Three-dimensional topological loops with solvable multiplication groups

Abstract

We prove that each 3-dimensional connected topological loop L having a solvable Lie group of dimension ≤ 5 as the multiplication group of L is centrally nilpotent of class 2. Moreover, we classify the solvable non-nilpotent Lie groups G which are multiplication groups for 3-dimensional simply connected topological loops L and dim $G \leq 5$. These groups are direct products of proper connected Lie groups and have dimension 5. We find also the inner mapping groups of L.

Keywords: Multiplication groups of loops, topological transformation group, solvable Lie group

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1. Introduction

The multiplication group $Mult(L)$ of a loop L introduced in [1], [2] connects the loop with the group theory since for any normal subloop of L there is a normal subgroup of $Mult(L)$ and conversely to every normal subgroup of $Mult(L)$ corresponds a normal subloop of L (cf. Lemma 3). Necessary and sufficient conditions for a group K to be the multiplication group $Mult(L)$ of a loop L are established in [14]. In this criterion there are two special transversals A and B with respect to a subgroup S (see Lemma 1) which results in being the stabilizer of the identity of L in $Mult(L)$ and it is called the inner mapping group $Inn(L)$ of L. For finite loops the importance of $Mult(L)$ and $Inn(L)$ as well as the transversals A and B is documented in ([3], [13] - [16], [19]).

In general the multiplication group $Mult(L)$ for a topological loop L has infinite dimension. If L has a Lie group as its multiplication group, then the structure of L as well as that of $Mult(L)$ is strongly restricted. Hence it is justified to investigate Lie groups which are multiplication groups of L ([4] - [6], [17]). In this case the criterion in [14] can be effectively used and the topological loop L is realized as a sharply transitive section in a subgroup G of $Mult(L)$. This subgroup G is the group topologically generated by the left translations of L.

If the group $Mult(L)$ of a 2-dimensional topological loop L is a Lie group, then it is an elementary filiform Lie group $\mathcal{F}_n$ with $n \geq 4$ ([4]). Classifying all at most 5-dimensional solvable non-nilpotent Lie groups K which are multiplication groups $Mult(L)$ of 3-dimensional connected simply connected topological loops L we see that for the structure of $Mult(L)$ one has more freedom. Moreover, knowing $Mult(L)$ one can describe the structure of L and determine the inner mapping group of L.

In Section 3 we give the precise structure of the 3-dimensional simply connected topological loops L such that the multiplication group $Mult(L)$ of L is a solvable Lie group and L has a 1-dimensional connected normal subloop (see Theorem 6). In this paper we prove that for each 3-dimensional simply connected topological loop L having a solvable Lie group of dimension ≤ 5 as the multiplication group $Mult(L)$ of L the group $Mult(L)$ is a semidirect product of a group $Q \cong \mathbb{R}^2$ with the group $M = Z \times Inn(L) \cong \mathbb{R}^n$, $n \in \{2, 3\}$, where $\mathbb{R} = Z$ is a central subgroup of $Mult(L)$. So we show that none of the 4-dimensional solvable Lie groups as well as none of the 5-dimensional solvable non-nilpotent indecomposable Lie groups are multiplication groups of 3-dimensional topological loops L (see Sections 4 and 5). But there are many loops L having a 4-dimensional solvable Lie group as the group generated by their left translations (Theorem 10).

To classify the 5-dimensional solvable decomposable Lie groups $Mult(L)$ of L we have to find special left transversals to a 2-dimensional subgroup S of $Mult(L)$ such that the core of S in $Mult(L)$ is trivial, S is included in a normal subgroup $M \cong \mathbb{R}^3$ of $Mult(L)$ with $Mult(L)/M \cong \mathbb{R}^2$ and the normalizer of S in $Mult(L)$ is the direct product of S and the centre of $Mult(L)$. The final result of our efforts is the following: If $Mult(L)$ has 1-dimensional centre, then it is either the group $\mathcal{F}_3 \times \mathcal{L}_2$ or the group $\mathbb{R} \times \mathcal{L}_2 \times \mathcal{L}_2$, or the direct product $\mathbb{R} \times \Sigma$, where Σ is a 4-dimensional indecomposable solvable Lie group having 2-dimensional commutator subgroup and at most one 1-dimensional normal subgroup. If $Mult(L)$ has 2-dimensional centre, then $Mult(L)$ is either the group $\mathcal{F}_4 \times \mathbb{R}$ or the direct product of $\mathbb{R}^2$ and a 3-dimensional Lie group having 2-dimensional commutator subgroup (see Theorem 18).

We want to mention that a Lie group need not to be the multiplication group of a topological loop if its universal covering has this property. We illustrate this for the direct product Ω of $\mathbb{R}^2$ and the group of orientation preserving motions of the euclidean plane and the universal covering of Ω (Theorem 18 case 6) and Proposition 19).

As our result did not give any example of a 3-dimensional topological loop L having an indecomposable solvable Lie group as the multiplication group of L, further investigations should be focused on this type of groups.

2. Preliminaries

A binary system $(L, \cdot)$ is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution, which we denote by $y = a \backslash b$ and $x = b/a$. A loop L is proper if it is not a group.

The left and right translations $\lambda_a : y \mapsto a \cdot y : L \times L \to L$ and $\rho_a : y \mapsto y \cdot a : L \times L \to L$, $a \in L$, are bijections of L. The permutation group $Mult(L)$ generated by all left and right translations of the loop L is called the multiplication group of L and the stabilizer of $e \in L$ in the group $Mult(L)$ is called the inner mapping group $Inn(L)$ of L.

Let K be a group, let $S \leq K$, and let A and B be two left transversals to S in K. We say that A and B are S-connected if $a^{-1}b^{-1}ab \in S$ for every $a \in A$ and $b \in B$. The core $Co_K(S)$ of S in K is the largest normal subgroup of K contained in S. If L is a loop, then $\Lambda(L) = \{\lambda_a;\ a \in L\}$

and $R(L) = \{\rho_a;\ a \in L\}$ are $Inn(L)$-connected transversals in the group $Mult(L)$, and the core of $Inn(L)$ in $Mult(L)$ is trivial. We often use the following (see [14], Theorem 4.1 and Proposition 2.7).

Lemma 1. *A group K is isomorphic to the multiplication group of a loop if and only if there exists a subgroup S with $Co_K(S) = 1$ and S-connected transversals A and B satisfying $K = \langle A, B \rangle$.*

Lemma 2. *Let L be a loop with multiplication group $Mult(L)$ and inner mapping group $Inn(L)$. Then the normalizer $N_{Mult(L)}(Inn(L))$ is the direct product $Inn(L) \times Z(Mult(L))$, where $Z(Mult(L))$ is the centre of the group $Mult(L)$.*

The kernel of a homomorphism $\alpha : (L, \cdot) \to (L', *)$ of a loop L into a loop L' is a normal subloop N of L. The centre $Z(L)$ of a loop L consists of all elements z which satisfy the equations $zx \cdot y = z \cdot xy,\ x \cdot yz = xy \cdot z,\ xz \cdot y = x \cdot zy,\ zx = xz$ for all $x, y \in L$. If we put $Z_0 = e$, $Z_1 = Z(L)$ and $Z_i/Z_{i-1} = Z(L/Z_{i-1})$, then we obtain a series of normal subloops of L. If Z_{n-1} is a proper subloop of L but $Z_n = L$, then L is centrally nilpotent of class n. The next assertion was proved by Albert in [1], Theorems 3, 4 and 5 and by Bruck in [2], IV.1, Lemma 1.3.

Lemma 3. *Let L be a loop with multiplication group $Mult(L)$ and identity element e.*

(i) Let α be a homomorphism of the loop L onto the loop $\alpha(L)$ with kernel

N. Then α induces a homomorphism of the group $Mult(L)$ onto the group $Mult(\alpha(L))$.

Let $M(N)$ be the set $\{m \in Mult(L);\ xN = m(x)N$ for all $x \in L\}$. Then $M(N)$ is a normal subgroup of $Mult(L)$ containing the multiplication group $Mult(N)$ of the loop N and the multiplication group of the factor loop L/N is isomorphic to $Mult(L)/M(N)$.

(ii) For every normal subgroup $\mathcal{N}$ of $Mult(L)$ the orbit $\mathcal{N}(e)$ is a normal subloop of L. Moreover, $\mathcal{N} \le M(\mathcal{N}(e))$.

A loop L is called topological if L is a topological space and the binary operations $(x,y) \mapsto x \cdot y,\ (x,y) \mapsto x \backslash y, (x,y) \mapsto y/x : L \times L \to L$ are continuous. Let G be a connected Lie group, let H be a subgroup of G. A continuous section $\sigma : G/H \to G$ is called sharply transitive, if the set $\sigma(G/H)$ operates sharply transitively on G/H, which means that for any xH and yH there exists precisely one $z \in \sigma(G/H)$ with $zxH = yH$. Every connected topological loop L having a Lie group G as the group topologically generated by the left translations of L is obtained on a homogeneous space G/H, where H is a closed subgroup of G with $Co_G(H) = 1$ and $\sigma : G/H \to G$ is a continuous sharply transitive section such that $\sigma(H) = 1 \in G$ and the subset $\sigma(G/H)$ generates G. The multiplication of L on the manifold G/H is defined by $xH * yH = \sigma(xH)yH$ and the group G is the group topologically generated by the left translations of L. Moreover, the subgroup H is the stabilizer of the identity element $e \in L$ in the group G. The following assertion is proved in [9], IX.1.

Lemma 4. *For any connected topological loop there is a universal covering loop. This loop is simply connected.*

The elementary filiform Lie group $\mathcal{F}_n$ is the simply connected Lie group of dimension $n \geq 3$ such that its Lie algebra has a basis $\{e_1, \cdots, e_n\}$ with $[e_1, e_i] = e_{i+1}$ for $2 \leq i \leq n-1$. A 2-dimensional simply connected loop $L_{\mathcal{F}}$ is called an elementary filiform loop if its multiplication group is an elementary filiform group $\mathcal{F}_n$, $n \geq 4$ ([5]).

Homogeneous spaces of solvable Lie groups are called solvmanifolds.

3. Three-dimensional topological loops with one-dimensional connected normal subloop

Let L be a topological loop on a connected 3-dimensional manifold such that the group $Mult(L)$ topologically generated by all left and right translations of L is a Lie group. The loop L is a 3-dimensional homogeneous space with respect to the transformation group $Mult(L)$ acting transitively and effectively on L. According to Theorem B and Theorem 1 in [10] the simply connected spaces $S^2 \times \mathbb{R}$ and S^3 are not solvmanifolds. Hence from [8] we get the following.

Lemma 5. *Let L be a 3-dimensional proper connected topological loop such that its multiplication group $Mult(L)$ is a solvable Lie group. If L is simply connected, then it is homeomorphic to $\mathbb{R}^3$.*

Assume that the multiplication group $Mult(L)$ of a topological loop L is solvable. Let K be a minimal non-trivial connected normal subgroup of $Mult(L)$. Then one has dim $K \in \{1, 2\}$. By Lemma 3 the orbit $K(e)$ is a connected normal subloop of L. Since the core $Co_{Mult(L)}(Inn(L))$ is trivial $K(e) \neq \{e\}$. Hence the dimension of $K(e)$ is 1 or 2. Now we deal with the case that dim $K(e) = 1$.

Theorem 6. *Let L be a 3-dimensional proper connected simply connected topological loop such that its multiplication group $Mult(L)$ is a solvable Lie group. If L has a 1-dimensional connected normal subloop N, then N is isomorphic to the group $\mathbb{R}$ and we have the following possibilities:*

(a) The factor loop L/N is isomorphic to $\mathbb{R}^2$. Then N is contained in the centre of L and the group $Mult(L)$ is a semidirect product of a group $Q \cong \mathbb{R}^2$ with the abelian group $M = Z \times Inn(L) \cong \mathbb{R}^m$, $m \geq 2$, where $\mathbb{R} = Z \cong N$ is a central subgroup of $Mult(L)$.

(b) The loop L/N is isomorphic either to the non-abelian 2-dimensional Lie group $\mathcal{L}_2$ or to a 2-dimensional elementary filiform loop $L_{\mathcal{F}}$. Then the group $Mult(L)$ has a normal subgroup S containing $Mult(N) \cong \mathbb{R}$ such that the factor group $Mult(L)/S$ is isomorphic to the direct product $\mathcal{L}_2 \times \mathcal{L}_2$ if $L/N \cong \mathcal{L}_2$ or to an elementary filiform Lie group $\mathcal{F}_n$, $n \geq 4$, if $L/N \cong L_{\mathcal{F}}$. Moreover, $Mult(L)$ has dimension at least 5.

Proof. By Lemma 5 the loop L is homeomorphic to $\mathbb{R}^3$. The connected normal subloop N of L is isomorphic to $\mathbb{R}$ because the multiplication group of N a Lie subgroup of $Mult(L)$ (Theorem 18.18 in [17]). The factor loop L/N is a 2-dimensional connected loop such that the multiplication group $Mult(L/N)$ is a factor group of $Mult(L)$ (Lemma 3). The manifold L is a fibering of $\mathbb{R}^3$ over L/N with fibers homeomorphic to $\mathbb{R}$. Hence L/N is homeomorphic to $\mathbb{R}^2$ and therefore it is either a 2-dimensional connected Lie group or an elementary filiform loop $L_{\mathcal{F}}$ (Theorem 1 in [4]).

If the factor loop L/N is the Lie group $\mathbb{R}^2$, then by Lemma 3 there exists a normal subgroup M of $Mult(L)$ such that $Mult(L)/M$ is isomorphic to the multiplication group of the loop L/N and hence to the group $\mathbb{R}^2$. Therefore the group M is connected and $Mult(L)/M$ operates sharply transitively on the orbits of N in L. The group M contains the multiplication group $Mult(N) \cong \mathbb{R}$ of N and leaves every orbit of N in the manifold L invariant. Every orbit of N is homeomorphic to $\mathbb{R}$. Hence the group M induces on the orbit $N(e)$ either the sharply transitive group $\mathbb{R}$ or the group Ω isomorphic to the Lie group $\mathcal{L}_2$ ([18], Lemma 1.10).

Assume first that the group induced by M on $N(e)$ is $\Omega \cong \mathcal{L}_2$. Then M induces a group isomorphic to Ω on every orbit $N(x)$, $x \in L$. Since all 1-dimensional connected subgroups of Ω different from the commutator subgroup are conjugate, the stabilizer Ω_e of $e \in L$ in Ω would fix on every orbit $N(x)$ precisely one point. The set of fixed points of Ω_e in L coincides with that of fixed points of the stabilizer $Inn(L)$ of $e \in L$ in $Mult(L)$. This latter is the centre Z of L (see [2], IV.1). Hence the centre Z of L would be at least 2-dimensional and we would have $L = N \cdot Z$. But then L would be an abelian group which is a contradiction.

Therefore the group M induces on every orbit $N(x)$, $x \in L$, the sharply transitive group $\mathbb{R}$. The stabilizer M_1 of $e \in L$ in M fixes every point of the orbit $N(e) = M(e)$. Hence M_1 is a normal subgroup of M. Since the factor group M/M_1 is isomorphic to $\mathbb{R}$ the commutator subgroup M' of M is contained in M_1 and M' is normal in $Mult(L)$. If M' were different from $\{1\}$, then $Mult(L)$ would contain the normal subgroup M' which has fixed points. This is a contradiction because the transitive group $Mult(L)$ acts effectively on L. Hence M is abelian. If M would contain a compact connected subgroup $K \neq \{1\}$, then K would be isomorphic to the group $SO_2(\mathbb{R})$ and it would be a normal subgroup of $Mult(L)$ which has a fixed point in L. This contradiction yields that M is isomorphic to $\mathbb{R}^n$. Since L is a proper loop of dimension 3 one has dim $Mult(L) \geq 4$ and hence $n \geq 2$. As the inner mapping group $Inn(L)$ has codimension 3 it is the group M_1. Since M_1 fixes every element of the loop $N(e)$ the normal subloop N is a central subgroup of L. The group consisting of the translations by elements of N is isomorphic to N and it is a central subgroup Z of $Mult(L)$. Then we have $M = Z \times Inn(L)$ and the assertion (a) is proved.

If the factor loop L/N is isomorphic to the Lie group $\mathcal{L}_2$, respectively to an elementary filiform loop $L_{\mathcal{F}}$, then the multiplication group $Mult(L/N)$ is isomorphic to the direct product $\mathcal{L}_2 \times \mathcal{L}_2$, respectively to an elementary filiform Lie group $\mathcal{F}_n$, $n \geq 4$. Moreover, there exists a normal subgroup S of $Mult(L)$ containing the group $Mult(N) \cong \mathbb{R}$ (see Lemma 3) such that $Mult(L)/S$ is isomorphic to the group $Mult(L/N)$ and the assertion (b) follows. □

4. Three-dimensional topological loops with four-dimensional solvable Lie group as multiplication group do not exist

The following Lemma follows from Theorem 18.18 in [17], Theorem 1 in [4] and Theorem 6 (a).

Lemma 7. *If there exists proper connected topological loop L having a* 4*-dimensional solvable non-nilpotent Lie group as its multiplication group* $Mult(L)$*, then L has dimension* 3*. Moreover, if L is simply connected and has a* 1*-dimensional normal subloop, then $Mult(L)$ is a semidirect product of $\mathbb{R}^2$ with a normal subgroup $M \cong \mathbb{R}^2$ containing a* 1*-dimensional central subgroup of $Mult(L)$.*

The 4-dimensional indecomposable Lie algebras are listed in [11], § 5. Among these solvable Lie algebras there are four with 1-dimensional centre: the filiform Lie algebra $g_{4,1}$ and the non-nilpotent Lie algebras $g_{4,3}$, $g_{4,8}$ with $h = -1$, $g_{4,9}$ with $p = 0$. Proposition 4.3 in [5] shows that the filiform Lie group $\mathcal{F}_4$ is not the multiplication group of 3-dimensional connected topological loops. Since the commutator Lie algebra of $g_{4,8}$ and $g_{4,9}$ has dimension 3 there is no connected topological loop L having these Lie algebras as the Lie algebra of $Mult(L)$ (see Lemma 7 and Theorem 6 (a)).

The commutator Lie algebra of $g_{4,3}$ has dimension 2. Hence for the corresponding simply connected Lie group G it seems to be more natural that G can be the multiplication group $Mult(L)$ of connected topological loops. Although, as we will show, there are four classes of 3-dimensional simply connected topological loops L having G as the group generated by their left translations (Theorem 10), for any of these loops the multiplication group $Mult(L)$ has dimension greater than 4 (Corollary 11). For the classification of these loops L we often use the following lemmata, the first of which is proved in [5] Lemma 4.2, and the second in [6] Lemma 3.1.

Lemma 8. *Let* $f : (x, y, z) \mapsto f(x, y, z) : \mathbb{R}^3 \to \mathbb{R}$ *be a continuous function. The function* $g : z \mapsto z + uf(x_0, y_0, z) : \mathbb{R} \to \mathbb{R}$ *is bijective for every* $x_0, y_0, u \in \mathbb{R}$ *if and only if* f *does not depend on the variable* z.

Lemma 9. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that for all* $z_1, z_2 \in \mathbb{R}$ *one has* $f(z_2) + e^{-z_2} f(z_1) = f(z_1 + z_2)$. *Then we get* $f(z) = c(1 - e^{-z})$, *where* c *is a real constant.*

Theorem 10. *Let* G *be the four-dimensional connected simply connected solvable Lie group the multiplication of which is represented on* $\mathbb{R}^4$ *by*

$$g(x_1, x_2, x_3, x_4)g(y_1, y_2, y_3, y_4) = g(x_1 + y_1 e^{x_4}, x_2 + y_2 + x_4 y_3, x_3 + y_3, x_4 + y_4).$$

Let H *be a non-normal subgroup of* G *isomorphic to* $\mathbb{R}$. *Using suitable* *automorphisms of G we may choose H as one of the following subgroups:*

$$H_1 = \{g(0,0,0,x_4); x_4 \in \mathbb{R}\},\ H_2 = \{g(0,0,x_3,0); x_3 \in \mathbb{R}\},$$

$$H_3 = \{g(x_3,0,x_3,0); x_3 \in \mathbb{R}\},\ H_4 = \{g(x_1,x_1,0,0); x_1 \in \mathbb{R}\}.$$

a) Every continuous sharply transitive section $\sigma : G/H_1 \to G$ with the properties that $\sigma(G/H_1)$ generates G and $\sigma(H_1) = 1$ is determined by the map $\sigma_f : g(x,y,z,0)H_1 \mapsto g(x,y,z,f(z))$, where $f : \mathbb{R} \to \mathbb{R}$ is a continuous non-linear function with $f(0) = 0$. The multiplication of the loop L_f given by σ_f can be written as

$$(x_1,y_1,z_1) * (x_2,y_2,z_2) = (x_1 + x_2 e^{f(z_1)}, y_1 + y_2 + z_2 f(z_1), z_1 + z_2). \quad (1)$$

b) Each continuous sharply transitive section $\sigma : G/H_2 \to G$ such that $\sigma(G/H_2)$ generates G and $\sigma(H_2) = 1$ has the form

$$\sigma_h : g(x,y,0,z)H_2 \mapsto g(x, y + h(x,z)z, h(x,z), z),$$

where $h : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function with $h(0,0) = 0$ such that h does not fulfil the identities $h(x,0) = 0$ and $h(0,z) = lz$, $l \in \mathbb{R}$, simultaneously. The multiplication of the loop L_h corresponding to σ_h is determined by

$$(x_1,y_1,z_1) * (x_2,y_2,z_2) = (x_1 + x_2 e^{z_1}, y_1 + y_2 - z_2 h(x_1,z_1), z_1 + z_2). \quad (2)$$

c) Every continuous sharply transitive section $\sigma : G/H_3 \to G$ such that $\sigma(G/H_3)$ generates G and $\sigma(H_3) = 1$ is given by the map

$$\sigma_f : g(x,y,0,z)H_3 \mapsto g(x + e^z f(x,y,z), y + z f(x,y,z), f(x,y,z), z)$$

with a continuous function $f : \mathbb{R}^3 \to \mathbb{R}$ *such that* $f(0,0,0) = 0$, f *does not satisfy either the identities*

$$f(x,y,0) = -x,\ f(0,0,z) = C(1-e^{-z}),\ C \in \mathbb{R}, \tag{3}$$

or the identities

$$f(x,y,0) = 0,\ f(0,0,z) = \lambda z,\ \lambda \in \mathbb{R}, \tag{4}$$

simultaneously and for all triples (x_1,y_1,z_1) *and* $(x_2,y_2,z_2) \in \mathbb{R}^3$ *the equations*

$$y = y_2 - y_1 + z_1 f(x,y,z_2 - z_1), \tag{5}$$

$$x = x_2 - e^{z_2 - z_1} x_1 + e^{z_2}(1 - e^{-z_1}) f(x,y,z_2 - z_1) \tag{6}$$

have a unique solution $(x,y) \in \mathbb{R}^2$. *The loop* L_f *corresponding to* σ_f *is defined by the multiplication*

$$(x_1,y_1,z_1) * (x_2,y_2,z_2) =$$

$$(x_1 + e^{z_1}(x_2 + f(x_1,y_1,z_1)(1-e^{z_2})), y_1 + y_2 - z_2 f(x_1,y_1,z_1), z_1 + z_2). \tag{7}$$

d) Any continuous sharply transitive section $\sigma : G/H_4 \to G$ *such that* $\sigma(G/H_4)$ *generates* G *and* $\sigma(H_4) = 1$ *is determined by the map*

$$\sigma_k : g(x,0,y,z)H_4 \mapsto g(x + e^z k(x,y,z), k(x,y,z), y, z),$$

where $k : \mathbb{R}^3 \to \mathbb{R}$ *is a continuous function with* $k(0,0,0) = 0$ *such that* k *does not fulfil the identities given by (3) in case c) simultaneously and such that for all triples* (x_1,y_1,z_1) *and* $(x_2,y_2,z_2) \in \mathbb{R}^3$ *the equation*

$$x + e^{z_2} k(x, y_2 - y_1, z_2 - z_1)[e^{-z_1} - 1] = x_2 - x_1 e^{z_2 - z_1} + e^{z_2}(z_2 - z_1) y_1 \tag{8}$$

has a unique solution $x \in \mathbb{R}$. The multiplication of the loop L_k corresponding to σ_k can be written as

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) =$$

$$(x_1 + e^{z_1}[x_2 + k(x_1, y_1, z_1) - e^{z_2}(z_1 y_2 + k(x_1, y_1, z_1))], y_1 + y_2, z_1 + z_2). \quad (9)$$

Proof. The linear representation of the group G is given in [7], Case 4.3. Let L be a 3-dimensional connected simply connected topological loop having G as the group topologically generated by its left translations. Then the stabilizer H of $e \in L$ in G is a 1-dimensional non-normal subgroup of G. As the Lie algebra $\mathbf{g}$ of G has a basis $\{e_1, e_2, e_3, e_4\}$ with $[e_1, e_4] = e_1$, $[e_3, e_4] = e_2$, the subgroup $\exp te_2$, $t \in \mathbb{R}$, is the centre of G, the subgroup $\exp(te_2 + se_1)$, $t, s \in \mathbb{R}$, is the commutator subgroup of G. Hence the automorphism group of $\mathbf{g}$ consists of the following linear mappings $\varphi(e_1) = ae_1$, $\varphi(e_2) = be_2$, $\varphi(e_3) = ke_2 + be_3$, $\varphi(e_4) = l_1 e_1 + l_2 e_2 + l_3 e_3 + e_4$, with $ab \neq 0$, $k, l_1, l_2, l_3 \in \mathbb{R}$. Since $\mathbb{R}e_1$ and $\mathbb{R}e_2$ are ideals of $\mathbf{g}$ the subalgebra $\mathbf{h}$ of H does not contain e_1, e_2. Hence H is a subgroup $\exp t(\alpha e_1 + \beta e_2 + \gamma e_3 + \delta e_4)$ with $t \in \mathbb{R}$ such that $\gamma^2 + \delta^2 = 1$ or $\alpha\beta \neq 0$. Then a suitable automorphism of G corresponding to an automorphism φ of $\mathbf{g}$ maps H onto one of the following subgroups

$$H_1 = \exp te_4, \quad H_2 = \exp te_3, \quad H_3 = \exp t(e_3 + e_1), \quad H_4 = \exp t(e_1 + e_2).$$

Every connected topological proper loop L having G as the group topologically generated by its left translations and H as the stabilizer of $e \in L$ in G

is determined by a continuous sharply transitive section $\sigma : G/H \to G$ with the properties that $\sigma(H) = 1 \in G$ and $\sigma(G/H)$ generates G.

First we assume that $H = H_1 = \{g(0,0,0,k);\ k \in \mathbb{R}\}$. Since all elements of G have a unique decomposition as $g(x,y,z,0)g(0,0,0,k)$, any continuous function $f : \mathbb{R}^3 \to \mathbb{R}; (x,y,z) \mapsto f(x,y,z)$ determines a continuous section $\sigma : G/H_1 \to G$ given by

$$\sigma : g(x,y,z,0)H_1 \mapsto g(x,y,z,0)g(0,0,0,f(x,y,z)) = g(x,y,z,f(x,y,z)).$$

The section σ is sharply transitive if and only if for any triple (x_1, y_1, z_1), $(x_2, y_2, z_2) \in \mathbb{R}^3$ there exists precisely one triple $(x, y, z) \in \mathbb{R}^3$ such that

$$g(x,y,z,f(x,y,z))g(x_1,y_1,z_1,0) = g(x_2,y_2,z_2,0)g(0,0,0,t)$$

for a suitable $t \in \mathbb{R}$. This provides the following equations $z = z_2 - z_1$, $t = f(x, y, z_2 - z_1)$,

$$y = y_2 - y_1 - z_1 f(x, y, z_2 - z_1), \tag{10}$$

$$x = x_2 - x_1 e^{f(x,y,z_2-z_1)}. \tag{11}$$

For $x_1 = 0$ equation (11) yields that $x = x_2$ and equation (10) has a unique solution for y if and only if the function $g : y \mapsto y + z_1 f(x_0, y, z_0) : \mathbb{R} \to \mathbb{R}$ is bijective for every $x_0 = x_2$, $z_0 = z_2 - z_1$ and $z_1 \in \mathbb{R}$. This is the case precisely if the function $f(x,y,z) = f(x,z)$ does not depend on the variable y (Lemma 8). Using this, equations (10) and (11) are reduced to

$$y = y_2 - y_1 - z_1 f(x, z_2 - z_1), \tag{12}$$

$$x = x_2 - x_1 e^{f(x, z_2 - z_1)}. \tag{13}$$

Applying Lemma 8 for the function $e^{f(x,z)} : \mathbb{R}^2 \to \mathbb{R}$ we obtain that equation (13) has a unique solution for x precisely if the function $f(x,z) = f(z)$ does not depend on x. Since in this case equation (12) has a unique solution $y = y_2 - y_1 - z_1 f(z_2 - z_1)$ each continuous function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ defines a loop L_f. This loop is proper if $\sigma(G/H_1)$ generates G. The set $\sigma(G/H_1) = \{g(x,y,z,f(z)); x,y,z \in \mathbb{R}\}$ contains the commutator subgroup $G' = \{g(x,y,0,0); x,y \in \mathbb{R}\}$ and the set $F = \{g(0,0,z,f(z)); z \in \mathbb{R}\}$. We have $G' \cap F = \{1\}$. Therefore $\sigma(G/H_1)$ does not generate G if the set FG'/G' is a one-parameter subgroup of G/G'. As

$$g(\mathbb{R},\mathbb{R},z_1,f(z_1))g(\mathbb{R},\mathbb{R},z_2,f(z_2)) = g(\mathbb{R},\mathbb{R},z_1+z_2,f(z_1)+f(z_2))$$

this is the case precisely if $f(z) = lz$, $l \in \mathbb{R}$. Hence for every non-linear function f there is a topological proper loop L_f.

In the coordinate system $(x,y,z) \mapsto g(x,y,z,0)H_1$ the multiplication of L_f is determined if we apply $\sigma(g(x_1,y_1,z_1,0)H_1) = g(x_1,y_1,z_1,f(z_1))$ to the left coset $g(x_2,y_2,z_2,0)H_1$ and find in the image coset the element of G which lies in the set $\{g(x,y,z,0)H_1;\ x,y,z \in \mathbb{R}\}$. A direct computation yields multiplication (1) and assertion a) is proved.

A similar consideration as in the previous case yields that for $H = H_2 = \{g(0,0,k,0);\ k \in \mathbb{R}\}$ an arbitrary continuous section $\sigma_2 : G/H_2 \to G$ may be given by $\sigma_2 : g(x,y,0,z)H_2 \mapsto$

$$g(x,y,0,z)g(0,0,h(x,y,z),0) = g(x,y+zh(x,y,z),h(x,y,z),z), \qquad (14)$$

for $H = H_3 = \{g(t,0,t,0);\ t \in \mathbb{R}\}$ a continuous section $\sigma_3 : G/H_3 \to G$ can be given by

$$\sigma_3 : g(x,y,0,z)H_3 \mapsto g(x,y,0,z)g(f(x,y,z),0,f(x,y,z),0) =$$

$$g(x+e^z f(x,y,z), y+zf(x,y,z), f(x,y,z), z), \qquad (15)$$

and for $H = H_4 = \{g(t,t,0,0);\ t \in \mathbb{R}\}$ a continuous section $\sigma_4 : G/H_4 \to G$ may be given by

$$\sigma_4 : g(x,0,y,z)H_4 \mapsto g(x,0,y,z)g(k(x,y,z),k(x,y,z),0,0) =$$

$$g(x+e^z k(x,y,z), k(x,y,z), y, z), \qquad (16)$$

where $h : \mathbb{R}^3 \to \mathbb{R}$, $f : \mathbb{R}^3 \to \mathbb{R}$, $k : \mathbb{R}^3 \to \mathbb{R}$ are continuous functions. These sections σ_i, $i = 2,3,4$, have the property $\sigma_i(H) = 1 \in G$ precisely if $h(0,0,0) = f(0,0,0) = k(0,0,0) = 0$.

The set $\sigma_i(G/H_i)$ given by (14), (15), (16) acts sharply transitively on G/H_i if and only if for $i = 2$ the equation

$$g(x,y+zh(x,y,z),h(x,y,z),z)g(x_1,y_1,0,z_1) = g(x_2,y_2,0,z_2)g(0,0,t,0), \qquad (17)$$

for $i = 3$ the equation

$$g(x+e^z f(x,y,z), y+zf(x,y,z), f(x,y,z), z)g(x_1,y_1,0,z_1) =$$

$$g(x_2, y_2, 0, z_2)g(t, 0, t, 0), \qquad (18)$$

for $i = 4$ the equation

$$g(x + e^z k(x, y, z), k(x, y, z), y, z)g(x_1, 0, y_1, z_1) = g(x_2, 0, y_2, z_2)g(t, t, 0, 0) \qquad (19)$$

has a unique solution $(x, y, z) \in \mathbb{R}^3$ with a suitable $t \in \mathbb{R}$ for any given triple (x_1, y_1, z_1), $(x_2, y_2, z_2) \in \mathbb{R}^3$. Equation (17) is equivalent to the following $z = z_2 - z_1$, $t = h(x, y, z_2 - z_1)$, $x = x_2 - e^{z_2 - z_1} x_1$ and

$$0 = y - y_2 + y_1 - z_1 h(x_2 - e^{z_2 - z_1} x_1, y, z_2 - z_1).$$

This last equation has a unique solution for y precisely if the function $h(x, y, z) = h(x, z)$ does not depend on the variable y (cf. Lemma 8). Equation (18) yields $z = z_2 - z_1$, $t = f(x, y, z_2 - z_1)$ and that equations (5), (6) in assertion c) have a unique solution $(x, y) \in \mathbb{R}^2$. Moreover, equation (19) gives $z = z_2 - z_1$, $y = y_2 - y_1$, $t = y_1(z_2 - z_1) + k(x, y_2 - y_1, z_2 - z_1)$ and that equation (8) in assertion d) has a unique solution $x \in \mathbb{R}$.

Now we investigate under which circumstances the set $\sigma_i(G/H_i)$, $i = 2, 3, 4$, generates the group G.

The set $\sigma_2(G/H_2) = \{g(x, y + zh(x, z), h(x, z), z); x, y, z \in \mathbb{R}\}$ contains the subgroup $K_2 = \{g(x, y, h(x, 0), 0);\ x, y \in \mathbb{R}\}$ and the subset $F_2 = \{g(0, zh(0, z), h(0, z), z);\ z \in \mathbb{R}\}$. The set $\sigma_3(G/H_3)$ given by (15) includes the subgroup $K_3 = \{g(x + f(x, y, 0), y, f(x, y, 0), 0);\ x, y \in \mathbb{R}\}$, and the subset $F_3 = \{g(e^z f(0, 0, z), zf(0, 0, z), f(0, 0, z), z);\ z \in \mathbb{R}\}$. The set $\sigma_4(G/H_4)$

given by (16) contains the subgroup $K_4 = \{g(x+k(x,y,0), k(x,y,0), y, 0);\ x, y \in \mathbb{R}\}$ and the subset $F_4 = \{g(e^z k(0,0,z), k(0,0,z), 0, z);\ z \in \mathbb{R}\}$. As for all these cases we have $K_i \cap F_i = \{1\}$ the set $\sigma_i(G/H_i)$, $i = 2,3,4$, does not generate G if the group K_i has dimension 2, for all $h \in F_i$ one has $h^{-1} K_i h = K_i$ and $F_i K_i / K_i$ is a one-parameter subgroup of G/K_i.

First we consider the pair (K_2, F_2). The group K_2 has dimension 2 if the subgroup $\{g(x, 0, h(x,0), 0); x \in \mathbb{R}\}$ is a one-parameter subgroup. This is the case precisely if $h(x,0) = bx$, $b \in \mathbb{R}$. For $h = g(0, zh(0,z), h(0,z), z) \in F_2$, $z \neq 0$ we get $h^{-1} g(x, y, bx, 0) h = g(xe^{-z}, y - bzx, bx, 0)$ is an element of K_2 if and only if $b = 0$. Then the group K_2 coincides with the commutator subgroup G' of G. The set $(F_2 G')/G'$ is a one-parameter subgroup precisely if $h(0,z) = lz$, $l \in \mathbb{R}$. Therefore any function $h : \mathbb{R}^2 \to \mathbb{R}$, which does not satisfy the identities $h(x,0) = 0$ and $h(0,z) = lz$, $l \in \mathbb{R}$, simultaneously determines a proper topological loop L_h. A direct computation yields that the multiplication of L_h corresponding to the section σ_2 in the coordinate system $(x,y,z) \mapsto g(x,y,0,z)H_2$ is given by (2). This proves assertion b).

Now we deal with the pair (K_3, F_3). The group K_3 has dimension 2 if and only if $f(x,y,0) = cx + dy$, $c, d \in \mathbb{R}$. For $h \in F_3$ with $z \neq 0$ we have

$$h^{-1} g(x+cx+dy, y, cx+dy, 0) h = g([(c+1)x+dy]e^{-z}, y-z(cx+dy), cx+dy, 0).$$

Hence $h^{-1} K_3 h = K_3$ if and only if one has either $c = -1$, $d = 0$ or $c = d = 0$. In the first case K_3 is the normal subgroup $\widetilde{G} = \{g(0, y, -x, 0);\ x, y \in \mathbb{R}\}$

of G, in the second case $K_3 = G'$. Since

$$g(e^{z_1} f(0,0,z_1), \mathbb{R}, \mathbb{R}, z_1) g(e^{z_2} f(0,0,z_2), \mathbb{R}, \mathbb{R}, z_2) =$$

$$g(e^{z_1+z_2} f(0,0,z_2) + e^{z_1} f(0,0,z_1), \mathbb{R}, \mathbb{R}, z_1 + z_2)$$

the set $F_3 \widetilde{G}/\widetilde{G}$ is a one-parameter subgroup of $G/\widetilde{G}$ if and only if for all z_1, $z_2 \in \mathbb{R}$ the identity $f(0,0,z_2) + e^{-z_2} f(0,0,z_1) = f(0,0,z_1 + z_2)$ holds. By Lemma 9 we obtain $f(0,0,z) = C(1 - e^{-z})$ with $C \in \mathbb{R}$. The set $F_3 G'/G'$ is a one-parameter subgroup of G/G' if and only if one has $f(0,0,z) = \lambda z$ for some $\lambda \in \mathbb{R}$. The set $\sigma_3(G/H_3)$ does not generate G if the function $f(x,y,z)$ satisfies either the identities given by (3) or the identities given by (4) in assertion c). A direct computation yields that the multiplication of the loop L_f corresponding to the section σ_3 in the coordinate system $(x,y,z) \mapsto g(x,y,0,z)H_3$ is given by (7) and the assertion c) is proved.

Finally we consider the pair (K_4, F_4). The group K_4 has dimension 2 if and only if $k(x,y,0) = ax + by$, $a, b \in \mathbb{R}$. For $h \in F_4$, $z \neq 0$ we have

$$h^{-1} g(x + ax + by, ax + by, y, 0) h = g([(a+1)x + by]e^{-z}, -zy + ax + by, y, 0).$$

Hence we obtain $h^{-1} K_4 h = K_4$ if and only if $a = -1$ and $b = 0$. Then the group K_4 coincides with the group $\widetilde{G}$ introduced in the previous case. Hence the same consideration as there proves that the set $\sigma_4(G/H_4)$ does not generate G if the function $k(x,y,z)$ satisfies the identities given by (3). A direct computation gives that in the coordinate system $(x,y,z) \mapsto g(x,0,y,z)H_4$

the multiplication of the loop L_k is given by (9) and the assertion d) is proved. □

Corollary 11. *There is no connected topological loop L such that the multiplication group of L is locally isomorphic to the group G in Theorem 10.*

Proof. By Lemmata 4, 7, 5 we may assume that L is homeomorphic to $\mathbb{R}^3$. Every Lie group locally isomorphic to the group G in Theorem 10 has a 1-dimensional centre Z. The orbit $Z(e)$ is a 1-dimensional normal subloop of L isomorphic to $\mathbb{R}$ (see Lemma 3). Hence the multiplication group of L is the simply connected group G (cf. Lemma 7) and the normal subgroup $M \cong \mathbb{R}^2$ of G given in Theorem 6 (a) is the commutator subgroup $G' = \{\exp(te_1 + ue_2); t, u \in \mathbb{R}\}$ of G. Moreover, the inner mapping group $Inn(L)$ of L is a 1-dimensional non-normal subgroup of G'. Hence $Inn(L)$ must be the subgroup H_4 (see Theorem 10). The normalizer of H_4 in G is the group $N = \{\exp(t_1e_1 + t_2e_2 + t_3e_3); t_i \in \mathbb{R}\}$. As the direct product $Z \times Inn(L) = G'$ we have a contradiction to Lemma 2. □

Now we treat 4-dimensional solvable Lie groups which are direct products.

Proposition 12. *There exists no connected topological loop L such that the multiplication group of L is a 4-dimensional solvable Lie group which is the direct product of proper connected Lie groups.*

Proof. By Lemmata 4, 7 and 5 we may assume that the loop L is homeomorphic to $\mathbb{R}^3$. Every 4-dimensional solvable decomposable Lie group has a

1-dimensional normal subgroup N. As the orbit $N(e)$ is a 1-dimensional normal subloop of L it follows from Lemma 7 that the group $Mult(L)$ is simply connected and its centre has dimension ≥ 1. Hence $Mult(L)$ has the form $C \times S$, where C is the group $\mathbb{R}$ and S is a 3-dimensional simply connected Lie group. The orbit $C(e)$ is a 1-dimensional central subgroup of L isomorphic to $\mathbb{R}$ (see Theorem 11 in [1]). By Theorem 6 (a) there is a 2-dimensional normal subgroup M containing the group $C \cong \mathbb{R}$ and the commutator subgroup $Mult(L)' = S'$ of $Mult(L)$. Hence one has dim $Mult(L)' = 1$. Then $Mult(L)$ is isomorphic either to $G_1 = \mathbb{R}^2 \times \mathcal{L}_2$ or to $G_2 = \mathbb{R} \times \mathcal{F}_3$, where $\mathcal{F}_3$ is the 3-dimensional filiform Lie group. Proposition 5.1 (i) in [5] shows that the group G_2 is not the multiplication group of a topological loop L homeomorphic to $\mathbb{R}^3$.

Now we suppose that the group $Mult(L)$ is the group G_1 which is given on $\mathbb{R}^4$ by the multiplication

$$g(x_1, x_2, x_3, x_4)g(y_1, y_2, y_3, y_4) = g(x_1 + y_1, x_2 + y_2, x_3 + y_3, y_4 + x_4 e^{y_3}).$$

Then the centre Z of G_1 is the group $Z = \{g(x, y, 0, 0), x, y \in \mathbb{R}\}$ and the commutator subgroup of G_1 is the group $G_1' = \{g(0, 0, 0, z), z \in \mathbb{R}\}$. By Theorem 11 in [1] the orbit $Z(e)$ is the centre of L isomorphic to $\mathbb{R}^2$. Since the multiplication group $Mult(L/Z(e))$ of the factor loop $L/Z(e)$ is a factor group of G_1 (see Lemma 3) we get $L/Z(e)$ is the group $\mathbb{R}$ (see Theorem 18.18 in [17]). Hence there is a normal subgroup P of G_1 such that Z is a subgroup of P and the factor group G_1/P is isomorphic to the group $Mult(L/Z(e)) \cong \mathbb{R}$ (Lemma 3). Then one has $G_1' < P$ and therefore $P = Z \times G_1'$. As G_1/P acts sharply transitively on the orbits $Z(x)$, $x \in L$, the inner mapping group $Inn(L)$ of the loop L is a 1-dimensional subgroup of P with $Co_{G_1}(Inn(L)) = 1$. The Lie algebra $\mathbf{g_1}$ of G_1 has a basis $\{e_1, e_2, e_3, e_4\}$ with $[e_4, e_3] = e_4$. Hence the Lie algebra $\mathbf{p}$ of P is given by $\mathbf{p} = \langle e_1, e_2, e_4 \rangle$ and we may choose $Inn(L)$ as the subgroup $\exp t(e_4 + ae_1 + be_2)$, $t \in \mathbb{R}$, with $a \neq 0$ or $b \neq 0$. Each automorphism φ of $\mathbf{g_1}$ has the form $\varphi(e_1) = k_1e_1 + k_2e_2$, $\varphi(e_2) = l_1e_1 + l_2e_2$, $\varphi(e_4) = ne_4$, $\varphi(e_3) = a_1e_1 + a_2e_2 + a_3e_4 + e_3$ such that $(k_1l_2 - l_1k_2)n \neq 0$, $k_i, l_i, n, a_j \in \mathbb{R}$, $i = 1, 2$, $j = 1, 2, 3$. Then we can change $Inn(L)$ by an automorphism of G_1 such that $Inn(L) = \{\exp t(e_4 + e_1), t \in \mathbb{R}\} = \{g(u, 0, 0, u), u \in \mathbb{R}\}$.

According to Lemma 1 the group G_1 is isomorphic to the multiplication group $Mult(L)$ of a topological proper loop L having the subgroup $Inn(L)$ as its inner mapping group precisely if there are two left transversals A and B to $Inn(L)$ in G_1 such that $\{a^{-1}b^{-1}ab;\ a \in A, b \in B\}$ is contained in $Inn(L)$ and the set $\{A, B\}$ generates the group G_1. Arbitrary left transversals to the group $Inn(L)$ in G_1 are: $A = \{g(x, y, z, f(x, y, z)); x, y, z \in \mathbb{R}\}$ and $B = \{g(k, l, m, h(k, l, m));\ k, l, m \in \mathbb{R}\}$, where $f : \mathbb{R}^3 \to \mathbb{R}$, $h : \mathbb{R}^3 \to \mathbb{R}$ are continuous functions with $f(0, 0, 0) = h(0, 0, 0) = 0$. The products $a^{-1}b^{-1}ab$ with $a \in A$ and $b \in B$ are elements of $Inn(L)$ if and only if the equation $h(k, l, m)(1 - e^z) = f(x, y, z)(1 - e^m)$ holds for all $x, y, z, k, l, m \in \mathbb{R}$. Since the left hand side of the last equation does not depend on the variables x and

y and the right hand side is independent of k, l we have $h(k,l,m) = h(m)$, $f(x,y,z) = f(z)$ and it follows that $\frac{h(m)}{1-e^m} = \frac{f(z)}{1-e^z} = k$, where k is a real constant. Then both sets A and B consist of the centre Z of G_1 and the one-parameter subgroup $F = \{g(0,0,z,k(1-e^z)), z \in \mathbb{R}\}$ with $Z \cap F = \{1\}$. Hence $\{A, B\}$ does not generate the group G_1. This contradiction proves the assertion. □

Proposition 13. *A* 4*-dimensional connected Lie group having no normal subgroup of dimension* 1 *cannot be the multiplication group of a connected topological proper loop* L*.*

Proof. We may suppose that L is homeomorphic to $\mathbb{R}^3$ (see Lemmata 4, 7 and 5). Any 4-dimensional connected Lie group having no 1-dimensional normal subgroup is locally isomorphic to the group G given in Case 4.12 of [7]. The Lie algebra $\mathbf{g}$ of G is given by $[e_1, e_3] = e_1,\ [e_2, e_3] = e_2,\ [e_1, e_4] = -e_2,\ [e_2, e_4] = e_1$ (see $g_{4,10}$ in [11]).

The commutator subgroup G' of G is the 2-dimensional abelian normal subgroup $G' = \{g(x,y,0,0), x, y \in \mathbb{R}\}$. The orbit $G'(e)$ is a connected normal subloop of L with dimension 1 or 2. As G has discrete centre one has $\dim G'(e) = 2$ (see Lemma 7). The multiplication group of the subloop $G'(e)$ is a subgroup of G (see Lemma 3). Then $G'(e)$ is isomorphic to $\mathbb{R}^2$ because none of the groups $Mult(\mathcal{L}_2) = \mathcal{L}_2 \times \mathcal{L}_2$ and $Mult(L_{\mathcal{F}}) = \mathcal{F}_n$, $n \geq 4$, are contained in G. As the multiplication group of the factor loop $L/G'(e)$ is a factor group of $Mult(L)$ the loop $L/G'(e)$ is isomorphic to $\mathbb{R}$ (see Theorem

18.18 in [17]). Then there is a normal subgroup K of G such that G/K is isomorphic to the multiplication group $Mult(L/G'(e)) \cong \mathbb{R}$ (cf. Lemma 3). Therefore the group K has dimension 3, it contains the subgroup G' and leaves every orbit $G'(x)$, $x \in L$, in L invariant. Hence the Lie algebra $\mathbf{k}$ of K has one of the following forms: $\mathbf{k_1} = \langle e_1, e_2, e_4 + l e_3 \rangle$, $l \in \mathbb{R}$, $\mathbf{k_2} = \langle e_1, e_2, e_3 \rangle$. The Lie group K_1 of $\mathbf{k_1}$ has no 1-dimensional normal subgroup. For this reason K_1 cannot induce on the orbit $G'(e)$ a 2-dimensional group. Any 1-dimensional normal subgroup S of the Lie group K_2 of $\mathbf{k_2}$ is contained in the commutator subgroup $K_2' = G'$. Hence K_2/S is isomorphic to $\mathcal{L}_2$. As G' acts sharply transitively on $G'(e)$, for every element $s \in S \setminus \{1\}$ one has $s(e) \neq e$ and K_2 cannot induce on the orbit $G'(e)$ a 2-dimensional group.

Hence the group induced by K_i, $i = 1, 2$, on the orbit $G'(e)$ is isomorphic to K_i. Then K_i induces a group isomorphic to K_i on every orbit $G'(x)$, $x \in L$. The same consideration as for the group $\Omega \cong \mathcal{L}_2$ discussed in the proof of Theorem 6 (a) is valid for the groups K_i, $i = 1, 2$. Therefore the centre of L would be at least 1-dimensional and we have a contradiction to the fact that G has discrete centre. □

5. Five-dimensional solvable indecomposable Lie groups

There are 39 classes of 5-dimensional solvable indecomposable Lie algebras ([12]). Among them precisely the Lie algebras $g_{5,1}$ to $g_{5,6}$ are nilpotent. The non-nilpotent Lie algebras have at most a 1-dimensional centre. In this section we prove that there does not exist 3-dimensional connected topological loop L such that the Lie algebra of the group $Mult(L)$ of L is a 5-dimensional solvable non-nilpotent indecomposable Lie algebra.

Proposition 14. *There exists no* 3*-dimensional connected topological proper loop* L *such that the Lie algebra of its multiplication group is a* 5*-dimensional solvable indecomposable Lie algebra with trivial centre.*

Proof. We may assume that L is homeomorphic to $\mathbb{R}^3$ (see Lemmata 4 and 5). In [12] the 5-dimensional solvable indecomposable Lie algebras $\mathbf{g}$ with trivial centre are the Lie algebras $g_{5,7}$, $g_{5,9}$, the Lie algebras $g_{5,11}$ to $g_{5,13}$, the Lie algebras $g_{5,16}$ to $g_{5,18}$, $g_{5,21}$, $g_{5,23}$, $g_{5,24}$, $g_{5,27}$, the Lie algebras $g_{5,31}$ to $g_{5,37}$, the Lie algebras $g_{5,19}$, $g_{5,20}$ and $g_{5,28}$ in the case of that $\alpha \neq -1$, $g_{5,15}$ in the case of that $\gamma \neq 0$, $g_{5,25}$ in the case of that $\beta \neq 0$, $p \neq 0$, $g_{5,26}$ in the case of that $p \neq 0$, $g_{5,30}$ in the case of that $h \neq -2$.

All Lie algebras $\mathbf{g}$ from this list with exceptions of the Lie algebras $g_{5,17}$, $g_{5,18}$ and $g_{5,33}$ have the 1-dimensional ideal $\mathbf{n}_1 = \langle e_1 \rangle$ such that the factor algebras $\mathbf{g}/\mathbf{n}_1$ are not isomorphic to the Lie algebras of the groups $\mathcal{L}_2 \times \mathcal{L}_2$ or $\mathcal{F}_4$. As the centre of $\mathbf{g}$ is trivial these Lie algebras cannot be the Lie algebras of the multiplication groups of 3-dimensional topological loops (Theorem 6). The Lie algebra $g_{5,33}$ is defined by $[e_1, e_4] = e_1$, $[e_3, e_4] = \beta e_3$, $[e_2, e_5] = e_2$, $[e_3, e_5] = \gamma e_3$, where $\gamma^2 + \beta^2 \neq 0$. The factor algebra $g_{5,33}/\langle e_1 \rangle$, respectively $g_{5,33}/\langle e_2 \rangle$ is isomorphic to the Lie algebra of $\mathcal{L}_2 \times \mathcal{L}_2$ precisely if $\gamma = 0$, respectively $\beta = 0$. But for $\gamma = \beta = 0$ the Lie algebra $g_{5,33}$ is decomposable. Hence it remains to investigate the Lie algebras $g_{5,17}$ and $g_{5,18}$ which have no

1-dimensional ideal. We denote by G the Lie group of the Lie algebra $g_{5,17}$, respectively of $g_{5,18}$ and assume that G is the multiplication group $Mult(L)$ of L. In both cases we consider the normal subgroup $N = \{\exp(t_1 e_1 + t_2 e_2); t_i \in \mathbb{R}, i = 1, 2\}$ of G.

First we suppose that the orbit $N(e)$ is a one-dimensional connected normal subloop of L. By Lemma 3 the group G has a connected normal subgroup M containing the group N such that the factor group G/M is isomorphic to the multiplication group of the factor loop $L/N(e)$. Since $\dim M \geq \dim N = 2$ the dimension of G/M is ≤ 3. Hence by Theorem 6 the factor group G/M would be isomorphic to $\mathbb{R}^2$. As G has discrete centre we have a contradiction to Theorem 6 (a).

Therefore $N(e)$ is a two-dimensional connected normal subloop of L. The multiplication group $Mult(N(e))$ of $N(e)$ is a subgroup of $Mult(L) = G$. As none of the groups $Mult(\mathcal{L}_2) = \mathcal{L}_2 \times \mathcal{L}_2$ and $Mult(L_{\mathcal{F}}) = \mathcal{F}_n$, $n \geq 4$, are subgroups of G the normal subloop $N(e)$ is isomorphic to the group $\mathbb{R}^2$. The multiplication group of the factor loop $L/N(e)$ is isomorphic to $\mathbb{R}$ (see Theorem 18.18 in [17]). There exists a normal subgroup K of G such that the factor group G/K is isomorphic to $Mult(L/N(e)) \cong \mathbb{R}$ (see Lemma 3). Hence K contains the commutator subgroup G' of G. Since $\dim K = \dim G' = 4$ the group K coincides with the abelian group G'. Hence K induces on the orbit $N(e)$ the group $\mathbb{R}^2$. The stabilizer K_e of $e \in L$ in K fixes every point on the orbit $N(e) = K(e)$. The inner mapping group $Inn(L)$ of L is the group K_e. Hence $N(e)$ would be a 1-dimensional central subgroup of L which contradicts the fact that G has discrete centre and the assertion follows. □

Proposition 15. *Let L be a connected simply connected topological proper loop of dimension 3 such that the Lie algebra of its multiplication group is a 5-dimensional solvable non-nilpotent indecomposable Lie algebra having a 1-dimensional centre. Then for the pair $(\mathbf{g}, \mathbf{m})$ of the Lie algebras of the multiplication group $Mult(L)$ of L and the abelian normal subgroup M given in Theorem 6 (a) one of the following cases can occur:*

(a) The Lie algebra $\mathbf{g_1}$ is defined by $[e_2, e_3] = e_1$, $[e_2, e_5] = e_3$, $[e_4, e_5] = e_4$ and $\mathbf{m_1} = \mathbf{g_1}'$.

(b) The Lie algebra $\mathbf{g_2}$ is defined by $[e_2, e_4] = e_1$, $[e_1, e_5] = e_1$, $[e_2, e_5] = e_2$, $[e_4, e_5] = e_3$ and $\mathbf{m_2} = \mathbf{g_2}'$.

(c) The Lie algebra $\mathbf{g_3}$ is defined by $[e_1, e_4] = e_1$, $[e_2, e_5] = e_2$, $[e_4, e_5] = e_3$ and $\mathbf{m_3} = \mathbf{g_3}'$.

(d) The Lie algebra $\mathbf{g_4}$ is defined by $[e_1, e_4] = e_1$, $[e_2, e_4] = e_2$, $[e_1, e_5] = -e_2$, $[e_2, e_5] = e_1$, $[e_4, e_5] = e_3$ and $\mathbf{m_4} = \mathbf{g_4}'$.

Proof. By Lemma 5 the loop L is homeomorphic to $\mathbb{R}^3$. According to [12] the 5-dimensional solvable non-nilpotent indecomposable Lie algebras $\mathbf{g}$ with 1-dimensional centre ζ are the Lie algebras $g_{5,8}$, $g_{5,10}$, $g_{5,14}$, $g_{5,22}$, $g_{5,29}$, $g_{5,38}$, $g_{5,39}$, the Lie algebras $g_{5,19}$, $g_{5,20}$ and $g_{5,28}$ in the case of that $\alpha = -1$, $g_{5,15}$ in the case of that $\gamma = 0$, $g_{5,25}$ in the case of that $\beta \neq 0$, $p = 0$, $g_{5,26}$

in the case of that $p = 0$, $\epsilon = \pm 1$ and $g_{5,30}$ in the case of that $h = -2$. If $\mathbf{g}$ is the Lie algebra of the multiplication group $Mult(L)$ of L, then the Lie group $Z = \exp \zeta$ is the centre of $Mult(L)$ and the orbit $Z(e)$, where e is the identity element of L, is the 1-dimensional centre of L (see Theorem 11 in [1]). If $Mult(L)$ does not belong to the Lie algebra $g_{5,38}$, then the factor algebras $\mathbf{g}/\zeta$ are different from the Lie algebras of the Lie groups $\mathcal{L}_2 \times \mathcal{L}_2$ or $\mathcal{F}_4$. Therefore the factor loop $L/Z(e)$ is isomorphic to $\mathbb{R}^2$ (cf. Theorem 6). The Lie algebra $g_{5,38}$ is defined by $[e_1, e_4] = e_1$, $[e_2, e_5] = e_2$, $[e_4, e_5] = e_3$. As $S = \{\exp(te_1); t \in \mathbb{R}\}$ is a connected normal subgroup of the Lie group of $g_{5,38}$ the orbit $S(e)$ is a 1-dimensional connected normal subloop of L. The factor algebra $g_{5,38}/\langle e_1 \rangle$ is also different from the Lie algebras of the groups $\mathcal{L}_2 \times \mathcal{L}_2$ and $\mathcal{F}_4$ and the factor loop $L/S(e)$ is again isomorphic to $\mathbb{R}^2$. Hence the Lie algebra $\mathbf{g}$ of $Mult(L)$ has a 3-dimensional abelian ideal $\mathbf{m}$ such that $\mathbf{m}$ contains the commutator ideal $\mathbf{g}'$ of $\mathbf{g}$ (cf. Theorem 6 (a)). The commutator ideal of the Lie algebras $g_{5,19}$, $g_{5,25}$, $g_{5,28}$ and $g_{5,30}$ has dimension 4. The commutator ideal of the Lie algebras $g_{5,20}$ and $g_{5,26}$ is non-abelian. Hence these Lie algebras cannot be the Lie algebras of the multiplication groups of 3-dimensional topological loops.

For the Lie algebras $g_{5,8}$, $g_{5,10}$, $g_{5,14}$ and $g_{5,15}$ the commutator ideal $\mathbf{g}'$ of $\mathbf{g}$ is isomorphic to $\mathbb{R}^3$ and contains the centre of $\mathbf{g}$. Hence one has $\mathbf{m} = \mathbf{g}'$. If $\mathbf{g}$ is the Lie algebra of the multiplication group of L, then the Lie algebra $\mathbf{inn(L)}$ of the inner mapping group $Inn(L)$ of L is a 2-dimensional subalgebra of $\mathbf{m}$ containing no ideal $\neq 0$ of $\mathbf{g}$ (see Theorem 6 (a)). The direct sum of the centre ζ of $\mathbf{g}$ and the Lie algebra $\mathbf{inn}(\mathbf{L})$ coincides with $\mathbf{m}$. The Lie algebra $\mathbf{n}$ of the normalizer of $Inn(L)$ in the Lie group of $\mathbf{g}$ is the 4-dimensional abelian nilradical $rad = \langle e_1, e_2, e_3, e_4 \rangle$ of $\mathbf{g}$. This contradiction to Lemma 2 yields that only the Lie algebras $g_{5,22}$, $g_{5,29}$, $g_{5,38}$ and $g_{5,39}$ can occur as the Lie algebras of the multiplication groups $Mult(L)$ of 3-dimensional topological loops L. The Lie algebra $g_{5,29}$ in [12] is isomorphic to the Lie algebra given in assertion (b). The ideal $\mathbf{m}$ of these Lie algebras is the commutator ideal and the assertion is proved. □

Now we exclude the Lie algebras in cases (a) to (d) of Proposition 15.

Proposition 16. *There does not exist* 3*-dimensional connected topological proper loop L such that the Lie algebra* $\mathbf{g}$ *of the multiplication group of L is one of the Lie algebras listed in cases (a) to (d) of Proposition 15.*

Proof. By Lemmata 4 and 5 we may assume that L is homeomorphic to $\mathbb{R}^3$. The linear representation of the Lie group G_i of $\mathbf{g_i}$ is: For $i = 1$

$$g(x_1, y_1, z_1, q_1, w_1)g(x_2, y_2, z_2, q_2, w_2) =$$

$$g(x_1 + w_1 y_2 + \frac{w_1^2 z_2}{2} + x_2, y_1 + w_1 z_2 + y_2, z_1 + z_2, q_1 + e^{z_1} q_2, w_1 + w_2)$$

for $i = 2$

$$g(q_1, x_1, y_1, z_1, w_1)g(q_2, x_2, y_2, z_2, w_2) =$$

$$g(q_1 + e^{w_1} q_2 + x_1 z_2, x_1 + e^{w_1} x_2, y_1 + w_1 z_2 + y_2, z_1 + z_2, w_1 + w_2)$$

for $i = 3$

$$g(q_1, x_1, y_1, z_1, w_1) g(q_2, x_2, y_2, z_2, w_2) =$$

$$g(q_1 + e^{z_1} q_2, x_1 + e^{w_1} x_2, y_1 + w_1 z_2 + y_2, z_1 + z_2, w_1 + w_2).$$

For $i = 4$ the group G_4 is the linear group of matrices

$$\left\{ g(x,y,q,w,z) = \begin{pmatrix} 1 & x & y & -w & q \\ 0 & e^w \cos z & e^w \sin z & 0 & 0 \\ 0 & -e^w \sin z & e^w \cos z & 0 & 0 \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, x, y, q, w, z \in \mathbb{R} \right\}$$

(cf. Cases 5.22, 5.29, 5.38, 5.39 in [7]). First we determine which subgroups of the group G_i can occur as the inner mapping group $Inn(L)_i$ of L. By Theorem 6 (a) the Lie algebra $\mathbf{inn(L)_i}$ of the inner mapping group $Inn(L)_i$ of L is a 2-dimensional subalgebra of the commutator ideal $\mathbf{m_i} = \mathbf{g_i}'$ given in Proposition 15 such that $\mathbf{inn(L)_i}$ does not contain any ideal $\neq \{0\}$ of $\mathbf{g_i}$. As $\langle e_1 \rangle$ is the centre of $\mathbf{g_1}$ and $\langle e_4 \rangle$ is an ideal of $\mathbf{g_1}$ we may choose the Lie algebra $\mathbf{inn(L)_1}$ as follows $\mathbf{inn(L)_1} = \langle e_3 + a_1 e_1, e_4 + a_2 e_1 \rangle$, $a_1, a_2 \in \mathbb{R}$, $a_2 \neq 0$. The automorphism group of $\mathbf{g_1}$ consists of the following mappings $\alpha(e_1) = c^2 e_1$, $\alpha(e_2) = b_1 e_1 + c e_2 + b_3 e_3$, $\alpha(e_3) = c f_3 e_1 + c e_3$, $\alpha(e_4) = d e_4$, $\alpha(e_5) = f_1 e_1 + f_3 e_3 + f_4 e_4 + e_5$, where $cd \neq 0$, $b_1, b_3, f_1, f_3, f_4 \in \mathbb{R}$. Using an automorphism of G_1 we may assume that

$$Inn(L)_1 = \{\exp(t e_3 + u(e_1 + e_4)), t, u \in \mathbb{R}\} = \{g(u, t, 0, u, 0), t, u \in \mathbb{R}\}.$$

The centre of the Lie algebras $\mathbf{g_i}$, $i = 2,3,4$, is $\langle e_3\rangle$. Moreover, $\langle e_1\rangle$, respectively $\langle e_1\rangle$ and $\langle e_2\rangle$, respectively $\langle e_1, e_2\rangle$ are ideals of $\mathbf{g_2}$, respectively $\mathbf{g_3}$, respectively $\mathbf{g_4}$. Hence we may choose $\mathbf{inn(L)_i}$, $i = 2,3,4$, in the following way $\mathbf{inn(L)_i} = \langle e_1 + k_1 e_3, e_2 + k_2 e_3\rangle$, $k_1, k_2 \in \mathbb{R}$, such that for $i = 2$ one has $k_1 \neq 0$, for $i = 3$ we get $k_1 k_2 \neq 0$ and for $i = 4$ at least one of the real parameters k_1, k_2 is different from 0. For $k_1 k_2 \neq 0$ the automorphism $\alpha(e_1) = k_1 e_1$, $\alpha(e_2) = k_2 e_2$, $\alpha(e_3) = e_3$, $\alpha(e_4) = e_4$, $\alpha(e_5) = e_5$ of $\mathbf{g_i}$, $i = 2,3,4$, maps the Lie algebra $\mathbf{inn(L)_i}$ onto $\mathbf{inn(L)}_{2,1} = \mathbf{inn(L)_3} = \mathbf{inn(L)}_{4,1} = \langle e_1 + e_3, e_2 + e_3\rangle$. For $k_2 = 0$ the automorphism $\gamma(e_1) = k_1 e_1$, $\gamma(e_2) = e_2$, $\gamma(e_3) = e_3$, $\gamma(e_4) = e_4$, $\gamma(e_5) = e_5$ maps the subalgebra $\mathbf{inn(L)_i}$ onto $\mathbf{inn(L)}_{2,2} = \mathbf{inn(L)}_{4,3} = \langle e_1 + e_3, e_2\rangle$. For $k_1 = 0$ the automorphism $\beta(e_1) = e_1$, $\beta(e_2) = k_2 e_2$, $\beta(e_3) = e_3$, $\beta(e_4) = e_4$, $\beta(e_5) = e_5$ maps $\mathbf{inn(L)_i}$ onto $\mathbf{inn(L)}_{4,2} = \langle e_1, e_2 + e_3\rangle$. The corresponding Lie groups are $Inn(L)_{2,1} = Inn(L)_3 = Inn(L)_{4,1} = \{g(t_1, t_2, t_1 + t_2, 0, 0), t_i \in \mathbb{R}, i = 1,2\}$, $Inn(L)_{2,2} = Inn(L)_{4,3} = \{g(t_1, t_2, t_1, 0, 0), t_i \in \mathbb{R}, i = 1,2\}$, $Inn(L)_{4,2} = \{g(t_1, t_2, t_2, 0, 0), t_i \in \mathbb{R}, i = 1,2\}$.

Arbitrary left transversals to the group $Inn(L)_i$ of G_i are: For $i = 1$

$$A_1 = \{g(k, f_1(k,l,m), l, f_2(k,l,m), m), k,l,m \in \mathbb{R}\},$$

$$B_1 = \{g(u, g_1(u,v,w), v, g_2(u,v,w), w), u,v,w \in \mathbb{R}\},$$

for $i = 2,3,4$

$$A = \{g(k_1(k,l,m), k_2(k,l,m), k, l, m), k,l,m \in \mathbb{R}\}$$

$$B = \{g(h_1(u,v,w), h_2(u,v,w), u, v, w), u, v, w \in \mathbb{R}\},$$

where $f_i(k,l,m) : \mathbb{R}^3 \to \mathbb{R}$, $k_i(k,l,m) : \mathbb{R}^3 \to \mathbb{R}$, $g_i(u,v,w) : \mathbb{R}^3 \to \mathbb{R}$, $h_i(u,v,w) : \mathbb{R}^3 \to \mathbb{R}$, $i = 1,2$, are continuous functions with $f_i(0,0,0) = k_i(0,0,0) = g_i(0,0,0) = h_i(0,0,0) = 0$. We prove that none of the groups G_i, $i = 1,2,3,4$, satisfies the condition that for all $a \in A_i$ and $b \in B_i$ one has $a^{-1}b^{-1}ab \in Inn(L)_i$. It means that the groups G_i, $i = 1,2,3,4$, are not multiplication groups of L (cf. Lemma 1).

The products $a^{-1}b^{-1}ab$ with $a = g(0, f_1(0,0,m), 0, f_2(0,0,m), m) \in A_1$ and $b = g(0, g_1(0,v,0), v, g_2(0,v,0), 0) \in B_1$ are elements of $Inn(L)_1$ if and only if the equation

$$f_2(0,0,m) = m\frac{g_1(0,v,0)e^v}{(1-e^v)} - \frac{m^2 v e^v}{2(1-e^v)} \qquad (20)$$

is satisfied for all $m, v \in \mathbb{R}$. Since the left hand side of (20) depends only on the variable m for all $v \in \mathbb{R} \setminus \{0\}$ the function $v \mapsto \frac{ve^v}{(1-e^v)}$ must be constant which is a contradiction.

The products $a^{-1}b^{-1}ab$ with $a = g(k_1(0,0,m), k_2(0,0,m), 0, 0, m) \in A$, $b = g(h_1(0,v,0), h_2(0,v,0), 0, v, 0) \in B$ are contained in $Inn(L)_3$, respectively in $Inn(L)_{4,i}$, $i = 1,2,3$, if and only if the equation

$$m = k_1(0,0,m)\frac{e^{-v}-1}{v} + \frac{h_2(0,v,0)}{v}(1-e^{-m}), \qquad (21)$$

respectively for $i = 1$ the equation

$$-m = k_1(0,0,m)\frac{1-e^v}{v} + k_2(0,0,m)\frac{1-e^v}{v} + \frac{h_2(0,v,0)}{v}(\cos m - \sin m - 1) +$$

$$\frac{h_1(0,v,0)}{v}(\cos m + \sin m - 1), \tag{22}$$

respectively for $i = 2$

$$-m = k_2(0,0,m)\frac{1-e^v}{v} + \frac{h_1(0,v,0)}{v}\sin m + \frac{h_2(0,v,0)}{v}(\cos m - 1), \tag{23}$$

respectively for $i = 3$

$$-m = k_1(0,0,m)\frac{1-e^v}{v} + \frac{h_1(0,v,0)}{v}(\cos m - 1) - \frac{h_2(0,v,0)}{v}\sin m \tag{24}$$

holds for all $m, v \in \mathbb{R}$. Since the left hand side of these equations depends only on the variable m and the function $v \mapsto \frac{1-e^{\varepsilon v}}{v}$, where $\varepsilon = 1$ or -1, is not constant we get $k_j(0,0,m) = 0$ and $h_j(0,v,0) = c_j v$, with $c_j \in \mathbb{R}$, $j = 1,2$. Then equation (21), respectively (22), respectively (23), respectively (24) yields that for all $m \in \mathbb{R}$ the identity $m = c_2(1 - e^{-m})$, respectively $-m = c_1(\cos m + \sin m - 1) + c_2(\cos m - \sin m - 1)$, respectively $-m = c_1 \sin m + c_2(\cos m - 1)$, respectively $-m = c_1(\cos m - 1) - c_2 \sin m$ is satisfied which is a contradiction.

The products $a^{-1}b^{-1}ab$ with $a = g(k_1(0,0,m), k_2(0,0,m), 0, 0, m) \in A$, $b = g(h_1(0,v,w), h_2(0,v,w), 0, v, w) \in B$ are contained in $Inn(L)_{2,1}$, respectively in $Inn(L)_{2,2}$ if and only if the equation

$$mv = \tag{25}$$

$$\frac{h_1(0,v,w)+h_2(0,v,w)}{e^w}(1-\frac{1}{e^m})+\frac{k_1(0,0,m)}{e^m}(\frac{1}{e^w}-1)+\frac{k_2(0,0,m)}{e^m}(\frac{1+v}{e^w}-1),$$

respectively

$$mv = \frac{h_1(0,v,w)}{e^w}(1-\frac{1}{e^m}) + \frac{k_1(0,0,m)}{e^m}(\frac{1}{e^w}-1) + \frac{vk_2(0,0,m)}{e^{m+w}} \tag{26}$$

is satisfied for all $m, v, w \in \mathbb{R}$. For $v = 0$ equation (25), respectively (26) gives $\frac{h_1(0,0,w)+h_2(0,0,w)}{1-e^w} = \frac{k_1(0,0,m)+k_2(0,0,m)}{1-e^m} = d$, respectively $\frac{h_1(0,0,w)}{1-e^w} = \frac{k_1(0,0,m)}{1-e^m} = d$ for a suitable constant $d \in \mathbb{R}$. If $w = 0$, then equation (25), respectively (26) yields

$$v = \frac{h_1(0,v,0) + h_2(0,v,0)}{me^m}(e^m - 1) + \frac{k_2(0,0,m)}{me^m}v, \tag{27}$$

respectively

$$v = \frac{h_1(0,v,0)}{me^m}(e^m - 1) + \frac{k_2(0,0,m)}{me^m}v. \tag{28}$$

As the function $g : m \mapsto \frac{e^m-1}{e^m m}$ is not constant the right hand side of equation (27), respectively (28) is equal to v precisely if $h_1(0,v,0) = -h_2(0,v,0)$ and $k_2(0,0,m) = me^m$, respectively $h_1(0,v,0) = 0$ and $k_2(0,0,m) = me^m$. Putting $k_1(0,0,m) = d(1-e^m) - k_2(0,0,m)$, $k_2(0,0,m) = me^m$ into (25) and $k_1(0,0,m) = d(1-e^m)$, $k_2(0,0,m) = me^m$ into (26) we have

$$v(e^w - 1) = \frac{e^m - 1}{me^m}[h_1(0,v,w) + h_2(0,v,w) - d(1-e^w)], \tag{29}$$

respectively

$$v(e^w - 1) = \frac{e^m - 1}{me^m}[h_1(0,v,w) - d(1-e^w)]. \tag{30}$$

Since the left hand side of equations (29) and (30) depends only on the variables v and w and the function $m \mapsto \frac{e^m-1}{me^m}$ is not constant we get $h_1(0,v,w) + h_2(0,v,w) = d(1-e^w)$ in equation (29) and $h_1(0,v,w) = d(1-e^w)$ in equation (30). But then in both cases one has $v(e^w - 1) = 0$ for all $v, w \in \mathbb{R}$ which is a contradiction. $\square$

6. Three-dimensional topological loops having five-dimensional solvable decomposable Lie groups as their multiplication groups

We classify all 5-dimensional connected solvable Lie groups which are direct products of proper connected subgroups and which are multiplication groups of 3-dimensional connected simply connected topological loops L. Moreover, we determine the inner mapping groups of L.

Proposition 17. *Let L be a connected simply connected topological proper loop of dimension 3 such that its multiplication group $Mult(L)$ is a 5-dimensional solvable Lie group which is the direct product of connected subgroups. Then L contains a central subgroup $C \cong \mathbb{R}$ such that the factor loop $L/C \cong \mathbb{R}^2$. Moreover:*

(I) If the centre of the group $Mult(L)$ has dimension 1, then for the pair $(\mathbf{mult(L)}, \mathbf{m})$ of the Lie algebras of $Mult(L)$ and the normal subgroup M in Theorem 6 (a) one of the following cases occurs:

(a) The group $Mult(L)_1$ is the group $\mathcal{F}_3 \times \mathcal{L}_2$. The Lie algebra $\mathbf{mult(L)_1}$ is defined by $[e_1, e_2] = e_3$, $[e_4, e_5] = e_4$ and $\mathbf{m_1} = \langle e_2, e_3, e_4 \rangle$.

(b) The group $Mult(L)_2$ is the group $\mathcal{L}_2 \times \mathcal{L}_2 \times \mathbb{R}$. The Lie algebra $\mathbf{mult(L)_2}$ is defined by $[e_1, e_2] = e_1$, $[e_3, e_4] = e_3$, $[e_5, e_i] = 0$ for all $i = 1, \cdots, 4$, and $\mathbf{m_2} = \langle e_1, e_3, e_5 \rangle$.

(c) The Lie algebra $\mathbf{mult(L)_3}$ is defined by $[e_2, e_3] = e_1$, $[e_1, e_4] = e_1$, $[e_2, e_4] = e_2$, $[e_5, e_i] = 0$ for all $i = 1, \cdots, 4$, and $\mathbf{m_3} = \langle e_1, e_2, e_5 \rangle$.

(d) The Lie algebra $\mathbf{mult(L)_4}$ is defined by $[e_1, e_3] = e_1$, $[e_2, e_3] = e_2$,

$[e_1, e_4] = -e_2$, $[e_2, e_4] = e_1$, $[e_5, e_i] = 0$ for all $i = 1, \cdots, 4$, and $\mathbf{m_4} = \langle e_1, e_2, e_5 \rangle$.

(II) If $Mult(L)$ has 2-dimensional centre, then it is either the group $\mathcal{F}_4 \times \mathbb{R}$ or the direct product of the group $\mathbb{R}^2$ and a 3-dimensional solvable Lie group S having 2-dimensional commutator subgroup. In the second case the Lie algebra $\mathbf{mult(L)}$ is the direct sum $\langle e_1, e_2, e_3 \rangle \oplus \langle e_4, e_5 \rangle$, where $\langle e_1, e_2, e_3 \rangle$ is the Lie algebra of S. The Lie algebra $\mathbf{m}$ has one of the following forms: $\mathbf{m}_{II,1} = \langle e_1, e_2, e_4 \rangle$, $\mathbf{m}_{II,2} = \langle e_1, e_2, e_5 + ke_4 \rangle$, $k \in \mathbb{R}$.

Proof. The loop L is homeomorphic to $\mathbb{R}^3$ (see Lemma 5). We assume that the multiplication group $Mult(L)$ of L is a 5-dimensional decomposable solvable Lie group. Then for $Mult(L)$ we have the following possibilities: $\mathcal{L}_2 \times \mathbb{R}^3$, $\mathcal{L}_2 \times \mathcal{L}_2 \times \mathbb{R}$, $\mathcal{L}_2 \times S$, $\mathbb{R}^2 \times S$, $\mathbb{R} \times K$, where S is a 3-dimensional and K is a 4-dimensional solvable indecomposable Lie group. All of these Lie groups have a normal subgroup $N \cong \mathbb{R}$ such that $Mult(L)/N$ is isomorphic neither to $\mathcal{L}_2 \times \mathcal{L}_2$ nor to $\mathcal{F}_4$. Then the factor loop $L/N(e)$ is isomorphic to $\mathbb{R}^2$ (see Theorem 6), the group $N(e)$ is central in L and the first assertion is proved. Moreover, $Mult(L)$ is simply connected because it is a semidirect product of $\mathbb{R}^2$ with a normal subgroup $M \cong \mathbb{R}^3$ such that M contains a 1-dimensional central subgroup of $Mult(L)$ (cf. Theorem 6 (a)).

Since L is not associative, the centre Z of $Mult(L)$ has dimension 1 or 2. If dim $Z = 1$, then $Mult(L)$ is either the group $\mathcal{F}_3 \times \mathcal{L}_2$ or the direct product $K \times Z$, where Z is the group $\mathbb{R}$ and K is a 4-dimensional solvable

Lie group with discrete centre.

If $Mult(L) = \mathcal{F}_3 \times \mathcal{L}_2$, then its Lie algebra $\mathbf{mult(L)}$ is given by $[e_1, e_2] = e_3$, $[e_4, e_5] = e_4$. The commutator ideal $\mathbf{mult(L)}' = \langle e_3, e_4 \rangle$ contains the centre $\langle e_3 \rangle$ of $\mathbf{mult(L)}$. Since all 2-dimensional subalgebras of the Lie algebra $\mathbf{f_3}$ of $\mathcal{F}_3$ containing the centre of $\mathbf{f_3}$ can be mapped under an element of $Aut(\mathbf{f_3})$ onto the subalgebra $\langle e_2, e_3 \rangle$ we may assume that the Lie algebra $\mathbf{m}$ of M has the form as in case (a) of assertion (I).

If $Mult(L) = K \times Z$, then $Mult(L)$ has a normal subgroup $M \cong \mathbb{R}^3$ such that M contains the commutator subgroup $Mult(L)' = K'$ and the centre Z of $Mult(L)$. Since there is no 4-dimensional solvable Lie group with discrete centre and 1-dimensional commutator subgroup, the dimension of K' must be 2. Hence the Lie algebra $\mathbf{k}$ of K is one of the following: the Lie algebra of $\mathcal{L}_2 \times \mathcal{L}_2$ or $g_{4,8}$ with $h = 0$ or $g_{4,10}$ in [11], § 5. If $\mathbf{k}$ is the Lie algebra of $\mathcal{L}_2 \times \mathcal{L}_2$, then we get case (b) in assertion (I). If $\mathbf{k}$ is the Lie algebra $g_{4,8}$ with $h = 0$, then we obtain case (c) of assertion (I). If $\mathbf{k}$ is the Lie algebra $g_{4,10}$, then we have case (d) in assertion (I).

Now we assume that $Mult(L)$ has a 2-dimensional centre. If $Mult(L)$ is nilpotent, then it is the group $\mathcal{F}_4 \times \mathbb{R}$ and Proposition 5.1 of [5] proves the assertion. If $Mult(L)$ is not nilpotent, then it is either the direct product $K \times N$, where $N \cong \mathbb{R}$ and K is a 4-dimensional solvable non-nilpotent indecomposable Lie group with 1-dimensional centre, or the direct product $S \times R$, where $R \cong \mathbb{R}^2$ and S is a 3-dimensional solvable Lie group with discrete centre.

If $Mult(L) = K \times N$, then the orbit $N(e)$ is a 1-dimensional central subgroup of L with $L/N(e) \cong \mathbb{R}^2$. Hence $Mult(L)$ has a normal subgroup $M \cong \mathbb{R}^3$ containing N and the commutator subgroup $Mult(L)' = K'$ of $Mult(L)$. Among the 4-dimensional solvable non-nilpotent Lie algebras only the Lie algebra $g_{4,3}$ has a 1-dimensional centre and an abelian commutator subalgebra (cf. § 5 of [11]). If $\mathbf{k}$ is the Lie algebra $g_{4,3}$, then the Lie algebra $\mathbf{mult(L)}$ of $Mult(L)$ is defined by $[e_1, e_4] = e_1$, $[e_3, e_4] = e_2$, $[e_5, e_i] = 0$ for all $i = 1, \cdots, 4$, and the Lie algebra $\mathbf{m}$ of M has the form $\langle e_1, e_2, e_5 \rangle$. The inner mapping group $Inn(L)$ of L is a 2-dimensional connected subgroup of M such that $Co_{Mult(L)}(Inn(L)) = 1$. As $\langle e_2, e_5 \rangle$ is the centre of $\mathbf{mult(L)}$ the Lie algebra $\mathbf{inn(L)}$ of $Inn(L)$ has the form $\mathbf{inn(L)} = \langle e_2 + a_1 e_1, e_5 + a_2 e_1 \rangle$ with $a_1 a_2 \neq 0$. Then the Lie algebra $\langle e_1, e_2, e_3, e_5 \rangle$ of the normalizer $N_{Mult(L)}(Inn(L))$ is different from the Lie algebra $\langle e_1, e_2, e_5 \rangle$ of $Z \times Inn(L)$. This contradiction to Lemma 2 excludes the Lie algebra $g_{4,3}$.

If $Mult(L) = S \times R$, then the commutator ideal $\mathbf{i} = \langle e_1, e_2 \rangle$ of the Lie algebra $\mathbf{s} = \langle e_1, e_2, e_3 \rangle$ of S is commutative (see [11], § 4). Let N be a 1-dimensional subgroup of the centre $R = \exp\{a e_4 + b e_5, a, b \in \mathbb{R}\}$ of $Mult(L)$. The Lie algebra $\mathbf{n}$ of N has one of the following forms: $\mathbf{n}_1 = \langle e_4 \rangle$, $\mathbf{n}_2 = \langle e_5 + k e_4 \rangle$, $k \in \mathbb{R}$. As the Lie algebra $\mathbf{m}$ of the normal subgroup $M \cong \mathbb{R}^3$ is the direct sum $\mathbf{i} \oplus \mathbf{n}$, the form of $\mathbf{m}$ is given in assertion (II). □

Theorem 18. *Let L be a connected simply connected topological proper loop* *of dimension* 3 *such that its multiplication group is a* 5*-dimensional solvable non-nilpotent Lie group which is the direct product of proper connected subgroups. Then the following Lie groups are the multiplication groups* $Mult(L)$ *and the following subgroups are the inner mapping groups* $Inn(L)$ *of* L*:*

1) $Mult(L)_1$ *is the Lie group* $\mathcal{F}_3 \times \mathcal{L}_2$ *the multiplication of which is given by* $g(x_1, x_2, x_3, x_4, x_5)g(y_1, y_2, y_3, y_4, y_5) =$

$$g(x_1 + y_1, x_2 + y_2, x_3 + y_3 - x_1 y_2, y_4 + x_4 e^{y_5}, x_5 + y_5).$$

$Inn(L)_1$ *is the following subgroup* $\{g(0, t, k, k, 0);\ t, k \in \mathbb{R}\}$.

2) $Mult(L)_2$ *is the Lie group* $\mathcal{L}_2 \times \mathcal{L}_2 \times \mathbb{R}$ *which is represented on* $\mathbb{R}^5$ *by the multiplication* $g(x_1, x_2, x_3, x_4, x_5)g(y_1, y_2, y_3, y_4, y_5) =$

$$g(y_1 + x_1 e^{y_2}, x_2 + y_2, y_3 + x_3 e^{y_4}, x_4 + y_4, x_5 + y_5).$$

$Inn(L)_2$ *is the following subgroup* $\{g(t, 0, k, 0, t + k);\ t, k \in \mathbb{R}\}$.

3) The multiplication of the group $Mult(L)_3$ *is defined by*

$$g(z_1, y_1, x_1, w_1, q_1)g(z_2, y_2, x_2, w_2, q_2) =$$

$$g(z_1 + e^{w_1} z_2 - x_1 e^{w_1} y_2, y_1 + e^{w_1} y_2, x_1 + x_2, w_1 + w_2, q_1 + q_2).$$

$Inn(L)_3$ *is one of the following groups:* $Inn(L)_{3,1} = \{g(z, y, 0, 0, z); z, y \in \mathbb{R}\}$, $Inn(L)_{3,2} = \{g(z, y, 0, 0, z + y); z, y \in \mathbb{R}\}$.

4) The multiplication group $Mult(L)_4$ is the group of matrices

$$\left\{ g(x,y,w,z,u) = \begin{pmatrix} 1 & x & y & u \\ 0 & e^w \cos z & e^w \sin z & 0 \\ 0 & -e^w \sin z & e^w \cos z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, x,y,w,z,u \in \mathbb{R} \right\}$$

(see Case 4.12 in [7]). Moreover, $Inn(L)_4$ is one of the following subgroups: $Inn(L)_{4,1} = Inn(L)_{3,1}$, $Inn(L)_{4,2} = \{g(x,y,0,0,y); x,y \in \mathbb{R}\}$ $Inn(L)_{4,3} = Inn(L)_{3,2}$.

5) The multiplication group $Mult(L)_5$ is the direct product of $\mathbb{R}^2$ and the connected Lie group of dimension 3 having precisely one 1-dimensional normal subgroup. The multiplication of $Mult(L)_5$ is given by

$$g(x_1,x_2,x_3,x_4,x_5)g(y_1,y_2,y_3,y_4,y_5) =$$

$$g(y_1 + x_1 e^{y_3}, y_2 + x_2 e^{y_3} + x_1 y_3 e^{y_3}, x_3 + y_3, x_4 + y_4, x_5 + y_5).$$

$Inn(L)_5$ is the following subgroup $\{g(x,y,0,y,0); x,y \in \mathbb{R}\}$.

6) The elements of the multiplication group $Mult(L)_6$ can be written in the following form

$$g(x,y,z,u,v) = \begin{pmatrix} 1 & x & y & u & v \\ 0 & e^{az} \cos z & e^{az} \sin z & 0 & 0 \\ 0 & -e^{az} \sin z & e^{az} \cos z & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, x,y,z,u,v \in \mathbb{R}, a > 0.$$

$Inn(L)_6$ is one of the subgroups: $Inn(L)_{6,1} = \{g(x, y, 0, x + y, 0); x, y \in \mathbb{R}\}$, $Inn(L)_{6,2} = \{g(x, y, 0, x, 0); x, y \in \mathbb{R}\}$, $Inn(L)_{6,3} = Inn(L)_5$.

7) $Mult(L)_7$ is the direct product of $\mathbb{R}^2$ and the connected Lie group of dimension 3 having precisely two 1-dimensional normal subgroups. The group $Mult(L)_7$ is represented on $\mathbb{R}^5$ by the following multiplication

$$g(x_1, x_2, x_3, x_4, x_5)g(y_1, y_2, y_3, y_4, y_5) =$$

$$g(y_1 + x_1 e^{ay_3}, y_2 + x_2 e^{by_3}, x_3 + y_3, x_4 + y_4, x_5 + y_5), \tag{31}$$

with fixed but different numbers $a, b \in \mathbb{R}\backslash\{0\}$.

8) $Mult(L)_8$ is the direct product of $\mathbb{R}^2$ and the connected Lie group of dimension 3 having infinitely many 1-dimensional normal subgroups. The multiplication of $Mult(L)_8$ is given by (31) with $a = b \in \mathbb{R}\backslash\{0\}$.

The inner mapping group $Inn(L)_i$, $i = 7, 8$, is the group $Inn(L)_{6,1}$.

Proof. By Lemma 5 the loop L is homeomorphic to $\mathbb{R}^3$. For $i = 1, 2, 3, 4$, the Lie algebras $\mathbf{mult(L)}_i$ of the groups $Mult(L)_i$ and the ideals $\mathbf{m}_i$ of $\mathbf{mult(L)}_i$ are given in Proposition 17, (I) cases (a) to (d). The Lie algebra $\mathbf{inn(L)}_i$ of the inner mapping group $Inn(L)_i$ of L is a 2-dimensional subalgebra of $\mathbf{m}_i$ containing no ideal $\neq \{0\}$ of $\mathbf{mult(L)}_i$, $i = 1, 2, 3, 4$. For $i = 1$ the Lie algebra $\mathbf{inn(L)}$ has the form $\mathbf{inn(L)}_{b_1,b_2} = \langle e_2 + b_1 e_3, e_4 + b_2 e_3 \rangle$, $b_1, b_2 \in \mathbb{R}$, $b_2 \neq 0$. The automorphism $\beta(e_1) = e_1$, $\beta(e_2) = e_2 - b_1 e_3$, $\beta(e_3) = e_3$, $\beta(e_4) = b_2 e_4$, $\beta(e_5) = e_5$ maps $\mathbf{inn(L)}_{b_1,b_2}$ onto $\mathbf{inn(L)}_1 = \langle e_2, e_4 + e_3 \rangle$. The corresponding group $Inn(L)_1$ is given in assertion 1).

As $\langle e_5\rangle$ is the centre of $\mathbf{mult(L)}_2$ the Lie algebra $\mathbf{inn(L)_2}$ has the form $\mathbf{inn(L)}_{a_1,a_2} = \langle e_1 + a_1e_5, e_3 + a_2e_5\rangle$, $a_1, a_2 \in \mathbb{R}$ with $a_1a_2 \neq 0$. Using the automorphism $\alpha(e_1) = a_1e_1$, $\alpha(e_2) = e_2$, $\alpha(e_3) = a_2e_3$, $\alpha(e_4) = e_4$, $\alpha(e_5) = e_5$ of $\mathbf{mult(L)}_2$ the Lie algebra $\mathbf{inn(L)}_{a_1,a_2}$ is reduced to $\mathbf{inn(L)}_2 = \langle e_1 + e_5, e_3 + e_5\rangle$. The corresponding group $Inn(L)_2$ is given in assertion 2).
As $\langle e_5\rangle$ is the centre and $\langle e_1, e_2\rangle$ is the commutator ideal of $\mathbf{mult(L)_i}$ for $i = 3, 4$, we can write $\mathbf{inn(L)_i}$ in the form $\mathbf{inn(L)}_{k_1,k_2} = \langle e_1+k_1e_5, e_2+k_2e_5\rangle$, k_1, $k_2 \in \mathbb{R}$. For $i = 3$ one has $k_1 \neq 0$ and for $i = 4$ at least one of the parameters k_1, k_2 is different from 0. Similarly to the automorphism α of $\mathbf{mult(L)}_2$ we can find suitable automorphisms of $\mathbf{mult(L)_i}$, $i = 3, 4$, which map the Lie algebra $\mathbf{inn(L)}_{k_1,0}$ onto $\mathbf{inn(L)}_{3,1} = \mathbf{inn(L)}_{4,1} = \langle e_1+e_5, e_2\rangle$, the Lie algebra $\mathbf{inn(L)}_{0,k_2}$ onto $\mathbf{inn(L)}_{4,2} = \langle e_1, e_2 + e_5\rangle$ and the Lie algebra $\mathbf{inn(L)}_{k_1,k_2}$, $k_1k_2 \neq 0$, onto $\mathbf{inn(L)}_{3,2} = \mathbf{inn(L)}_{4,3} = \langle e_1+e_5, e_2+e_5\rangle$. The corresponding Lie groups are the groups $Inn(L)_{3,1} = Inn(L)_{4,1}$, $Inn(L)_{3,2} = Inn(L)_{4,3}$, $Inn(L)_{4,2}$ given in assertions 3) and 4).
The sets $A_1 = \{g(x, e^z-1, y, 0, z); x, y, z \in \mathbb{R}\}$ and $B_1 = \{g(n, 0, l, -n, m); l, m, n \in \mathbb{R}\}$ are $Inn(L)_1$-connected left transversals in $Mult(L)_1$. The sets $A_2 = \{g(2 - e^{x_2} - e^{x_4}, x_2, 0, x_4, x_5 + 2 - e^{x_2} - e^{x_4}); x_2, x_4, x_5 \in \mathbb{R}\}$ and $B_2 = \{g(1 - e^{y_2}, y_2, 1 - e^{y_2}, y_4, y_5); y_2, y_4, y_5 \in \mathbb{R}\}$ are $Inn(L)_2$-connected transversals in $Mult(L)_2$. The sets $A_3 = \{g((e^w - 1)(x + 2) - x, 1 - e^w, x, w, q); x, w, q \in \mathbb{R}\}$ and $B_3 = \{g((2 - e^l)k, e^l - 1, k, l, m); k, l, m \in \mathbb{R}\}$, respectively the sets B_3 and $C_3 = \{g(x(e^w - 2), 1 - e^w, x, w, q); x, w, q \in \mathbb{R}\}$

are $Inn(L)_{3,1}$-, respectively $Inn(L)_{3,2}$-connected transversals in $Mult(L)_3$. The set $A_4 = B_4 = \{g(1 - e^u \cos v, -e^u \sin v, u, v, w); u, v, w \in \mathbb{R}\}$ is a left transversal to the subgroups $Inn(L)_{4,i}$ for every $i = 1, 2, 3$ in $Mult(L)_4$. Moreover, the sets $\{A_i, B_i\}$ for all $i = 1, 2, 3, 4$ as well as $\{B_3, C_3\}$ generate the group $Mult(L)_i$. This proves assertions 1) to 4) (cf. Lemma 1).

The Lie algebra $\mathbf{mult(L)_5}$ of the group $Mult(L)_5$ in assertion 5) is defined by $[e_1, e_3] = pe_1 - e_2$, $[e_2, e_3] = e_1 + pe_2$, $[e_4, e_i] = [e_5, e_i] = [e_4, e_5] = 0$, $i = 1, 2, 3$, $p > 0$ (see $g_{3,5}$ in [11], § 4). The Lie algebra $\mathbf{mult(L)_6}$ of the group $Mult(L)_6$ in assertion 6) is given by $[e_2, e_3] = e_2$, $[e_1, e_3] = e_1 + e_2$, $[e_1, e_2] = [e_4, e_5] = [e_4, e_i] = [e_5, e_i] = 0$, $i = 1, 2, 3$ (see [17], Lemma 23.16). The Lie algebra $\mathbf{mult(L)_7}$ of the group $Mult(L)_7$ in assertion 7) is defined by $[e_1, e_3] = ae_1$, $[e_2, e_3] = be_2$, $[e_1, e_2] = [e_4, e_i] = [e_5, e_i] = [e_4, e_5] = 0$, $i = 1, 2, 3$, where $a \neq b \in \mathbb{R}\backslash\{0\}$. For $a = b$ we get the Lie algebra $\mathbf{mult(L)_8}$ of the group $Mult(L)_8$ in assertion 8) (see [17], Section 23.1).

For $i = 5, 6, 7, 8$, the Lie algebra $\mathbf{inn(L)_i}$ of the inner mapping group $Inn(L)_i$ of L is a 2-dimensional subalgebra of $\mathbf{m}_{II,j}$, $j = 1, 2$, given in Proposition 17 (II) containing no ideal $\neq 0$ of $\mathbf{mult(L)_i}$. The Lie algebra $\mathbf{inn(L)_i}$ has one of the following forms: $\mathbf{inn(L)}_{a_1,a_2} = \langle e_1 + a_1 e_4, e_2 + a_2 e_4 \rangle$, $a_1, a_2 \in \mathbb{R}$ and $\mathbf{inn(L)}_{b_1,b_2} = \langle e_1 + b_1(e_5 + ke_4), e_2 + b_2(e_5 + ke_4) \rangle$, $b_1, b_2, k \in \mathbb{R}$, such that for $i = 5$ one has $a_2 b_2 \neq 0$, for $i = 6$ at least one of the parameters a_1, a_2, respectively b_1, b_2 is different from 0, for $i = 7, 8$, one has $a_1 a_2 b_1 b_2 \neq 0$.

For $i = 5$ using the automorphism $\alpha(e_1) = e_1 + \frac{a_1}{a_2} e_2$, $\alpha(e_2) = e_2$, $\alpha(e_3) = e_3$,

$\alpha(e_4) = \frac{1}{a_2}e_4$, $\alpha(e_5) = e_5$, respectively $\beta(e_1) = e_1 + \frac{b_1}{b_2}e_2$, $\beta(e_2) = e_2$, $\beta(e_3) = e_3$, $\beta(e_4) = e_4 + e_5$, $\beta(e_5) = \left(\frac{1}{b_2} - k\right)e_4 - ke_5$ of $\mathbf{mult(L)_5}$ we can change $\mathbf{inn(L)}_{a_1,a_2}$, respectively $\mathbf{inn(L)}_{b_1,b_2}$ onto the Lie algebra $\mathbf{inn(L)_5} = \langle e_1, e_2 + e_4\rangle$.

For $i = 6, 7, 8$ the automorphism $\gamma(e_1) = a_1e_1$, $\gamma(e_2) = a_2e_2$, $\gamma(e_3) = e_3$, $\gamma(e_4) = e_4$, $\gamma(e_5) = e_5$, respectively $\delta(e_1) = b_1e_1$, $\delta(e_2) = b_2e_2$, $\delta(e_3) = e_3$, $\delta(e_4) = e_4 + e_5$, $\delta(e_5) = (1 - k)e_4 - ke_5$ of $\mathbf{mult(L)_i}$ maps the Lie algebra $\mathbf{inn(L)}_{a_1,a_2}$, respectively $\mathbf{inn(L)}_{b_1,b_2}$ onto $\mathbf{inn(L)}_{6,1} = \mathbf{inn(L)}_7 = \mathbf{inn(L)}_8 = \langle e_1+e_4, e_2+e_4\rangle$. The automorphism γ, respectively δ of $\mathbf{mult(L)}_6$ with $a_2 = 1 = b_2$ maps the Lie algebra $\mathbf{inn(L)}_{a_1,0}$, respectively $\mathbf{inn(L)}_{b_1,0}$ onto $\mathbf{inn(L)}_{6,2} = \langle e_1 + e_4, e_2\rangle$. The automorphism γ, respectively δ of $\mathbf{mult(L)}_6$ with $a_1 = 1 = b_1$ maps $\mathbf{inn(L)}_{0,a_2}$, respectively $\mathbf{inn(L)}_{0,b_2}$ onto $\mathbf{inn(L)}_{6,3} = \mathbf{inn(L)}_5$. The corresponding Lie groups $Inn(L)_5 = Inn(L)_{6,3}$, $Inn(L)_{6,2}$, $Inn(L)_{6,1} = Inn(L)_7 = Inn(L)_8$ are given in assertions 5) to 8).

The sets $A_5 = \{g(0, 1 - e^{k_1}(1 + k_1), k_1, k_2 + 1 - e^{k_1}(1 + k_1), k_3); k_i \in \mathbb{R}, i = 1, 2, 3\}$ and $B_5 = \{g(1 - e^{l_1}, 1 - e^{l_1}, l_1, l_2, l_3); l_i \in \mathbb{R}, i = 1, 2, 3\}$ are $Inn(L)_5$-connected transversals in $Mult(L)_5$.

The set $A_6 = B_6 = \{g(1+e^{ak_1}(\sin k_1 - \cos k_1), 1 - e^{ak_1}(\sin k_1 + \cos k_1), k_1, k_2, k_3); k_i \in \mathbb{R}\}$ is for every $i = 1, 2, 3$, a left transversal to $Inn(L)_{6,i}$ in $Mult(L)_6$. The set $A_7 = B_7 = \{g(2 - e^{bk_1} - e^{ak_1}, 2 - e^{bk_1} - e^{ak_1}, k_1, k_2, k_3); k_i \in \mathbb{R}, i = 1, 2, 3\}$ is a left transversal to $Inn(L)_7$ in $Mult(L)_7$. The set $A_8 = B_8 = \{g(1 - e^{ak_1} - k_1, k_1, k_1, k_2, k_3); k_i \in \mathbb{R}, i = 1, 2, 3\}$ is a left transversal to

$Inn(L)_8$ in $Mult(L)_8$. Since Lemma 1 is satisfied for all these transversals, assertions 5) to 8) is proved. $\square$

By the previous theorem only a classification of connected simply connected 5-dimensional solvable Lie groups which are the multiplication groups of connected topological loops L with dimension 3 is given. The next proposition shows that Lie groups which cannot be the multiplication groups of L can have universal coverings which are multiplication groups of L.

Proposition 19. *The direct product G of $\mathbb{R}^2$ and the connected component of the euclidean motion group of $\mathbb{R}^2$ cannot be the multiplication group of a 3-dimensional topological loop L.*

Proof. The group G is represented in case 6) of Theorem 18 such that $a = 0$. The subgroups of G which can occur as the inner mapping group of L are also listed in case 6) of Theorem 18. Arbitrary left transversals to $Inn(L)_{6,i}$, $i = 1, 2, 3$, are $A = \{g(f_1(k_1, k_2, k_3), f_2(k_1, k_2, k_3), k_1, k_2, k_3); k_i \in \mathbb{R}\}$ and $B = \{g(h_1(l_1, l_2, l_3), h_2(l_1, l_2, l_3), l_1, l_2, l_3); l_i \in \mathbb{R}\}$ such that for the continuous functions $f_j(k_1, k_2, k_3) : \mathbb{R}^3 \to \mathbb{R}$, $h_j(l_1, l_2, l_3) : \mathbb{R}^3 \to \mathbb{R}$, $j = 1, 2$, one has $f_j(0, 0, 0) = h_j(0, 0, 0) = 0$. The set $\{a^{-1}b^{-1}ab,\ a \in A, b \in B\}$ is contained in $Inn(L)_{6,i}$ if and only if for $i = 1$

$$h_1(l_1, l_2, l_3)(1 - \cos k_1 - \sin k_1) + h_2(l_1, l_2, l_3)(1 + \sin k_1 - \cos k_1) =$$

$$f_1(k_1, k_2, k_3)(1 - \cos l_1 - \sin l_1) + f_2(k_1, k_2, k_3)(1 + \sin l_1 - \cos l_1) \qquad (32)$$

for $i = 2$

$$h_1(l_1, l_2, l_3)(1 - \cos k_1) + h_2(l_1, l_2, l_3) \sin k_1 =$$

$$f_1(k_1, k_2, k_3)(1 - \cos l_1) + f_2(k_1, k_2, k_3) \sin l_1 \qquad (33)$$

for $i = 3$

$$h_2(l_1, l_2, l_3)(1 - \cos k_1) - h_1(l_1, l_2, l_3) \sin k_1 =$$

$$f_2(k_1, k_2, k_3)(1 - \cos l_1) - f_1(k_1, k_2, k_3) \sin l_1 \qquad (34)$$

holds for all $k_1, k_2, k_3, l_1, l_2, l_3 \in \mathbb{R}$. As the right hand side of equations (32), (33) and (34) does not depend on the variables l_2, l_3 and the left hand side of (32), (33) and (34) is independent of k_2, k_3 we get $h_j(l_1, l_2, l_3) = h_j(l_1)$ and $f_j(k_1, k_2, k_3) = f_j(k_1)$ for all $j = 1, 2$. In this case the function $h_j(l_1)$, respectively $f_j(k_1)$, $j = 1, 2$, has the form $a_{1,j}(1 - \cos l_1) + a_{2,j} \sin l_1$, respectively $b_{1,j}(1 - \cos k_1) + b_{2,j} \sin k_1$, where $a_{1,j}, a_{2,j}, b_{1,j}, b_{2,j} \in \mathbb{R}$. Then the set $A \cup B$ does not generate G. This contradiction to Lemma 1 yields the assertion. $\square$

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