \tau_3^1 + 2 = \dim \operatorname{Ker} \tau_3^1 + \dim T^* - 2 = \dim \operatorname{Ker} \tau_4^1,$$

which proves that the sequence (6.109) is exact.

Let $p = j_3(E)_x$ be a 3rd order solution of P_4^1 in $x \in TM \setminus \{0\}$. The conditions of the compatibility of the operator P_4^1 are given by $\mu_i(\nabla P_4^1 E) = 0$, where

$$\begin{aligned}
\mu_1(\nabla P_4^1 E) = {} & \chi_1(v_1\Omega_E([v_1, S], h_1) - v_1\Omega_E([h_1, S], v_1) + [v_1, S]\Omega_E(v_1, h_1)) \\
& + \chi_2(v_2\Omega_E([v_2, S], h_2) - v_2\Omega_E([h_2, S], v_2) + [v_2, S]\Omega_E(v_2, h_2)) \\
& - (S\chi_1)v_1\Omega_E(v_1, h_1) - (S\chi_2)v_2\Omega_E(v_2, h_2), \\
\mu_2(\nabla P_4^1 E) = {} & (S\chi_1)v_1\Omega_E(v_1, h_1) + (S\chi_2)v_2\Omega_E(v_2, h_2) - \chi_1(v_1\Omega_E([v_1, S], h_1) \\
& - v_1\Omega_E([h_1, S], v_1) + [v_1, S]\Omega_E(v_1, h_1) - \xi^S_{h_2}[h_2, v_1]\Omega_E(v_1, h_1) \\
& - \xi^S_{v_2}[v_1, v_2]\Omega_E(v_1, h_1)) + \chi_2(\xi^S_{h_1}[h_1, v_2]\Omega_E(v_1, h_1) + \xi^S_{v_1}[v_1, v_2]\Omega_E(v_1, h_1)) \\
& + \chi_1(\xi^S_{h_2} v_1 \sum_{cycl} \Omega_E([v_1, h_1], h_2) - \xi^S_{v_2} v_1 \sum_{cycl} \Omega_E([v_1, v_2], h_1)) \\
& - \chi_2(\xi^S_{h_1} v_2 \sum_{cycl} \Omega_E([v_2, h_1], h_2) + \xi^S_{v_1} v_2 \sum_{cycl} \Omega_E([v_2, v_1], h_2))
\end{aligned}$$

which can be written as

$$\mu_i(\nabla P_4^1) = \sum_{j=1,2} r_j^i v_j \Omega_E(v_j, h_j) + \sum_{j=1,2} s_j^i \Omega_E(v_j, h_j) \tag{6.110}$$

$i = 1, 2$, where there is no summation for the repeated index, and the coefficients r_j^i and s_j^i $i, j = 1, 2$ are functions completely determined by S. Its definition is given in the Appendix (A.6)

Using the conditions (6.105), the system (6.110) can be written as

$$\begin{aligned}
r_1 v_1 \Omega_E(v_1, h_1) + s_1^1 \Omega_E(v_1, h_1) - s_1^2 \Omega_E(v_2, h_2) = 0, \\
s_2^1 \Omega_E(v_1, h_1) - s_2^2 \Omega_E(v_2, h_2) = 0.
\end{aligned} \tag{6.111}$$

In order to lift the third order solutions of P_4^1 we have to explore on the coefficients r_1, s_j^i of this system.

First case: $r_1 = 0$, $s_j^i = 0$

In this case, obviously, any third order solution of P_4^1 can be lifted into a fourth order solution.

Second case: $r_1 = 0$ and $\operatorname{rank}(s_j^i) > 1$.

Since in this case (6.111) contains two linearly independent equations relating $\Omega_E(v_1, h_1)$ and $\Omega_E(v_2, h_2)$, there is no regular second order formal solution satisfying the above compatibility conditions, and therefore S is non-variational.

Third case: $r_1 = 0$ and $\operatorname{rank}(s_j^i) = 1$.

If $\operatorname{rank}(s_j^i) = 1$, then (6.111) gives a new second order equation, which we shall denote as

$$s_1 \Omega_E(v_1, h_1) + s_2 \Omega_E(v_2, h_2) = 0. \tag{6.112}$$

In others words we have to introduce into the system the second order operator $P_s : C^\infty(TM) \to C^\infty(TM)$ defined by

$$P_s E := s_1 \Omega_E(v_1, h_1) + s_2 \Omega_E(v_2, h_2) \tag{6.113}$$

into the system. Using Lemma 6.6 we find the necessary and sufficient conditions for S to be variational: see at the end of the paragraph.

Fourth case: $r_1 \neq 0$.

If $r_1 \neq 0$, then the first equation of the system (6.111) gives a new 3rd order obstruction, represented by the operator $P_r : C^\infty(TM) \longrightarrow C^\infty(TM)$, where

$$P_r(E) := r_1 c_1 \Omega_E(v_1, h_1) + s_1^1 \Omega_E(v_1, h_1) + s_1^2 \Omega_E(v_2, h_2). \tag{6.114}$$

Now we have to study the integrability of the differential operator

$$P_5^1 := (P_4^1, P_r). \tag{6.115}$$

The symbol of P_r is given by

$$\sigma_3(P_r) : S^3T^* \longrightarrow \mathbb{R}, \qquad [\sigma_3(P_r)B_3] = r_1 B_3(v_1^3), \tag{6.116}$$

and the symbol of the prolongation is

$$\sigma_4(P_r) : S^4T^* \longrightarrow T^*, \qquad [\sigma_4(P_r)B_4](X) = r_1 B_4(X, v_1^3). \tag{6.117}$$

where $B_3 \in S^3T^*$, $B_4 \in S^4T^*$ and $X \in T$. The equation defining $g_3(P_r)$ is independent of the equations of $g_3(P_4^1)$. Thus $\dim g_3(P_5^1) = \dim g_3(P_4^1) - 1$.

Considering the prolongation, it is interesting to remark that $G_4(P_5^1) = G_4(P_4^1)$ and therefore we also have $g_4(P_5^1) = g_4(P_4^1)$. Indeed, an element B_4 in $G_4(P_5^1)$ is determined by the components $B_4(h_2^3, v_2)$. Therefore the equations $\sigma_4(P_r)(h_i) = 0$ and $\sigma_4(P_r)(v_i) = 0$, $i = 1, 2$ can be expressed with the help of the equations which define $g_4(P_4^1)$. Consequently $\dim g_4(P_5^1) = \dim g_4(P_4^1)$ that is

$$\operatorname{rank} \sigma_4(P_5^1) = \operatorname{rank} \sigma_4(P_4^1).$$

Let us consider the map

$$\tau_5^1 : S^2T^* \otimes F_3 \oplus T^* \oplus T^* \longrightarrow K_4^1 \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R},$$

defined by $\tau_5^1 := (\tau_4^1, (\rho_r^i)_{i=1,\dots,4})$ where

$$\tau_4^1(B, C_\Gamma, C_A, C_{[h,A]}, C_r) = \tau_4^1(B, C_\Gamma, C_A, C_{[h,A]}),$$
$$\rho_r^1(B, C_\Gamma, C_A, C_{[h,A]}, C_r) = C_r(S) - \tau_1 B(v_1, v_1, h_1),$$
$$\rho_r^2(B, C_\Gamma, C_A, C_{[h,A]}, C_r) = C_r(h_2) + \frac{\chi_1}{2} C_\Gamma(v_1, v_1, h_1, h_2) - \frac{\chi_1}{\lambda} C_A(v_1, h_1, h_1, h_2),$$
$$\rho_r^3(B, C_\Gamma, C_A, C_{[h,A]}, C_r) = C_r(v_2) - \frac{\chi_1}{\lambda} C_A(v_1, v_1, h_1, h_2),$$
$$\rho_r^4(B, C_\Gamma, C_A, C_{[h,A]}, C_r) = \chi_1 C_r(v_1) - \tau_1 C_{[h,A]}(v_1) + \frac{\tau_1 \chi_2}{\lambda} C_A(v_2, v_2, h_1, h_2).$$

The sequence

$$S^3T^* \xrightarrow{\sigma_4(P_5^1)} (T^* \otimes F_4) \oplus T^* \xrightarrow{\tau_5^1} K_5^1 \longrightarrow 0 \qquad (6.118)$$

with $K_5^1 := \operatorname{Im} \tau_5^1$, is exact.

Indeed it is easy to see that $\operatorname{Im} \sigma_4(P_4^1) \subset \operatorname{Ker} \tau_4^1$. On the other side, if we consider the system defined by $\tau_5^1 = 0$, then $C_r(h_1)$, $C_r(h_2)$, $C_r(v_2)$ and $C_r(v_1)$ are pivot terms in the equations $\rho_r^1 = 0$, $\rho_r^2 = 0$, $\rho_3 = 0$, and $\rho_4 = 0$ respectively. Using the exactness of the sequence (6.109) we find that

$$\operatorname{rank} \sigma_4(P_5^1) = \operatorname{rank} \tau_4^1 = \dim \operatorname{Ker} \tau_4^1 + \dim T^* - 4 = \dim \operatorname{Ker} \tau_5^1,$$

which proves that the sequence (6.118) is exact.

We will now compute the obstructions arising from the maps ρ_x^i. Let $j_3(E)_x$ be a third order formal solution of P_3^1 at x. We have.

$$\begin{aligned}
\rho_x^1(\nabla P_3^1 E) &= S(P, E) - r_1 v_1(\nabla \omega_E(v_1, h_1)) \\
&\quad + (S s_1^1)\Omega_E(v_1, h_1) + (S s_1^2)\Omega_E(v_2, h_2) + s_1^1 S\Omega_E(v_1, h_1) + s_1^2 S\Omega_E(v_2, h_2), \\
\rho_x^2(\nabla P_3^1 E) &= h_2(P, E) + r_1 v_1(\nabla i_2 \Omega_E(v_1, h_1, h_2)) - \frac{r_1}{\lambda} v_1(\nabla i_A \Omega_E(h_2, h_1, h_2)) \\
&= r_1 v_1\Big(\sum_{cycl} \Omega_E([v_1, h_1], h_2)\Big) + (h_2 r_1) v_1 \Omega_E(v_1, h_1) + r_1 [h_2, v_1] \Omega_E(v_1, h_1) + \\
&\quad - s_1^1 h_2 \Omega_E(v_1, h_1) + s_1^2 h_2 \Omega_E(v_2, h_2) + (h_2 s_1^1)\Omega_E(v_{[illegible]}, h_1) + (h_2 s_1^2)\Omega_E(v_2, h_2), \\
\rho_x^3(\nabla P_3^1 E) &= v_2(P, E) - \frac{r_1}{\lambda} v_1(\nabla i_A \Omega_E(v_1, h_1, h_2)) \\
&= r_1 v_1\Big(\sum_{cycl} \Omega_E([v_1, h_1], v_2)\Big) + (v_2 r_1) v_1 \Omega_E(v_1, h_1) + r_1 [v_2, v_1] \Omega_E(v_1, h_1) + \\
&\quad - (v_2 s_1^1)\Omega_E(v_1, h_1) + (v_2 s_1^2)\Omega_E(v_2, h_2) + s_1^1 v_2 \Omega_E(v_1, h_1) + s_1^2 v_2 \Omega_E(v_2, h_2), \\
\rho_x^4(\nabla P_3^1 E) &= (\chi_1(v_1 s_1^1) - r_1(v_1 k_1))\Omega(v_1, h_1) + (\chi_1(v_1 s_1^2) - r_1(v_1 k_2))\Omega(v_2, h_2) \\
&\quad + (\chi_1(v_1 r_1) + \chi_1 s_1^1 - r_1(v_1 \chi_1) - r_1 k_1) v_1 \Omega(v_1, h_1) - r_1(v_1 \chi_2) v_2 \Omega(v_2, h_2) \\
&\quad + (\chi_1 s_1^2 - r_1 k_2) v_1 \Omega(v_2, h_2) + r_1 \chi_1 ([v_2, v_1]\Omega(v_1, h_2) + v_2\Big(\sum_{cycl} \Omega([v_2, v_1], h_2)\Big))
\end{aligned}$$

Since $\chi_1 \neq 0$, $\chi_2 \neq 0$ and $r_1 \neq 0$, the 3rd order derivative of E appearing in the above expressions can be expressed in terms of the second order jet of E using the equations (6.47), (6.105) and (6.111 a). Therefore the new obstructions can be written in the form

$$\rho_x^i(\nabla P_3^1) = q_1^i \Omega(v_1, h_1) + q_2^i \Omega(v_2, h_2), \tag{6.119}$$

$i = 1, \ldots, 4$, where the coefficients q_j^i can be easily computed.

Let us consider the matrix

$$M_5 := \begin{pmatrix} q_1^1 & q_2^1 \\ q_1^2 & q_2^2 \\ q_1^3 & q_2^3 \\ s_1^1 & s_1^2 \end{pmatrix} \tag{6.120}$$

defined with the help of the coefficients of the equations (6.111 b) and the equations (6.119).

If $M_5 = 0$, the conditions of compatibility are identically satisfied for every third order solution of P_3^1. Then any third order solution of P_3^1 can be lifted into a 4th order solution.

If rank $M_5 = 2$, then $\Omega_E(v_1, h_1) = 0$ and $\Omega_E(v_2, h_2) = 0$, which is excluded if S is variational and E is a regular Lagrangian associated to S (see Lemma 6.1).

If rank $M_5 = 1$, then the equations (6.119) and the equations (6.111 b) are linearly dependent and one of them can be removed. Let us denote

$$q_1 \Omega(v_1, h_1) + q_2 \Omega(v_2, h_2) = 0 \tag{6.121}$$

the remaining equation. If $q_1 = 0$ or $q_2 = 0$, the spray cannot be variational. Assuming $q_1 \neq 0$ and $q_2 \neq 0$, the equation (6.121) gives a new second order condition of compatibility. In order to introduce it into the system, we define the differential operator $P_q : C^\infty(TM) \longrightarrow C^\infty(TM)$ by the formula

$$P_q(E) := q_1 \Omega_E(v_1, h_1) + q_2 \Omega_E(v_2, h_2), \tag{6.122}$$

and we consider the new system defined by the operator

$$(P_6^1, P_{v_1}, \nabla P_q). \tag{6.123}$$

Using the coefficients of the operator $(P_{[h..4]}, P_v(v_1), \nabla P_q(v_2), P_q)$ we define the matrix

$$M_6 := \begin{pmatrix} \chi_1 & \chi_2 & k_1 & k_2 \\ r_1 & 0 & s_1^1 & s_1^2 \\ q_1 & 0 & \bar{q}_1^1 & \bar{q}_2^1 \\ 0 & q_2 & \bar{q}_1^2 & \bar{q}_2^2 \\ 0 & 0 & q_2 & q_2 \end{pmatrix} \tag{6.124}$$

where

$$\begin{aligned} \bar{q}_1^1 &= (v_1 q_1) + q_2 \chi^1_{v_2}, & \bar{q}_2^1 &= (v_1 q_2) + q_2 \chi^2_{v_1} \\ \bar{q}_1^2 &= (v_2 q_1) + q_1 \chi^1_{v_2}, & \bar{q}_2^2 &= (v_2 q_2) + q_1 \chi^2_{v_2}. \end{aligned} \tag{6.125}$$

According to the same reason that we have already used, we see that if rank $M_6 = 4$, then the system has no regular 3rd order solution, so the spray S is non-variational.

If rank $M_6 = 3$, then the two first rows can be expressed in terms of the others, and this means that the differential operators $P_{[h..4]}$ and P_v can be removed from the system. Then the system is equivalent to the system defined by

$$\Gamma_6 := (\Gamma_5, \Gamma_7), \tag{6.126}$$

whose condition of compatibility can be expressed in terms of the functions $\Theta^i_{q_1,q_2}$ using the Completion Lemma

Finally, by a computation analogous to that of the above sections, one can check that the operators Γ^1_4, Γ^1_5 and Γ^1_6 are 2-acyclic. Now we can state

Theorem 6.5 *Let S be a rank one atypical spray, suppose that $\tilde{A}$ is diagonalizable and the spray is irreducible. Then,*

(1) If $r_1 = 0$ and $s^i_j = 0$ $(i, j = 1, 2)$, then S is locally variational.
(2) If $r_1 = 0$, then S is locally variational if and only if

(a) $\det(s^i_j)_{i,j=1,2} = 0$,
(b) $M_4 = 0$,
(c) $s_1 \neq 0$, $s_2 \neq 0$ and $\Theta^1_{s_1,s_2} = 0$, $\Theta^2_{s_1,s_2} = 0$

(3) If $r_1 \neq 0$ and $M_5 = 0$, then S is locally variational .
(4) If $r_1 \neq 0$ and $M_5 \neq 0$, then S is locally variational if and only if

(a) rank $M_5 = 1$,
(b) rank $M_6 = 3$,
(c) $q_1 \neq 0$, $q_2 \neq 0$, and $\Theta^i_{q_1,q_2} = 0$, $i = 1, 2$.

□

6.3.3 *The inverse problem when $\tilde{A}$ is non-diagonalizable*

The study of the inverse problem in the case where $\tilde{A}$ is non-diagonalizable is very close to the study in the diagonalizable case. We will give only the results here, referring to [Mu] for detailed demonstrations. The explicit formulae are given in the Appendix.

Let us return to the Section 6.3.1. As we have seen, the supplementary condition of compatibility to lift a 3rd order formal solution $p = j_3(E)_x$ of the prolongation P_1^1 of the system

$$\begin{cases} \omega = 0, & \text{i.e. the Euler-Lagrange equation} \\ i_T\Omega = 0, \\ i_A\Omega = 0. \end{cases}$$

is $\varphi_E = 0$ (cf. Proposition 6.3). We have the following Proposition, which corresponds to Lemma 6.5 in the non-diagonalizable case.

Lemma 6.9 *In an adapted Jordan basis* $\{h_i, v_i\}_{\{i=1,2\}}$ *(i.e.* h_1 *and* $v_1 = Jh_1$ *span the eigenspace* Δ *associated to the eigenvalue* λ *of* $\tilde{A}$*, and* h_2*,* $v_2 := Jh_2$ *span the other characteristic space) we have*

$$\varphi_E(h_1, h_2, h_1, h_2) = \eta_1\, v_2 \Omega_E(v_1, h_2) + \sum_{i=1,2} p_i\, \Omega_E(v_i, h_2). \qquad (6.137)$$

where the functions η_1*,* p_1 *and* p_2 *depend only on the spray. Their definition is given in the Appendix (A.7). In particular,* η_1 *vanishes if and only if the distribution* Δ^1 *is reducible. In this case we will say that the spray is reducible.*

Proof. If $j_3(E)_x$ is a 3rd order solution at x of P_1^1, then we have $(\nabla P_3 E)_x = 0$ and hence

$$(\nabla P_T)(X, h_1, h_j) = X\Omega_E(h_j, v_1) = 0,$$
$$(\nabla P_A)(X, h_1, h_j) = \lambda X i_J \Omega_E(h_j, v_1) - X\Omega(v_1, h_j) = -X\Omega(v_1, h_j) = 0.$$

for every $X \in T_x$. But $d\Omega_E = 0$, and so:

$$\begin{aligned} &\Omega_E([h_1, v_1], h_2) + \Omega_E([v_1, h_2], h_1) + \Omega_E([h_2, h_1], v_1) - \\ &\qquad - h_1\Omega_E(v_1, h_2) - v_1\Omega_E(h_2, h_1) - h_2\Omega_E(h_1, v_1) = 0. \\ &\Omega_E([v_1, h_1], v_2) + \Omega_E([h_1, v_2], v_1) + \Omega_E([v_2, v_1], h_1) \\ &\qquad - v_1\Omega_E(h_1, v_2) - h_1\Omega_E(v_2, v_1) - v_2\Omega_E(v_1, h_1) = 0. \end{aligned}$$

Since $\Omega_E|_{T_v \times T_v} = 0$, $h_1\Omega_E(v_1, h_2)$ and $v_1\Omega_E(h_1, v_2)$ can be expressed by Ω_E without its derivatives, i.e. by the second order derivatives of the Lagrangian E:

$$h_1\Omega_E(v_1, h_2) = \Omega_E([h_1, v_1], h_2) + \Omega_E([v_1, h_2], h_1) - \Omega_E([h_2, h_1], v_1),$$
$$v_1\Omega_E(h_1, v_2) = \Omega_E([v_1, h_1], v_2) + \Omega_E([h_1, v_2], v_1) + \Omega_E([v_2, v_1], h_1).$$

Thus we get

$$\begin{aligned} v_1\Omega_E(v_1,h_2) &= \eta^1_{v_1}\Omega_E(v_1,h_2) + \eta^2_{v_1}\Omega_E(v_2,h_2),\\ h_1\Omega_E(v_1,h_2) &= \eta^1_{h_1}\Omega_E(v_1,h_2) + \eta^2_{h_1}\Omega_E(v_2,h_2), \end{aligned} \tag{6.128}$$

and

$$\begin{aligned} v_1\Omega_E(v_2,h_2) &= v_2\Omega_E(v_1,h_2) + \eta^1_{v_2}\Omega_E(v_1,h_2) + \eta^2_{v_2}\Omega_E(v_2,h_2),\\ h_1\Omega_E(v_2,h_2) &= h_2\Omega_E(v_1,h_2) + \eta^1_{h_2}\Omega_E(v_1,h_2) + \eta^2_{h_2}\Omega_E(v_2,h_2), \end{aligned} \tag{6.129}$$

where the coefficients are defined in by (A.8) and (A.9).

On the other hand, $(\mathcal{L}_S\omega_E)_x = (d\omega_E)_x$ and $(d\omega_E)_x = 0$ imply that

$$S\Omega_E(X,Y)_x = \Omega_E([S,X],Y)_x + \Omega_E(X,[S,Y])_x \tag{6.130}$$

So

$$\begin{aligned} S\Omega_E(v_1,h_2) &= \left(\xi^{[S,v_1]}_{v_1} + \xi^{[S,h_2]}_{h_2}\right)\Omega_E(v_1,h_2) + \xi^{[S,h_2]}_{v_2}\Omega_E(v_2,h_2),\\ S\Omega_E(v_2,h_2) &= \left(\xi^{[S,v_2]}_{v_1} + \xi^{[S,h_2]}_{h_1}\right)\Omega_E(v_1,h_2) + \left(\xi^{[S,v_2]}_{v_2} + \xi^{[S,h_2]}_{h_2}\right)\Omega_E(v_2,h_2). \end{aligned}$$

and

$$\begin{aligned} h_2\Omega_E(v_1,h_2) &= \nu_1 v_2\Omega_E(v_1,h_2) + \nu^1_1\Omega_E(v_1,h_2) + \nu^2_1\Omega_E(v_2,h_2),\\ h_2\Omega_E(v_2,h_2) &= \sum_{i=1,2}\nu_i v_2\Omega_E(v_i,h_2) + \sum_{i=1,2}\nu^i_2\Omega_E(v_i,h_2), \end{aligned} \tag{6.131}$$

where the coefficients ν_i and ν^j_i, $i,j = 1,2$, depend only on the spray (their definition is given by (A.5)). If we collect the second and third order terms which appear in the compatibility condition $\varphi_E = 0$, we obtain the first part of the Lemma.

For the second part, we notice that we have $\eta_1(x) = 0$ if and only if $\xi^{[v_1,h_1]}_{h_2}v_1 + \xi^{[v_1,h_1]}_{v_2} = 0$ at x, that is

$$\det\begin{pmatrix} \xi^S_{h_2} & \xi^{[v_1,h_1]}_{h_2} \\ \xi^S_{v_2} & \xi^{[v_1,h_1]}_{v_2} \end{pmatrix}(x) = 0$$

So $\eta_1(x) = 0$ if and only if $pr_2 S_x$ and $pr_2[v_1,h_1]_x$ are linearly dependent, where pr_2 denotes the projection on to the characteristic distribution complementary to the eigenspace Δ in the characteristic splitting of T_x, that is there exists $\mu \in \mathbb{R}$, such that

$$pr_2([v_1,h_1] - \mu S)_x = 0$$

Since $S_x \notin \Delta_x$, $\eta_1(x)$ vanishes if and only if either Δ is integrable ($\mu = 0$), or $S_x \in \Delta^2_x$, that is Δ^2 is reducible at x. □

There are two cases to study, according to wheter S is reducible or not.

6.3.3.1 *Reducible case*

One can check, in a way completely analogous to that in the diagonalisable case, that the operator P^1_3 is 2-acyclic (cf. [Mu]). So if the coefficients p_1 and p_2 vanish, the operator P^1_3 and P_3 are formally integrable. It follows that the spray is variational. An example of a spray satisfying these conditions is the following:

Example 6.5 Let us consider the system:

$$\begin{cases} \ddot{x}_1 = x_2, \\ \ddot{x}_2 = 0. \end{cases} \tag{6.132}$$

We have $\Gamma^i_j = 0$, $A^1_1 = A^2_1 = A^2_2 = 0$, and $A^1_2 = 1$. Thus $A^2 = 0$, and the rank of the spray is 1. The eigenvectors of A adapted to the connection Γ are $\left\{h_1 = \frac{\partial}{\partial x_1}, v_1 = \frac{\partial}{\partial y_1}\right\}$. In particular, the eigenspace Δ is integrable, the spray is not typical and a Jordan basis of A is $\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}\right\}$. Hence the brackets which appear in the expressions of the function p_i vanish identically. Therefore the system (6.132) is variational.

Let us suppose that p_1 and p_2 are not both zero. In order to give the conditions of compatibility of the operator

$$P_3 E := (\omega_E, i_\Gamma \Omega_E, i_A \Omega_E, P_{(h,A)} E) \tag{6.133}$$

where

$$P_{(h,A)}(E) = p_1 \Omega_E(v_1, h_2) + p_2 \Omega_E(v_2, h_2), \tag{6.134}$$

we introduce the following notation, analogous to that of the functions Θ^i (see page 122) introduced in the diagonalizable case.

Definition 6.3 Let ϑ_1 and ϑ_2 two function on TM, with $\vartheta_1 \neq 0$. We can define the functions $\Psi^i_{\vartheta_1,\vartheta_2}$ on $T(TM)$ by the formulae

$$\begin{aligned} \Psi^1_{\vartheta_1,\vartheta_2} :=& \Big(\vartheta_1 \xi^S_{h_1} \xi^{[h_1,v_1]}_{v_1} - \vartheta_1 \xi^S_{v_1} \xi^{[h_1,v_1]}_{h_1} + 2\vartheta_1 \xi^{[v_1,S]}_{h_1} - (S\vartheta_1)\Big) \\ & - \frac{\vartheta_2}{\vartheta_1}\Big(\vartheta_2 \xi^S_{h_2} \xi^{[h_2,v_2]}_{v_2} - \vartheta_2 \xi^S_{v_2} \xi^{[h_2,v_2]}_{h_2} + 2\vartheta_2 \xi^{[v_2,S]}_{h_2} - (S\vartheta_2)\Big), \\ \Psi^2_{\vartheta_1,\vartheta_2} =& \Big(2\vartheta_1 \xi^{[v_1,S]}_{h_1} + \vartheta_1 \xi^{[S_2,h_1]}_{h_1} + \vartheta_1 \xi^{[S_2,v_1]}_{v_1} - (S_1\vartheta_1)\Big) \\ & - \frac{\vartheta_2}{\vartheta_1}\Big(\vartheta_2 \xi^{[S_1,h_2]}_{h_2} + \vartheta_2 \xi^{[S_1,v_2]}_{v_2} - (S_1\vartheta_2)\Big), \end{aligned} \qquad (6.136)$$

where $S_1 := pr_1 S$ and $S_2 := pr_2 S$ are the projections of S on to Δ and on to the other characteristic space.

As the formula (6.127) shows, the higher order compatibility condition in the reducible case gives a new second order condition. The analyses of the different possibilities in the reducible case are possible with the help of Lemma 6.10, which corresponds to the Completion Lemma 6.6 of the diagonalizable case.

Lemma 6.10 *(Completion Lemma in the non-diagonalizable case.) Let $(h_i, v_i)_{i=1,2}$ be an adapted basis of $\mathcal{A}$, $x \in TM \setminus \{0\}$, ϑ_1, ϑ_2 smooth functions in a neighborhood of x not both zero. Let us consider the second order differential operator $P_\vartheta : C^\infty(TM) \to C^\infty(TM)$ defined on a neighborhood of x by*

$$P_\vartheta E = \vartheta_1 \Omega_E(v_1, h_2) + \vartheta_2 \Omega_E(v_2, h_2),$$

and the operator $\tilde{P}_\vartheta := (P_1, P_\vartheta)$, and let us denote by $N_{\vartheta_1,\vartheta_2}$ the matrix

$$N_{\vartheta_1,\vartheta_2} := \begin{pmatrix} \eta_1 & p_1 & p_2 \\ \vartheta_2 & \tilde{\vartheta}_1 & \tilde{\vartheta}_2 \\ 0 & \vartheta_1 & \vartheta_2 \end{pmatrix} \qquad (6.136)$$

defined by the coefficients of the rows of $P_{(h_1,A)}$, $\nabla P_\vartheta(v_1)$, and P_ϑ, where

$$\begin{aligned} \tilde{\vartheta}_1 &= (v_1 \vartheta_1) + \vartheta_2 \eta^1_{v_1} - \vartheta_2 \big(\xi^{[h_1,v_2]}_{v_1} - \xi^{[v_2,h_2]}_{h_2} - \xi^{[h_2,v_1]}_{h_1}\big), \\ \tilde{\vartheta}_2 &= (v_1 \vartheta_2) + \vartheta_1 \eta^2_{v_1} - \vartheta_2 \big(\xi^{[h_1,v_2]}_{v_1} - \xi^{[h_2,v_1]}_{h_2}\big). \end{aligned} \qquad (6.137)$$

Then

(1) If $\vartheta_2(x) = 0$, then there are no regular second order formal solutions of $\tilde{P}_\vartheta$ at x.

(2) If $\theta_2(x) \neq 0$, then

(a) there are regular formal solutions of $\bar{P}_\theta$ on a neighborhood U of x if and only if $\Psi^1_{\theta_1,\theta_2} = 0$, $\Psi^2_{\theta_1,\theta_2} = 0$, and $\det(N_{\theta_1,\theta_2}) = 0$ on U.

(b) Moreover, the operator $\bar{P}_\theta$ is "complete" in the sense that if we add to $\bar{P}_\theta(E) = 0$ a new differential equation of the type

$$a\, v_2 \Omega_E(v_1, h_2) + b\, v_2 \Omega_E(v_2, h_2) + r\, \Omega(v_1, h_2) + s\, \Omega(v_2, h_2) = 0$$

which is independent of $\bar{P}_\theta(E) = 0$ and has prolongation at x, then the new system has no regular second order solutions at x.

The statements 1) and 2b) can easily be checked by a simple computation.

The proof of 2a) is very similar to the proof of Lemma 6.5. As in the diagonalizable case, one can see that any 2nd order *formal* solution of $\bar{P}_\theta$ can be lifted into a 3rd order solution if and only if $\Psi^i_{\theta_1,\theta_2} = 0$, $i = 1, 2$. However, $H^2_2(\bar{P}_\theta) \neq 0$, that is $\bar{P}_\theta$ is not 2-acyclic. Thus there is an extra compatibility condition for the prolonged system. An analysis analogous to that of the diagonalizable case allows us to show that this obstruction appears to lift a 3rd order solution into a 4th order solution. In fact, any 3rd order solution of $\bar{P}_\theta$ can be lifted into a 4th order solution, and there also exists a regular second order solution if and only if $j_{1,x}(\Psi^i_{\theta_1,\theta_2}) = 0$, $i = 1, 2$ and $\det N_{\theta_1,\theta_2} = 0$.

On the other side, the first prolongation of $\bar{P}_\theta$ being 2-acyclic, the operator $\bar{P}_\theta$ and therefore P_θ are formally integrable and have a regular 2nd order solution, hence 2a) of the Lemma holds.

In the reducible case we arrive at the following

Theorem 6.6 *Let S be an atypical spray of rank 1 and suppose that A is non-diagonalizable and S reducible. Then S is locally variational if and only if $p_1 = p_2 = 0$, or $p_2 \neq 0$ and $\Psi^i_{p_1,p_2} = 0$, $i = 1, 2$.*

Example 6.6 Let us consider the system

$$\begin{cases} \ddot{x}_1 = F(x_1, x_2, \dot{x}_1, \dot{x}_2), \\ \ddot{x}_2 = 0. \end{cases} \tag{6.138}$$

where $\frac{\partial^2 F}{\partial y_1^2} \neq 0$. We have $\Gamma^2_j = 0$, and $A^2_1 = A^2_2 = 0$. The rank of the spray is 1. The eigenvectors of A adapted to the connection Γ are $\left\{h_1 = \frac{\partial}{\partial x_1}, v_1 = \frac{\partial}{\partial y_1}\right\}$, hence the spray is non-typical. The Jordan basis for $\tilde{A}$ is given by the vectors

$$h_1 = \frac{\partial}{\partial x_1} - \Gamma^1_1 \frac{\partial}{\partial y_1}, \qquad v_1 = \frac{\partial}{\partial y_1},$$
$$h_2 = \frac{\partial}{\partial x_2} - \Gamma^1_2 \frac{\partial}{\partial y_1}, \qquad v_2 = \frac{\partial}{\partial y_2}.$$

The computation gives $\eta_1 = 0$, $p_2 = 0$ and

$$p_1 = \Gamma^1_1 \frac{\partial \Gamma^1_{11}}{\partial x_1} - \frac{\partial \Gamma^1_{11}}{\partial x_1} + \frac{\partial \Gamma^1_{12}}{\partial x_1} - \Gamma^1_{11}\Gamma^1_{12} - \Gamma^1_{11}\Gamma^1_1 \left(\Gamma^1_1 \frac{\partial \Gamma^1_{21}}{\partial y_1} + \frac{\partial \Gamma^1_{11}}{\partial y_2} - \Gamma^1_1 \frac{\partial \Gamma^1_{11}}{\partial x_1} - \Gamma^1_{11}\Gamma^1_{11}\right).$$

where $\Gamma^i_{jk} := \frac{\partial \Gamma^i_j}{\partial y_k}$. So, generically $p_1 \neq 0$ and then the spray is non-variational.

6.3.3.2 *Irreducible case*

If S is irreducible, then we have to study the integrability of the differential operator

$$P_S E = (\omega_E, i_\Gamma \Omega_E, i_A \Omega_E, P_{(h,A)} E), \tag{6.139}$$

where

$$P_{(h,A)}(E) := \eta_1 \eta_2 \Omega_E(v_1, h_2) + p_1 \Omega_E(v_1, h_2) + p_2 \Omega_E(v_2, h_2). \tag{6.140}$$

Let

$$\tau^1_4 : (S^2T^* \otimes F_1) \oplus T^* \longrightarrow K^1_3 \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \tag{6.141}$$

be the map defined by $\tau^1_4 := (\tau^1_3, \tau_{(h,A)}, \rho_1, \rho_2, \rho_3)$, where

$\tilde{\tau}_3(B, C_\Gamma, C_A, C_{(h,A)}) = \tau_3(B, C_\Gamma, C_A)$,
$\rho_1(B, C_\Gamma, C_A, C_{(h,A)}) = \eta_1 B(v_2, v_1, h_2) - C_{(h,A)}(S)$,
$\rho_2(B, C_\Gamma, C_A, C_{(h,A)}) = C_{(h,A)}(h_1) - \frac{\eta_1}{2} C_\Gamma(v_2, v_1, h_1, h_2) - \eta_1 C_A(v_2, h_2, h_1, h_2)$
$\rho_3(B, C_\Gamma, C_A, C_{(h,A)}) = C_{(h,A)}(v_1) + \eta_1 C_A(v_2, v_1, h_1, h_2)$.

It is easy to show that the sequence

$$S^4T^* \xrightarrow{\sigma_4(P_4^1)} (S^2T^* \otimes F_3) \oplus T^* \xrightarrow{\tau_4^1} K_4^1 \longrightarrow 0$$

with $K_4^1 := \operatorname{Im} \tau_4^1$, is exact.

The new conditions of compatibility for a 3rd order formal solution $j_3(E)_x$ of P_4 in $x \in TM \setminus \{0\}$, are given by the equations $\rho_i(\nabla P_4^1 E)_x = 0$, $i = 1, 2, 3$:

$$\begin{aligned}
\rho_1(\nabla P_4^1 E) =& (S\eta_1)v_2\Omega(v_1, h_2) + S(p_1\Omega(v_1, h_2) + p_2\Omega(v_2, h_2)) \\
&+ \eta_1[S, v_2]\Omega(v_1, h_2) + \eta_1 v_2(\Omega([S, v_1], h_2) + \Omega(v_1, [S, h_2])). \\
\rho_2(\nabla P_4^1 E) =& (h_1\eta_1)v_2\Omega(v_1, h_2) + h_1(p_1\Omega(v_1, h_2) + p_2\Omega(v_2, h_2)) \\
&+ \eta_1[h_1, v_2]\Omega(v_1, h_2) - \eta_1 v_2\Big(\sum_{cycl}\Omega([v_1, h_1], h_2)\Big). \\
\rho_3(\nabla P_4^1 E) =& (v_1\eta_1)v_2\Omega(v_1, h_2) + v_1(p_1\Omega(v_1, h_2) + p_2\Omega(v_2, h_2)) \\
&+ \eta_1[v_1, v_2]\Omega(v_1, h_2) - \eta_1 v_2\Big(\sum_{cycl}\Omega([v_2, v_1], h_1)\Big).
\end{aligned}$$

With the help of the equations (6.128), (6.129) and (6.131) the obstruction can be written as

$$\rho_i(\nabla P_4^1 E) = \sum_{j=1,2} \eta_j^i v_2 \Omega_E(v_j, h_2) + \sum_{j=1,2} p_j^i v_1 \Omega_E(v_j, h_1). \tag{6.142}$$

The explicit expression of the coefficients η_j^i and p_j^i is given in the Appendix (A.11). Let

$$N_1 := \begin{pmatrix} \eta_1 & 0 & p_1 & p_2 \\ \eta_1^1 & \eta_2^1 & p_1^1 & p_2^1 \\ \eta_1^2 & \eta_2^2 & p_1^2 & p_2^2 \\ \eta_1^3 & \eta_2^3 & p_1^3 & p_2^3 \end{pmatrix} \tag{6.143}$$

be the matrix of the coefficients of the operator $P_{(\lambda,\mu)}$ and of the equations $\rho_i(\nabla P_4) = 0$, $i = 1, 2, 3$. Using a line of reasoning completely analogous to the one we have developed in the precedent sections (cf. for example the proof of Lemma 6.7, page 138), we find

- If $\operatorname{rank} N_1 = 1$, then every third order solution of P_4^1 can be lifted into a fourth order solution.

- If rank $N_1 = 2$, then a new condition of compatibility has to be introduced into the system.
- If rank $N_1 = 3$, then:
  (1) if $\eta_2^i = 0$, $i = 1, 2, 3$, the spray is non-variational;
  (2) if one of the η_2^i, $i = 1, 2, 3$ does not vanish then a new 3rd order condition of compatibility has to be introduced in the system.
- If rank $N_1 = 4$, then the spray is non-variational.

The new conditions of compatibility can be written in the form

$$\begin{aligned} \eta_2 v_2 \Omega_E(v_2, h_2) + q_1^1 \Omega_E(v_1, h_2) + q_1^2 \Omega_E(v_2, h_2) = 0, \\ q_2^1 \Omega_E(v_1, h_2) + q_2^2 \Omega_E(v_2, h_2) = 0, \end{aligned} \tag{6.144}$$

where the coefficients η_2, q_i^j can easily be computed from the matrix (6.143).

a) Case $\eta_2 = 0$

Let us assume that $\eta_2 = 0$ and the rank of the matrix N_1 is two. In this case the two equations of (6.144) are linearly dependent, and give a new relationship between the terms $\Omega_E(v_1, h_2)$ and $\Omega_E(v_2, h_2)$. Let us denote it by

$$q_1 \Omega_E(v_1, h_2) + q_2 \Omega_E(v_2, h_2) = 0 \tag{6.145}$$

According to the Completion Lemma (Lemma 6.10) we obtain the necessary and sufficient conditions for S to be variational.

b) Case $\eta_2 \neq 0$

If $\eta_2 \neq 0$, then the first equation of the compatibility conditions (6.144) gives a new 3rd order condition, and therefore we have to consider the operator

$$P_5^1 E := (P_4^1, P_5), \tag{6.146}$$

where $P_5 : C^\infty(TM) \to C^\infty(TM)$ denotes the operator

$$P_5 E := \eta_2 v_2 \Omega_E(v_2, h_2) + q_1^1 \Omega_E(v_1, h_2) + q_1^2 \Omega_E(v_2, h_2). \tag{6.147}$$

Let us consider the map

$$\tau_5^1 : S^2T^* \otimes F_5 \oplus T^* \oplus T^* \xrightarrow{\tau_5^1} K_4^1 \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}, \qquad (6.148)$$

defined by $\tau_5^1 = (\bar{\tau}_4^1, \rho_q^1, \rho_q^2, \rho_q^3)$, where

$$\bar{\tau}_4^1(B, C_\Gamma, C_A, C_{(h,A)}, C_q) = \tau_4^1(B, C_\Gamma, C_A, C_{(h,A)})$$
$$\rho_q^1(B, C_\Gamma, C_A, C_{(h,A)}, C_q) = C_q(S) - \eta_2 B(v_2, v_2, h_2),$$
$$\rho_q^2(B, C_\Gamma, C_A, C_{(h,A)}, C_q) = C_q(h_1) - \frac{\eta_2}{2} C_\Gamma(v_2, v_2, h_1, h_2) + \frac{\eta_2}{\eta_1} B_{(h,A)}(h_2),$$
$$\rho_q^3(B, C_\Gamma, C_A, C_{(h,A)}, C_q) = \eta_1 C_q(v_1) - \eta_2[\sigma_2(P_{(h,A)})B](v_2),$$

and τ_4^1 is defined on page 158. By a standard computation one can check that the sequence

$$S^2T^* \xrightarrow{\sigma_2(P_4^1)} (T^* \otimes F_4) \oplus T^* \xrightarrow{\tau_5^1} K_5^1 \longrightarrow 0 \qquad (6.149)$$

is exact, where $K(P_5^1)$ denotes the image of τ_5^1. Let $j_3(E)_x$ be a 3rd order formal solution of P_5^1 on $x \in TM \setminus \{0\}$. Computing the obstructions, we arrive at:

$$\begin{aligned}
\rho_q^1(\nabla P_5^1 E) &= (S\eta_2)v_2\Omega_E(v_2, h_2) + S(q_1\Omega_E(v_1, h_2) + q_2\Omega_E(v_2, h_2)) \\
&\quad + \eta_2[S, v_2]\Omega_E(v_2, h_2) + \eta_2 v_2\Big(\Omega_E([S, v_2], h_2) + \Omega_E(v_2, [S, h_2])\Big) \\
\rho_q^2(\nabla P_5^1 E) &= (h_1\eta_2)v_2\Omega_E(v_2, h_2) + h_1(q_1\Omega_E(v_1, h_2) + q_2\Omega_E(v_2, h_2)) \\
&\quad - \frac{\eta_2}{\eta_1}\Big\{(h_2\eta_1)v_2\Omega_E(v_1, h_2) + h_2(q_1\Omega_E(v_1, h_2) + q_2\Omega_E(v_2, h_2))\Big\} \\
&\quad + \eta_2[h_1, v_2]\Omega_E(v_2, h_2) - \eta_2[h_2, v_2]\Omega_E(v_2, h_2) + \eta_2 v_2\Big(\sum_{cycl}\Omega_E([h_1, v_2], h_2)\Big), \\
\rho_q^3(\nabla P_5^1 E) &= \eta_1(v_1\eta_2)v_2\Omega_E(v_2, h_2) - \eta_2(v_1\eta_1)v_2\Omega_E(v_1, h_2) \\
&\quad + \eta_1 v_1\{q_1\Omega_E(v_1, h_2) + q_2\Omega_E(v_2, h_2)\} - \eta_1 v_2\{p_1\Omega_E(v_1, h_2) + p_2\Omega_E(v_2, h_2)\} \\
&\quad + \eta_1\eta_2[v_1, v_2]\Omega_E(v_2, h_2) + \eta_1\eta_2 v_2\Big(\sum_{cycl}\Omega_E([v_1, v_2], h_2)\Big).
\end{aligned}$$

Moreover, using the equations (6.128), (6.129), and taking into account that $(P_{(h,A)}E)_x = 0$, $(P_q E)_x = 0$, we can eliminate the 3rd order terms from the expression of the conditions of compatibility, which can be written in the form:

$$\rho_q^1(\nabla P_4^1) = r_1^1\Omega_E(v_1, h_1) + r_2^1\Omega_E(v_2, h_2), \qquad (6.150)$$

$i = 1, 2, 3$. Let us consider the matrix

$$N_2 := \begin{pmatrix} r_1^1 & r_2^1 \\ r_1^2 & r_2^2 \\ r_1^3 & r_2^3 \\ q_2^1 & q_2^2 \end{pmatrix} \tag{6.151}$$

defined by the coefficients of the second order conditions of compatibility (6.150) and (6.144 b). We have the following possibilities:

(1) If $N_2 = 0$, then every third order solution of the operator P_3 can be lifted into a fourth order solution;

(2) If rank $N_2 = 1$*, then the equations (6.150) are equivalent to one of them, which we denote*

$$P_r := r_1 \Omega_E(v_1, h_1) + r_2 \Omega_E(v_2, h_2) = 0. \tag{6.152}$$

(3) If rank $N_2 = 2$*, then $\Omega_E(v_1, h_1) = \Omega_E(v_2, h_2) = 0$ and S cannot be variational (see Lemma 6.1).*

Indeed, 1) and 3) can easily be checked by a simple computation. Let us consider the case, when rank $N_2 = 1$.

If $r_2 = 0$, then there are no regular second order solutions of P_3^1 which satisfy the compatibility conditions (6.144 b) and (6.150), so the spray S is non-variational.

If $r_2 \neq 0$ we must study the integrability of the system:

$$(P_3^1, P_r, \nabla P_r). \tag{6.153}$$

Let

$$N_4 := \begin{pmatrix} \eta_1 & 0 & p_1 & p_2 \\ 0 & \eta_2 & q_1 & q_2 \\ r_2 & 0 & \bar{r}_1^1 & \bar{r}_2^1 \\ r_1 & r_2 & \bar{r}_1^2 & \bar{r}_2^2 \\ 0 & 0 & r_1 & r_2 \end{pmatrix} \tag{6.154}$$

be the matrix defined by the coefficients of $P_{(h,A)}$, $P_r(v_1)$, $\nabla P_r(v_2)$ and P_r. The coefficients $\bar{r}_i^j$ are given explicitly in the Appendix (A.10).

Since $r_2 \neq 0$, we have rank $N_4 \geq 3$. Obviously, if rank $N_4 = 4$ then S is non-variational, because there are no second order regular solutions satisfying the compatibility conditions.

If rank $N_1 = 3$, then considering the system $P_6 := (P_5, P_7)$, i.e. the second order part of the system (6.153), the Completion Lemma (Lemma 6.10) get us the necessary and sufficient condition for S to be variational. Therefore we can state the following

Theorem 6.7 *Let S be a rank one atypical spray and suppose that $\dot{A}$ is non-diagonalizable and S reducible.*

(1) If rank $N_1 = 1$, then S is locally variational;

(2) If rank $N_1 = 2$, and

(a) if $\eta_2 = 0$, then S is locally variational if and only if

$$\det N_{q_1,q_2} = 0, \quad q_2 \neq 0, \quad \text{and} \quad \Psi^i_{q_1,q_2} = 0, \; i = 1, 2,$$

(b) if $\eta_2 \neq 0$, and

i. if rank $N_2 = 0$, then S is locally variational,

ii. if rank $N_2 = 1$, then S is locally variational if and only if rank $N_3 = 3$, $r_2 \neq 0$, and $\Psi^i_{r_1,r_2} = 0$, $i = 1, 2$

iii. if rank $N_2 = 2$, then S is non-variational.

(3) If rank $N_1 = 3$, and

(a) if $\eta_2 = 0$, then S is non-variational,

(b) if $\eta_2 \neq 0$, and

i. rank $N_2 = 1$, then S is locally variational if and only if rank $N_3 = 3$, $r_2 \neq 0$, and $\Psi^i_{r_1,r_2} = 0$, $i = 1, 2$.

ii. rank $N_2 = 2$, then S is non-variational.

(4) If rank $N_1 = 4$, then S is non-variational.

6.4 Rank $S = 2$

6.4.1 *Typical sprays*

In this section we suppose that the spray has rank 2. This means that the tensor fields J, A, A' give a basis of the $C^\infty(TM)$-module spanned by $\{J, A, A', \ldots, A^{(n)}, \ldots\}$.

Let us return to the study of the operator P_3, that is of the system

$$\begin{cases} \omega_E &= 0, \\ i_\Gamma \Omega_E &= 0, \\ i_A \Omega_E &= 0. \end{cases}$$

As we have seen (cf. page 90 and 103), a second order formal solution $j_2(E)_x$ of P_3 in $x \in TM \setminus \{0\}$ can be lifted into a third order solution if and only if $i_{A'}\Omega_E = 0$. If $\operatorname{rank} S = 2$, this gives a new obstruction which has to be introduced into the system. Then we have to study the differential operator

$$\tilde{P}_4 : C^\infty(TM) \longrightarrow Sec\,(T_v^* \oplus \Lambda^2 T_v^* \oplus \Lambda^2 T_v^* \oplus \Lambda^2 T_v^*),$$

defined by $\tilde{P}_4 = (P_3, P_{A'})$, where $P_{A'} = i_{A'} dd_J$.

Proposition 6.5 *[Dou] Let* $\operatorname{rank} S = 2$. *If* $\bar{A}$ *and* $\bar{A}'$ *have a common eigenvector, then the spray is not variational.*

Indeed, if $\bar{A}$ and $\bar{A}'$ have a common eigenvector, then they also have a common horizontal eigenvector h_1 and a vertical one $v_1 = Jh_1$. Let h_2 and $v_2 = Jh_2$ be such that $\{h_1, v_1, h_2, v_2\}$ is an adapted Jordan basis for $\bar{A}$ and denote by a_{ij} the components of the matrix of A' in this basis, that is,

$$A'h_1 = a_{11}v_1 + a_{21}v_2$$

Of course, $a_{12} = 0$. Note that, since $\operatorname{rank} S = 2$, we have $a_{21} \neq 0$ if $\bar{A}$ is diagonalizable and $a_{11} - a_{22} \neq 0$ if $\bar{A}$ is not diagonalizable.

Suppose that the spray S is variational and let E be a regular Lagrangian associated to S. Since $i_{A'}\Omega_E = 0$, we have

$$i_{A'}\Omega(h_1, h_2) = a_{11}\Omega(v_1, h_2) - a_{21}\Omega(v_1, h_1) - a_{22}\Omega(v_2, h_1) = 0$$

Now $a_{21}\Omega(v_1, h_1) = 0$ if $\bar{A}$ is diagonalizable and $(a_{11} - a_{22})\,\Omega(v_1, h_2) = 0$ if $\bar{A}$ is not diagonalizable. Then $\Omega_E(v_1, h_1) = 0$ in the diagonalizable case and $\Omega_E(v_1, h_2) = 0$ in the non-diagonalizable case and this is excluded (cf. Lemma (5.1)).

□

Corollary 6.2 *The typical sprays of rank 2 are not variational.*

Indeed, suppose that S is in the eigenspace Δ_λ corresponding to the eigenvalue λ. In this case hS and $C = Jh$ are also found in Δ_λ (cf. Proposition 3.7). Then $vS \in \Delta_\lambda$, hence there exists $\mu \in \mathbb{R}$, such that $vS_x = \mu C_x$. We have

$$\tilde{A}'(hS) = F(A'hS + A'FhS) = FA'hS = Fv[AS, S] - FA[hS, S] =$$
$$= Fv[\lambda C, S] + FA[vS, S] = (S\lambda)FC + \lambda Fv[C, S] + \mu FA[C, S],$$

but

$$[C, S] = [J, S](S) = hS - vS = hS - \mu C,$$

and so

$$\tilde{A}'(hS) = (S\lambda)FC - \lambda FvS + \mu FA(hS) = (S\lambda)FC = (S\lambda)hS.$$

This means that hS is a common eigenvector for $\tilde{A}$ and $\tilde{A}'$, whereby we can conclude that the spray is non-variational.

6.4.2 *Atypical sprays*

We will now consider the cases where the spray S is atypical. Using the results of the Completion Lemmas (cf. Lemmas 6.6 and 6.10) we can easily formulate the necessary and sufficient conditions for the spray to be variational.

a) $\tilde{A}$ is diagonalizable

Let $\{h_1, h_2, v_1, v_2\}$ be an adapted basis and $A'h_i = a_{i1}v_1 + a_{i2}v_2$. The compatibility condition which has to be introduced into the system $P_3 = 0$ is

$$i_{A'}\Omega_E(h_1, h_2) = -a_{21}\Omega_E(v_1, h_1) + a_{12}\Omega_E(v_2, h_2) = 0.$$

Note that a_{12} and a_{21} are not both zero, because rank $S = 2$. The situation is the one described in the Completion Lemma 6.6. Thus we can state the following

Theorem 6.8 *Let S be an atypical spray of rank 2, with $\tilde{A}$ diagonalizable. Then S is locally variational if and only if*

1) $\bar{A}$ *and* $\bar{A}'$ *have no common eigenvector (i.e.* $a_{12}a_{21} \neq 0$),

2) $\Theta^i_{-a_{21},a_{12}} = 0$, $i = 1, 2$,

3) $\operatorname{rank}(M_{-a_{21},a_{12}}) = 3$,

where the matrix $M_{-a_{21},a_{12}}$ *is defined in (6.56) with* $g_1 = -a_{21}$ *and* $g_2 = a_{12}$.

b) $\bar{A}$ *is non-diagonalizable*

If $\bar{A}$ is non-diagonalizable, then the computations and results are similar to those in the diagonalizable case. In a Jordan basis adapted to $\bar{A}'$ the compatibility condition $i_{A'}\Omega_E = 0$ is

$$(a_{11} - a_{22})\Omega_E(v_1, h_2) + a_{12}\Omega_E(v_2, h_2) = 0.$$

Using the Completion Lemma 6.10 we arrive at the following

Theorem 6.9 *Let S be an atypical spray of rank 2 and suppose that* $\bar{A}$ *is non-diagonalizable. Then S is locally variational if and only if*

1) $\bar{A}$ *and* $\bar{A}'$ *have no common eigenvector (i.e.* $a_{12} \neq 0$),

2) $\Psi^i_{a_{11}-a_{22},a_{12}} = 0$, $i = 1, 2$,

3) $\operatorname{rank}(N_{a_{11}-a_{22},a_{12}}) = 2$,

where the matrix $N_{a_{11}-a_{22},a_{12}}$ *is defined by (6.136) with* $\vartheta_1 = a_{11} - a_{22}$ *and* $\vartheta_2 = a_{12}$

Chapter 7

Euler-Lagrange Systems in the Isotropic Case

In the previous chapter we gave the complete classification of the variational sprays on 2-dimensional manifolds. Despite the fact that the dimension of these manifolds is low, the complete analysis is complex, as we have seen. In the higher dimensional cases the situation is, of course, much more complicated since the conditions of integrability involves not only the Douglas tensor, but also the curvature tensor and its derivatives and the higher order elements of the graded Lie-algebra associated to the spray (see Section 4.2). Therefore it is not really reasonable to expect a complete classification of variational sprays on n-dimensional manifolds where $n \in \mathbb{N}$ is arbitrary, unless we consider a particular class of sprays.

Natural restrictions can be imposed on the curvature of the natural connection associated to the spray. In this chapter we will consider isotropic sprays, whose geometrical meaning was explained in Section 3.5: if they are variational, the associated Lagrangian has isotropic curvature. They are analogous to the geodesic of a Riemann manifold with constant curvature for non quadratic second order equations.

As in the previous chapter, manifolds and the other objects (tensors, functions etc.) are assumed to be analytic. If an object lives on the tangent bundle, then it is assumed to be analytic away from the zero section.

7.1 The flat case

The simplest case of isotropic sprays is when the semi-basic 1-form α in the Douglas tensor (3.33) vanishes, and so the spray is flat (see Definition

3.28]. The following theorem is a generalization to the n-dimensional case of the Theorem I of Douglas.

Theorem 7.1 *Every flat spray is locally variational on $TM \setminus \{0\}$.*

Remark. This Theorem has also been proved by I.M. Anderson and G. Thomson in [AT] using Cartan's Theory of exterior differential systems, and recently by M. Crampin, E. Martinez and W. Sarlet in [SCM] using Riquier's Theory of partial differential systems. Our result has already been published in [GM].

Proof. Recall that a second order solution $j_2(E)_x$ of the Euler-Lagrange operator P_1 can be lifted into a third order solution if and only if $(i_\Gamma \Omega_E)_x = 0$, where $\Gamma = [J, S]$ and $\Omega_E = dd_J E$ (cf. Paragraph 5.1). Thus we have to study the integrability of the differential operator $P_2 = (P_1, i_\Gamma dd_J)$. We have already showed that P_2 is a regular operator on $TM \setminus \{0\}$, and that at any $x \in TM \setminus \{0\}$, the set of second order formal solutions, $R_{2,x}(P_2)$, contains regular 2nd order formal solutions (see Paragraph 5.2).

On the other hand, the compatibility conditions for P_2 are given by the equations

$$i_A \Omega_E = 0,$$
$$i_R \Omega_E = 0,$$

(cf. Proposition 5.2). Now

$$i_A \Omega_E = i_{\lambda J} \Omega_E = \lambda i_J \Omega_E = \lambda d_J^2 E = \lambda d_{[J,J]} E = 0,$$

and

$$i_R \Omega_E = -i_{\frac{1}{3}[\lambda J, J]} \Omega_E = -\frac{1}{3} d_{[\lambda J, J]} d_J E = -\frac{1}{3}(d_{\lambda J} d_J^2 E + d_J \lambda d_J^3) = 0.$$

Thus the conditions of compatibility are satisfied.

Let us now prove that P_2 is involutive. Let $B \in S^2 T^*$ be a symmetric tensor. Since $g_2(P_2) = g_2(P_1) \cap g_2(P_\Gamma)$, we have $B \in g_2(P_2)$ if and only if

$$B(S, JX) = 0, \tag{7.1}$$
$$B(hX, JY) - B(hY, JX) = 0. \tag{7.2}$$

for any $X, Y \in T$ (cf. (5.2) and (5.8)). In an adapted basis $\{h_i, v_i\}_{i=1,\dots,n}$ with $h_1, \dots, h_n$ horizontal and $v_i = Jh_i$, these equations are

$$B(S, v_i) = 0, \tag{7.3}$$

$$B(h_i, v_j) - B(h_j, v_i) = 0, \tag{7.4}$$

where $i \leq j$, $i, j = 1, \dots, n$. Since these equations are independent, we find

$$\dim g_2(P_2) = \frac{n(n+1)}{2} + n^2.$$

On the other hand, as we have seen in Section 5.2, we have

$$\dim g_3(P_2) = \frac{4n(n+1)(2n+1)}{3}.$$

To give a quasi-regular basis, we will consider the homogeneous and the non-homogeneous case separately. Note that the spray is homogeneous if and only if it is horizontal. Indeed

$$\Gamma S = [J, S]S = [C, S] - J[S, S] = [C, S],$$

so $vS = \frac{1}{2}(S - [C, S])$ and hence $vS = 0$ if and only if $[C, S] = S$.

Homogeneous case

Let us first consider a basis $\mathcal{B} = \{h_i, v_i\}_{i=1,\dots,n}$ of T_x with $h_1, \dots, h_n$ horizontal, $h_n := S$ and $Jh_i = v_i$ for $i = 1, \dots, n$. We have $v_n = C$. Let $B \in S^2T^*$ and put forward

$$a_{ij} := B(h_i, h_j), \quad b_{ij} := B(h_i, v_j), \quad \text{and} \quad c_{ij} := B(v_i, v_j).$$

Note that

a) $a_{ij} = a_{ji}$ and $c_{ij} = c_{ji}$, $i, j = 1, \dots, n$.

b) $b_{ni} = 0$, $i = 1, \dots, n$.

c) $b_{ij} = b_{ji}$, $i, j = 1, \dots, n$.

The relation *a*) comes from the symmetry of B, while the equations *b*) and *c*) correspond to the equations (7.3) and (7.4) respectively. An easy computation shows that this basis is not quasi-regular. Let us now consider the basis $\tilde{B} = \{e_i, v_i\}_{i=1,\dots,n}$ where

$$e_i := \begin{cases} h_i - i\, v_{ii} & \text{for} \quad i = 1,\dots,n-1, \\ h_n + \sum_{k=1}^{n} v_{kk} & \text{for} \quad i = n. \end{cases}$$

We shall prove that this basis is quasi-regular. Let us denote

$$\tilde{a}_{ij} = B(e_i, e_j), \quad \tilde{b}_{ij} = B(e_i, v_j), \quad \text{and} \quad \tilde{c}_{ij} = B(v_i, v_j).$$

We have $\tilde{c}_{ij} = c_{ij}$ for $i, j = 1, \dots, n$, and also

$$\tilde{b}_{ij} = \begin{cases} b_{ij} + i\, c_{ij}, & i, j = 1, \dots, n-1; \\ c_{in}, & j = n; \\ \sum_{k=1}^{n} c_{kj}, & i = n \end{cases}$$

These relations allow us to express the components of the block $(\tilde{c}_{ij})$ in terms of the components $\tilde{b}_{ij}$:

$$\begin{aligned}
\tilde{c}_{in} &= \frac{1}{i}\, \tilde{b}_{in}, && i < n, \\
\tilde{c}_{ij} &= \frac{1}{i-j}\, (\tilde{b}_{ij} - \tilde{b}_{ji}), && 1 \le i < j < n, \\
\tilde{c}_{ii} &= \tilde{b}_{ni} - \sum_{k \ne i} \frac{1}{k-i}\, (\tilde{b}_{ik} - \tilde{b}_{ki}), && i < n, \\
\tilde{c}_{nn} &= \tilde{b}_{nn} - \sum_{k \ne n} \frac{1}{k}\, \tilde{b}_{kn}.
\end{aligned}$$

Thus an element B of $g_2(F_Y)$ is completely determined by its components $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$. Taking into account that the matrix $(\tilde{a}_{ij})$ is symmetric, we obtain

$$\dim g_2(F_Y)_{e_1 \dots e_k} = \frac{(n-k)(n-k+1)}{2} + n(n-k).$$

On the other hand $\dim g_2(P_2)_{v_1,\dots,v_n,w_1,\dots,w_k} = 0$, for $k = 1,\dots,n$, whereby

$$\begin{aligned}
&\dim g_2(P_2) + \sum_{k=1}^{n} \dim(g_2(P_2))_{v_1,\dots,v_k} + \sum_{k=1}^{n} \dim(g_2(P_2))_{v_1,\dots,v_n,w_1,\dots,w_k} \\
&\quad = n^2 + \frac{n(n+1)}{2} + \sum_{k=1}^{n} \frac{(n-k)(n-k+1)}{2} + \sum_{k=1}^{n} n(n-k) \\
&\quad = n^2 + \frac{n(n+1)}{2} + \frac{n(n-1)(n+1)}{6} + \frac{n^2(n-1)}{2} = \frac{4n(n+1)(2n+1)}{3} \\
&\quad = \dim g_3(P_2),
\end{aligned}$$

which proves that the basis $\tilde{B} = \{e_j, w_j\}_{j=1,\dots,n}$ is quasi-regular.

Non-homogeneous case

In this case we have $vS \neq 0$. First consider a basis $\{h_i, v_i\}_{i=1,\dots,n}$ with $v_i = Jh_i$, $h_n = hS$ such that the vectors h_i, for $i = 1,\dots,n-1$, are horizontal and the equation $vS = \sum_{k=1}^{n} v_k$ holds. In this basis for an element $B \in g_2(P_2)$ we have the following relations:

a) $a_{ij} = a_{ji}$ and $c_{ij} = c_{ji}$,

b) $b_{ij} = b_{ji}$,

c) $b_{ni} = -\sum_{k=1}^{n} c_{ki}$,

for $i = 1,\dots,n$. The relations *a*) show that B is symmetric, the equation *b*) comes from the equation (7.3), while the property *c*) comes from the equation (7.4), because

$$b_{ni} = B(hS, v_i) = B(S, v_i) - B(vS, v_i) \overset{(7.4)}{=} -B(vS, v_i) = -\sum_{k=1}^{n} c_{ki}.$$

Let us now consider the basis $\tilde{B} = \{e_i, v_i\}_{i=1,\dots,n}$ where $e_i = h_i + iv_i$ for $i = 1,\dots,n$ and denote by $\tilde{a}_{ij}$, $\tilde{b}_{ij}$ and $\tilde{c}_{ij}$, $i, j = 1,\dots,n$ the components of

B in this basis. We have $\bar{c}_{ij} = c_{ij}$ and

$$\begin{aligned}
\bar{b}_{ij} &= b_{ij} + i\, c_{ij}, & i,j &= 1,\dots,n-1, \\
\bar{b}_{in} &= -\sum_{k=1}^{n} c_{kn} + i c_{in}, & i &= 1,\dots,n-1, \\
\bar{b}_{ni} &= -\sum_{k=1}^{n} c_{ki} + n\, c_{in}, & i &= 1,\dots,n-1.
\end{aligned}$$

Hence, as in the homogeneous case, the block $(\bar{c}_{ij})$ can be expressed in terms of the elements of the block $(\hat{b}_{ij})$. We arrive at the following relations:

$$\bar{c}_{ij} = \frac{1}{i-j}\left(\hat{b}_{ij} - \hat{b}_{ji}\right), \qquad 1 \le j < i \le n, \tag{7.5}$$

$$\bar{c}_{ii} = \frac{n-1}{n-i}\left(\bar{b}_{ni} - \bar{b}_{in}\right) - \bar{b}_{ni} + \sum_{k=1,k\neq i}^{n-1} \frac{1}{k-i}\left(\bar{b}_{ki} - \bar{b}_{ik}\right), \qquad 1 \le i < n, \tag{7.6}$$

$$\bar{c}_{nn} = \frac{1}{n-1}\left(\hat{b}_{nn} + \sum_{k=1}^{n-1} \frac{1}{n-k}\left(\hat{b}_{nk} - \hat{b}_{kn}\right)\right). \tag{7.7}$$

Equation (7.5) is obvious. To check (7.6), note that

$$\bar{b}_{ni} = b_{ni} + n c_{in} = \sum_{k=1,k\neq i}^{n-1} c_{ii} - c_{ki} - (n-1) c_{ii}$$

and hence, using (7.5)

$$\bar{c}_{ii} = \frac{n-1}{n-i}\left(\bar{b}_{ni} - \bar{b}_{in}\right) - \bar{b}_{ni} - \sum_{k=1,k\neq i}^{n-1} \frac{\left(\hat{b}_{ki} - \hat{b}_{ik}\right)}{k-i}.$$

To prove (7.6) note that $\bar{b}_{nn} = b_{nn} + n\, c_{nn}$, so

$$\bar{b}_{nn} = -\sum_{k=1}^{n-1} c_{kn} + (n-1) c_{nn},$$

and thus

$$c_{nn} = \frac{1}{n-1}\left(\hat{b}_{nn} + \sum_{k=1}^{n-1} c_{kn}\right) = \frac{1}{n-1}\left(\hat{b}_{nn} + \sum_{k=1}^{n-1} \frac{1}{n-k}\left(\hat{b}_{nk} - \hat{b}_{kn}\right)\right).$$

Now, as in the homogeneous case, an element B of $g_2(P_2)$ is determined by the components of the blocs $\bar{a}_{ij}$ and $\bar{b}_{ij}$ where $\bar{a}_{ij}$ is symmetric, and therefore we find, as in the homogeneous case

$$\dim g_2(P_2)_{v_1 \ldots v_k} = \frac{(n-k)(n-k+1)}{2} + n(n-k),$$

and $\dim g_2(P_2)_{v_1 \ldots v_k, v_1 \ldots v_k} = 0$, for $k = 1, \ldots, n$. Hence the same computation as in the homogeneous case shows that the basis $\bar{B}$ is quasi-regular. The Theorem is proved □

The computation of the Cartan characters shows that the general solution depends on $n+1$ functions with n variables *.

7.2 The non-flat case

We suppose in this section that the spray is isotropic, i.e. $A = \lambda J + \alpha \otimes C$, where α is a non-zero semi-basic 1-form.

Note that if L is a semi-basic (1-1) tensor, its matrix in the natural basis $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$ is

$$L = \begin{pmatrix} 0 & 0 \\ L_\alpha^\beta & 0 \end{pmatrix},$$

where (x^i) is a local coordinate system on M and (x^i, y^i) is a local coordinate system on TM. To give an intrinsic definition of the Jordan blocks of the matrix (L_α^β), we put forward $\bar{L} = FL + LF$ (cf. page 58): it is clear, from the matrix of $\bar{L}_x$, that $\bar{L}_x$ has the same Jordan blocks as $(L_\alpha^\beta(x))$ of the components of L. Now we can formulate the following intrinsic

Definition 7.1 A semi-basic (1-1) tensor L has a *constant algebraic type* on an open set U, if the degrees of the elementary divisors in the Jordan decomposition of $\bar{L}$ are constant on U.

Lemma 7.1 *Let S be a non-flat isotropic spray ($\alpha \neq 0$) and $x_0 \in TM$. Then:*

(1) $\bar{A}_{x_0}$ is diagonalisable if and only if $i_S\alpha_{x_0} \neq 0$.
(2) If A has a constant algebraic type on U and S is variational on U, then $i_S\alpha_x \neq 0$ for every $x \in U$. In particular $0 \notin U$.

*This result is different from the results of ([Dou], [AT], and [SCM]) because we consider Lagrangians to be time-independent and, moreover, in our problem the unknown function is the Lagrangian E, whereas in the works mentioned the unknowns are $\frac{\partial^2 E}{\partial y^i \partial y^j}$.

Proof. Let us denote by Span(X) the line bundle in T spanned by the vector field $X \in \mathfrak{X}(TM)$, and more generally Span$(X_1, \ldots, X_k)$ denotes the distribution spanned by the vector fields $X_i \in \mathfrak{X}(TM)$, $i = 1, \ldots, k$.

Then the horizontal eigenspaces of $\dot{A}$ are $\mathcal{H} = \alpha^{\perp} \cap T^h$ and Span(hS) corresponding respectively to the eigenvalues λ and $\lambda + \alpha(S)$. If $\alpha(S)_{x_0} = 0$, then λ_{x_0} has a multiplicity of $2n$. Now if $\dot{A}_{x_0}$ is diagonalizable, we have $\dot{A}_{x_0} = \lambda I$; but this is excluded because $\alpha_{x_0} \neq 0$.

Conversely, if $\alpha(S)_{x_0} \neq 0$ then

$$T^h_{x_0} = \mathcal{H}_{x_0} \oplus \mathrm{Span}(hS)_{x_0},$$

and we have a splitting of T_{x_0} into eigenspaces corresponding to the eigenvalues λ_{x_0} and $(\lambda + i_S \alpha_{x_0})$:

$$T_{x_0} = (\mathcal{H} \oplus J\mathcal{H})_{x_0} \oplus (Span(hS) \oplus Span(C))_{x_0}$$

which proves that $\dot{A}_{x_0}$ is diagonalisable. Therefore we find (1).

(2) Since $\dot{A}$ is an algebraically constant type on U, either $\dot{A}$ is diagonalizable or non-diagonalisable at any point of U. This means that either $i_S\alpha\,|_U = 0$ or $i_S\alpha\,|_U \neq 0$. Suppose that S is variational and E is a regular Lagrangian associated to S. The condition of compatibility $i_A\Omega_E = 0$ (cf. Proposition 5.2) gives $i_{\lambda J + \alpha\otimes C}\Omega_E = 0$ i.e. $\alpha \wedge i_C\Omega_E = 0$. Since $\alpha \neq 0$, there exists $\rho_E \in C^\infty(U, \mathbb{R})$ such that $i_C\Omega_E = \rho_E\alpha$ on U, and hence $\Omega(C, S) = \rho_E\alpha(S)$. So, if $i_S\alpha = 0$ on U, then every $x \in U$ has null length. But this is excluded by Lemma (3.1). Then we have $i_S\alpha\,|_U \neq 0$. Finally it is easy to see, for example using local coordinates, that $i_S\alpha(0) = 0$, so $0 \notin U$. $\Diamond$

Let us now assume that the spray S is variational, and let E be a regular Lagrangian associated to S. The compatibility condition $i_A\Omega = 0$ gives $i_{\lambda J + \alpha\otimes C}\Omega_E = 0$, i.e. $\alpha \wedge i_C\Omega_E = 0$. Now $i_S\alpha\, i_C\Omega_E - \Omega_E(C, S)\,\alpha = 0$ at x_0. Taking into account that $\Omega_E(C, S) \neq 0$, if $(i_S\alpha)_{x_0} = 0$ then $\alpha_{x_0} = 0$ which is excluded, because the spray is not flat. Then we have:

Corollary 7.1 *Let S be a non-flat isotropic spray with $A = \lambda J + \alpha\otimes C$, and let $\dot{A}$ be algebraically constant. Then if $(i_S\alpha)_x = 0$, then S is not variational on a neighborhood of x.*

In the following we suppose that $\dot{A}$ has an *algebraic constant type* and $i_S\alpha \neq 0$ on a neighborhood of $x \in TM$.

As we have seen in Lemma 5.2, a necessary condition to lift a second order formal solution $j_2(E)_x$ of P_2 into a 3rd order formal solution is that $(i_A\Omega_E)_x = 0$, i.e. $\alpha_x \wedge (i_C\Omega_E)_x = 0$. Then we have to introduce this equation into the system and consider the differential operator

$$P_3 := (P_1, P_T, P_A) : \quad C^\infty(TM) \longrightarrow Sec\,(T_v^* \oplus \Lambda_v^2 \oplus \Lambda_\alpha^2),$$

where $\Lambda_\alpha^2 = \{ \Theta \in \Lambda_v^2 \mid \exists\, \theta \in T_v^* : \Theta = \alpha \wedge \theta \}$, and $P_A : C^\infty(TM) \longrightarrow Sec\Lambda_\alpha^2$ is defined by

$$P_A = i_A dd_J.$$

As we will see, the study of the integrability of this system will depend on the degree of non holonomy of the distribution $\mathcal{D}$ spanned by S and C. The first case, when $\mathcal{D}$ is integrable, arises if and only if S is typical:

Proposition 7.1 *An isotropic non-flat spray is typical if and only if the distribution $\mathcal{D}$ spanned by S and C is integrable.*

Proof. The integrability of $\mathcal{D}$ implies that then the spray is typical (cf. Proposition 3.6) .

Conversely, let S be an isotropic spray with $i_S\alpha \neq 0$ and suppose that it is typical. Note that $[C, S] = S - 2vS$. To prove that $\mathcal{D}$ is integrable, we only need to prove that vS and C are linearly dependent. As we have seen, the eigenvalues of $\tilde{A}$ are λ and $\lambda + i_S\alpha$. The dimension of the vertical eigenspace of A corresponding to $\lambda + i_S\alpha$ is 1. But S is typical, that is it is an eigenvector of $\tilde{A}$. Thus, by Proposition 3.7, hS and $C = JS$ are eigenvectors corresponding to the same eigenvalue as S. Now, if $vS = 0$, then vS and C are linearly dependent; if $vS \neq 0$, then vS is also an eigenvector corresponding to the same eigenvalue as C, because S and hS are also eigenvector corresponding to the same eigenvalue as C. Hence vS is proportional to C

□

7.2.1 *Typical sprays*

In this section we suppose that the distribution $\mathcal{D} = \text{Span}\{S, C\}$ is integrable, that is S is typical. The following Theorem contains the case of homogeneous sprays in particular.

Theorem 7.2 *Let S be a non-flat isotropic spray, with $A = \lambda J + \alpha \otimes C$. We suppose that A has an algebraic constant type on a neighborhood of x and that S is typical. Then S is locally variational on a neighborhood of $x \in TM$ if and only if*

1. $i_S \alpha \neq 0$, (7.8)
2. $\alpha \wedge d_J \alpha = 0$, (7.9)
3. $\alpha \wedge \alpha' = 0$, (where $\alpha' = i_h \mathcal{L}_S \alpha$), (7.10)
4. $\alpha \wedge D_{hX} \alpha = 0$ for every $X \in \operatorname{Ker} \alpha$, (7.11)

where D is the Berwald connection on TM associated to the spray S.

Proof. First we notice that at any $x \in TM \setminus \{0\}$ there exists a regular 2nd order formal solution.

Indeed, using the notation introduced on page 87, a second order jet $(j_2 E)_x \in J_2(\mathcal{E})$ is a regular second order solution of P_3 in $x = (x^i, y^i) \in TM$ if and only if it satisfies the inequality (5.10), the linear equations (5.11), (5.12), and the linear equation $(P_4 E)_x = 0$ which is

$$p_{\underline{ij}} A^j_k = p_{\underline{kj}} A^j_i. \tag{7.12}$$

where $p_{\underline{ij}} = \frac{\partial^2 E}{\partial y^i \partial y^j}(x)$. Let $\{v_1, \dots, v_n\}$ now be a basis of T_x^v so that the matrix A^j_i is diagonal, and let g_x be a scalar product of T_x^v so that the basis $\{v_1, \dots, v_n\}$ is orthogonal. We then have $g_x(\bar{A}_x X, Y) = g(X, \bar{A}_x Y)$. If $(p_{\underline{ij}})$ is the matrix of g_x with respect to the basis $\{\frac{\partial}{\partial y^i}\}_{i=1,\dots,n}$, we find that (5.10) and (7.12) are satisfied. Solving the system (5.11), (5.12) with respect to the pivot terms p_i and p_{ji}, we arrive at a regular second order formal solution of P_3 at $x \in TM$.

The proof of the formal integrability of the operator P_3 involves two steps.

Step I. *First compatibility conditions.*

We have already computed the symbol of P_3 and its first prolongation in Section 6.2. We will now compute $\dim g_3(P_3)$. Let $B \in S^3 T^*$; since

$g_3(P_3) = g_3(P_1) \cap g_3(P_T) \cap g_3(P_A)$, we have $B \in g_3(P_3)$ if and only if

$$B(X, S, JY) = 0, \tag{7.13}$$

$$B(X, hY, JZ) - B(X, hZ, JY) = 0, \tag{7.14}$$

$$B(X, AY, JZ) - B(X, AZ, JY) = 0, \tag{7.15}$$

for every $X, Y, Z \in T$. Let $\mathcal{B} = \{h_i, v_i\}_{i=1,\dots,n}$ be a basis adapted to $\bar{A}$ so that $h_1, \dots, h_{n-1}$ are horizontal eigenvectors corresponding to the eigenvalue λ, $h_n := hS$ (which is a horizontal eigenvector corresponding to the eigenvalue $\lambda + i_S\alpha$), and $v_i = Jh_i$, for $i = 1, \dots, n$. The equation (7.13) yields the system

$$B(h_i, S, v_j) = 0, \tag{7.16}$$

$$B(v_i, S, v_j) = 0, \tag{7.17}$$

for $i, j = 1, \dots, n$, while the equation (7.14) gives

$$B(h_i, h_j, v_k) - B(h_i, h_k, v_j) = 0, \tag{7.18}$$

$$B(v_i, h_j, v_k) - B(v_i, h_k, v_j) = 0, \tag{7.19}$$

for $i, j, k = 1, \dots, n$, and the equation (7.15) gives

$$B(h_i, C, v_j) = 0, \tag{7.20}$$

$$B(v_i, C, v_j) = 0, \tag{7.21}$$

for $i = 1, \dots, n$, and $j = 1, \dots, n-1$. Since the spray is typical, there exists $\mu \in C^\infty(TM)$ so that $vS = \mu C$. Thus the equation (7.20) can be expressed with the help of the other equations. Indeed, noting that $C = v_n$ and using (7.19), we arrive at

$$B(h_i, C, v_j) = B(v_j, h_i, v_n) = B(v_j, hS, v_i) = B(v_j, S, v_i) - \mu B(v_j, C, v_i) = 0,$$

for $i, j = 1, \dots, n-1$, and

$$B(h_n, C, v_j) = B(hS, C, v_j) = B(C, hS, v_j) = B(C, S, v_j) - \mu B(C, C, v_j) = 0.$$

for $i = n$, $j = 1, \dots, n-1$ according to (7.17) and (7.21). Now there are four blocks in the tensor B: $B_1 = B(h_i, h_j, h_k)$, $B_2 = B(h_i, h_j, v_k)$, $B_3 = B(h_i, v_j, v_k)$ and $B_4 = B(v_i, v_j, v_k)$. Of course, B_1 and B_4 are symmetric in the indices i, j, k, because B is symmetric. By (7.18) and (7.19), B_2 and B_3 are also symmetric. Thus

$$\dim g_3(P_\Gamma) = \frac{4n(n+1)(n+2)}{6}, \tag{7.22}$$

A simple computation shows that the system (7.16), (7.17) contains $\frac{n(n-1)}{2}$ equations which are independent of the equations (7.18) and (7.19). Moreover, each equation of (7.21) is independent of the equations (7.16), (7.17), (7.18), and (7.19). Hence

$$\dim g_3(P_3) = \frac{4n(n+1)(n+2)}{6} - \Big(\frac{2n(n+1)}{2} + \frac{(n-1)n}{2} + (n-1)\Big),$$

and so

$$\operatorname{rang} \sigma_3(P_3) = \dim S^3T^* - \dim g_3(P_3) = \frac{4n^3 + 9n^2 + 5n - 6}{6}. \tag{7.23}$$

In order to find the conditions of compatibility for P_3 we consider the maps

$$\begin{aligned}
\tau_{[J,A]} &: (T^* \otimes T_0^*) \oplus (T^* \otimes \Lambda^2_0) \oplus (T^* \otimes \Lambda^2_{\mathcal H}) \longrightarrow \Lambda^2_{-} \\
\tau_{A'} &: (T^* \otimes T_0^*) \oplus (T^* \otimes \Lambda^2_0) \oplus (T^* \otimes \Lambda^2_{\mathcal H}) \longrightarrow \Lambda^3_0 \\
\tau_{\mathcal H} &: (T^* \otimes T_0^*) \oplus (T^* \otimes \Lambda^2_0) \oplus (T^* \otimes \Lambda^2_{\mathcal H}) \longrightarrow T_0^* \otimes T_0^*
\end{aligned}$$

defined by

$$\tau_{[J,A]}(B_S, B_\Gamma, B_A)(X, Y, Z) := \sum_{X,Y,Z}^{\text{cycl}} B_A(JX, Y, Z),$$

$$\tau_{A'}(B_S, B_\Gamma, B_A)(X, Y) := B_A(S, X, Y) - \big(B_S(AX, Y) - B_S(AY, X)\big),$$

where $X, Y, Z \in T$, and

$$\begin{aligned}
\tau_{\mathcal H}(B_S, B_\Gamma, B_A)(X, Y) := \tfrac{1}{2} B_\Gamma(JY, X, S) + B_S(JY, X) \\
- \frac{\mu}{4\gamma\alpha} B_A(JY, S, X) - \frac{1}{4\gamma\alpha} B_A(hX, S, Y)
\end{aligned}$$

for $X, Y \in \mathcal H$ where $\mathcal H = \alpha^\perp \cap T^h$. Let us consider $\tau_3 = (\tau_1, \tau_{[J,A]}, \tau_{A'}, \tau_{\mathcal H})$ where the morphism τ_1 is defined as in the Paragraph 5.2 (on page 89), and let $K_3 = \operatorname{Im} \tau_3$. Then the sequence

$$S^3T^* \xrightarrow{\sigma_3(P_3)} (T^* \otimes T_0^*) \oplus (T^* \otimes \Lambda^2_0) \oplus (T^* \otimes \Lambda^2_{\mathcal H}) \xrightarrow{\tau_3} K_3 \longrightarrow 0$$

is exact.

Indeed, it is easy to see that $\tau_3 \circ \sigma_1(P_3) = 0$. On the other hand, one can check that Ker $\tau_{[J,A]}$ is defined by $\frac{(n-1)(n-2)}{2}$ equations, while Ker $\tau_{A'}$ is defined by $(n-1)$ equations and Ker τ_H by $(n-1)^2$ equations. Now these equations are independent and they are also independent of the equations which define Ker $\tau_2(P_2)$. Thus

$$\dim \operatorname{Ker} \tau_3(P_3) = \dim \operatorname{Ker} \tau_2(P_2) + \dim(T^* \otimes \Lambda^2_\alpha) - \frac{(n-1)(n-2)}{2} - (n-1) - (n-1)^2,$$

and so

$$\dim \operatorname{Ker} \tau_3(P_3) = \frac{4n^3 + 9n^2 + 5n - 6}{6} = \operatorname{rang} \sigma_1(P_3)$$

which shows that the sequence is exact.

Let ∇ be a linear connection on TM, and $p = j_2(E)_x$ a regular 2nd order formal solution of P_2. p can be lifted into a 3rd order solution if and only if $[\tau_3 \nabla(P_3 E)]_x = 0$, that is $(\tau_2, \tau_{[J,A]}, \tau_{A'}, \tau_H)\, [\nabla(P_3 E)]_x = 0$. Taking into account that $(\omega_E)_x = 0$, $(i_C \Omega_E)_x = 0$ and $(i_A \Omega_E)_x = 0$, and using the equation (5.18) we have

$$\tau_3[\nabla(P_3 E)]_x = \tau_2[\nabla(P_2 E)]_x = (0,\ 0,\ i_B \Omega_E,\ 0)_x.$$

Now

$$3R = [J, A] = [J, \lambda J + \alpha \otimes C] = d_J \lambda \wedge J + d_J \alpha \otimes C - \alpha \wedge J,$$

so

$$3(i_B \Omega_E)_x = (d_J \alpha)_x \wedge (i_C \Omega_E)_x$$

But $(i_A \Omega_E)_x = \alpha_x \wedge (i_C \Omega_E)_x = 0$ and $\alpha_x \neq 0$ thus, there exists $\rho_E \in \mathbb{R}$ such that

$$(i_C \Omega_E)_x = \rho_E \alpha_x. \tag{7.24}$$

So

$$3(i_B \Omega_E)_x = \rho_E (d_J \alpha \wedge \alpha)_x.$$

Noting that $\rho_E \neq 0$, because Ω_E is non-degenerated, we have $(i_B \Omega_E)_x = 0$ and therefore $\tau_3[\nabla(P_3 E)]_x = 0$, if and only if the condition (7.9) holds

Let us now compute the condition of compatibility given by $\tau_{[J,A]}$. We have

$$\tau_{[J,A]}[\nabla(P_3E)]_x = (d_J i_A \Omega_E)_x = (i_{[J,A]}\Omega_E)_x - (d_A d_J^2 E)_x = 3\,(i_h \Omega_E)_x,$$

so $\tau_{[J,A]}$ does not give a new condition: if $d_J\alpha \wedge \alpha = 0$, then the condition $\tau_{[J,A]}[\nabla(P_3E)]_x = 0$ is satisfied.

To compute the obstruction coming from the equation $\tau_{A'}[\nabla(P_3E)]_x = 0$, note that we have

$$\tau_{A'}[\nabla(P_3E)]_x = (\mathcal{L}_S i_A \Omega_E - d_A P_1 E)_x = (i_{A'}\Omega_E)_x.$$

Since $vS = \mu C$, we arrive at

$$\begin{aligned} A' &= v[S,A]h = v(\mathcal{L}_S\lambda)J + \lambda[S,J] - \mathcal{L}_S\alpha \otimes C + \alpha \otimes [S,C])\,h \\ &= \lambda' J + \alpha' \otimes C + \alpha \otimes vS = \lambda' J + (\alpha' + \mu\alpha) \otimes C, \end{aligned}$$

where we set

$$\lambda' = \mathcal{L}_S\lambda \qquad \text{and} \qquad \alpha' = h^*(\mathcal{L}_S\alpha).$$

Thus

$$i_{A'}\Omega = (\alpha' + \mu\alpha) \wedge i_C\Omega_E = \rho_E(\alpha' + \mu\alpha) \wedge \alpha = \rho_E\alpha' \wedge \alpha.$$

Hence $\tau_{A'}[\nabla(P_3E)]_x = 0$ if and only if the condition (7.10) of the Theorem is satisfied.

Finally let $X, Y \in \mathcal{H}$. We have

$$\begin{aligned} \tau_{\mathcal{H}}[\nabla(P_3E)]_x(X,Y) &= \frac{1}{2}\nabla(i_C\Omega_E)(JY,X,S) + \nabla\omega_E(JY,X) \\ &- \frac{\mu}{i_S\alpha}\nabla(i_A\Omega_E)(JY,S,X) - \frac{1}{i_S\alpha}\nabla(i_A\Omega_E)(hX,S,Y) = \frac{1}{2}JY(i_{\nabla_0 C}\Omega_E(X,S)) \\ &+ JY\omega_E(X) - \frac{\mu}{i_S\alpha}JY(i_A\Omega_E)(S,X) - \frac{1}{i_S\alpha}hX(i_A\Omega_E(S,Y)) \\ &= JY(i_h\Omega_E - \Omega_E)(X,S)) + d\omega_E(JY,X) - \mu JY(\Omega_E(C,X)) - hX(\Omega_E(C,Y)) \end{aligned}$$

at x. Now $i_C\Omega_E(X)_x = 0$ because $X \in \alpha_x^{\perp}$, then

$$\mu\, JY\,\Omega_E(C,X) = JY\,\Omega_E(\mu C,X) = JY\,\Omega_E(vS,X),$$

hence

$$\tau_{\mathcal{H}}[\nabla(P_3E)]_x(X,Y) = JY\,\Omega_E(X,S) + d\omega_E(JY,X) + hX\,\Omega_E(S,JY).$$

On the other hand $d\omega_E = \mathcal{L}_S \Omega_E$, so

$$d\omega_E(JY, X) = JY\,\Omega_E(S, X) - X\,\Omega_E(S, JY) - \Omega_E(S, [JY, X]),$$

hence

$$\tau_{\mathcal{H}}[\nabla(P_3 E)]_x(X, Y) = \Omega_E(S, [X, JY])_x.$$

Consider the basis $B = \{h_i, Jh_i\}_{i=1,\dots,n}$, where $h_i \in \mathcal{H}$ for $i = 1, \dots, n-1$ and $h_n := hS$. We have at $x \in TM$:

$$\Omega_E(S, h_i) = \Omega_E(vS, h_i) = \mu\Omega_E(C, h_i) = \frac{\mu}{i_S\alpha} i_A\Omega_E(S, h_i) - \frac{\mu\lambda}{i_S\alpha} i_J\Omega_E(S, h_i) = 0,$$
$$\Omega_E(S, Jh_i) = \Omega_E(hS, Jh_i) = \Omega_E(C, h_i) = \frac{1}{i_S\alpha} i_A\Omega_E(S, h_i) - \frac{\lambda}{i_S\alpha} i_J\Omega_E(S, h_i) = 0,$$

for $i = 1, \dots, n-1$. It shows that $\Omega_E(S, [X, JY])_x = \Omega_E(S, [X, JY]_C)_x$, where $[X, JY]_C$ denotes the component of the vector $[X, JY]$ on $C = Jh_n$ in the basis B. Therefore

$$\tau_{\mathcal{H}}[\nabla(P_3 E)]_x = 0 \quad \text{if and only if} \quad [X, JY]_C = 0$$

by Lemma 7.1. Now, writing the spectral decomposition of A, we can easily obtain the projection on to the eigenspace corresponding to the eigenvalue $\lambda + i_S\alpha$, i.e. the projection on to the distribution spanned by S and C:

$$\frac{1}{\lambda + i_S\alpha - \lambda}(A - \lambda I) - \frac{1}{i_S\alpha}(i_F\alpha \otimes C + \alpha \otimes hS).$$

Therefore the projection on to the space spanned by C is

$$\frac{1}{i_S\alpha} i_F\alpha \otimes C. \tag{7.25}$$

Thus

$$[X, JY]_C = \frac{1}{i_S\alpha}\alpha(F[X, JY]) \otimes C.$$

Let D be the Berwald connection associated to S. Taking into account that $X, Y \in \mathcal{H} = \alpha^{\perp} \cap T^h$ and that $\alpha(F[X, JY])_x$ depends only on the values of the vectors X and Y at x, we have

$$\alpha(F[X, JY])_x = \alpha(F[[h, JY], X])_x = \alpha(F D_{hX} JY)_x = D_{hX}\alpha(Y)_x$$

for every $Y \in \mathcal{H}$, which shows that

$$\tau_{\mathcal{H}}[\nabla(P_3 E)]_x = 0 \quad \text{if and only if} \quad (D_{hX}\alpha \wedge \alpha)_x = 0 \ \ \forall X \in \operatorname{Ker}\alpha.$$

So we have proved that a regular second order formal solution of P_3 can be lifted into a 3rd order formal solution if and only if the equations (7.9)–(7.11) hold. In order to prove the Theorem, we only need to prove that P_3 is involutive.

Step II.: *P_3 is involutive.*

Since $g_2(P_3) = g_2(P_1) \cap g_2(P_2) \cap g_2(P_4)$, an element $B \in S^2T^*$ is found in $g_2(P_3)$ if and only if the equations (7.1), (7.2) and

$$B(AX, JY) - B(AY, JX) = 0 \tag{7.26}$$

hold for any $X, Y \in T$. Let $\mathcal{B} = \{h_i, v_i\}_{i=1,\dots,n}$ be a basis of T_x with $h_i \in H_\alpha = T^h \cap \alpha^\perp$ for $i = 1, \dots, n-1$, $h_n = S$ and $Jh_i = v_i$. Set $a_{ij} := B(h_i, h_j)$, $b_{ij} := B(h_i, v_j)$ and $c_{ij} := B(v_i, v_j)$, first we will prove that B is found in $g_2(P_3)$ if and only if

$$\begin{cases} c_{ni} = 0, & i = 1, \dots, n-1, \\ b_{ni} = 0, & i = 1, \dots, n-1, \\ b_{nn} = \mu c_{nn}, & \\ b_{ij} = b_{ji}, & i, j = 1, \dots, n. \end{cases} \tag{7.27}$$

Indeed,

- using the equation (7.26) computed on $X = S$ and $Y = h_i$ for $i = 1, \dots, n-1$ we find

$$B(AS, v_i) - B(Ah_i, C) = \operatorname{tg}\alpha \, B(v_n, v_i) = (\operatorname{tg}\alpha)\, c_{ni}.$$

- If $i < n$, then using (7.1) we find

$$b_{ni} = B(S, v_i) - B(\nu S, v_i) = B(S, v_i) - \mu B(v_n, v_i) = B(S, v_i) = 0.$$

- For b_{nn} we have

$$b_{nn} = B(S, C) - B(\nu S, C) = B(S, v_n) - \mu B(v_n, v_n) = \mu c_{nn}.$$

Hence

$$\dim g_2(P_3) = \dim S^2T^* - n - (n-1) - \frac{n(n-1)}{2} = \frac{3n^2 - n + 2}{2}.$$

Let us now consider the basis $\tilde{\mathcal{B}} = \{e_i, v_j\}_{i,j=1,\dots,n}$, where

$$
\begin{aligned}
e_1 &= h_1 + v_n, \\
e_i &= h_i + (i-1)\,v_i, \qquad \text{for} \quad i = 2, \dots, n-1, \\
e_n &= h_n + \sum_{i=1}^{n} v_i .
\end{aligned}
$$

If $\bar a_{ij} = B(e_i, e_j)$, $\bar b_{ij} = B(e_i, v_j)$ and $c_{ij} = B(v_i, v_j)$ are the components of B, the block $[\bar b_{ij}]$ is

$$
\begin{bmatrix}
b_{11} & \cdot & b_{1,n-1} & c_{n,n} \\
b_{12} + c_{12} & & b_{2,n-1} + c_{2,n-1} & 0 \\
\vdots & & \vdots & \vdots \\
b_{1,n-1} + (n-1)c_{1,n-1} & & b_{n-1,n-1} + (n-1)c_{n-1,n-1} & 0 \\
\sum_{i=1}^{n-1} c_{i1} & & \sum_{i=1}^{n-1} c_{i,n-1} & c_{nn}
\end{bmatrix},
$$

and, of course, $\bar c_{ij} = c_{ij}$. Then $\bar c_{ij}$ can be expressed in terms of the block $(\bar b_{ij})$ in the following way:

$$
\begin{aligned}
\bar c_{in} &= 0, & 1 \le i < n; \\
\bar c_{nn} &= \bar b_{1n}; \\
\bar c_{ij} &= \frac{1}{(i-j)}\,(\bar b_{ij} - \bar b_{ji}), & 1 < i < j < n, \\
\bar c_{ii} &= \frac{1}{(i-1)}\,(\bar b_{ii} - \bar b_{1i}), & i = 2, \dots, (n-1), \\
\bar c_{11} &= -\sum_{k=1,\,k\neq i}^{n-i} \frac{1}{k-i}\,(\bar b_{kk} - \bar b_{ik}), & i = j < n.
\end{aligned}
$$

Therefore the elements of $g_2(P_3)$ are determined by the $\bar a_{ij}$ and the $\bar b_{ij}$. Thus

$$
\dim g_2(P_3)_{e_1 \dots e_k} = \tfrac{1}{2}(n-k)(n-k+1) + (n-1)(n-k).
$$

and

$$
\dim g_2(P_3)_{e_1, \dots, e_n, v_1 \dots v_k} = 0.
$$

for $k = 1, \ldots, n$, so

$$\begin{aligned}\dim g_2(P_3) &+ \sum_{k=1}^{n} \dim g_2(P_3)_{e_1 \ldots e_k} + \sum_{k=1}^{n} \dim g_2(P_3)_{e_1,\ldots,e_n,e_{n+1},\ldots,e_{n+k}} \\ &= \frac{3n^2 - n + 2}{2} + \sum_{k=1}^{n} \frac{(n-k)(n-k+1)}{2} + \sum_{k=1}^{n}(n-1)(n-k) \\ &= \frac{1}{6}(4n^3 + 3n^2 - n + 6) = \dim g_3(P_3),\end{aligned}$$

which shows that P_3 is involutive. The Theorem is proved. $\square$

Note that the Theorem holds for homogeneous and quadratic sprays.

Taking into account a result of Szenthe (cf. [Sze]) which states, that if a homogeneous (resp. quadratic) spray is variational, then there exists also a homogeneous (resp. quadratic) regular associated Lagrangian, we can state:

Theorem 7.3 *Let Γ be a homogeneous (resp. linear) connection on an analytical manifold. Locally there is a Finsler (resp. Riemann) structure with isotropic curvature so that the canonical (resp. Levi-Civita) connection is Γ if and only if the spray of Γ is isotropic, and the Douglas tensor satisfies the conditions of Theorem 7.2.*

7.2.2 *Atypical sprays*

When the spray is atypical, the distribution $\mathcal{D}$ spanned by S and C is non-integrable (cf. Proposition 7.1). This is equivalent to the fact that the distribution $\bar{\mathcal{D}}$ spanned by vS and C is 2-dimensional. As we will see, the study of the atypical case greatly depends on the degree of non-holonomy of $\bar{\mathcal{D}}$, that is on the length of the sequence $\bar{\mathcal{D}} \subset \bar{\mathcal{D}}^2 \subset \bar{\mathcal{D}}^3 \subset \ldots$, where $\bar{\mathcal{D}}^1 = \bar{\mathcal{D}}$, and $\bar{\mathcal{D}}^{k+1} := [\bar{\mathcal{D}}^k, \bar{\mathcal{D}}^k]$. In this last section we will study the case where the holonomy is *weak*, that is $\bar{\mathcal{D}}^2 = \bar{\mathcal{D}}$, or, in other words, $\mathcal{D}$ is integrable.

We recall that if $A = \lambda J + \alpha \otimes C$ with $i_S\alpha = 0$, then S is non-variational (cf. Corollary 7.1). Thus we can assume that $i_S\alpha \neq 0$. We shall prove the following

Theorem 7.4 *Let S be an isotropic atypical spray, with $A = \lambda J + \alpha \otimes C$ and suppose that the distribution $\bar{D} = \mathrm{Span}(C, vS)$ is integrable. Then S is variational if and only if*

$$
\begin{aligned}
\alpha \wedge \alpha' &\neq 0, \\
\alpha \wedge d_J \alpha &= 0, \\
\alpha'' \wedge \alpha + \frac{2\, i_{\nu S}\alpha}{i_S \alpha}\, \alpha' \wedge \alpha &= 0, \\
i_S \alpha' \, (d_J \alpha \wedge \alpha') &= 0, \\
D_{hX}\alpha \wedge \alpha &= 0, \quad \forall X \in \alpha^{\perp},
\end{aligned}
$$

where, for a scalar form β, we put forward

$$\beta' = h^*(\mathcal{L}_S \beta). \tag{7.28}$$

The proof of the theorem will be carried out in 4 steps.

STEP I: *First lift of the second order solutions of P_3.*

Lemma 7.2 *A 2nd order solution $(j_2 E)_x$ of P_3 at $x \in TM$ can be lifted into a 3rd order solution if and only if*

$$
\begin{aligned}
(i_A \Omega_E)_x &= 0, \\
(\alpha \wedge d_J \alpha)_x &= 0, \\
(\alpha \wedge d_J \alpha)'_x &= 0.
\end{aligned} \tag{7.29}
$$

Proof. We recall that $\mathrm{Ker}\,\sigma_3(P_3)$ is defined by the equations (7.13)–(7.15), that is, in an adapted base, by the equations (7.16) – (7.21). Note that the equations (7.21) are independent of the others, whereas some of the equations (7.20) are related to the others by

$$[\sigma_3(P_3)B](S, S, h_j) = B(S, C, v_j) = B(C, S, v_j) \overset{(7.17)}{=} 0,$$

for $j = 1, \ldots, n-1$. Thus

$$
\begin{aligned}
\dim \mathrm{Ker}\,\sigma_3(P_3) &= \dim \mathrm{Ker}\,\sigma_3(P_2) - \left[\frac{n(n-1)}{2} + (n-1) + \frac{n(n-1)}{2} \right] \\
&= \dim \mathrm{Ker}\,\sigma_3(P_2) - (n^2 - 1),
\end{aligned}
$$

and therefore

$$\operatorname{rank}\sigma_3(P_3) = \operatorname{rank}\sigma_3(P_2) + (n^2 - 1).$$

Let

$$(T^*\otimes T_v^*)\oplus(T^*\otimes\Lambda^2T_v^*)\oplus(T^*\otimes\Lambda^2_\alpha) \xrightarrow{\ \tau_3\ } K_2\oplus\Lambda^2T_v^*\oplus\Lambda^3T_v^*\oplus\Lambda^3T_v^*$$

be the morphism defined by $\tau_3 = (\bar{\tau}_3, \tau_{A'}, \eta_{[J,A]}, \eta_{[h,A]})$, where

$$\begin{aligned}
\bar{\tau}_3(B_S, B_\Gamma, B_A)(X,Y) &= \tau_2(B_S, B_\Gamma)(X,Y),\\
\tau_{A'}(B_S, B_\Gamma, B_A)(X,Y) &= B_A(S,X,Y) - (B_S(AX,Y) - B_S(AY,X)),
\end{aligned}$$

$$\eta_{[J,A]}(B_S, B_\Gamma, B_A)(X,Y,Z) = B_A(JX,Y,Z) + B_A(JY,Z,X) + B_A(JZ,X,Y),$$

$$\eta_{[h,A]}(B_S, B_\Gamma, B_A)(X,Y,Z) = \sum_{X,Y,Z}^{\text{cycl}} B_A(hX,Y,Z) + \frac{1}{2}\sum_{X,Y,Z}^{\text{cycl}} B_\Gamma(AX,Y,Z).$$

We will prove that the sequence

$$S^3T^* \xrightarrow{\ \sigma_3(P_3)\ } T^*\otimes T_v^*\oplus T^*\otimes\Lambda^2_v\oplus T^*\otimes\Lambda^2_\alpha \xrightarrow{\ \tau_3\ } K_3 \longrightarrow 0$$

is exact, where $K_3 := \operatorname{Im}\tau_3$.

It is easy to show that $\operatorname{Im}\sigma_3(P_3)\subset\operatorname{Ker}\tau_3$. On the other hand, the equations $\tau_{A'} = 0$, $\eta_{[J,A]} = 0$, and $\eta_{[h,A]} = 0$ yield

$$(n-1) + \tfrac{1}{2}(n-1)(n-2) + \tfrac{1}{2}(n-1)(n-2)$$

equations which are independent of the system $\tau_2 = 0$.

Indeed, let us consider an adapted base $B = \{h_i, v_j\}$ with $h_i\in\mathcal{H}$ for $i = 1,\dots,n-1$, $h_n := hS$ and $v_j = Jh_j$.

i) Taking $X = h_n$ and $Y = h_i$ in the equation $\tau_{A'} = 0$ we obtain $(n-1)$ new equations independent of the system $\tau_2 = 0$. There is no other independent equation of $\tau_2 = 0$: if $X, Y\in\mathcal{H}_\alpha$, then $B_A(S,X,Y) = 0$, since $B_A\in T^*\otimes\Lambda_\alpha$. However, since $A|_{\mathcal{H}_\alpha} = \lambda J$, the equation $\tau_{A'}(B_S, B_\Gamma, B_A)(X,Y) = 0$ is related to the equations $\tau_\Gamma = 0$ and then to $\tau_2 = 0$, because

$$\tau_{A'}(B_S, B_\Gamma, B_A)(X,Y) = B_S(AX,Y) - B_S(AY,X) = \lambda\tau_\Gamma(B_S)(X,Y).$$

ii) On the other hand, once again using the fact that $A|_{\mathcal{H}_\alpha} = \lambda J$, the equation $\eta_{[J,A]} = 0$ restricted to $\mathcal{H}_\alpha$ does not give new equations with respect to the system $\tau_2 = 0$. It gives independent equations when it is computed on the vectors S, h_i and h_j with $1\le i<j<n$, and then $\eta_{[J,A]} = 0$ adds $\frac{1}{2}(n-1)(n-2)$ new equations to the system $(\tau_2, \tau_{A'}) = 0$.

iii) A similar argument shows that $\tau_{[h,A]} = 0$ gives new equations if one of the arguments is S, and the other two vectors are h_i, h_j for $1 \le i < j < n$. Therefore $\tau_{[h,A]} = 0$ gives $\frac{1}{2}(n-1)(n-2)$ new equations.

Note that these equations are independent, because the $(n-1)$ components $B_A(S, S, h_i)$ are pivot terms for the equations $\tau_{A'} = 0$ and $B_A(v_i, S, h_j)$ and $B_A(h_i, S, h_j)$ are pivot terms for the equation $\tau_{[J,A]} = 0$, and $\tau_{[h,A]} = 0$ respectively $i < j$, $i, j = 1, \ldots, n-1$. Therefore

$$\begin{aligned}\dim \operatorname{Ker} \tau_3 &= \dim \operatorname{Ker} \tilde{\tau}_3 - [(n-1) + (n-1)(n-2)] \\ &= \dim \operatorname{Ker} \tau_3 + \dim\left(T^* \oplus \Lambda_0^2\right) - (n-1)^2 = \operatorname{rank} \sigma_3(P_3),\end{aligned}$$

which proves that the sequence is exact.

We can compute the conditions of compatibility for P_3. Let ∇ be an arbitrary linear connection on TM and $j_2(E)_x$ a 2nd order regular formal solution of P_3 at x. $j_2(E)_x$ can be lifted into a 3rd order solution if and only if $(\tau_2 \nabla(P_3 E))_x = 0$. Note that $(\omega_E)_x = 0$, $(i_C \Omega_E)_x = 0$, and $(i_A \Omega_E)_x = 0$. From $(i_A \Omega_E)_x = 0$ we get $(\alpha \wedge i_C \Omega_E)_x = 0$; thus there exists $\rho_1 \neq 0$ such that

$$\rho_1 \alpha_x = (i_C \Omega_E)_x. \tag{7.30}$$

Let us compute now the compatibility conditions. We have:

- $\tilde{\tau}_2[\nabla(P_3 E)]_x = \tau_2[\nabla(P_2 E)]_x = 0$,
- $\tau_{A'}[\nabla(P_3 E)]_x = \mathcal{L}_C i_A \Omega_E - d_A P_1 E = (i_{A'} \Omega_E)_x$,
- (cf. section 7.2.1)
  $$\begin{aligned}\tau_{[J,A]}[\nabla(P_3 E)]_x &= (i_{[J,A]} \Omega_E)_x = (i_{3JR} \Omega_E)_x = 3(d_J \alpha \wedge i_C \Omega_E)_x \\ &= 3\rho_E (d_J \alpha \wedge \alpha)_x,\end{aligned}$$
- $\tau_{[h,A]}[\nabla(P_3 E)]_x = \frac{1}{2}(d_A i_F \Omega_E)_x - (d_h i_A \Omega_E)_x = [d_A, d_h] d_J E_x = d_{[A,h]} d_J E_x$.

  Now $[h, A]$ is semi-basic, hence $d_{[A,h]} d_J E = i_{[A,h]} \Omega_E$, and

$$\begin{aligned}[h, A] &= [h, [h, S] - h[h, S]] = [h, [h, S]] + [h, F + J] = \\ &= [h, [h, S]] + [h, F] + [h, J] \overset{(*)}{=} -[R, S] + FR - RKF = R'.\end{aligned} \tag{7.31}$$

Thus

$$\tau_{[h,A]}[\nabla(P_3 E)]_x = i_{R'} \Omega_E.$$

Now $A = \lambda J + \alpha \otimes C$, so

$$R' = h^* v[S, R] = h^* v((d_J\lambda - \alpha)' \wedge J + (d_J\alpha)' \otimes C + d_J\alpha \otimes vS) \quad (7.32)$$

and then

$$\begin{aligned} i_{R'}\Omega_E &= (d_J\alpha)' \wedge i_C\Omega_E + d_J \wedge i_{vS}\Omega_E \\ &= (d_J\alpha)' \wedge \rho_1\alpha + d_J\alpha \wedge (\rho_1\alpha' + \rho_2\alpha) \overset{(7.29)}{=} \quad (7.33) \\ &= \rho_1 \left[(d_J\alpha)' \wedge \alpha + (d_J\alpha) \wedge \alpha'\right] = \rho_1 (d_J\alpha \wedge \alpha)' \end{aligned}$$

Thus

$$\eta_{6,4}[\mathcal{V}(P^1_3 E)]_x = \rho_1 (d_J\alpha \wedge \alpha)'.$$

Therefore $\eta_3[\mathcal{V}(P^1_3 E)]_x = 0$ if and only if $(i_{A'}\Omega_E)_x = 0$, $(d_J\alpha \wedge \alpha)_x = 0$, and $(d_J\alpha \wedge \alpha)'_x = 0$, which proves the Lemma.

◇

Note that

$$A' = h^*(v[S, A]) = h^* v((\mathcal{L}_S\lambda) J + \lambda\Gamma + (\mathcal{L}_S\alpha) \otimes C + \alpha \otimes [S, C]).$$

Since $h^* v(\Gamma) = v \circ h = 0$ and $v[S, C] = v(-\Gamma(S)) = vS$, we have

$$A' = \lambda' J + \alpha' \otimes C + \alpha \otimes vS. \quad (7.34)$$

Thus

$$i_{A'}\Omega_E = \alpha' \wedge i_C\Omega_E + \alpha \wedge i_{vS}\Omega_E.$$

Therefore the condition $i_{A'}\Omega = 0$ is equivalent to the equation

$$\alpha' \wedge i_C\Omega_E + \alpha \wedge i_{vS}\Omega_E = 0. \quad (7.35)$$

Since, by hypothesis, vS is not proportional to C, this condition is a new equation on $j_2(E)_x$ which we will introduce into the system.

In order to simplify the computations, we put forward $\tilde\alpha' := \alpha' - \frac{i_S\alpha'}{i_S\alpha}\alpha$, and

$$\tilde A' := \alpha \otimes vS + \tilde\alpha' \otimes C. \quad (7.36)$$

We have

$$\tilde A' = A' + \mu_1 A + \mu_2 J. \quad (7.37)$$

with $\mu_1 = \frac{i_S\alpha'}{i_S\alpha}$ and $\mu_2 = \frac{i_S\alpha'}{i_S\alpha}\lambda - \lambda'$. So the equation $i_{A'}\Omega_E = 0$ is equivalent to $i_{\tilde{A}}.\Omega_E = 0$.

Now we must study the integrability of the differential operator

$$P_4 = (P_3,\ P_{\tilde{A}'})\ :\quad C^{\infty}(TM) \longrightarrow Sec\,(T_1^* \oplus \Lambda_4^2 \oplus \Lambda_\alpha^2 \oplus \Lambda_{\alpha,\alpha'}^2).$$

where

$$\Lambda^2_{\alpha,\alpha'} := \left\{\Theta \in \Lambda^2 T_v^* \mid \exists\, \theta, \theta' \in T_v^* :\ \Theta = \alpha \wedge \theta + \alpha' \wedge \theta'\right\},$$

and

$$P_{\tilde{A}'} := i_{\tilde{A}'} dd_J\ :\qquad C^{\infty}(TM) \longrightarrow Sec\,\Lambda^2_{\alpha,\alpha'}.$$

STEP 1 *First lift of the second order solutions of P_4*

Remark. If S is a variational atypical spray in the neighborhood of $x \in TM \setminus \{0\}$, then

$$(\alpha \wedge \alpha')_x \neq 0. \tag{7.38}$$

Indeed, if S is variational and E is a regular Lagrangian associated to S, then E satisfies the condition of compatibility of P_3: $i_{A'}\Omega_E = 0$. Now $A' = \lambda J - \alpha' \otimes C + \alpha \otimes vS$, so

$$0 = i_{A'}\Omega_E = \alpha' \wedge i_C\Omega + \alpha \wedge i_{vS}\Omega.$$

If $(\alpha \wedge \alpha')_x = 0$, then $\alpha \wedge i_{vS}\Omega_x = 0$; so $i_{vS}\Omega_x$ is proportional to α_x and hence to $i_C\Omega_x$, that is $(vS)_x$ is proportional to C_x. But this is excluded by hypothesis.

From now on we shall suppose that $\alpha \wedge \alpha' \neq 0$.

The following Lemma highlights the role of the graded Lie algebra $\mathcal{A}_S$ spanned by J, and A, containing A', ..., R, R', ... in the inverse problem [1].

[1] For the definition of $\mathcal{A}_S$, see Section 4.2

Lemma 7.3 *Let S be an atypical isotropic spray. A 2nd order solution $(j_2E)_x$ at $x \neq 0$ of P_1 can be lifted into a 3rd order solution if and only if*

$$
\begin{aligned}
i_H \Omega_E &= 0, \\
i_{H'} \Omega_E &= 0, \\
i_{H''} \Omega_E &= 0, \\
i_{[\dot A', \dot A']} \Omega_E &= 0, \qquad (7.39)\\
i_{[\dot A, \dot A']} \Omega_E &= 0, \\
i_{\ddot A'} \Omega_E &= 0, \\
D\alpha|_x &= 0,
\end{aligned}
$$

where D denotes the Berwald connection

Proof. The symbol $\sigma_2(P_{\dot A'}) : S^2T^* \longrightarrow \Lambda^2_1$ of $P_{\dot A'}$ is given by

$$(\sigma_2(P_{\dot A'})B_2)(X,Y) = B_2(\dot A'X, JY) - B_2(\dot A'X, JY).$$

and the symbol of the first prolongation $\sigma_3(P_{\dot A'}) : S^3T^* \longrightarrow T^* \otimes \Lambda^2_1$ is

$$(\sigma_3(P_{\dot A'})B_3)(X,Y,Z) = B_3(X, \dot A'Y, JZ) - B_3(X, \dot A'Z, JY)),$$

where $B_i \in S^iT^*$, $i = 2, 3$, and $X, Y, Z \in T$. Thus $g_3(P_1)$, that is $\operatorname{Ker}\sigma_3(P_1) \cap \operatorname{Ker}\sigma_3(P_{\dot A'})$, is characterized by the equations (7.13) – (7.15) and

$$B(X, \dot A'Y, JZ) - B(X, \dot A'Z, JY) = 0, \qquad (7.40)$$

$X, Y, Z \in T$. Let us set

$$\mathcal{H}_{\alpha,\alpha'} := (\dot\alpha')^\perp \cap \mathcal{H}_\alpha,$$

and consider the base $\mathcal{B} = (h_i, v_i)$ where $h_i \in \mathcal{H}_{\alpha,\alpha'}$ for $i = 2, \ldots, n-1$, $h_n = S$ and $v_i = Jh_i$. In this basis $\dot\alpha'(h_1) \neq 0$ if and only if $\dot\alpha' \neq 0$, and the equation (7.40) gives the following system:

$$\begin{cases} (i_{h_1}\dot\alpha')\,B(h_i, C, v_j) = 0, \\ (i_{h_1}\dot\alpha')\,B(v_i, C, v_j) = 0, \\ (i_S\alpha)\,B(h_i, vS, v_j) = 0, \\ (i_S\alpha)\,B(v_i, vS, v_j) = 0, \\ (i_S\alpha)\,B(h_i, vS, v_1) - (i_{h_1}\dot\alpha')\,B(h_i, C, C) = 0, \\ (i_S\alpha)\,B(v_i, vS, v_1) - (i_{h_1}\dot\alpha')\,B(v_i, C, C) = 0, \end{cases}$$

where $i = 1, \ldots, n$, and $j = 2, \ldots, n-1$. It is not difficult to verify that among these equations only

$$\begin{aligned} &(i_S\alpha)\,B(h_i, vS, v_j) = 0, \\ &(i_S\alpha)\,B(h_1, vS, v_1) + (i_{h_1}\dot\alpha')\,B(h_1, C, C) = 0, \\ &(i_S\alpha)\,B(v_1, vS, v_1) + (i_{h_1}\dot\alpha')\,B(v_1, C, C) = 0, \\ &(i_S\alpha)\,B(v_n, vS, v_1) + (i_{h_1}\dot\alpha')\,B(v_n, C, C) = 0, \end{aligned}$$

for $i = 1, \ldots, n$, and $j = 2, \ldots, n-1$, $i \le j$, are independent of the system (7.13) – (7.15). It follows that

$$\operatorname{rank}\sigma_1(F_4) = \operatorname{rank}\sigma_1(F_3) + \tfrac{1}{2}(n-1)(n+2). \tag{7.41}$$

Let $\tau_4 := (\tau_S, \tau_{[J,\dot A']}, \tau_{\dot A',\dot A'}, \tau_{[A,\dot A']}, \tau_{\dot A'}, \eta_1, \eta_2, \eta_3)$ be the morphism defined for $(B_S, B_\Gamma, B_A, B_{\dot A'}) \in T^* \otimes (T_\Gamma^* \oplus \Lambda^2 T_S^* \oplus \Lambda^2_A \oplus \Lambda^2_{\dot A'})$ by

$$\tau_{[J,\dot A']}(B_S, B_\Gamma, B_A, B_{\dot A'})(X, Y, Z) = \sum_{X,Y,Z}^{\text{cycl}} B_{\dot A'}(JX, Y, Z),$$

$$\tau_{[\dot A',\dot A']}(B_S, B_\Gamma, B_A, B_{\dot A'})(X, Y, Z) = \sum_{X,Y,Z}^{\text{cycl}} B_{\dot A'}(\dot A'X, Y, Z),$$

$$\tau_{[A,\dot A']}(B_S, B_\Gamma, B_A, B_{\dot A'})(X, Y, Z) = \frac{1}{2}\Big(\sum_{X,Y,Z}^{\text{cycl}} B_{\Gamma}(AX, Y, Z)\Big) + \sum_{X,Y,Z}^{\text{cycl}} B_{\dot A'}(AX, Y, Z)$$

and

$$\begin{aligned} \tau_{\dot A'}(B_S, B_\Gamma, B_A, B_{\dot A'})(X, Y) &= B_S(\dot A'X, Y) - B_S(\dot A'Y, X) - B_{\dot A'}(S, X, Y), \\ \eta_1(B_S, B_\Gamma, B_A, B_{\dot A'})(V, W) &= (i_{h_1}\dot\alpha')\,B_A(V, S, W) - (i_S\alpha)\,B_{\dot A'}(V, h_1, W), \\ \eta_2(B_S, B_\Gamma, B_A, B_{\dot A'})(V, W) &= (i_{h_1}\dot\alpha')\,B_A(JV, S, W) - (i_S\alpha)\,B_{\dot A'}(JV, h_1, W), \\ \eta_3(B_S, B_\Gamma, B_A, B_{\dot A'})(V, W) &= B_\Gamma(JV, S, W) - B_{\dot A'}(W, S, V) \\ &\quad - \tfrac{1}{2}\big((i_S\alpha)\,B_\Gamma(JV, W, S) - (i_S\alpha)\,B_S(JV, W)\big). \end{aligned}$$

for $X, Y, Z \in T$, $V \in \mathcal{H}_a$ and $W \in \mathcal{H}_{a,\hat{A}'}$.

We shall prove that the sequence

$$S^3 T^* \xrightarrow{\sigma_3(P_4)} T^* \otimes (T_1^* \oplus \Lambda^2 T_0^* \oplus \Lambda^2_a \oplus \Lambda^2_{a,\hat{A}'}) \xrightarrow{\tau_4} K_4 \to 0$$

where $K_4 := \operatorname{Im} \tau_4$, is exact.

It easy to check that $\operatorname{Im} \sigma_3(P_4) \subset \operatorname{Ker} \tau_4$. On the other hand a computation of the number of the equations of $\tau_4 = 0$ independent of the equations of $\tau_3 = 0$ gives:

(1) Using the fact that $\hat{A}'|_{\mathcal{H}_{a,\hat{A}'}} = 0$, we find that $\tau_{[J,\hat{A}']} = 0$ (resp. $\eta_{[h,A]} = 0$) does not give new equations if $X, Y, Z \in \mathcal{H}_{a,\hat{A}'}$, and it gives $\frac{1}{2}(n-2)(n-3)$ new equations for $X = h_i$, $Y = h_j$, $1 < i < j < n$ and $Z = S$ or $Z = h_1$, and $n-2$ equations for $X = h_i$, $1 < i < n$, $Y = h_1$ and $Z = S$.

(2) $\tau_{[\hat{A}',\hat{A}']} = 0$ gives new equations only for $X = h_1$, $Y = S$ and $Z = h_i$, $1 < i < n$; then it gives $n-2$ new equations.

(3) $\tau_{\hat{A}'} = 0$ gives new equations only when one of the vectors is found in $\mathcal{H}_{a,\hat{A}'}$ and the other is S or h_1, so we obtain $2(n-2)$ equations. If the vectors are S and h_1 it again gives one new equation.

(4) In order to find the number of equations given by the η_i, $i = 1, 2, 3$, note that we have, modulo $\tau_3 = 0$

$$\begin{aligned}
\operatorname{Antisym} \eta_1\big|_{\mathcal{H}_{a,\hat{A}'}} &= \tau_{[h,\hat{A}']}\big|_{\mathcal{H}_a}, \\
\operatorname{Antisym} \eta_2\big|_{\mathcal{H}_{a,\hat{A}'}} &= \tau_{[J,\hat{A}']}\big|_{\text{[illegible]}\ \mathcal{H}_{a,\hat{A}'}}, \\
\operatorname{Antisym} \eta_3\big|_{\mathcal{H}_{a,\hat{A}'}} &= \tau_{[J,\hat{A}']}\big|_{\mathcal{H}_a};
\end{aligned}$$

hence η_1 and η_2 give $(n-2)(n-1) + (n-2)$ new equations and η_3 gives $\frac{1}{2}(n-2)(n-1) + 2(n-2)$ new equations.

It is not difficult to check that these equations are independent, so we arrive at

$$\begin{aligned}
\dim \operatorname{Ker} \tau_4 &= \dim \operatorname{Ker} \tau_3 + \dim (T^* \otimes \Lambda^2_{a,\hat{A}'}) - \tfrac{1}{2}(7n^2 - 11n - 4) \\
&= \operatorname{rank} \sigma_3(P_3) + \tfrac{1}{2}(n^2 - n + 4) = \operatorname{rank} \sigma_3(P_4),
\end{aligned}$$

which proves that the sequence is exact.

Now we can compute the compatibility conditions for P_4. Let ∇ be a linear connection on TM and $j_2(E)_x$ a regular 2nd order solution of P_4 at $x \in TM \setminus \{0\}$. Then the equations $\omega_E = 0$, $i_\Gamma \Omega_E = 0$, $i_A \Omega_E = 0$, $i_{\dot{A}} \Omega_E = 0$, and hence $i_{\ddot{A}} \Omega_E = 0$, hold at x. Let us compute $[\tau_4 \nabla(P_4 E)]_x = 0$:

(1) $\tau_3[\nabla(P_4 E)]_x = \tau_3[\nabla(P_3 E)]_x = [0, 0, i_H \Omega_E, i_{H'} \Omega_E]_x$.

(2) $\tau_{[J,\dot{A}]}[\nabla(P_4 E)]_x = (d_J i_{\dot{A}} \Omega_E)_x = (i_{[J,\dot{A}]} \Omega_E)_x$,
since $(d_{\dot{A}} d_J \Omega_E)_x = 0$.

(3) $\tau_{[h,\dot{A}]}[\nabla(P_4 E)]_x = (d_h i_{\dot{A}} \Omega_E + \frac{1}{2} d_{\dot{A}} i_\Gamma \Omega)_x - (d_h d_{\dot{A}} E)_x + (d_{\dot{A}} d_h d_J E)_x$
$= (d_{[h,\dot{A}]} d_J E)_x = (i_{[h,\dot{A}]} \Omega_E)_x$
Now $h - \frac{1}{2}[J, \Gamma] = 0$, and so, by (7.37), we arrive at

$$
\begin{aligned}
[h, \dot{A}'] &= [h, A'] + (d_h \mu_1) \wedge A + d\mu_1 \wedge hA + \mu_1 [h, A] + (d_h \mu_2) \wedge J \\
&\quad + d\mu_2 \wedge hJ + \mu_2 [h, J] = [h, A'] + (d_h \mu_1) \wedge A + \mu_1 [h, A] + (d_h \mu_2) \wedge J,
\end{aligned}
$$

and therefore $i_{[h,\dot{A}]} \Omega_E = i_{[h,A']} \Omega_E$. On the other hand

$$
[h, A'] = [h, A]' - [h', A] + R \bar{\wedge} FA - AF,
$$

so

$$
i_{[h,\dot{A}]} \Omega_E = i_{H'} \Omega_E - i_{[A,A]} \Omega_E.
$$

Let us first consider the term $i_{[A,A]} \Omega_x$. We have

$$
\begin{aligned}
[A, A] &= [\lambda J, \lambda J] + 2[\lambda J, \alpha \otimes C] + [\alpha \otimes C, \alpha \otimes C] = 2\lambda (d_J \lambda) \wedge J \\
&\quad + 2\big(\lambda (d_J \alpha) \otimes C + (\mathcal{L}_C \lambda) \alpha \wedge J - \lambda \alpha \wedge J\big) + \mathcal{L}_S (\alpha \wedge \alpha) \otimes C,
\end{aligned}
$$

so

$$
i_{[A,A]} \Omega_E = 2\lambda (d_J \alpha) \wedge i_C \Omega \overset{(7.30)}{=} 2\rho_1 \lambda (d_J \alpha) \wedge \alpha \overset{(7.28)}{=} 0,
$$

and thus

$$
\tau_{[h,\dot{A}]}[\nabla(P_4 E)]_x = (i_{[h,\dot{A}]} \Omega_E)_x = (i_{H'} \Omega_E)_x.
$$

(4) $\tau_{[\dot{A},\dot{A}]}[\nabla(P_4 E)]_x = (d_{\dot{A}} i_{\dot{A}} \Omega_E)_x = (d_{\dot{A}}^2 d_J E)_x = (d_{[\dot{A},\dot{A}]} d_J E)_x$
$= i_{[\dot{A},\dot{A}]} \Omega_{E,x}$

(5) $\tau_{\dot{A}''}[\nabla(P_4E)]_x = (d_{\dot{A}''}P_1E - \mathcal{L}_S i_{\dot{A}'}\Omega_E)_x = (i_{(\dot{A}')'}\Omega_E)_x$,

but

$$(\dot{A}')' = [S, \dot{A}'] \overset{(7.38)}{=} [S, A'] + (\mathcal{L}_S\mu_1)A + \mu_1[S, A] + \mathcal{L}_S\mu_2 J + \mu_2[S, J],$$

hence $i_{(\dot{A}')'}\Omega_E = i_{A''}\Omega_E$, so

$$\tau_{\dot{A}''}[\nabla(P_4E)]_x = i_{A''}\Omega_{E\,x}.$$

(6) In order to compute the obstructions given by η_i, $i = 1, 2, 3$, let us take $X \in \mathcal{H}$, $Y \in \mathcal{H}_{\alpha,\dot{\alpha}'}$ and $Z \in T$. We have at x:

$$\begin{aligned}\eta_1\nabla(P_4E) &= \dot{\alpha}'(h_1)X(i_A\Omega_E(S,Y)) - \alpha(S)Xi_{\dot{A}'}(\Omega(h_1, Y))\\ &= \alpha(S)\,\dot{\alpha}'(h_1)\,X(i_C\Omega(Y)) - i_C\Omega(Y)) = 0,\end{aligned}$$

$$\begin{aligned}\eta_2\nabla(P_4E) &= JXi_A\Omega_E(S,Y) \quad Yi_{\dot{A}'}\Omega_E(S,X) - \tfrac{1}{2}i_S\alpha\, JXi_Y\Omega_E(Y,S)\\ &- \alpha(S)JX\omega_E(Y) = \alpha(S)\{JX\Omega(vS,Y) + Y\Omega_E(C,X)\\ &+ JX\Omega(vS,Y) + d\omega_E(JX,Y)\} - \Omega_E(S,[JX,Y])\end{aligned}$$

$$\begin{aligned}\eta_3\nabla(P_4E) &= \dot{\alpha}'(h_1)JZ(i_A\Omega_E(S,Y)) - \alpha(S)JZi_{\dot{A}'}(\Omega(h_1, Y)) =\\ &= \alpha(S)\,\dot{\alpha}'(h_1)JZ(i_C\Omega_E(Y)) - i_C\Omega(Y)) = 0.\end{aligned}$$

As we already computed in section 7.2, we find

$$\eta_2[\nabla(P_4E)]_x = 0 \text{ if and only if } (D_{hX}\alpha \wedge \alpha)_x = 0 \ \ \forall X \in \operatorname{Ker}\alpha.$$

So the condition $[\tau_4\nabla(P_4E)]_x = 0$ is equivalent to the system (7.39). ☐

Step III: *Expression of the compatibility conditions (7.39) in terms of the spray*

We shall now prove that the conditions (7.39) can be expressed in terms of the spray without the 2nd order solution (only in this case every 2nd order solution can be lifted into a 3rd order solution.)

Let $j_2(E)_x$ be a 2nd order solution of P_4 at x. Since $(i_{A'}\Omega_E)_x = 0$, there exists $\rho_2 \in \mathbb{R}$ such that

$$(i_{vS}\Omega_E)_x = \rho_1\dot{\alpha}'_x + \rho_2\alpha_x. \tag{7.42}$$

Therefore we also have

$$(i_{vS}\Omega_E)_x = \rho_1\alpha'_x + \bar{\rho}_2\alpha_x. \tag{7.43}$$

where $\tilde{\rho}_2 = \rho_2 - \rho_1 \frac{i_S\alpha'}{i_S\alpha}$.

- As we have already showed (see page 187), the condition $i_E\Omega_x = 0$ is equivalent to $\alpha_x \wedge (d_J\alpha)_x = 0$, and $i_{E'}\Omega_x = 0$ is equivalent to $(\alpha\wedge d_J\alpha)'_x = 0$

- From (7.34) we have

$$\begin{aligned} A'' &= h^* v[S, A'] = \lambda'' J + \alpha' \otimes C + 2\alpha' \otimes vS + \alpha \otimes [S, vS] \\ &= \lambda'' J + (\alpha'' + (\lambda + i_S\alpha)\alpha)\otimes C + 2\alpha' \otimes vS \end{aligned} \tag{7.44}$$

Thus, at x :

$$\begin{aligned} i_{A''}\Omega_E &= (\alpha'' + (\lambda + i_S\alpha)\alpha)\wedge i_C\Omega_E + 2\alpha' \wedge i_{vS}\Omega \overset{(7.39)}{=} \\ &- \rho_1\Big(\alpha'' \wedge \alpha + \frac{2\rho_2}{\rho_1}\alpha' \wedge \alpha\Big) \end{aligned}$$

This expression shows that if S is variational, then

$$\alpha \wedge \alpha' \wedge \alpha'' = 0. \tag{7.45}$$

Let us suppose that this condition is satisfied. Then there exist $a, b \in \mathbb{R}$ so that $\alpha'' = a\alpha_x - b\alpha'$. We obtain at x :

$$i_{A''}\Omega_E = \rho_1\Big(\alpha'' \wedge \alpha + \frac{2\rho_2}{\rho_1}\alpha' \wedge \alpha\Big) = \rho_1\Big(b + 2\frac{\rho_2}{\rho_1}\Big)\alpha' \wedge \alpha.$$

Since $\rho_1 \neq 0$, $i_{A''}\Omega_{E_x} = 0$ if and only if

$$b + 2\,\frac{\rho_2}{\rho_1} = 0. \tag{7.46}$$

If vS_C and vS_{JH} denote the components of vS in the splitting $T_v = \mathrm{Span}(C) \oplus JH$, then we have

$$\begin{aligned} i_{vS}\Omega(S) &= i_{vS_C}\Omega(S) + i_{vS_{JH}}\Omega(S) = \xi_C^{vS} i_C\Omega(S) + \Omega(vS_{JH}, S) \overset{i_v\Omega=0}{=} \\ &= \xi_C^{vS} i_C\Omega(S) + i_C\Omega(vS_{JH}) \overset{(7.39)}{=} \xi_C^{vS}\rho_1\alpha(S), \end{aligned}$$

because $\alpha|_x = 0$. On the other hand

$$i_{vS}\Omega(S) = \rho_1 i_S\alpha' + \rho_2 i_S\alpha = \tilde{\rho}_2 i_S\alpha$$

by (7.42). Thus

$$\xi_C^{vS} = \frac{\tilde{\rho}_2}{\rho_1}. \tag{7.47}$$

and therefore $b = -2\,\xi_C^{vS}$. Since $\xi_C^{vS} = \frac{\alpha(FS)}{\alpha(S)}$, the condition of compatibility $i_{A''}\Omega = 0$ is equivalent to

$$\alpha'' \wedge \alpha + \frac{2 i_{FS}\alpha}{i_S\alpha}\alpha' \wedge \alpha = 0. \tag{7.48}$$

- From (7.36) we have

$$\begin{aligned}[J, \dot{A}'] = [J, A'] + d\mu_1 \wedge JA + \mu_1 [J, A] + (d_J\mu_2) \wedge J + d\mu_2 \wedge J^2 \\ + \mu_2 [J, J] = [J, A'] + (d_J\mu_1) \wedge A + \mu_1 [J, A] + (d_J\mu_2) \wedge J,\end{aligned}$$

thus

$$i_{[J,\dot{A}']}\Omega_E = i_{[J,A']}\Omega_E + \mu_1 i_{[J,A]}\Omega_E + (d_J\mu_1) \wedge i_A\Omega_E, \tag{7.49}$$

and then

$$i_{[J,\dot{A}']}[\nabla(P_1 E)]_x = i_{[J,A']}\Omega_E = i_{[J,A']}\Omega_E \overset{(7.31)}{=} 4\, i_{H'}\Omega_E.$$

Using the computation (7.33), we find that $i_{[J,\dot{A}']}[\nabla(P_1 E)]_x = 0$ if and only if $(d_J\alpha \wedge \alpha)_x = 0$.

- Let us now consider the condition $i_{H''}\Omega_E = 0$. We have

$$R'' = h^* v[S, R'] = h^* v\Big(\mathcal{L}_S\big((d_J\lambda)' + \alpha'\big)\wedge J + \mathcal{L}_S(d_J\alpha)' \otimes C$$
$$+ \big((d_J\lambda)' + \alpha'\big) \wedge [S, J] + \mathcal{L}_S(d_J\alpha) \otimes [S, C] + \mathcal{L}_S d_J\alpha \otimes vS + d_J \otimes [S, vS]\Big).$$

But

$$v[S, vS] = v[hS, S] = v[h, S](S) = v[h, S](hS) - A(S) = (\lambda + i_S\alpha)\,C,$$

and

$$v[S, C] = v(-\Gamma(S)) = v(-hS + vS) = vS,$$

hence

$$R'' = \big((d_J\lambda)'' + \alpha''\big)\wedge J + \big((d_J\alpha)'' + (\alpha(S) + \lambda)\, d_J\alpha\big) \otimes C + 2(d_J\alpha)' \otimes vS.$$

Therefore we have at x :

$$i_{R''}\Omega_E = \big((d_J\alpha)'' + (i_S\alpha + \lambda)\, d_J\alpha\big) \wedge i_C\Omega_E + 2(d_J\alpha)' \wedge i_{vS}\Omega_E.$$

Using (7.30) and (7.43) we arrive at

$$\begin{aligned} i_{R''}\Omega_E &= \rho_1\left(\left((d_J\alpha)'' + (i_S\alpha + \lambda)d_J\alpha\right)\wedge\alpha + 2(d_J\alpha)'\wedge\alpha'\right) + 2\dot\rho_1(d_J\alpha)'\wedge\alpha \\ &= \rho_1\left(\left((d_J\alpha)'' + (i_S\alpha + \lambda)d_J\alpha\right)\wedge\alpha + 2(d_J\alpha)'\wedge\alpha' + 2\frac{\dot\rho_1}{\rho_1}(d_J\alpha)'\wedge\alpha\right). \end{aligned}$$

On the other hand from (7.47) we have

$$\frac{\dot\rho_1}{\rho_1} = \frac{i_{FS}\alpha - i_S\alpha'}{i_S\alpha},$$

so the condition $i_{R''}\Omega_E = 0$ is equivalent to

$$\left((d_J\alpha)'' + (i_S\alpha + \lambda)d_J\alpha\right)\wedge\alpha + 2(d_J\alpha)'\wedge\alpha' + \frac{2(i_{FS}\alpha - i_S\alpha')}{i_S\alpha}(d_J\alpha)'\wedge\alpha = 0$$

Modulo the compatibility condition $i_R\Omega = 0$, i.e. the equation $\alpha\wedge d_J\alpha = 0$, we find that

$$\begin{aligned} i_{R''}\Omega_E &= \rho_1\left(\left((d_J\alpha)'' + (i_S\alpha + \lambda)d_J\alpha\right)\wedge\alpha + 2(d_J\alpha)'\wedge\alpha'\right) + 2\dot\rho_1(d_J\alpha)'\wedge\alpha \\ &= \rho_1\left((d_J\alpha)''\wedge\alpha + 2(d_J\alpha)'\wedge\alpha'\right) + 2\dot\rho_1(d_J\alpha)'\wedge\alpha \\ &= \rho_1(d_J\alpha\wedge\alpha)'' - d_J\alpha\wedge[\rho_1\alpha'' + 2\dot\rho_1\alpha'] \overset{(7.29)}{=} \rho_1\, d_J\alpha\wedge\left(\alpha'' + 2\frac{\dot\rho_1}{\rho_1}\alpha'\right). \end{aligned}$$

Now the condition $i_{R''}\Omega_E = 0$ is equivalent to

$$d_J\alpha\wedge\left(\alpha'' + 2\frac{i_{FS}\alpha - i_S\alpha'}{i_S\alpha}\alpha'\right) = 0, \qquad (7.50)$$

which, taking into account (7.48), can be expressed by the equation

$$i_S\alpha'(d_J\alpha\wedge\alpha') = 0. \qquad (7.51)$$

- The last condition of integrability is given by the equation $i_{[h,A]}\Omega_E = 0$. It gives a new condition only if it is computed on the vectors S, h_1 and X, with $X\in\mathcal{H}_{\alpha,\alpha'}$:

$$\begin{aligned} i_{[h,A]}\nabla(P,E)(X,S,h_1) &= \sum_{X,S,h_1}^{\mathrm{cycl}} JX(i_{\tilde A}\Omega(S,h_1)) + \alpha(S)\nu S(\dot\alpha'(h_1)\Omega(C,X)) \\ &\quad + \dot\alpha'(h_1)C(\alpha(S)\Omega(\nu S,X)) = \alpha(S)\dot\alpha'(h_1)(\nu S\Omega(C,X) - C\Omega_E(\nu S,X)) \\ &= \alpha(S)\dot\alpha'(h_1)\left(d\Omega(\nu S,C,X) + \sum_{\nu S,C,X}^{\mathrm{cycl}}\Omega_E([\nu S,C],X)\right) \\ &= \alpha(S)\dot\alpha'(h_1)(i_{[\nu S,C]}\Omega_E(X) + i_{\nu S}\Omega_E([X,C]) + i_C\Omega([\nu S,X])). \end{aligned}$$

Taking into account the equations (7.30) and (7.42), this expression vanishes if and only if

$$i_{[vS,C]}\Omega_E(X) = \frac{\rho_1}{i_S\alpha\, i_{h_1}\bar{\alpha}'}\left(\alpha([vS,X]) + \alpha'([X,C]) + \frac{\rho_2}{\rho_1}\alpha([X,C])\right).$$

These computations show that all the obstructions can be expressed without the second order solution $j_2(E)_x$ except the last one. However, if *the distribution spanned by* vS *and* C *is integrable*, then this condition can be expressed uniquely in terms of the spray. Indeed, in this case there exist λ_1 and λ_2 such that $[vS, C] = \lambda_1 C + \lambda_2 vS$, hence

$$i_{[vS,C]}\Omega_E = (\lambda_1\rho_1 + \lambda_2\rho_2)\alpha + \lambda_2\rho_1\alpha'.$$

But

$$\mathcal{H}_{\alpha,\alpha'} = T^A \cap \alpha^\perp \cap \alpha'^\perp,$$

and thus $\alpha|_{\mathcal{H}} \equiv 0$ and $\alpha'|_{\mathcal{H}} \equiv 0$. So

$$i_{[vS,C]}\Omega_E(X) = 0$$

because $X \in \mathcal{H}_{\alpha\alpha'}$. Therefore

$$\begin{aligned}\tau_{[A',A']}\nabla(P_4E)(X,S,h_1)\\ &= i_S\alpha\, i_{h_1}\bar{\alpha}'\left(\lambda_1\rho_1\alpha([vS,X]) + \lambda_2\rho_2\alpha([X,C]) + \lambda_2\rho_1\alpha'([vS,X])\right).\end{aligned}$$

On the other hand we have

$$i_{[X,C]}\alpha = -\mathcal{L}_C\alpha(X), \quad i_{[vS,X]}\alpha = \mathcal{L}_{vS}\alpha(X), \quad i_{[vS,X]}\alpha' = \mathcal{L}_{vS}\alpha'(X),$$

so $\tau_{[A',A']}\nabla(P_4E)(X,S,h_1) = 0$ if and only if

$$\left(\lambda_1(\mathcal{L}_{vS}\alpha) - \lambda_2\frac{\rho_2}{\rho_1}(\mathcal{L}_C\alpha) + \lambda_2(\mathcal{L}_{vS}\alpha')\right)\Big|_{\mathcal{H}_{\alpha,\alpha'}} = 0,$$

that is

$$\left(\lambda_1(\mathcal{L}_{vS}\alpha) - \lambda_2\frac{i_{FS}\alpha - i_S\alpha'}{i_S\alpha}(\mathcal{L}_C\alpha) + \lambda_2(\mathcal{L}_{vS}\alpha')\right)\wedge\alpha\wedge\alpha' = 0.$$

Thus we have the following result.

If the distribution spanned by vS and C is integrable, then all the conditions of compatibility for P_4 given by (7.39) can be expressed with the help of the spray S.

STEP IV: *Involutivity of P_4*

We shall now prove that P_4 is involutive. Since $g_2(P_4) = g_2(P_3) \cap g_2(P_{\bar{A}}) \cap g_2(P_A) \cap g_2(P_{A'})$, an element $B \in S^2T^*$ is in $g_2(P_3)$ if and only if the equations

$$\begin{cases} a) & B(S, JX) = 0, \\ b) & B(hX, JY) - B(hY, JX) = 0, \\ c) & B(AX, JY) - B(AY, JX) = 0, \\ d) & B(A'X, JY) - B(A'Y, JX) = 0, \end{cases} \tag{7.52}$$

hold. Let us denote by $vS_{J\mathcal{H}} := vS - \frac{\alpha(FS)}{\alpha(S)} C$ the projection of vS on $J\mathcal{H}$ with respect to the splitting $T^v = \operatorname{Span} C \oplus J\mathcal{H}$. Note that $vS_{J\mathcal{H}} \neq 0$, because S is not typical, (and so vS is not proportional to C), and that $F(vS_{J\mathcal{H}}) \in \alpha^{\perp}$. Let us consider the base $\mathcal{B} = \{h_i, v_i\}_{i=1,\dots,n}$, with

a) $h_i \in T^h \cap \alpha^{\perp}$ for $i = 1, \dots, n-1$,
b) $h_{n-1} := F(vS_{J\mathcal{H}})$;
c) $h_n := hS$,
d) $v_i = Jh_i$.

Using the notation $B = \begin{pmatrix} a_{ij} & b_{ij} \\ b_{ji} & c_{ij} \end{pmatrix}$ for the matrix of $B \in g_2(P_4)$ in this basis, i.e. $a_{ij} = B(h_i, h_j)$, $b_{ij} = B(h_i, v_j)$ and $c_{ij} = B(v_i, v_j)$, for $i, j = 1, \dots, n$, we have the following relations between the components.

$$\begin{cases} 1) & c_{ni} & = & 0, & i = 1, \dots, n-1; \\ 2) & c_{n-1,i} & = & 0, & i = 1, \dots, n; \\ 3) & c_{n-1,n-1} & = & [illegible]\, c_{nn}, & \\ 4) & b_{ni} & = & 0, & i = 1, \dots, n-2; \\ 5) & b_{n,n-1} & = & -[illegible]\, c_{nn}; & \\ 6) & b_{nn} & = & [illegible]\, c_{nn}; & \\ 7) & b_{ij} - b_{ji} & = & 0, & i, j = 1, \dots, n \end{cases} \tag{7.53}$$

Indeed,

(1) Since $i_S\alpha \neq 0$, we obtain from (7.52c) computed on $X = S$ and $Y = h_i$, $i = 1, \dots, n-1$,

$$B(AS, v_i) - B(Ah_i, C) \overset{S=h_n}{=} \alpha(S) \cdot B(v_n, v_i) \overset{(7.52.c)}{=} 0.$$

(2) If $i < n-1$:

$$c_{n-1,i} = B(v_{n-1}, v_i) \overset{(7.52.c)}{=} B(vS, v_i) \overset{(7.52.d)}{=} 0.$$

(3) Using (7.52.c) and then (7.52.d) we find

$$c_{n-1,n-1} = B(vS, v_{n-1}) = \frac{i_{h_1}\alpha'}{i_S\alpha} B(C, C) = c_{nn}.$$

(4) If $i < n-1$, then using (7.52.a) and also (7.53.b) we find

$$b_{ni} = B(S, v_i) - B(vS, v_i) = -B(vS_{|H}, v_i) - \xi_C^{vS} B(v_n, v_i) = 0.$$

(5) $b_{n,n-1} = B(hS, v_{n-1}) = B(S, v_{n-1}) - B(vS, v_{n-1}) \overset{(7.52.e)}{=}$

$= -B(vS, v_{n-1}) = -\frac{i_{h_1}\alpha'}{i_S\alpha} B(C, C)$

(6) Using the equations (7.52.a) and (7.53.b) we find

$$b_{nn} = B(hS, C) = B(S, C) - B(vS, C) = -B(vS, v_n) = \frac{i_{FS}\alpha'}{i_S\alpha} c_{nn}.$$

(7) $b_{ij} - b_{ji} = B(h_i, v_j) - B(h_j, h_i) \overset{(7.52.b)}{=} 0$

It follows that

$$\dim g_2(P_4) = \tfrac{1}{2}(3n^2 - 3n + 4).$$

Let us now consider the base $\bar{B} = \{e_i, v_i\}_{i=1,\dots,n}$, where

$$\begin{aligned} e_1 &:= h_1 + v_1 + v_n, \\ e_i &:= h_i + i\, v_i, \qquad \text{for } i = 2, \dots, n-2, \\ e_{n-1} &:= v_{n-2} + h_{n-1}, \\ e_n &:= h_n + \frac{i_{FS}\alpha'}{i_S\alpha} C. \end{aligned}$$

We will show that $\bar{B}$ is quasi-regular. Putting forward $\bar{a}_{ij} = B(e_i, e_j)$, $\bar{b}_{ij} = B(e_i, v_j)$ and $\bar{c}_{ij} = B(v_i, v_j) = c_{ij}$ we can express the matrix (b_{ij}) in terms of the components b_{ij} and c_{ij}:

$$\begin{pmatrix} b_{11} + c_{11} & \cdots & b_{1,n-2} + c_{1,n-2} & b_{1,n-1} & c_{nn} \\ b_{12} + 2c_{12} & \cdots & b_{2,n-2} + 2c_{2,n-2} & b_{2,n-1} & 0 \\ b_{13} + 3c_{13} & & b_{3,n-2} + 3c_{3,n-2} & b_{3,n-1} & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ b_{1,n-2} + (n-2)c_{1,n-2} & \cdots & b_{n-2,n-2} + (n-2)c_{n-2,n-2} & b_{n-2,n-1} & 0 \\ b_{1,n-1} + c_{1,n-1} & & b_{n-2,n-1} + c_{n-2,n-2} & b_{n-1,n-1} & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

On the other hand

$$\begin{aligned}
\bar{a}_{ij} &= \bar{a}_{ji}, & i < j; \\
\bar{b}_{n-1,i} &= \bar{b}_{i,n-1} + \frac{1}{(n-2-i)} (\bar{b}_{n-2,i} - \bar{b}_{i,n-2}), & 1 \leq i \leq n-3; \\
\bar{b}_{ni} &= 0, & 2 \leq i \leq n; \\
\bar{b}_{in} &= 0, & 1 \leq i \leq n \\
\bar{c}_{in} &= 0, & 1 \leq i < n; \\
\bar{c}_{i,n-1} &= 0, & 1 \leq i \leq n; \\
\bar{c}_{n-2,n-2} &= \bar{b}_{n-1,n-2} - \bar{b}_{n-2,n-1}, \\
\bar{c}_{n-1,n-1} &= \text{[illegible]}\, \bar{b}_{n1}, \\
\bar{c}_{nn} &= \bar{b}_{1n}, \\
\bar{c}_{ij} &= \frac{1}{(i-j)} (\bar{b}_{ij} - \bar{b}_{ji}), & 1 \leq i < j \leq n-2;
\end{aligned}$$

Therefore an element $B \in g_2(F_3)$ is determined by the following free components:

$$\begin{aligned}
&\bar{a}_{ij}, \quad i, j = 1, \ldots, n, \qquad i \leq j, \\
&\bar{b}_{ij}, \quad i = 1, \ldots, n-2, \quad j = 1, \ldots, n-1, \qquad i \leq j; \\
&\bar{c}_{ii}, \quad i = 1, \ldots, n-3; \\
&\bar{b}_{1n}, \quad \bar{b}_{n-2,n-2}, \quad \bar{b}_{n-1,n-1}, \quad \bar{b}_{n-1,n-2}.
\end{aligned}$$

If we note $\tilde{B} = \begin{pmatrix} \bar{a}_{ij} & \bar{b}_{ij} \\ \bar{b}_{ji} & \bar{c}_{ij} \end{pmatrix}$ the matrix of an element $B \in g_2(F_4)$, the blocks of $\bar B$ are given by:

$$(\bar a_{ij}) = \begin{pmatrix} \bar a_{11} & \bar a_{12} & & \bar a_{1n} \\ * & \bar a_{22} & & \bar a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ * & & * & \bar a_{nn} \end{pmatrix},$$

$$(\bar b_{ij}) = \begin{pmatrix} \bar b_{11} & \cdots & \bar b_{1,n-3} & \bar b_{1,n-2} & \bar b_{1,n-1} & \bar b_{1n} \\ \bar b_{21} & \cdots & \bar b_{2,n-3} & \bar b_{2,n-2} & \bar b_{2,n-1} & * \\ \vdots & & & \vdots & \vdots & \vdots \\ \bar b_{n-3,1} & & \bar b_{n-3,n-3} & \bar b_{n-3,n-2} & \bar b_{n-3,n-1} & * \\ * & \cdots & * & \bar b_{n-2,n-2} & \bar b_{n-2,n-1} & * \\ * & \cdots & * & * & * & * \end{pmatrix},$$

and

$$(\bar c_{ij}) = \begin{pmatrix} \bar c_{11} & * & \cdots & * & * & * \\ * & \bar c_{22} & * & * & * & * \\ \vdots & \ddots & \ddots & \vdots & * & * \\ * & \cdots & * & \bar c_{n-3,n-3} & * & * \\ * & \cdots & * & * & * & * \\ * & \cdots & * & * & * & * \end{pmatrix},$$

where we wrote only the free parameters explicitly, and "$*$" denotes the determined components. Now

$$\dim g_2(P_k)_{e_1,\dots,e_k} = \begin{cases} \dfrac{(n-k)(n-k+1)}{2} + (n-(k+2))(n-1) + 2 + (n-3), \\ \qquad\qquad\qquad\qquad \text{for } 1 \le k < n-2; \\ \dfrac{2 \cdot 3}{2} + 2 + (n-3), \qquad \text{for } k = n-2, \\ \dfrac{1 \cdot 2}{2} + (n-3), \qquad \text{for } k = n-1, \\ (n-3), \qquad \text{for } k = n, \end{cases}$$

and

$$\dim g_2(P_k)_{e_2,\dots,e_k,v_1,\dots,v_k} = \begin{cases} n-(k+3), & \text{for } \quad k = 1,\dots,n-3, \\ 0, & \text{for } \quad k = n-2,\ n-1,\ n, \end{cases}$$

so

$$
\begin{aligned}
&\dim g_2(P_4) + \sum_{k=1}^{n} \dim g_2(P_4)_{e_1 \ldots e_k} + \sum_{k=1}^{n} \dim g_2(P_4)_{e_1 \ldots e_k, b_1 \ldots b_k} \\
&\quad = \frac{3n^2 - 3n + 4}{2} + \sum_{k=1}^{n} \frac{(n-k)(n-k+1)}{2} + \sum_{k=1}^{n} (n - (k+2))(n-1) \\
&\qquad + 2(n-2) + n(n-3) + \frac{1}{2}(n-2)(n-3) = \frac{1}{6}(4n^3 - 4n + 6) \\
&\quad = \dim g_3(P_4),
\end{aligned}
$$

which shows that the base B is quasi-regular. Theorem 7.4 is proved. □

Appendix A

Formulae

A.1 Formulae of the Frölicher-Nijenhuis Theory

If $K \in \Psi^k(M)$, $L \in \Psi^l(M)$, $N \in \Psi^n(M)$ and $\omega \in \Lambda^q(M)$, then

(1) $i_L d_K - (-1)^{k(l-1)} d_K i_L = d_{K\bar{\wedge} L} + (-1)^k i_{[L,K]}$.

In particular, if $X, Y \in \mathfrak{X}(M)$, and $K, L \in \Psi^1(M)$, then

a) $i_X d_K = -d_K i_X + \mathcal{L}_{KX} - i_{[K,X]}$.

b) $i_X \mathcal{L}_Y = \mathcal{L}_Y i_X + i_{[X,Y]}$.

c) $i_K \mathcal{L}_X = \mathcal{L}_X i_K + i_{[K,X]}$.

d) $i_K d_L = d_L i_K + d_{LK} - i_{[K,L]}$.

(2) $$[K, L]\bar{\wedge} N = [K\bar{\wedge} N, L] - (-1)^{l(k-1)} K\bar{\wedge}[N, L] + (-1)^{kl+l} \left([L\bar{\wedge} N, K] - (-1)^{k(l-1)} L\bar{\wedge}[N, K]\right).$$

In particular, if $X, Y \in \mathfrak{X}(M)$ and $K, L, N \in \Psi^1(M)$, then

a) $i_X[K, N] = [KX, N] - K[X, N] + [NX, K] - N[X, K]$,

b) $[X, NK] = [X, N]K + N[X, K]$,

c) $[K, L]\bar{\wedge} N = [KN, L] - K[N, L] + [LN, K] - L[N, K]$,

d) $$\tfrac{1}{2}[K, K](X, Y) = [KX, KY] + K^2[X, Y] - K[KX, Y] - K[X, KY].$$

e) $[K,N](X,Y) = [KX,NY] + [NX,KY] - K[X,NY]$
$$- K[NX,Y] - N[X,KY] - N[KX,Y]$$
$$+ KN[X,Y] + NK[X,Y].$$

(3) $i_N i_L \omega = (-1)^{(n-1)(l-1)} i_L i_N \omega - (-1)^{(n-1)(l-1)} i_{i_N L}\omega + i_{i_L N}\omega.$

In particular, if $X, Y \in \mathfrak{X}(M)$, $N, L \in \Psi^1(M)$ and $K \in \Psi^k(M)$, then

a) $i_{KX} = i_X i_K - i_K i_X$,

b) $i_K i_L - i_L i_K = i_{LK} - i_{KL}$,

c) $i_{K\wedge X} = i_X i_K + (-1)^k i_K i_X$.

(4) $i_{\omega\wedge K}\pi = \omega \wedge i_K \pi$.

(5) $d_{\omega\wedge K}\pi = \omega \wedge d_K \pi + (-1)^{q+k} d\omega \wedge i_K \pi$.

(6) $[L, \omega\wedge K] = d_L\omega \wedge K - (-1)^{(l-1)(q+k)} d\omega \wedge (i_L K) + (-1)^{lq}\omega \wedge [L,K]$.

In particular, if $f, g \in C^\infty(M)$, $X, Y \in \mathfrak{X}(M)$, $K, L \in \Psi^1(M)$ and $\omega, \pi \in \Lambda^1(M)$, then

a) $[X, fK] = (\mathcal{L}_X f)K + f[X,K]$

b) $[K, fX] = d_K f \otimes X - df \otimes KX + f[X,K]$

c) $[gK, fX] = g(d_K f \otimes X - df \otimes KX) - (f\mathcal{L}_X g)K + gf[K,X]$

d) $[K, gL] = (d_K g) \wedge L - dg \wedge KL + g[K,L]$

e) $[fK, gL] = f(d_K g \wedge L - dg \wedge KL)$
$$+ g(d_L f \wedge K - df \wedge LK) + fg[K,L]$$

f) $[X, \omega\wedge K] = (\mathcal{L}_X\omega)\wedge K + \omega\wedge[X,K]$

g) $[K, \omega\wedge L] = d_K\omega \wedge L - d\omega \wedge KL + (-1)^q \omega\wedge[K,L]$

h) $[K, \omega\otimes X] = d_K\omega \otimes X - d\omega \otimes KX + (-1)^q \omega \wedge [K,X]$

i) $\frac{1}{2}[\omega\otimes X, \omega\otimes X] = (\omega\wedge\mathcal{L}_X\omega - (i_X\omega)d\omega)\otimes X$

j) $[\omega\otimes X, \pi\otimes Y] = (\omega\wedge\mathcal{L}_X\pi - \pi(X)d\omega)\otimes Y$
$$+ (\pi\wedge\mathcal{L}_Y\omega - \omega(Y)d\pi)\otimes X + (\omega\wedge\pi)\otimes[X,Y].$$

A.2 Formulae for Chapter 5

$$
\begin{aligned}
\chi_1^1 &= \chi_1\mu_{h_1}\xi_{h_2}^{[S,v_1]} + \chi_1\xi_{v_1}^{[S,v_1]} + \chi_1\mu_{h_1}\xi_{h_1}^{[S,v_1]} + (S\chi_1),\\
\chi_1^2 &= \xi_{v_2}^S\chi_1^3 - k_2(\mu_{h_1}\xi_{h_1}^S + \xi_{v_1}^S)\\
&\quad + \xi_{h_2}^S\left(\chi_1\xi_{h_1}^{[h_2,h_1]} + \chi_1(v_1\xi_{v_2}^{[h_2,v_1]}) + (v_2\chi_1) + \chi_1\xi_{v_1}^{[Z,v_1]}\right)\\
\chi_1^3 &= \chi_1\xi_{h_1}^{[v_2,h_1]} + \chi_1\xi_{v_1}^{[Z,v_1]} + (v_2\chi_1).
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
k_1^1 &= \chi_1\mu_{h_1}^1\xi_{h_2}^{[S,v_1]} - 2\chi_1(v_1\xi_{h_1}^{[S,v_1]}) + (Sk_1) + 2k_1\xi_{v_1}^{[S,v_1]},\\
k_2^1 &= \chi_1\mu_{h_1}^2\xi_{h_1}^{[S,v_1]} + (Sk_2) + 2k_2\xi_{v_2}^{[S,v_2]},\\
k_1^2 &= \xi_{h_2}^S\left(\chi_1\xi_{v_2}^{[v_1,h_1]}\chi_{v_1}^1 + \chi_1(v_1\xi_{h_1}^{[h_2,h_1]}) + \chi_1(v_1\xi_{h_1}^{[h_2,v_1]}) + \chi_1\xi_{v_2}^{[Z,v_1]}\chi_{v_2}^1\right.\\
&\quad \left. + k_1\chi_{h_2}^1 + (h_2k_1)\right) + \xi_{v_2}^S k_1^3 + k_1(2\xi_{v_1}^{[S,v_1]} - \mu_{h_1}^1\xi_{h_1}^S),\\
k_2^2 &= \xi_{h_2}^S\left(\chi_1(v_1\xi_{v_2}^{[v_1,h_1]}) + \chi_1\xi_{v_2}^{[v_1,h_1]}\chi_{v_1}^2 + \chi_1\xi_{v_2}^{[h_2,v_1]}\chi_{v_2}^2 + k_1\chi_{h_2}^2\right.\\
&\quad \left. + (h_2k_2)\right) + \xi_{v_2}^S k_2^3 - k_2\mu_{h_1}^2\xi_{h_1}^{[S,v_1]},\\
k_1^3 &= \chi_1\xi_{h_2}^{[h_1,v_1]}\chi_{v_1}^1 + \chi_1(v_1\xi_{h_1}^{[v_2,h_1]}) - \chi_1(v_1\xi_{v_1}^{[Z,v_1]}) + \chi_1\xi_{v_2}^{[Z,v_1]}\chi_{v_2}^1\\
&\quad + (v_2k_1) + k_1\chi_{v_2}^1,\\
k_2^3 &= \chi_1(v_1\xi_{h_2}^{[h_1,v_1]}) + \chi_1\xi_{h_2}^{[h_1,v_1]}\chi_{v_1}^2 - \chi_1\xi_{v_2}^{[v_2,v_1]}\chi_{v_2}^2 + (v_2k_2) + k_2\chi_{v_2}^2.
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
c_1^i &= k_1^i - \left(\frac{\chi_1^i}{\chi_1}\right)k_1, & i &= 1,2,\\
c_2^i &= k_2^i - \left(\frac{\chi_1^i}{\chi_1}\right)k_2, & i &= 1,2,\\
c_1^{i+2} &= a_{h_2}^i - \left(\frac{\chi_{h_2}^i}{\chi_{h_2}}\right)k_{h_2}^1, & i &= 1,2,\\
c_2^{i+2} &= b_{h_2}^i - \left(\frac{\chi_{h_2}^i}{\chi_{h_2}}\right)k_{h_2}^2, & i &= 1,2,\\
c_1^5 &= a_{h_2}^3 - \left(\frac{b_1}{\chi_1}\right)k_1 - \left(\frac{\chi_{h_2}^3}{\chi_{h_2}}\right)k_{h_2}^1,\\
c_2^5 &= b_{h_2}^3 - \left(\frac{b_1}{\chi_1}\right)k_2 - \left(\frac{\chi_{h_2}^3}{\chi_{h_2}}\right)k_{h_2}^2
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
\chi^1_{h_2} &= \chi_{h_2}\mu h_2 \xi^{[S,h_2]}_{h_2} - \chi_{h_2}\xi^{[S,h_2]}_{h_2} + \chi_{h_2}\mu h_2 \xi^{[S,v_2]}_{h_2} + (S\chi_{h_2}),\\
\chi^2_{h_2} &= \xi^S_{h_1}\big(\chi_{h_2}\xi^{[h_1,h_2]}_{h_2} + \chi_{h_2}(v_2\xi^{[h_1,v_2]}_{v_2}) + (v_2\chi_{h_2}) + \chi_{h_2}\xi^{[v_1,v_2]}_{v_2}\big)\\
&\qquad + \xi^S_{v_1}\chi^{+2}_{h_2} - k^1_{h_2}\big(\mu h_2 \xi^S_{h_1} + \xi^S_{v_2}\big),\\
\chi^3_{h_2} &= \chi_{h_2}\xi^{[v_1,h_2]}_{h_2} + 2\chi_{h_2}\xi^{[v_1,v_2]}_{v_2} + (v_2\chi_{h_2}).
\end{aligned}
$$

$$
\begin{aligned}
a^1_{h_2} &= \chi_{h_2}\mu^2_{h_2}\xi^{[S,v_2]}_{h_2} + (Sk^1_{h_2}) + 2k^1_{h_2}\xi^{[S,v_1]}_{v_1}.\\
a^2_{h_2} &= \xi^S_{h_1}\big(\chi_{h_2}v_2\xi^{[v_2,h_1]}_{v_1} + \chi_{h_2}\xi^{[v_2,h_2]}_{h_1}\chi^2_{v_2} + \chi_{h_2}\xi^{[v_1,v_2]}_{v_2}\chi^2_{v_1} - k^2_{h_2}\chi^2_{h_1}\\
&\qquad - h_1 k^1_{h_2}\big) + \xi^S_{v_2}a^2_{h_2} - k^1_{h_2}\mu^2_{h_2}\xi^{[S,v_2]}_{h_2}. && \text{(A.4)}\\
a^3_{h_2} &= \chi_{h_2}v_2\xi^{[h_2,v_2]}_{h_1} - \chi_{h_2}\xi^{[h_2,h_2]}_{h_1}\chi^2_{v_2} + \chi_{h_2}\xi^{[v_1,v_2]}_{v_2}\chi^2_{v_1} - v_1 k^1_{h_2} + k^2_{h_2}\chi^2_{v_1}.
\end{aligned}
$$

$$
\begin{aligned}
b^1_{h_2} &= \chi_{h_2}\mu^1_{h_2}\xi^{[S,v_2]}_{h_2} - 2\chi_{h_2}(h_2\xi^{[S,v_2]}_{v_2}) + (Sk^2_{h_2}) + 2k^2_{h_2}\xi^{[S,v_2]}_{v_2},\\
b^2_{h_2} &= \xi^S_{h_1}\big(\chi_{h_2}\xi^{[v_2,h_2]}_{v_1}\chi^1_{v_2} + \chi_{h_2}(v_2\xi^{[h_2,h_2]}_{h_2}) + \chi_{h_2}(v_2\xi^{[h_1,v_2]}_{v_2}) + (h_1 k^2_{h_2})\\
&\qquad + \chi_{h_2}\xi^{[v_1,v_1]}_{v_1}\chi^1_{v_1} + k^2_{h_2}\chi^1_{h_1}\big) + \xi^S_{v_1}b^2_{h_2} + k^1_{h_2}\big(2\xi^{[S,v_2]}_{h_2} - \mu^1_{h_2}\xi^2_{h_2}\big).\\
b^3_{h_2} &= \chi_{h_2}\xi^{[h_2,v_2]}_{h_1}\chi^1_{v_2} + \chi_{h_2}(v_2\xi^{[v_1,h_2]}_{h_2}) + \chi_{h_2}(v_2\xi^{[v_1,v_2]}_{v_2}) + \chi_{h_2}\xi^{[v_1,v_2]}_{v_1}\chi^1_{v_1}\\
&\qquad + (v_1 k^2_{h_2}) + k^2_{h_2}\chi^1_{v_1}.
\end{aligned}
$$

$$
\begin{aligned}
\mu_2 &= -\frac{\xi^S_{v_2}}{\xi^S_{h_2}},\\
\nu^1_1 &= -\frac{1}{\xi^S_{h_2}}\Big(\xi^{[S,v_1]}_{v_2} + \xi^{[S,h_2]}_{h_2} - \xi^S_{h_1}\eta^1_{h_2} - \xi^S_{v_1}\eta^1_{v_1}\Big),\\
\nu^2_1 &= -\frac{1}{\xi^S_{h_2}}\Big(\xi^{[S,v_1]}_{v_2} - \xi^S_{h_1}\eta^2_{h_1} - \xi^S_{v_1}\eta^2_{v_1}\Big),\\
\nu_1 &= -\Big(\frac{\xi^S_{h_1}\nu_1}{\xi^S_{h_2}} + \frac{\xi^S_{v_1}}{\xi^S_{h_2}}\Big), && \text{(A.5)}\\
\mu^1_2 &= \xi^{[S,v_2]}_{v_1} + \xi^{[S,h_2]}_{h_1} - \frac{\xi^S_{h_1}}{\xi^S_{h_2}}\big(\nu^1_1 + \eta^1_{h_2}\big) - \frac{\xi^S_{v_2}}{\xi^S_{h_2}}\\
\mu^2_2 &= \xi^{[S,v_2]}_{v_2} + \xi^{[S,h_2]}_{h_2} - \frac{\xi^S_{h_1}}{\xi^S_{h_2}}\big(\nu^2_1 + \eta^2_{h_2}\big) - \frac{\xi^S_{v_2}}{\xi^S_{h_2}}\eta^2_{v_2}.
\end{aligned}
$$

$$
\begin{aligned}
r_1^1 &= \chi_1\left(\xi^{[\nu_1,S]}_{\nu_1} + \xi^{[\lambda_1,S]}_{\lambda_1} + \xi^{[\lambda_1,S]}_{\lambda_1}\mu_{\lambda_1} + \xi^{[\nu_1,S]}_{\nu_1}\right) - (S\chi_1)\\
r_2^1 &= \chi_2\left(\xi^{[\nu_2,S]}_{\nu_2} + \xi^{[\lambda_2,S]}_{\lambda_2}\nu_2 + \xi^{[\nu_2,S]}_{\lambda_2}\mu_{\lambda_2} + \xi^{[\nu_2,S]}_{\nu_2}\right) - (S\chi_2)\\
r_1^2 &= (S\chi_1) - \chi_1\Big(\xi^{[\nu_1,S]}_{\nu_1} + \xi^{[\lambda_1,S]}_{\lambda_1} + \xi^S_{\lambda_2}\left(\xi^{[\lambda_2,\lambda_1]}_{\lambda_1} + \xi^{[\lambda_2,\nu_1]}_{\nu_1}\right)\\
&\qquad + \xi^S_{\nu_2}\left(\xi^{[\nu_2,\lambda_1]}_{\lambda_1} + \xi^{[\nu_2,\nu_1]}_{\nu_1}\right) + \xi^{[\nu_1,S]}_{\nu_1} + \xi^{[\lambda_1,S]}_{\lambda_1}\mu_{\lambda_1} + \xi^S_{\lambda_2}\xi^{[\lambda_2,\nu_1]}_{\nu_1}\\
&\qquad + \xi^S_{\lambda_2}\xi^{[\lambda_2,\nu_1]}_{\lambda_1}\mu_{\lambda_1} + \xi^S_{\nu_2}\xi^{[\nu_2,\nu_1]}_{\nu_1}\Big)\\
r_2^2 &= (S\chi_2) - \chi_2\Big(\xi^S_{\lambda_1}\xi^{[\lambda_1,\lambda_2]}_{\nu_2} + \xi^S_{\nu_1}\xi^{[\nu_1,\lambda_2]}_{\lambda_2} + \xi^S_{\nu_1}\xi^{[\nu_1,\nu_2]}_{\nu_2} + \xi^S_{\lambda_1}\xi^{[\lambda_1,\nu_2]}_{\nu_2}\\
&\qquad + \xi^S_{\lambda_1}\xi^{[\lambda_1,\lambda_2]}_{\lambda_2}\mu_{\lambda_2} + \xi^S_{\lambda_1}\xi^{[\lambda_1,\lambda_2]}_{\nu_2} + \xi^S_{\nu_1}\xi^{[\nu_1,\nu_2]}_{\nu_2}\Big)\\
s_1^1 &= \chi_1\left((\nu_1\xi^{[\nu_1,S]}_{\nu_1}) + (\nu_1\xi^{[\lambda_1,S]}_{\lambda_1}) + \xi^{[\lambda_1,S]}_{\lambda_1}\mu^1_{\lambda_1}\right) + \chi_2\xi^{[\nu_2,S]}_{\lambda_2}\mu^1_{\lambda_2}\\
s_2^1 &= \chi_2\left((\nu_2\xi^{[\nu_2,S]}_{\nu_2}) + (\nu_2\xi^{[\lambda_2,S]}_{\lambda_2}) + \xi^{[\lambda_2,S]}_{\lambda_2}\mu^2_{\lambda_2}\right) + \chi_1\xi^{[\lambda_1,S]}_{\lambda_1}\mu^2_{\lambda_1}\\
s_1^2 &= -\chi_1\Big((\nu_1\xi^{[\nu_1,S]}_{\nu_1}) + (\nu_1\xi^{[\lambda_1,S]}_{\lambda_1}) + \xi^S_{\lambda_2}\big(\xi^{[\nu_1,\lambda_1]}_{\nu_1}\chi^1_{\nu_1} + (\nu_1\xi^{[\lambda_2,\lambda_1]}_{\lambda_1})\\
&\qquad + \nu_1\xi^{[\lambda_2,\nu_1]}_{\nu_1}\big) + \xi^S_{\nu_2}\big(\xi^{[\nu_2,\lambda_1]}_{\lambda_1}\chi^1_{\nu_1} + (\nu_1\xi^{[\nu_2,\lambda_1]}_{\lambda_1}) + \nu_1\xi^{[\nu_2,\nu_1]}_{\nu_1}\big) + \xi^{[\lambda_1,S]}_{\lambda_1}\mu^1_{\lambda_1}\\
&\qquad + \xi^S_{\lambda_2}\xi^{[\lambda_2,\nu_1]}_{\lambda_1}\mu^1_{\lambda_1} + \xi^S_{\lambda_2}\xi^{[\lambda_2,\nu_1]}_{\lambda_2}\chi^1_{\lambda_2} + \xi^S_{\lambda_2}\xi^{[\lambda_2,\nu_2]}_{\nu_2}\chi^1_{\nu_2} + \xi^S_{\nu_2}\xi^{[\nu_2,\nu_2]}_{\nu_2}\chi^1_{\nu_2}\Big)\\
&\quad - \chi_2\Big(\xi^S_{\lambda_1}\nu_2\xi^{[\lambda_1,\lambda_2]}_{\lambda_2} + \xi^S_{\nu_1}\xi^{[\lambda_1,\nu_2]}_{\lambda_2}\chi^1_{\nu_2} + \xi^S_{\nu_1}\nu_2\xi^{[\nu_1,\lambda_2]}_{\lambda_2} + \xi^S_{\nu_1}\xi^{[\nu_1,\lambda_2]}_{\lambda_1}\chi^1_{\nu_2}\\
&\qquad + \xi^S_{\lambda_1}\xi^{[\nu_2,\lambda_1]}_{\nu_1}\chi^1_{\nu_1} + \xi^S_{\lambda_1}\xi^{[\nu_2,\lambda_1]}_{\lambda_2}\chi^1_{\lambda_1} + \xi^S_{\lambda_1}\xi^{[\nu_2,\lambda_1]}_{\lambda_2}\mu^1_{\lambda_2} + \xi^S_{\nu_1}\xi^{[\nu_2,\nu_1]}_{\nu_1}\chi^1_{\nu_1}\Big)\\
s_2^2 &= -\chi_1\Big(\xi^S_{\lambda_2}\nu_1\xi^{[\nu_1,\lambda_1]}_{\nu_2} + \xi^S_{\lambda_2}\xi^{[\nu_1,\lambda_1]}_{\nu_2}\chi^2_{\nu_1} + \xi^S_{\nu_2}\nu_1\xi^{[\nu_1,\lambda_1]}_{\lambda_2} + \xi^S_{\nu_2}\xi^{[\lambda_1,\lambda_1]}_{\lambda_2}\chi^2_{\nu_1}\\
&\qquad + \xi^{[\nu_1,S]}_{\lambda_1}\mu^2_{\lambda_1} + \xi^S_{\lambda_2}\xi^{[\lambda_2,\lambda_1]}_{\lambda_1}\mu^2_{\lambda_1} + \xi^S_{\lambda_2}\xi^{[\lambda_2,\lambda_1]}_{\lambda_2}\chi^2_{\lambda_2} + \xi^S_{\lambda_2}\xi^{[\lambda_2,\nu_1]}_{\nu_2}\chi^2_{\nu_2}\\
&\qquad + \xi^S_{\nu_2}\xi^{[\nu_2,\nu_1]}_{\nu_2}\chi^2_{\nu_2}\Big) - \chi_2\Big(\xi^S_{\lambda_1}(\nu_2\xi^{[\nu_2,\lambda_1]}_{\nu_2}) + \xi^S_{\lambda_1}\nu_2(\xi^{[\lambda_2,\lambda_1]}_{\lambda_2})\\
&\qquad + \xi^S_{\nu_1}\xi^{[\lambda_2,\nu_2]}_{\lambda_2}\chi^2_{\nu_2} + \xi^S_{\nu_1}(\nu_2\xi^{[\lambda_2,\nu_1]}_{\nu_2}) + \xi^S_{\nu_1}(\nu_2\xi^{[\nu_2,\nu_1]}_{\nu_2}) + \xi^S_{\lambda_1}\xi^{[\nu_2,\lambda_1]}_{\lambda_1}\chi^2_{\nu_2}\\
&\qquad + \xi^S_{\lambda_1}\xi^{[\lambda_2,\lambda_1]}_{\nu_1}\chi^2_{\nu_1} + \xi^S_{\lambda_1}\xi^{[\nu_2,\lambda_1]}_{\lambda_1}\chi^2_{\lambda_1} + \xi^S_{\lambda_1}\xi^{[\lambda_2,\lambda_1]}_{\lambda_2}\mu^2_{\lambda_2} + \xi^S_{\nu_1}\xi^{[\nu_2,\nu_1]}_{\nu_1}\chi^2_{\nu_1}\Big)
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
p_1 &= 2\left(\xi_{h_2}^{[v_2,h_1]} v_1 + \xi_{h_2}^{[v_1,h_1]}\right) \\
p_1 &= 2\,\xi_{h_2}^{[v_1,h_1]} v_1^1 + \xi_{v_2}^{[h_1,v_2]} \eta_{v_2}^1 + \xi_{h_2}^{[h_1,v_1]} \eta_{h_2}^1 + \left(v_2 \xi_{h_2}^{[h_1,h_2]}\right) \quad \left(v_1 \xi_{v_2}^{[h_2,v_1]}\right) \\
&\quad - v_2 \xi_{v_1}^{[v_1,h_1]} + h_1 \xi_{v_2}^{[v_2,v_1]} - h_1 \xi_{h_1}^{[v_2,h_1]} - h_1 \xi_{h_2}^{[h_1,v_2]} + \eta_{h_1}^1 \Big(2\xi_{h_1}^{[h_1,v_2]} \\
&\quad + \xi_{v_2}^{[v_2,v_1]} - \xi_{h_2}^{[h_1,v_2]}\Big) + \eta_{v_2}^1 \Big(2\xi_{v_1}^{[h_2,v_1]} 2\xi_{h_1}^{[h_1,v_1]} + \xi_{h_2}^{[h_1,h_2]} \xi_{v_2}^{[h_2,v_2]}\Big) \qquad \text{(A.7)} \\
p_2 &= 2\xi_{h_2}^{[v_1,h_1]} v_1^2 + \xi_{v_2}^{[h_1,v_2]} \eta_{v_2}^2 + v_1 \xi_{v_2}^{[h_1,v_1]} + h_1 \xi_{h_2}^{[h_1,v_1]} + \xi_{h_2}^{[h_1,v_1]} \eta_{h_2}^2 \\
&\quad + \left(2\xi_{h_1}^{[h_1,v_1]} + \xi_{v_2}^{[v_2,v_1]} - \xi_{h_2}^{[h_1,v_2]}\right) \eta_{h_1}^2 + \left(2\xi_{v_1}^{[h_1,v_1]} + \xi_{h_2}^{[h_1,h_2]} + \xi_{v_2}^{[h_2,v_1]}\right) \eta_{v_1}^2
\end{aligned}
$$

$$
\begin{aligned}
\eta_{v_1}^1 &= \xi_{h_1}^{[v_2,h_1]} + \xi_{h_2}^{[h_1,v_2]} - \xi_{v_2}^{[h_2,v_1]}, \\
\eta_{v_1}^2 &= \xi_{h_2}^{[v_1,h_1]}, \\
\eta_{h_1}^1 &= \xi_{v_1}^{[h_1,v_1]} + \xi_{v_2}^{[v_2,h_2]} - \xi_{h_2}^{[h_2,h_1]}, \\
\eta_{h_1}^2 &= \xi_{v_2}^{[h_1,v_1]}.
\end{aligned} \qquad \text{(A.8)}
$$

$$
\begin{aligned}
\eta_{v_2}^1 &= \xi_{v_1}^{[v_1,v_2]} - \xi_{h_2}^{[h_2,h_2]} - \xi_{h_1}^{[h_2,v_1]}, \\
\eta_{v_2}^2 &= \xi_{v_2}^{[v_1,v_2]} - \xi_{h_2}^{[h_2,v_1]}, \\
\eta_{h_2}^1 &= \xi_{v_1}^{[h_1,v_2]} + \xi_{v_2}^{[v_2,h_2]} - \xi_{h_1}^{[h_2,h_1]}, \\
\eta_{h_2}^2 &= \xi_{v_2}^{[h_1,v_2]} - \xi_{h_2}^{[h_2,h_1]}.
\end{aligned} \qquad \text{(A.9)}
$$

$$
\begin{aligned}
\dot r_1 &= (v_1 r_1) + r_1 \eta_{h_1}^1 + r_2\left(\xi_{v_1}^{[v_1,v_2]} - \xi_{h_2}^{[v_2,h_2]} - \xi_{h_1}^{[h_2,v_1]}\right), \\
\dot r_2 &= (v_1 r_2) + r_1 \eta_{h_1}^2 + r_2\left(\xi_{v_2}^{[v_1,v_2]} - \xi_{h_2}^{[h_2,v_1]}\right), \\
\dot r_1^2 &= (v_2 r_1), \\
\dot r_2^2 &= (v_2 r_2).
\end{aligned} \qquad \text{(A.10)}
$$

$$
\begin{aligned}
\eta_1^1 &= (S\eta_1) + \eta_1\left(\xi_{v_2}^{[S,v_2]} + \xi_{h_2}^{[S,v_2]} + \nu_1 \xi_{v_1}^{[S,v_1]} - \xi_{h_2}^{[S,h_2]}\right),\\
\bar{\eta}_2^1 &= \eta_1 \xi_{v_2}^{[S,v_1]},\\
p_1^1 &= (Sp_1) + p_1\left(\xi_{v_1}^{[S,v_1]} + \xi_{h_2}^{[S,h_2]}\right) + p_2\left(\xi_{v_2}^{[S,v_2]} + \xi_{h_2}^{[S,h_2]}\right)\\
&\quad + \eta_1\left(\eta^2_{h_1}\xi_{h_1}^{[S,v_2]} + \nu_1^1 \xi_{h_2}^{[S,v_2]} - \eta^1_{v_1}\xi_{v_1}^{[S,v_2]} + v_1 \xi_{v_1}^{[S,v_1]} + v_2 \xi_{h_2}^{[S,h_2]}\right),\\
p_2^1 &= (Sp_2) + p_1 \xi_{v_2}^{[S,h_2]} + p_2\left(\xi_{v_2}^{[S,v_2]} + \xi_{h_2}^{[S,h_2]}\right)\\
&\quad + \eta_1\left(\eta^2_{h_1}\xi_{h_1}^{[S,v_2]} + \nu_1^1 \xi_{h_2}^{[S,v_2]} + \eta^2_{v_1}\xi_{v_1}^{[S,v_2]} + (v_2 \xi_{v_2}^{[S,h_1]})\right).
\end{aligned}
$$

$$
\begin{aligned}
\eta_1^2 &= h_1 \eta_1 + p_2 \nu_1\\
&\quad + \eta_1\left(\xi_{v_2}^{[h_1,v_2]} + \xi_{h_2}^{[h_1,v_2]}\nu_1 - \xi_{v_1}^{[v_1,h_1]} - \xi_{h_2}^{[h_2,h_1]} - \xi_{v_2}^{[h_2,v_1]}\right),\\
\bar{\eta}_2^2 &= \eta_1 \xi_{v_2}^{[v_1,h_1]},\\
p_1^2 &= h_1 p_1 + p_1 \eta^1_{h_1} - p_2\left(\nu_1^1 + \eta^1_{h_2}\right) + \eta_1\left(\eta^1_{h_1}\xi_{h_1}^{[h_1,v_2]} + \nu_1^1 \xi_{h_2}^{[h_1,v_2]}\right.\\
&\quad \left. + \eta^1_{v_1}\xi_{v_1}^{[h_1,v_2]} + (v_1 \xi_{v_1}^{[h_1,h_1]}) + (v_2 \xi_{h_2}^{[h_1,h_2]}) + (v_1 \xi_{v_2}^{[h_1,h_2]})\right),\\
p_2^2 &= (h_1 p_2) + p_1 \eta^2_{h_1} + p_2\left(\nu_1^2 + \eta^2_{h_2}\right)\\
&\quad + \eta_1\left(\eta^2_{h_1}\xi_{h_1}^{[h_1,v_2]} + \nu_1^2 \xi_{h_2}^{[h_1,v_2]} + \eta^2_{v_1}\xi_{v_1}^{[h_1,v_2]} + (v_2 \xi_{v_2}^{[h_1,v_1]})\right).
\end{aligned}
\qquad \text{(A.11)}
$$

$$
\begin{aligned}
\eta_1^3 &= (v_1\eta_1) + p_2 + \eta_1\left(\xi_{v_2}^{[v_1,v_2]} + \nu_1 \xi_{h_2}^{[v_1,v_2]} + \xi_{v_2}^{[v_1,v_2]} + \xi_{h_2}^{[h_1,v_2]} - \xi_{h_1}^{[v_1,h_1]}\right),\\
\bar{\eta}_2^3 &= \eta_1 \xi_{h_2}^{[v_1,h_1]},\\
p_1^3 &= (v_1 p_1) + p_1 \eta^1_{v_1} + p_2 \eta^1_{v_2} + + \eta_1\left(\eta^1_{h_1}\xi_{h_1}^{[v_1,v_2]} + \nu_1^1 \xi_{h_2}^{[v_1,v_2]}\right.\\
&\quad \left. + \eta^1_{v_1}\xi_{v_1}^{[v_1,v_2]} + (v_1 \xi_{v_2}^{[v_1,v_2]}) + (v_2 \xi_{h_2}^{[v_1,v_2]}) + (v_1 \xi_{h_1}^{[v_1,h_1]})\right),\\
p_2^3 &= (v_1 p_2) + p_1 \eta^2_{v_1} + p_2 \eta^2_{v_2}\\
&\quad + \eta_1\left(\eta^2_{h_1}\xi_{h_1}^{[v_1,v_2]} + \nu_1^2 \xi_{h_2}^{[v_1,v_2]} + \eta^2_{v_1}\xi_{v_1}^{[v_1,v_2]} + (v_2 \xi_{h_2}^{[v_1,h_1]})\right).
\end{aligned}
$$

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