Extensions and tangent prolongations of differentiable loops

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Abstract

The tangent Akivis algebra and Sabinin algebra of degree 3 of a differentiable loop is the tangential object determined by the third-order Taylor polynomial of the multiplication function of the loop. It is endowed with a bilinear skew-symmetric and a trilinear operation defined by the infinitesimal commutator and associator of the loop. The aim of our work is to study tangent algebras of degree 3 of abelian extensions of differentiable loops, which are affine extensions of the tangent algebras of the loop by abelian algebras. This class of loop extensions has previously been studied in terms of computational complexity and in terms of universal algebra. We apply the obtained results to the determination of tangent algebras of degree 3 of tangent prolongation of differentiable loops.

1 Introduction

We remember the main constructions and results of the theory of tangent prolongation of a Lie group G with its Lie algebra $\mathfrak{g}$. Denoting by $\lambda_x : G \to G$, $\rho_x : G \to G$ the left, respectively, right multiplication and $e \in G$ the identity element, the map $(x, \xi) \mapsto (x, \mathrm{d}_x \lambda_x^{-1} \xi)$, $\xi \in \mathrm{T}_e(G)$, identifies the tangent bundle $\mathrm{T}(G)$ with the product $G \times \mathrm{T}_e(G)$. The manifold $G \times \mathrm{T}_e(G)$ has a natural Lie group structure, called *tangent prolongation of* G, determined by the multiplication

$$(x, X) \cdot (y, Y) = \big(xy, \mathrm{d}_{xy}\lambda_{xy}^{-1} \frac{d}{dt}\big|_{t=0}(x \exp tX \cdot y \exp tY)\big) = \big(xy, \mathrm{Ad}_y^{-1} X + Y\big),$$

where $x, y \in G$, $X, Y \in \mathrm{T}_e(G)$ and $\mathrm{Ad}_g = \mathrm{d}_e(\lambda_g \rho_g^{-1}) : \mathrm{T}_e(G) \to \mathrm{T}_e(G)$ is the adjoint action of $g \in G$ on $\mathrm{T}_e(G)$. This means that the tangent prolongation is a semidirect product $G \ltimes \mathrm{T}_e(G)$ determined by the adjoint representation. The Lie algebra of the tangent prolongation is the semidirect sum $\mathfrak{g} \oplus_\alpha \mathfrak{a}$ of $\mathfrak{g}$ with the abelian Lie algebra $\mathfrak{a}$ on $\mathrm{T}_e(G)$, which is determined by the homomorphism $\mathfrak{g} \to \mathrm{End}(\mathfrak{a})$ given by $\alpha : \xi \mapsto \theta^{-1} \cdot \mathrm{ad}_\xi \cdot \theta$, where $\theta : \mathfrak{a} \to \mathfrak{g}$ is the identity map of the underlying vector space. The Lie bracket of $\mathfrak{g} \oplus_\alpha \mathfrak{a}$ is given by

$$[(\xi, X), (\eta, Y)] = ([\xi, \eta], \alpha_\xi Y - \alpha_\eta X) = ([\xi, \eta], \theta^{-1}([\xi, \theta(Y)] + [\theta(X), \eta])), \qquad (1)$$

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cf. [35], §V.1. and [34], §3.15.
The aim of our research project was to study the tangent prolongation of differentiable loops, giving a natural generalization of the tangent prolongation of Lie groups. Our initial research showed that the prolonged loop structure to the tangent bundle belongs to a special category of loop extensions. This class of loop extensions has previously been studied in terms of computational complexity in [20], [18], called *polyabelian loops*, and then in terms of universal algebra in [32], [33], called *abelian extensions*. The latter name is justified by the discovered interesting properties of such loops, which allow us to consider them as non-associative variants of extensions of abelian groups, yielding a somewhat broader class of extensions than the Schreier-type theory of loop extensions studied in [25]. We have previously studied the abstract construction of abelian extensions of loops that have some weak inverse property [10], and we have shown that the tangent prolongation of loops inherits the classically weak associativity properties of the base loop [11].
The basic idea of Lie theory of groups is to associate with an analytic group a Lie algebra defined on its tangent space at the identity, whose operation is the infinitesimal commutator of the group, determined by a second-order Taylor polynomial of the multiplication function. This construction establishes a one-to-one correspondence between local Lie groups and Lie algebras, this assertion is called Lie's third theorem.
L. V. Sabinin, P. O. Mikheev ([19], [28], [29]) generalized Lie's third theorem to local analytic loops by introduction an infinite number of multilinear operations on the tangent space at the identity element, corresponding to higher-order terms of the Taylor series decomposition of the local loop multiplication. They proved that for any analytic local loop the tangent space at the identity inherits two families of multilinear operations $\langle x_1, \dots, x_m; y, z\rangle$, $m \geq 0$ and $(x_1, \dots, x_m; y_1, \dots, y_n)$, $m \geq 1$, $n \geq 2$, satisfying an infinite number of identities, that make a tangent algebra, determining the local loop uniquely. The abstract version of this tangent algebra is called *Sabinin algebra*. For a Lie group, the tangent Sabinin algebra is a Lie algebra, and if it is a Moufang loop then this tangent algebra is a Malcev algebra. The theory of Sabinin algebras developed intensively in the last decades (see e.g. [5], [7], [22], [23], [27]).
An essential difference between group theory and loop theory is that differentiable loops can belong to any differentiability class, and the tangent algebra defined on the tangent space up to the order of differentiability does not uniquely determine the corresponding local loop. This justifies the investigation of local loops of any finite differentiability class and their connection with their tangent algebras determined by the terms of their Taylor polynomial (cf. [12], [13]). The development of Lie theory on differentiable local loops and their tangent algebras of degree 3 in the framework of web geometry was started by M. A. Akivis (see [1], [2], [3]). Their method, applied to the third-order Taylor polynomial of the multiplication function of a non-associative differentiable loop, leads to a binary-ternary algebra whose operations are the tangent commutator and associator, measuring the non-commutativity, respectively, the non-associativity of the loop. These algebras were later called the Akivis algebra and were studied by many authors (see e.g. [12], [13], [21], [31], [30]).
Each Akivis algebra is closely related to a so-called *Sabinin algebra of degree* 3, which is obtained by considering only the bilinear and trilinear operations together with polynomial identities up to degree 3 of the Sabinin algebra. A Sabinin algebra of degree 3 is a vector space together with a skew-symmetric bilinear operation and two trilinear operations. For a tangent Sabinin

algebra of degree 3 of a differentiable local loop, the bilinear operation up to the sign is the infinitesimal commutator of the local loop and the two trilinear operators are the symmetric and skew-symmetric parts in the last two variables of the infinitesimal associator of the loop. Hence the variety of Sabinin algebras of degree 3 is equivalent to the variety of Akivis algebras (cf. Remark 13, [5], p. 7). In our following investigation, we will study the infinitesimal properties of local loops of differentiability class 3 using their power series up to order 3 of the loop multiplication function. The results will be formulated first in Akivis algebras and then interpreted them in Sabinin algebras of degree 3.

The development of the theory of loop extensions has recently attracted much attention and has been applied to the construction and study of loops with special properties. (e.g. [6], [8], [9], [10], [11], [14], [15], [18], [21], [24], [25], [26], [33]). A loop extension L of a loop K by the loop M is a short exact sequence

$$1 \to M \overset{\iota}{\to} L \overset{\pi}{\to} K \to 1,$$

of loops, where $\iota(M)$ is a normal subloop of L and π induces an isomorphism of the factor loop $L/\iota(M)$ to K. The objectives of this paper are to give a systematic investigation of abelian extensions of differentiable loops and the corresponding extension theory of the tangent algebras of degree 3, and to find algebraic characterizations of these tangent algebras of the tangent prolongation of differentiable loops.

In §2 we introduce the basic concepts and methods of our research, in particular the tools of Taylor expansions of differentiable local loop operations, the construction of tangent commutator and associator of differentiable local loops, and the necessary notations and definitions of multilinear algebra and of binary-ternary algebra, particularly of Akivis and Sabinin algebras of degree 3. In §3, we define a set of multilinear maps, called the data system of the binary-ternary algebra extension, and characterize extensions of Akivis algebras leading to Akivis algebras in terms of their data system. In §4 we compute the data system of the tangent Akivis algebra extension of abelian loop extensions and show that the data system of these extensions are so-called affine extensions, containing only linear and constant terms with respect to the variables belonging to the abelian ideal of the extension. We express also the operations of the tangent Sabinin algebra of degree 3 of the abelian loop extension. We say that an extension of a differentiable local loop by an abelian group is *almost abelian* if the tangent Akivis algebra of the extension is an affine extension and the multiplication function is given by a third order polynomial. We prove a characterization of almost abelian extensions from which it follows that this class is larger than the class of abelian extensions. §5 is devoted to the characterization of abelian extensions of local loops associated with a given affine extension of Akivis algebras. It is shown that the monomial terms of the third-order Taylor polynomial of the multiplication map satisfy a nonlinear underdetermined system of equations. We find the solutions depending on arbitrarily chosen tensors and the dimension of the vector space of these tensors. The multitude of solutions allows us to consider affine extensions of Akivis algebras by abelian algebras as infinitesimal versions of abelian extensions of loops. In §6 we apply the results on tangent algebras of degree 3 of abelian extensions to the case of tangent prolongation of differentiable loops. We obtain a remarkable form of the commutator and the associator of the tangent Akivis algebra and prove that both operations of the tangent Akivis algebra are given by analogous formulas to the expression (1) of the commutator of the Lie algebra of the tangent prolongation of a Lie group. We determine the operations of the Sabinin algebra of degree 3 of tangent prolongation of differentiable loops.

2 Preliminaries

A *loop* is a set L with three binary operations $\cdot, \backslash, / : L \times L \to L$ in which the identities

$$(x/y) \cdot y = x,\ y\backslash(y \cdot x) = x,\ (x \cdot y)/y = x,\ y \cdot (y\backslash x) = x, \quad x, y \in L \tag{2}$$

are fulfilled and there is an identity element $e \in L$ satisfying

$$e \cdot x = x \cdot e = x/e = e\backslash x = x \quad \text{for all} \quad x \in L. \tag{3}$$

Left translations $\lambda_x : L \to L$, $\lambda_x y = x \cdot y$, and *right translations* $\rho_x : L \to L$, $\rho_x y = y \cdot x$, of the multiplication operation $x \cdot y$ are bijective maps and the *left* and *right division* operations of L satisfy $x\backslash y = \lambda_x^{-1} y$, respectively $x/y = \rho_y^{-1} x$. A commutative loop is called *abelian*.

$\mathcal{C}^r$-differentiable local loops

For a differentiable map $\varphi : M \to N$ between differentiable manifolds M and N we denote by $\mathrm{d}_x\varphi : \mathrm{T}_x(M) \to \mathrm{T}_{\varphi(x)}(N)$ the linear differential map between the tangent spaces at a point $x \in M$. Let V^n be a real vector space of dimension n and F a k-variable differentiable map defined in a neighbourhood of $(0, \ldots, 0) \in V^n \times \cdots \times V^n$, then $F'_i(u)$, $i = 1, \ldots, k$, will denote the linear differential map of F at the point $(0, \ldots, 0)$ with respect to the i-th vector variable, applied to the vector $u \in V^n$. Similarly, $F''_{ij}(u, v)$ denotes the bilinear second, respectively, $F'''_{ijk}(u, v, w)$ the trilinear third differential map at $(0, \ldots, 0)$ with respect to the i-th and j-th, respectively, the i-th, j-th and k-th vector variables, applied to the adequate number of vectors $u, v \in V^n$ or $u, v, w \in V^n$.

An n-dimensional $\mathcal{C}^r$-differentiable manifold L equipped with a $\mathcal{C}^r$-differentiable partial operation $(x, y) \mapsto x \cdot y$ (called partial multiplication) that is defined in an open domain $(e, e) \in U \subset L \times L$ and satisfies $e \cdot x = x \cdot e = x$ for all $x \in L$ with a fixed $e \in L$ is called *$\mathcal{C}^r$-differentiable local H-space* with identity element $e \in L$.

If two more $\mathcal{C}^r$-differentiable partial operations $\backslash, / : U \to L$ (called left and right partial divisions) are defined in a $\mathcal{C}^r$-differentiable local H-space L, and $\cdot, \backslash, / : U \to L$ satisfy (2) and (3), if the terms connected by equal sign have meaning, then L is a *$\mathcal{C}^r$-differentiable local loop* with identity element $e \in L$.

Let L be a $\mathcal{C}^r$-differentiable local H-space ($r \geq 3$) of dimension n covered by a coordinate neighbourhood. We identify L with its coordinate chart in the euclidean vector space $(V^n, \langle ., . \rangle)$ and the identity element $e \in L$ with the zero element $0 \in V^n$. The coordinate function of the local multiplication has the Taylor expansion

$$x \cdot y = x + y + q(x, y) + r(x, x, y) + s(x, y, y) + M(x, y), \tag{4}$$

in a neighbourhood of $(0, 0) \in V^n \times V^n$ with an error term $M(x, y)$ satisfying

$$\lim_{x,y \to 0} \frac{M(x, y)}{(|x| + |y|)^3} = 0.$$

The bilinear and trilinear monomials in (4) are expressed by

$$q = (x \cdot y)''_{xy}(0, 0), \quad r = \frac{1}{2}(x \cdot y)'''_{xxy}(0, 0), \quad s = \frac{1}{2}(x \cdot y)'''_{xyy}(0, 0), \tag{5}$$

on the vector space V^n, (e.g. Corollary 4.4. in [17]), hence r and s are symmetric in the first, respectively, in the last two variables.

Remark 2.1. We notice that $q(x,y)$ is skew-symmetric in canonical coordinate systems (cf. [16] and [4]), having the same differentiability property as the local multiplication. This property of the bilinear form q can also be provided by a locally invertible coordinate change $\phi(x) = x - \frac{1}{2}q(x,x)$ in a neighbourhood of $0 \in V^n$. Indeed, denoting the multiplication by $\phi(x) \star \phi(y)$ with respect to the coordinates $\phi(x) \in V^n$ we have

$$\phi(x) \star \phi(y) = \phi(x \cdot y) = x - \frac{1}{2}q(x,x) + y - \frac{1}{2}q(y,y) + q(x,y) - \frac{1}{2}q(x,y) - \frac{1}{2}q(y,x) + \cdots .$$

The inverse of the map $\phi(x) = x - \frac{1}{2}q(x,x)$ is of the form $\phi^{-1}(x) = x + \frac{1}{2}q(x,x) + o(2)$, hence with $\tilde{x} = \phi(x)$ and $\tilde{y} = \phi(y)$ we get the expansion

$$\tilde{x} \star \tilde{y} = \tilde{x} + \tilde{y} + \frac{1}{2}\left(q(\tilde{x},\tilde{y}) - q(\tilde{y},\tilde{x})\right) + o(2).$$

In the following we assume that the bilinear map $q : V^n \times V^n \to V^n$ in (4) is skew-symmetric. According to the implicit mapping theorem the partial left and right division operations are implicitly determined by the equation $x \cdot y - z = 0$ in a neighbourhood of $(0,0,0)$ in $V^n \times V^n \times V^n \to V^n$ and have the same differentiability properties as the multiplication $(x,y) \mapsto x \cdot y$, since the tangent maps $(x \cdot y - z)'_y(0,0,0)$ and $(x \cdot y - z)'_x(0,0,0)$ are invertible (see e.g. Theorem 5.9. in [17]). It follows

Proposition 2.2. Any $\mathcal{C}^r$-differentiable local H-space L is a $\mathcal{C}^r$-differentiable local loop on a neighbourhood of the identity element $e \in L$ (cf. [4] (1.3) Proposition).

An immediate computation shows that the Taylor expansions of the coordinate functions of the left and right divisions are of the form

$$\begin{aligned} y/x =& y - x - q(y-x,x) + q(q(y-x,x),x) - r(y-x,y-x,x) - s(y-x,x,x) + o(3), \\ x\backslash y =& y - x - q(x,y-x) + q(x,q(x,y-x)) - r(x,x,y-x) - s(x,y-x,y-x) + o(3), \end{aligned}$$

where $o(3)$ is an error term up to order 3.

Definition 2.1. Let L be a $\mathcal{C}^r$-differentiable local loop and $\alpha(t)$, $\beta(t)$, $\gamma(t)$ differentiable curves in L with initial data

$$\alpha(0) = \beta(0) = \gamma(0) = e,\ \alpha'(0) = X,\ \beta'(0) = Y,\ \gamma'(0) = Z, \quad X, Y, Z \in \mathrm{T}_e(L).$$

The bilinear *tangent commutator* $(X,Y) \mapsto [X,Y]$ of the local loop L on the tangent space $\mathrm{T}_e(L)$ is defined by

$$[X,Y] = \frac{1}{2}\frac{d^2 t}{d\,t^2}\Big|_{t=0}(\alpha(t)\cdot\beta(t))/(\beta(t)\cdot\alpha(t)) = \frac{1}{2}\frac{d^2 t}{d\,t^2}\Big|_{t=0}(\beta(t)\cdot\alpha(t))\backslash(\alpha(t)\cdot\beta(t)). \tag{6}$$

The trilinear *tangent associator* $(X,Y,Z) \mapsto \langle X,Y,Z\rangle$ of L on the tangent space $\mathrm{T}_e(L)$ is defined by

$$\begin{aligned} \langle X,Y,Z\rangle =& \frac{1}{6}\frac{d^3 t}{d\,t^3}\Big|_{t=0}\left((\alpha(t)\cdot\beta(t))\cdot\gamma(t)\right) / \left(\alpha(t)\cdot(\beta(t)\cdot\gamma(t))\right) = \\ =& \frac{1}{6}\frac{d^3 t}{d\,t^3}\Big|_{t=0}\left(\alpha(t)\cdot(\beta(t)\cdot\gamma(t))\right) \backslash \left((\alpha(t)\cdot\beta(t))\cdot\gamma(t)\right). \end{aligned} \tag{7}$$

Let L be a local loop identified with a neighbourhood W^n of 0 in the vector space V^n, the identity element of L with $0 \in V^n$ and the tangent space $\mathrm{T}_e(L)$ with V^n. The bilinear and trilinear maps $q : \mathrm{T}_e(L) \times \mathrm{T}_e(L) \to \mathrm{T}_e(L)$ and $r, s : \mathrm{T}_e(L) \times \mathrm{T}_e(L) \times \mathrm{T}_e(L) \to \mathrm{T}_e(L)$ are well defined by (5), using the Taylor expansion (4). According to 2.1.Lemma and 2.2.Lemma in [12] (or IX.6.6. Theorem in [13]) the commutator (6) and the associator (7) of the local multiplication (4) are expressed by the first non-vanishing term of the Taylor series of

$$(x \cdot y)/(y \cdot x) \text{ or } (y \cdot x)\backslash(x \cdot y) \text{ and } \big((x \cdot y) \cdot z\big)/\big(x \cdot (y \cdot z)\big) \text{ or } \big(x \cdot (y \cdot z)\big)\backslash\big((x \cdot y) \cdot z\big),$$

respectively. It follows for any $X, Y, Z \in \mathrm{T}_e(L)$

$$\begin{aligned} [X, Y] &= q(X, Y) - q(Y, X) (= 2q(X, Y) \text{ if } q(X, Y) \text{ is skew symmetric}), \\ \langle X, Y, Z\rangle &= q(q(X, Y), Z) - q(X, q(Y, Z)) + 2r(X, Y, Z) - 2s(X, Y, Z). \end{aligned} \tag{8}$$

Tensor products

In the following we investigate bilinear and trilinear maps between vector spaces V and U for the description of algebraic models of tangent commutators and associators. We identify the bilinear maps $V \times V \to U$ with elements of the tensor product $V_* \otimes V_* \otimes U$ and the trilinear maps $V \times V \times V \to U$ with elements of $V_* \otimes V_* \otimes V_* \otimes U$. We will denote by $V_* \wedge V_* \otimes U \subset V_* \otimes V_* \otimes U$ the subspace of $V_* \otimes V_* \otimes U$ consisting of skew-symmetric tensors and by $V_* \odot V_* \otimes U \subset V_* \otimes V_* \otimes U$ the subspace of $V_* \otimes V_* \otimes U$ consisting of symmetric tensor.

Definition 2.2. The map $\mathrm{Alt} : V_* \otimes V_* \otimes V_* \otimes U \to V_* \wedge V_* \wedge V_* \otimes U$ of trilinear maps defined by

$$\mathrm{Alt}(T)(\xi, \eta, \zeta) = \frac{1}{6}(T(\xi, \eta, \zeta) - T(\eta, \xi, \zeta) + T(\eta, \zeta, \xi) - T(\zeta, \eta, \xi) + T(\zeta, \xi, \eta) - T(\xi, \zeta, \eta)),$$

$T \in V_* \otimes V_* \otimes V_* \otimes U$, will be called *alternator map.*

The map $\mathrm{Sym} : V_* \otimes V_* \otimes V_* \otimes U \to V_* \odot V_* \odot V_* \otimes U$ of trilinear maps defined by

$$\mathrm{Sym}(T)(\xi, \eta, \zeta) = \frac{1}{6}(T(\xi, \eta, \zeta) + T(\eta, \xi, \zeta) + T(\eta, \zeta, \xi) + T(\zeta, \eta, \xi) + T(\zeta, \xi, \eta) + T(\xi, \zeta, \eta)),$$

$T \in V_* \otimes V_* \otimes V_* \otimes U$, will be called *symmetrizer map.*

Clearly, the trilinear map $\mathrm{Alt}(T)$ is skew-symmetric and $\mathrm{Sym}(T)$ is symmetric in all pairs of variables. Moreover $\mathrm{Alt}^2(T) = \mathrm{Alt}(\mathrm{Alt}(T)) = \mathrm{Alt}(T)$ and $\mathrm{Sym}^2(T) = \mathrm{Sym}(\mathrm{Sym}(T)) = \mathrm{Sym}(T)$, hence the maps $\mathrm{Alt} : V_* \otimes V_* \otimes V_* \otimes U \to V_* \wedge V_* \wedge V_* \otimes U$ and $\mathrm{Sym} : V_* \otimes V_* \otimes V_* \otimes U \to V_* \odot V_* \odot V_* \otimes U$ are projections onto subspaces. Hence one has the direct sum decompositions

$$\begin{aligned} V_* \wedge V_* \wedge V_* \otimes U \,\oplus\, \mathrm{Ker}(\mathrm{Alt}) &= V_* \otimes V_* \otimes V_* \otimes U, \\ V_* \odot V_* \odot V_* \otimes U \oplus \mathrm{Ker}(\mathrm{Sym}) &= V_* \otimes V_* \otimes V_* \otimes U \end{aligned}$$

of vector spaces, where $\mathrm{Ker}(\mathrm{Alt})$ and $\mathrm{Ker}(\mathrm{Sym})$ denotes the subspaces in $V_* \otimes V_* \otimes V_* \otimes U$, annihilated by the projections Alt and Sym, respectively.

Akivis algebra and Sabinin algebra of degree 3

In the following we consider non-associative algebras over a field $\mathbb{F}$ of characteristic $\neq 2, 3$. In particular, we will study tangent algebras of differentiable local loops, which are algebras over the real field $\mathbb{R}$.

Definition 2.3. A *binary-ternary algebra* $\mathcal{A} = (A, [.,.]_{\mathcal{A}}, \langle .,.,. \rangle_{\mathcal{A}})$ is a vector space A over a field $\mathbb{F}$ equipped with a skew-symmetric bilinear and a trilinear operation:

$$k_{\mathcal{A}} : (X, Y) \mapsto [X, Y]_{\mathcal{A}}, \quad m_{\mathcal{A}} : (X, Y, Z) \mapsto \langle X, Y, Z \rangle_{\mathcal{A}}.$$

In the following the bilinear and the trilinear operations of a binary-ternary algebra $\mathcal{M}$ will be denoted by $[.,.]_{\mathcal{M}}$ and $\langle .,.,. \rangle_{\mathcal{M}}$, respectively.
$\mathcal{A}$ is called *abelian* if $[X, Y]_{\mathcal{A}} = 0$ and $\langle X, Y, Z \rangle_{\mathcal{A}} = 0$ for any $X, Y, Z \in \mathcal{A}$.
A *homomorphism* between binary-ternary algebras is a linear map preserving the operations. A subalgebra $\mathcal{I} \subseteq \mathcal{A}$ is *ideal* if it is the kernel of a homomorphism to some binary-ternary algebra, i.e. one has $[a, \mathcal{I}]_{\mathcal{A}} \subseteq \mathcal{I}$ and $\langle a, b, \mathcal{I} \rangle_{\mathcal{A}} \subseteq \mathcal{I}$, $\langle a, \mathcal{I}, b \rangle_{\mathcal{A}} \subseteq \mathcal{I}$, $\langle \mathcal{I}, a, b \rangle_{\mathcal{A}} \subseteq \mathcal{I}$ for all $a, b \in \mathcal{A}$.

Definition 2.4. A binary-ternary algebra $\mathcal{A}$ over $\mathbb{F}$ is called *Akivis algebra* if the operations $[X, Y]_{\mathcal{A}}$ and $\langle X, Y, Z \rangle_{\mathcal{A}}$ satisfy the so-called *Akivis identity*:

$$\mathrm{Alt}(m_{\mathcal{A}})(X, Y, Z) = \frac{1}{6}([[X, Y]_{\mathcal{A}}, Z]_{\mathcal{A}} + [[Y, Z]_{\mathcal{A}}, X]_{\mathcal{A}} + [[Z, X]_{\mathcal{A}}, Y]_{\mathcal{A}}).$$

Definition 2.5. A vector space S over a field $\mathbb{F}$ together with a skew-symmetric bilinear operation $\{a, b\} : S \times S \to S$ and two trilinear operations $(a, b, c) : S \times S \times S \to S$, $\Phi_{1,2}(a, b, c) : S \times S \times S \to S$ satisfying the identities

$$\begin{aligned}
&(a, b, c) + (a, c, b) = 0, \\
&(a, b, c) + \{\{b, c\}, a\} + (b, c, a) + \{\{c, a\}, b\} + (c, a, b) + \{\{a, b\}, c\} = 0, \\
&\Phi_{1,2}(a, b, c) = \Phi_{1,2}(a, c, b)
\end{aligned}$$

for all $a, b, c \in S$ is called the Sabinin algebra $\mathcal{S} = (S, \{.,.\}, (.,.,.), \Phi_{1,2}(.,.,.))$ of degree 3.

According to Remark 13 in [5] the variety of Sabinin algebras of degree 3 is equivalent to the variety of Akivis algebras.

Lemma 2.3. Any Akivis algebra $\mathcal{A} = (A, [.,.]_{\mathcal{A}}, \langle .,.,. \rangle_{\mathcal{A}})$ is a Sabinin algebra
$\mathcal{S} = (S, \{.,.\}, (.,.,.), \Phi_{1,2}(.,.,.))$ of degree 3 with

$$\{a, b\} = -[a, b]_{\mathcal{A}}, \quad (a, b, c) = \langle a, c, b \rangle_{\mathcal{A}} - \langle a, b, c \rangle_{\mathcal{A}}, \quad \Phi_{1,2}(a, b, c) = \frac{1}{2}(\langle a, b, c \rangle_{\mathcal{A}} + \langle a, c, b \rangle_{\mathcal{A}})$$

for all $a, b, c \in A$.
Conversely, any Sabinin algebra $\mathcal{S} = (S, \{.,.\}, (.,.,.), \Phi_{1,2}(.,.,.))$ of degree 3 is an Akivis algebra $\mathcal{A} = (A, [.,.]_{\mathcal{A}}, \langle .,.,. \rangle_{\mathcal{A}})$ with

$$[a, b]_{\mathcal{A}} = -\{a, b\}, \quad \langle a, b, c \rangle_{\mathcal{A}} = \frac{1}{2}(2\Phi_{1,2}(a, b, c) - (a, b, c))$$

for all $a, b, c \in S$.

Let L be a local loop identified with a neighbourhood W^n of 0 in the vector space V^n, the identity element of L with $0 \in V^n$ and the tangent space $\mathrm{T}_e(L)$ with V^n. The formulas (8) for the tangent commutator (6) and the tangent associator (7) of L define a binary-ternary algebra on the tangent space $\mathrm{T}_e(L)$, this binary-ternary algebra is a real Akivis algebra on the tangent space $\mathrm{T}_e(L)$, cf. [12], [13].

Definition 2.6. The tangent space $\mathrm{T}_e(L)$ of a $\mathcal{C}^r$-differentiable local loop L equipped with the tangent commutator (6) and tangent associator (7) operations expressed by (8) is called the *tangent Akivis algebra* of L and denoted by $\mathrm{Ak}(L)$.
A loop L is said to be *associated* with an Akivis algebra $\mathcal{A}$ if $\mathcal{A}$ is isomorphic to the tangent Akivis algebra $\mathrm{Ak}(L)$ of L.

3 Extension of binary-ternary algebras

Definition 3.1. Let $\mathcal{G}$, $\mathcal{B}$ be binary-ternary algebras. A binary-ternary *extension* $\mathcal{C}$ of $\mathcal{G}$ by $\mathcal{B}$ is a short exact sequence

$$0 \to \mathcal{B} \xrightarrow{\iota} \mathcal{C} \xrightarrow{\pi} \mathcal{G} \to 0.$$

Assume that $s : \mathcal{G} \to \mathcal{C}$ is an injective linear map such that $\pi \circ s = \mathrm{Id}_{\mathcal{G}}$. Then for $x, y, z \in \mathcal{G}$ denote

$$\widetilde{\psi}(x,y) = [s(x), s(y)]_{\mathcal{C}} - s([x,y]_{\mathcal{G}}), \quad \widetilde{\Psi}(x,y,z) = \langle s(x), s(y), s(z)\rangle_{\mathcal{C}} - s(\langle x,y,z\rangle_{\mathcal{G}}). \tag{9}$$

If $\iota : \mathcal{B} \to \mathcal{C}$ is an embedding of the ideal $\mathcal{B} \subset \mathcal{C}$, and we identify $\mathcal{G}$ with the factor algebra $\mathcal{C}/\mathcal{B}$, then $\mathcal{C}$ is a vector space direct sum $\mathcal{C} = s(\mathcal{G}) \oplus \mathcal{B}$. Let $\mathcal{A}$ be the binary-ternary algebra defined on the subspace $s(\mathcal{G})$ such that $s : \mathcal{G} \to s(\mathcal{G}) = \mathcal{A}$ becomes an algebra isomorphism. Then in $\mathcal{C} = \mathcal{A} \oplus \mathcal{B}$ the maps $\pi : \mathcal{A} \oplus \mathcal{B} \to \mathcal{A}$ and $s : \mathcal{A} \to \mathcal{A} \oplus \mathcal{B}$ are expressed by

$$\pi(\xi, X) = \xi, \quad s(\xi) = (\xi, 0), \qquad \xi \in \mathcal{A},\ X \in \mathcal{B}.$$

Denoting the bilinear skew-symmetric map $\psi(\xi,\eta) = \widetilde{\psi}((\xi,0),(\eta,0))$ and the trilinear map $\Psi(\xi,\eta,\zeta)$ $= \widetilde{\Psi}((\xi,0),(\eta,0),(\zeta,0))$ we obtain from (9):

$$\begin{aligned} [(\xi,0),(\eta,0)]_{\mathcal{C}} =& [(\xi,0),(\eta,0)]_{\mathcal{A}} + \psi(\xi,\eta), \\ \langle(\xi,0),(\eta,0),(\zeta,0)\rangle_{\mathcal{C}} =& \langle(\xi,0),(\eta,0),(\zeta,0)\rangle_{\mathcal{A}} + \Psi(\xi,\eta,\zeta). \end{aligned} \tag{10}$$

Since $\pi(\psi(\xi,\eta)) = \pi(\Psi(\xi,\eta,\zeta)) = 0$ one has $\psi(\xi,\eta), \Psi(\xi,\eta,\zeta) \in \{0\} \oplus \mathcal{B}$. Hence (10) determines the maps

$$\begin{aligned} \psi \in \mathcal{A}_* \wedge \mathcal{A}_* \otimes \mathcal{B}, \quad \Psi \in \mathcal{A}_* \otimes \mathcal{A}_* \otimes \mathcal{A}_* \otimes \mathcal{B}, \\ [(\xi,0),(\eta,0)]_{\mathcal{C}} = \left([(\xi,0),(\eta,0)]_{\mathcal{A}}, \psi(\xi,\eta)\right), \\ \langle(\xi,0),(\eta,0),(\zeta,0)\rangle_{\mathcal{C}} = \left(\langle(\xi,0),(\eta,0),(\zeta,0)\rangle_{\mathcal{A}}, \Psi(\xi,\eta,\zeta)\right). \end{aligned} \tag{11}$$

Now, we can compute

$$\begin{aligned} &[(\xi,X),(\eta,Y)]_{\mathcal{C}} = \\ &\big([(\xi,0),(\eta,0)]_{\mathcal{A}}, [(0,X),(0,Y)]_{\mathcal{C}} + [(\xi,0),(0,Y)]_{\mathcal{C}} - [(\eta,0),(0,X)]_{\mathcal{C}} + \psi(\xi,\eta)\big) = \\ &\big([\xi,\eta]_{\mathcal{A}}, [X,Y]_{\mathcal{B}} + [(\xi,0),(0,Y)]_{\mathcal{C}} - [(\eta,0),(0,X)]_{\mathcal{C}} + \psi(\xi,\eta)\big). \end{aligned} \tag{12}$$

Similarly we get

$$\begin{aligned}&\langle(\xi,X),(\eta,Y),(\zeta,Z)\rangle_{\mathcal{C}}=\big(\langle\xi,\eta,\zeta\rangle_{\mathcal{A}},\langle X,Y,Z\rangle_{\mathcal{B}}+\langle(0,X),(\eta,0),(\zeta,0)\rangle_{\mathcal{C}}+\\&\langle(\xi,0),(0,Y),(\zeta,0)\rangle_{\mathcal{C}}+\langle(\xi,0),(\eta,0),(0,Z)\rangle_{\mathcal{C}}+\langle(\xi,0),(0,Y),(0,Z)\rangle_{\mathcal{C}}+\\&\langle(0,X),(\eta,0),(0,Z)\rangle_{\mathcal{C}}+\langle(0,X),(0,Y),(\zeta,0)\rangle_{\mathcal{C}}+\Psi(\xi,\eta,\zeta)\big).\end{aligned}\tag{13}$$

We define the multilinear maps

$$\alpha\in\mathcal{A}_*\otimes\mathcal{B}_*\otimes\mathcal{B},\quad \lambda,\mu,\nu\in\mathcal{A}_*\otimes\mathcal{A}_*\otimes\mathcal{B}_*\otimes\mathcal{B},\quad \sigma,\tau,\rho\in\mathcal{A}_*\otimes\mathcal{B}_*\otimes\mathcal{B}_*\otimes\mathcal{B}$$

by the equations

$$\begin{aligned}\alpha(\xi,X)=[(\xi,0),(0,X)]_{\mathcal{C}},\quad &\lambda(\xi,\eta,Z)=\langle(0,Z),(\xi,0),(\eta,0)\rangle_{\mathcal{C}},\\ \mu(\xi,\eta,Z)=\langle((\eta,0),(0,Z),(\xi,0)\rangle_{\mathcal{C}},\quad &\nu(\xi,\eta,Z)=\langle(\xi,0),(\eta,0),(0,Z)\rangle_{\mathcal{C}},\\ \sigma(\xi,Y,Z)=\langle(\xi,0),(0,Y),(0,Z)\rangle_{\mathcal{C}},\quad &\tau(\xi,Y,Z)=\langle(0,Z),(\xi,0),(0,Y)\rangle_{\mathcal{C}},\\ \rho(\xi,Y,Z)=\langle(0,Y),(0,Z),(\xi,0)\rangle_{\mathcal{C}}.\end{aligned}\tag{14}$$

Definition 3.2. The collection of maps $\Delta=\{\alpha,\lambda,\mu,\nu,\sigma,\tau,\rho,\psi,\Psi\}$ defined by (11) and (14) is called the *data system* of the extension $0\to\mathcal{B}\to\mathcal{C}\to\mathcal{A}\to 0$. The binary-ternary algebra $\mathcal{C}$ will be denoted by $\mathcal{A}\oplus_\Delta\mathcal{B}$.

The following assertion is obtained from the previous equations (12), (13) and (14):

Proposition 3.1. The operations of the binary-ternary algebra $\mathcal{A}\oplus_\Delta\mathcal{B}$ determined by the data system Δ are expressed by

$$\begin{aligned}&[(\xi,X),(\eta,Y)]_{\mathcal{C}}=\big([\xi,\eta]_{\mathcal{A}},[X,Y]_{\mathcal{B}}+\alpha(\xi,Y)-\alpha(\eta,X)+\psi(\xi,\eta)\big)\\&\langle(\xi,X),(\eta,Y),(\zeta,Z)\rangle_{\mathcal{C}}=\big(\langle\xi,\eta,\zeta\rangle_{\mathcal{A}},\langle X,Y,Z\rangle_{\mathcal{B}}+\\&\lambda(\eta,\zeta,X)+\mu(\zeta,\xi,Y)+\nu(\xi,\eta,Z)+\sigma(\xi,Y,Z)+\tau(\eta,Z,X)+\rho(\zeta,X,Y)+\Psi(\xi,\eta,\zeta)\big).\end{aligned}\tag{15}$$

Conversely, for arbitrary collection of maps $\Delta=\{\alpha,\lambda,\mu,\nu,\sigma,\tau,\rho,\psi,\Psi\}$, where

$$\begin{gathered}\alpha\in\mathcal{A}_*\otimes\mathcal{B}_*\otimes\mathcal{B},\quad \lambda,\mu,\nu\in\mathcal{A}_*\otimes\mathcal{A}_*\otimes\mathcal{B}_*\otimes\mathcal{B},\quad \sigma,\tau,\rho\in\mathcal{A}_*\otimes\mathcal{B}_*\otimes\mathcal{B}_*\otimes\mathcal{B}\\ \psi\in\mathcal{A}_*\wedge\mathcal{A}_*\otimes\mathcal{B},\quad \Psi\in\mathcal{A}_*\otimes\mathcal{A}_*\otimes\mathcal{A}_*\otimes\mathcal{B},\end{gathered}$$

the equations (15) define a binary-ternary algebra $\mathcal{A}\oplus_\Delta\mathcal{B}$ with data system Δ.

Definition 3.3. A binary-ternary algebra $\mathcal{A}\oplus_\Delta\mathcal{B}$ is called

affine extension if $\sigma,\tau,\rho=0$,

semidirect sum if $\psi,\Psi=0$,

linear semidirect sum if $\sigma,\tau,\rho,\psi,\Psi=0$.

Theorem 3.2. A binary-ternary algebra $\mathcal{A}\oplus_\Delta\mathcal{B}$ with $\Delta=\{\alpha,\lambda,\mu,\nu,\sigma,\tau,\rho,\psi,\Psi\}$ is an Akivis algebra if and only if $\mathcal{A}$ and $\mathcal{B}$ are Akivis algebras and

(i) for all $\xi \in \mathcal{A}$, $Y, Z \in \mathcal{B}$

$$\begin{aligned} &[\alpha(\xi, Y), Z]_{\mathcal{B}} + [Y, \alpha(\xi, Z)]_{\mathcal{B}} - \alpha(\xi, [Y, Z]_{\mathcal{B}}) = \\ &\rho(\xi, Y, Z) - \rho(\xi, Z, Y) + \sigma(\xi, Y, Z) - \sigma(\xi, Z, Y) + \tau(\xi, Y, Z) - \tau(\xi, Z, Y), \end{aligned} \tag{16}$$

(ii) for all $\xi, \eta \in \mathcal{A}$, $Z \in \mathcal{B}$

$$\begin{aligned} &\lambda(\xi, \eta, Z) - \lambda(\eta, \xi, Z) + \mu(\xi, \eta, Z) - \mu(\eta, \xi, Z) + \nu(\xi, \eta, Z) - \nu(\eta, \xi, Z) = \\ &\alpha([\xi, \eta]_{\mathcal{A}}, Z) - \alpha(\xi, \alpha(\eta, Z)) + \alpha(\eta, \alpha(\xi, Z)), \end{aligned} \tag{17}$$

(iii) for all ξ, η, $\zeta \in \mathcal{A}$

$$\begin{aligned} &\psi([\xi, \eta]_{\mathcal{A}}, \zeta) + \psi([\eta, \zeta]_{\mathcal{A}}, \xi) + \psi([\zeta, \xi]_{\mathcal{A}}, \eta) - \\ &-\alpha(\zeta, \psi(\xi, \eta)) - \alpha(\xi, \psi(\eta, \zeta)) - \alpha(\eta, \psi(\zeta, \xi)) = \mathrm{Alt}(\Psi)(\xi, \eta, \zeta). \end{aligned} \tag{18}$$

Proof. We will use the following technical assertion:

Lemma 3.3. Let U, V, W be vector spaces and $F : U \oplus V \to W$ a trilinear map, which is invariant with respect to cyclic permutations of variables. Then $F((\xi, X), (\eta, Y), (\zeta, Z)) = 0$ for any $\xi, \eta, \zeta \in U$, $X, Y, Z \in V$ if and only if

$$\begin{aligned} &F((\xi, 0), (\eta, 0), (\zeta, 0)) = 0, \quad F((0, X), (0, Y), (0, Z)) = 0, \\ &F((\xi, 0), (0, Y), (0, Z)) = 0, \quad F((\xi, 0), (\eta, 0), (0, Z)) = 0 \end{aligned} \tag{19}$$

for any $\xi, \eta, \zeta \in U$, $X, Y, Z \in V$.

Proof. Using the trilinear property of the map F and applying cyclic permutations of variables we obtain the identity

$$\begin{aligned} 0 = {} & F((\xi, 0), (\eta, 0), (\zeta, 0)) + F((0, X), (0, Y), (0, Z)) + \\ &+ \big(F((\xi, 0), (\eta, 0), (0, Z)) + F((\zeta, 0), (\xi, 0), (0, Y)) + F((\eta, 0), (\zeta, 0), (0, X))\big) + \\ &+ \big(F((\xi, 0), (0, Y), (0, Z)) + F((\eta, 0), (0, Z), (0, X)) + F((\zeta, 0), (0, X), (0, Y))\big) \end{aligned}$$

for any $\xi, \eta, \zeta \in U$, $X, Y, Z \in V$. This is true if and only if the constant, linear, bilinear and trilinear terms in the variables X, Y, Z are vanishing, that is we get the equivalent system of identities

$$\begin{aligned} &F((\xi, 0), (\eta, 0), (\zeta, 0)) = 0, \quad F((0, X), (0, Y), (0, Z)) = 0 \\ &\big(F((\xi, 0), (\eta, 0), (0, Z)) + F((\zeta, 0), (\xi, 0), (0, Y)) + F((\eta, 0), (\zeta, 0), (0, X))\big) = 0, \\ &\big(F((\xi, 0), (0, Y), (0, Z)) + F((\eta, 0), (0, Z), (0, X)) + F((\zeta, 0), (0, X), (0, Y))\big) = 0 \end{aligned}$$

Putting 0 into the variables X, Y in the third identity, respectively, into η, ζ in the fourth one, we get (19). Conversely, it follows from the equations

$$F((\xi, 0), (0, Y), (0, Z)) = 0, \quad F((\xi, 0), (\eta, 0), (0, Z)) = 0, \quad \text{for all} \quad \xi, \eta \in U, \ Y, Z \in V,$$

by substitutions $\eta \mapsto \xi$, $\zeta \mapsto \eta$, $Z \mapsto Y$, $X \mapsto Z$, respectively, $\zeta \mapsto \xi$, $\xi \mapsto \eta$, $X \mapsto Y$, $Y \mapsto Z$ that

$$\begin{aligned} 0 = {} & \big(F((\xi, 0), (\eta, 0), (0, Z)) + F((\zeta, 0), (\xi, 0), (0, Y)) + F((\eta, 0), (\zeta, 0), (0, X))\big) + \\ &+ \big(F((\xi, 0), (0, Y), (0, Z)) + F((\eta, 0), (0, Z), (0, X)) + F((\zeta, 0), (0, X), (0, Y))\big). \end{aligned}$$

Hence the lemma is true. □

For the proof of the theorem we notice that if $\mathcal{A} \oplus_\Delta \mathcal{B}$ is an Akivis algebra then the ideal $\mathcal{B}$ and $\mathcal{A}$, isomorphic to the factor algebra of $\mathcal{A} \oplus_\Delta \mathcal{B}$, are Akivis algebras. Putting the terms of the Akivis identity on one side, the map obtained is invariant with respect to cyclic permutations of variables, so we can apply Lemma 3.3. Substituting $X = Y = Z = 0$ in the Akivis identity, we get constant terms with respect to X, Y, Z, giving condition (iii) in the assertion. The replacement $X = Y = \zeta = 0$ yields linear terms in Z, giving condition (ii). By putting $X = \eta = \zeta = 0$ we obtain bilinear terms in Y, Z, hence condition (i) follows and the theorem is proved. □

We now formulate a special construction of linear semidirect sum of Akivis algebras.

Proposition 3.4. Let $\mathcal{A} = (A, [.,.]_\mathcal{A}, \langle .,.,. \rangle_\mathcal{A})$, $\mathcal{A}^* = (A, [.,.]_\mathcal{A}, \langle .,.,. \rangle_{\mathcal{A}^*})$ be Akivis algebras, $\mathcal{A}^+$ the abelian Akivis algebra on the vector space A and $\theta : \{0\} \oplus A \to A \oplus \{0\}$ a bijective linear map. The data system Δ defined by $\sigma = \tau = \rho = \psi = \Psi = 0$ and

$$\alpha(\xi, Z) = \theta^{-1}[\xi, \theta Z]_\mathcal{A},\ \lambda(\xi, \eta, Z) = \theta^{-1}\langle \theta Z, \xi, \eta \rangle_{\mathcal{A}^*},\ \mu(\xi, \eta, Z) = \theta^{-1}\langle \eta, \theta Z, \xi \rangle_{\mathcal{A}^*},$$

$$\nu(\xi, \eta, Z) = \theta^{-1}\langle \xi, \eta, \theta Z \rangle_{\mathcal{A}^*}$$

determine a linear semidirect sum $\mathcal{A} \oplus_\Delta \mathcal{A}^+$ of Akivis algebras.

Proof. The identity (16) of Theorem 3.2 is satisfied since $\mathcal{A}^+$ is abelian and $\sigma = \tau = \rho = 0$. The identity (17) can be obtained by conjugation with the map θ of the Akivis identity $\mathcal{A}^*$. For example:

$$\alpha(\xi, \alpha(\eta, Z)) = \theta^{-1}[\xi, \theta \cdot \theta^{-1}[\eta, \theta Z]_\mathcal{A}]_\mathcal{A} = \theta^{-1}[\xi, [\eta, \theta Z]_\mathcal{A}]_\mathcal{A}, \quad \lambda(\xi, \eta, Z) = \theta^{-1}\langle \theta Z, \xi, \eta \rangle_{\mathcal{A}^*}.$$

Likewise, we get the same conjugation relationship for the other terms of the Akivis identity. □

4 Akivis algebra and Sabinin algebra of degree 3 of abelian extensions of local loops

Consider a $\mathcal{C}^r$-differentiable local loop $L = (L, \cdot, /, \backslash)$ of dimension n with $r \geq 3$, a vector space U^k and a $\mathcal{C}^r$-differentiable map

$$\Omega : ((x, X), (y, Y)) \mapsto \Omega(x, y, X, Y), \quad \Omega : (L \times U^k) \times (L \times U^k) \to U^k, \tag{20}$$

satisfying

$$\Omega(x, e, X, 0) = X,\ \Omega(e, y, 0, Y) = Y,\ \Omega(e, e, X, Y) = X + Y \text{ for any } (x, X), (y, Y) \in L \times U^k. \tag{21}$$

Let $L \times_\Omega U^k$ denote the local H-space defined on $L \times U^k$ by the multiplication

$$(x, X)(y, Y) = (xy, \Omega(x, y, X, Y)), \quad x, y \in L, \quad X, Y \in U^k \tag{22}$$

having $(e, 0) \in L \times U^k$ as identity element. It is easy to see that $L \times_\Omega U^k$ satisfies the short exact sequence

$$0 \to U^k \to L \times_\Omega U^k \to L \to e,$$

hence $L \times_\Omega U^k$ is an extension of the local loop L by the vector group U^k. According to Proposition 2.2 the multiplication (22) induces a $\mathcal{C}^r$-differentiable local loop $\mathcal{F}(\Omega)$ on a neighbourhood of $(e, 0)$ in $L \times_\Omega U^k$. A particular case of this construction is formulated in the following

Definition 4.1. A $\mathcal{C}^r$-differentiable *loop cocycle* on the product manifold $L \times U^k$ is a triple of $\mathcal{C}^r$-differentiable maps:

$$P, Q : L \times L \to \mathrm{GL}(U^k), \quad \Theta : L \times L \to U^k \quad \text{with} \tag{23}$$

$$P(x,e) = \mathrm{Id} = Q(e,y), \quad \Theta(x,e) = 0 = \Theta(e,y) \quad \text{for all} \quad x, y \in L.$$

The *abelian extension* $\mathcal{F}(P,Q,\Theta)$ is the $\mathcal{C}^r$-differentiable local loop on $L \times U^k$ defined by the $\mathcal{C}^r$-differentiable operations

$$\begin{aligned}
(x,X)\cdot(y,Y) &= (xy, P(x,y)X + Q(x,y)Y + \Theta(x,y)),\\
(y,Y)/(x,X) &= \left(y/x, P(y/x,x)^{-1}(Y - Q(y/x,x)X - \Theta(y/x,x))\right),\\
(x,X)\backslash(y,Y) &= \left(x\backslash y, Q(x,x\backslash y)^{-1}(Y - P(x,x\backslash y)X - \Theta(x,x\backslash y))\right)
\end{aligned}$$

and identity element $(e,0)$. If Θ is trivial, i.e. $\Theta(x,y) = 0$ for all $x, y \in L$, then $\mathcal{F}(P,Q)$ is called *linear abelian extension.*

We identify L with a coordinate chart $W^n \subset V^n$ containing $0 \in V^n$ such that $e \in L$ corresponds to $0 \in W^n$ and investigate the abelian extension $\mathcal{F}(P,Q,\Theta)$ determined by the cocycle (23). The power series expansion of the maps $P, Q : W^n \times W^n \to \mathrm{GL}(U^k)$ in a neighbourhood of $(0,0)$ has the form

$$\begin{aligned}
P(x,y) &= \mathrm{Id} + P_2'(y) + P_{12}''(x,y) + \frac{1}{2}P_{22}''(y,y) + o(2),\\
Q(x,y) &= \mathrm{Id} + Q_1'(x) + \frac{1}{2}Q_{11}''(x,x) + Q_{12}''(x,y) + o(2),
\end{aligned} \tag{24}$$

since $P(x,0) = \mathrm{Id} = Q(0,y)$, $x, y \in W^n$, where $P_{22}''(x,y)$ and $Q_{11}''(x,y)$ are symmetric bilinear forms. The power series expansion

$$\begin{aligned}
\Theta(x,y) = \Theta_1'(x) + \Theta_2'(y) + \frac{1}{2}(\Theta_{11}''(x,x) + 2\Theta_{12}''(x,y) + \Theta_{22}''(y,y)) +\\
\frac{1}{3!}(\Theta_{111}'''(x,x,x) + 3\Theta_{112}'''(x,x,y) + 3\Theta_{122}'''(x,y,y) + \Theta_{222}'''(y,y,y)) + o(3)
\end{aligned}$$

of $\Theta : W^n \times W^n \to U^k$ in a neighbourhood of $(0,0)$ satisfies

$$\begin{aligned}
\Theta(x,0) &= \Theta_1'(x) + \frac{1}{2}\Theta_{11}''(x,x) + \frac{1}{3!}\Theta_{111}'''(x,x,x) + o(3) = 0,\\
\Theta(0,y) &= \Theta_2'(y) + \frac{1}{2}\Theta_{22}''(y,y) + \frac{1}{3!}\Theta_{222}'''(y,y,y) + o(3) = 0,
\end{aligned}$$

hence the power series expansion gets the form

$$\Theta(x,y) = \Theta_{12}''(x,y) + \frac{1}{2}\left(\Theta_{112}'''(x,x,y) + \Theta_{122}'''(x,y,y)\right) + o(3), \quad x, y \in W^n, \tag{25}$$

where $\Theta_{112}'''(x,y,z)$, respectively $\Theta_{122}'''(x,y,z)$ are symmetric in the first, respectively, last two variables. Consequently we have the expansion at $(0,0)$ with respect to the variables $(x,X), (y,Y) \in$

$W^n \times U^k$:

$$
\begin{aligned}
P(x,y)X + Q(x,y)Y + \Theta(x,y) = X + Y + P_2'(y)X + Q_1'(x)Y + \\
+P_{12}''(x,y)X + \frac{1}{2}P_{22}''(y,y)X + \frac{1}{2}Q_{11}''(x,x)Y + Q_{12}''(x,y)Y + \\
\Theta_{12}''(x,y) + \frac{1}{2}(\Theta_{112}'''(x,x,y) + \Theta_{122}'''(x,y,y)) + o(3).
\end{aligned} \tag{26}
$$

The trilinear maps

$$
\begin{aligned}
((x,X),(y,Y),(z,Z)) \mapsto P_{12}''(y,z)X + P_{12}''(x,z)Y + Q_{11}''(x,y)Z + \frac{1}{2}\Theta_{112}'''(x,y,z), \\
((x,X),(y,Y),(z,Z)) \mapsto P_{22}''(y,z)X + Q_{12}''(x,z)Y + Q_{12}''(x,y)Z + \frac{1}{2}\Theta_{122}'''(x,y,z)
\end{aligned}
$$

are symmetric in the first, respectively, last two variables. Introducing the notations

$$
\begin{aligned}
\mathcal{Q}\left((x,X),(y,Y)\right) &= \left(q(x,y), P_2'(y)X + Q_1'(x)Y + \Theta_{12}''(x,y)\right), \\
\mathcal{R}\left((x,X),(y,Y),(z,Z)\right) &= \left(r(x,y,z), \frac{1}{2}\left(P_{12}''(y,z)X + P_{12}''(x,z)Y + Q_{11}''(x,y)Z + \Theta_{112}'''(x,y,z)\right)\right), \\
\mathcal{S}\left((x,X),(y,Y),(z,Z)\right) &= \left(s(x,y,z), \frac{1}{2}\left(P_{22}''(y,z)X + Q_{12}''(x,z)Y + Q_{12}''(x,y)Z + \Theta_{122}'''(x,y,z)\right)\right),
\end{aligned} \tag{27}
$$

we obtain the expansion

$$
\begin{aligned}
(x,X)\cdot(y,Y) = (x,X) + (y,Y) + \mathcal{Q}\left((x,X),(y,Y)\right) + \mathcal{R}\left((x,X),(x,X),(y,Y)\right) + \\
+\mathcal{S}\left((x,X),(y,Y),(y,Y)\right) + o(3), \quad (x,X),(y,Y) \in W^n \times U^k
\end{aligned}
$$

of the multiplication of $\mathcal{F}(P,Q,\Theta)$. Now we can compute the commutator:

$$
\begin{aligned}
[(x,X),(y,Y)] = \mathcal{Q}\left((x,X),(y,Y)\right) - \mathcal{Q}\left((y,Y),(x,X)\right) = \\
=([x,y], \left(P_2'(y) - Q_1'(y)\right)X + \left(Q_1'(x) - P_2'(x)\right)Y + \Theta_{12}''(x,y) - \Theta_{12}''(y,x))).
\end{aligned} \tag{28}
$$

and the associator:

$$
\begin{aligned}
&\langle (x,X),(y,Y),(z,Z)\rangle = \mathcal{Q}(\mathcal{Q}((x,X),(y,Y)),(z,Z)) - \mathcal{Q}((x,X),\mathcal{Q}((y,Y),(z,Z))) + \\
&+2\mathcal{R}((x,X),(y,Y),(z,Z)) - 2\mathcal{S}((x,X),(y,Y),(z,Z)) = \\
&= \left(q(q(x,y),z), P_2'(z)\left(P_2'(y)X + Q_1'(x)Y + \Theta_{12}''(x,y)\right) + Q_1'(q(x,y))Z + \Theta_{12}''(q(x,y),z)\right) - \\
&- \left(q(x,q(y,z)), P_2'(q(y,z))X + Q_1'(x)\left(P_2'(z)Y + Q_1'(y)Z + \Theta_{12}''(y,z)\right) + \Theta_{12}''(x,q(y,z))\right) + \\
&+ \left(2r(x,y,z), P_{12}''(y,z)X + P_{12}''(x,z)Y + Q_{11}''(x,y)Z + \Theta_{112}'''(x,y,z)\right) - \\
&- \left(2s(x,y,z), P_{22}''(y,z)X + Q_{12}''(x,z)Y + Q_{12}''(x,y)Z + \Theta_{122}'''(x,y,z)\right) = \\
&= \left(\langle x,y,z\rangle, \left(P_2'(z)P_2'(y) - P_2'(q(y,z)) + P_{12}''(y,z) - P_{22}''(y,z)\right)X + \right. \\
&+ \left(P_2'(z)Q_1'(x) - Q_1'(x)P_2'(z) + P_{12}''(x,z) - Q_{12}''(x,z)\right)Y + \\
&+ \left(Q_1'(q(x,y)) - Q_1'(x)Q_1'(y) + Q_{11}''(x,y) - Q_{12}''(x,y)\right)Z + \\
&+P_2'(z)\Theta_{12}''(x,y) - Q_1'(x)\Theta_{12}''(y,z) + \Theta_{12}''(q(x,y),z) - \Theta_{12}''(x,q(y,z)) + \\
&\left. +\Theta_{112}'''(x,y,z) - \Theta_{122}'''(x,y,z)\right)
\end{aligned} \tag{29}
$$

of the Akivis algebra of the abelian extension $\mathcal{F}(P,Q,\Theta)$.

Theorem 4.1. The tangent Akivis algebra $\mathrm{Ak}(\mathcal{F}(P,Q,\Theta))$ of the abelian extension $\mathcal{F}(P,Q,\Theta)$ is an affine extension $\mathrm{Ak}(L) \oplus_\Delta (U^k)^+$ of the Akivis algebra $\mathrm{Ak}(L)$ of L by the abelian Akivis algebra $(U^k)^+$ on the vector space U^k. The data system Δ consists of the multilinear maps

$$\alpha : \mathrm{T}_e(L) \to \mathrm{End}(U^k), \quad \lambda,\mu,\nu : \mathrm{T}_e(L)\times\mathrm{T}_e(L) \to \mathrm{End}(U^k), \;\; \psi : \mathrm{T}_e(L)\times\mathrm{T}_e(L) \to U^k,$$

$$\Psi : \mathrm{T}_e(L)\times\mathrm{T}_e(L)\times\mathrm{T}_e(L) \to U^k$$

expressed by

$$\begin{aligned}
\alpha(\xi,Y) &= Q_1'(\xi)Y - P_2'(\xi)Y,\\
\lambda(\xi,\eta,Z) &= (P_2'(\eta)P_2'(\xi) - P_2'(q(\xi,\eta)) + P_{12}''(\xi,\eta) - P_{22}''(\xi,\eta))Z,\\
\mu(\xi,\eta,Z) &= (P_2'(\xi)Q_1'(\eta) - Q_1'(\eta)P_2'(\xi) + P_{12}''(\eta,\xi) - Q_{12}''(\eta,\xi))Z,\\
\nu(\xi,\eta,Z) &= (Q_1'(q(\xi,\eta)) - Q_1'(\xi)Q_1'(\eta) + Q_{11}''(\xi,\eta) - Q_{12}''(\xi,\eta))Z,
\end{aligned} \tag{30}$$

$$\begin{aligned}
\psi(\xi,\eta) &= \Theta_{12}''(\xi,\eta) - \Theta_{12}''(\eta,\xi),\\
\Psi(\xi,\eta,\zeta) &= P_2'(\zeta)\Theta_{12}''(\xi,\eta) - Q_1'(\xi)\Theta_{12}''(\eta,\zeta) + \Theta_{12}''(q(\xi,\eta),\zeta) -\\
&\quad - \Theta_{12}''(\xi,q(\eta,\zeta)) + \Theta_{112}'''(\xi,\eta,\zeta) - \Theta_{122}'''(\xi,\eta,\zeta),
\end{aligned} \tag{31}$$

for any $\xi,\eta,\zeta \in \mathrm{T}_e(L)$, $Y,Z \in U^k$.

Proof. Putting $x=\xi$, $y=\eta$, $z=\zeta$ in the equations (28) and (29) and using the definitions (11), (14) and taking into account (15) we obtain the assertion. □

It follows from Lemma 2.3:

Corollary 4.2. The Sabinin algebra $\mathcal{S}(\mathcal{F}(P,Q,\Theta))$ of degree 3 of the abelian extension $\mathcal{F}(P,Q,\Theta)$ is expressed by the operations:

$$\begin{aligned}
&\{(x,X),(y,Y)\} = \big(\{x,y\}, \big(Q_1'(y) - P_2'(y)\big)X + \big(P_2'(x) - Q_1'(x)\big)Y + \Theta_{12}''(y,x) - \Theta_{12}''(x,y)\big)\\
&\big((x,X),(y,Y),(z,Z)\big) = \big((x,y,z),\\
&\big(P_2'(y)P_2'(z) - P_2'(z)P_2'(y) - 2P_2'(q(z,y)) + P_{12}''(z,y) - P_{12}''(y,z)\big)X+\\
&\big(Q_1'(q(x,z)) + Q_{11}''(x,z) - Q_1'(x)(Q_1'(z) - P_2'(z)) - P_2'(z)Q_1'(x) - P_{12}''(x,z)\big)Y+\\
&\big(P_2'(y)Q_1'(x) + P_{12}''(x,y) - Q_1'(x)(P_2'(y) - Q_1'(y)) - Q_1'(q(x,y)) - Q_{11}''(x,y)\big)Z+\\
&P_2'(y)\Theta_{12}''(x,z) - P_2'(z)\Theta_{12}''(x,y) - Q_1'(x)(\Theta_{12}''(z,y) - \Theta_{12}''(y,z)) + \Theta_{12}''(q(x,z),y) -\\
&\Theta_{12}''(q(x,y),z) - 2\Theta_{12}''(x,q(z,y)) + \Theta_{112}'''(x,z,y) - \Theta_{112}'''(x,y,z)\big)\\
&\Phi_{1,2}((x,X),(y,Y),(z,Z)) = \big(\Phi_{1,2}(x,y,z),\\
&\frac{1}{2}\big(\big(P_2'(z)P_2'(y) + P_2'(y)P_2'(z) + P_{12}''(y,z) + P_{12}''(z,y) - 2P_{22}''(y,z)\big)X+\\
&\big(P_2'(z)Q_1'(x) + Q_1'(q(x,z)) - Q_1'(x)(P_2'(z) + Q_1'(z)) + P_{12}''(x,z) + Q_{11}''(x,z) - 2Q_{12}''(x,z)\big)Y+\\
&\big(P_2'(y)Q_1'(x) + Q_1'(q(x,y)) - Q_1'(x)(P_2'(y) + Q_1'(y)) + Q_{11}''(x,y) + P_{12}''(x,y) - 2Q_{12}''(x,y)\big)Z+\\
&P_2'(z)\Theta_{12}''(x,y) + P_2'(y)\Theta_{12}''(x,z) - Q_1'(x)(\Theta_{12}''(y,z) + \Theta_{12}''(z,y)) + \Theta_{12}''(q(x,y),z)+\\
&\Theta_{12}''(q(x,z),y) + \Theta_{112}'''(x,y,z) + \Theta_{112}'''(x,z,y) - 2\Theta_{122}'''(x,y,z)\big)\big).
\end{aligned}$$

Now, we investigate local loop extensions whose multiplication map is a third order polynomial such that the tangent algebra is an affine extension of an Akivis algebra by an abelian algebra.

Definition 4.2. Let $\mathcal{F}(\Omega)$ be a local loop with multiplication (22) defined on the coordinate chart $W^n \subset V^n$, where the map (20) satisfies the conditions (21). The loop $\mathcal{F}(\Omega)$ is called *almost abelian extension* of L by U^k if Ω is a polynomial of third order and the tangent Akivis algebra $\mathrm{Ak}(\mathcal{F}(\Omega))$ is an affine extension of an Akivis algebra by an abelian algebra.

Proposition 4.3. The local loop extension $\mathcal{F}(\Omega)$ of L by U^k is almost abelian if and only if the polynomial $\Omega(x, y, X, Y)$ can be expressed in the form

$$\begin{aligned}
&\Omega(x,y,X,Y) = P(x,y)X + Q(x,y)Y + \Theta(x,y) = \\
&= X + Y + P'_2(y)X + Q'_1(x)Y + P''_{12}(x,y)X + \frac{1}{2}P''_{22}(y,y)X + \\
&+ \frac{1}{2}Q''_{11}(x,x)Y + Q''_{12}(x,y)Y + \\
&+ \Theta''_{12}(x,y) + \frac{1}{2}(\Theta'''_{112}(x,x,y) + \Theta'''_{122}(x,y,y)) + \\
&+ \Lambda(x,X,Y) + \Lambda(x,Y,Y) + \Lambda(y,X,X) + \Lambda(y,X,Y)
\end{aligned} \tag{32}$$

where $\Lambda(x, Y, Z)$ is an arbitrary trilinear form symmetric in Y, Z.

Proof. Consider the polynomial $\Omega(x, y, X, Y)$ of order 3. According to the condition (21) we have $\Omega(x, 0, X, 0) = X$ and $\Omega(0, y, 0, Y) = Y$, hence we obtain, using the notation of the expansion (24), that the linear terms in the variable X are only of the form

$$X, \quad P'_2(y)X = \Omega''_{23}(y,X), \quad P''_{12}(x,y)X = \Omega'''_{123}(x,y,X), \quad P''_{22}(y,y)X = \Omega'''_{223}(y,y,X)$$

and the corresponding terms in the variable Y are only

$$Y, \quad Q'_1(x)Y = \Omega''_{14}(x,Y), \quad Q''_{11}(x,x)Y = \Omega'''_{114}(x,x,Y), \quad Q''_{12}(x,y)Y = \Omega'''_{124}(x,y,Y).$$

We get their sum as

$$\begin{aligned}
P(x,y)X &= X + P'_2(y)X + P''_{12}(x,y)X + \frac{1}{2}P''_{22}(y,y)X, \\
Q(x,y)Y &= Y + Q'_1(x)Y + \frac{1}{2}Q''_{11}(x,x)Y + Q''_{12}(x,y)Y.
\end{aligned}$$

Using the notation of the expansion (25) the constant terms in the Taylor polynomial with respect to the variables X and Y are

$$\Theta''_{12}(x,y) = \Omega''_{12}(x,y), \ \Theta'''_{112}(x,x,y) = \Omega'''_{112}(x,x,y), \ \Theta'''_{122}(x,y,y) = \Omega'''_{122}(x,y,y),$$

and their sum is

$$\Theta(x,y) = \Theta''_{12}(x,y) + \frac{1}{2}\left(\Theta'''_{112}(x,x,y) + \Theta'''_{122}(x,y,y)\right), \quad x,y \in W^n. \tag{33}$$

In addition, we get the terms that are quadratic or bilinear in the variables X, Y of the form

$$\Omega'''_{134}(x,X,Y), \quad \Omega'''_{144}(x,Y,Y), \quad \Omega'''_{233}(y,X,X), \quad \Omega'''_{234}(y,X,Y).$$

Clearly, the bilinear forms $\Omega'''_{144}(x,X,Y)$, $\Omega'''_{233}(y,X,Y)$ corresponding to the quadratic forms $\Omega'''_{144}(x,Y,Y)$, $\Omega'''_{233}(y,X,X)$ are symmetric in X, Y. We obtain the trilinear maps

$$\begin{aligned}((x,X),(y,Y),(z,Z)) \mapsto P''_{12}(y,z)X + P''_{12}(x,z)Y + Q''_{11}(x,y)Z + \frac{1}{2}\Theta'''_{112}(x,y,z) + \\ +\Omega'''_{134}(x,Y,Z) + \Omega'''_{134}(y,X,Z) + \Omega'''_{233}(z,X,Y), \\ ((x,X),(y,Y),(z,Z)) \mapsto P''_{22}(y,z)X + Q''_{12}(x,z)Y + Q''_{12}(x,y)Z + \frac{1}{2}\Theta'''_{122}(x,y,z) + \\ +\Omega'''_{144}(x,Y,Z) + \Omega'''_{234}(z,X,Y) + \Omega'''_{234}(y,X,Z)\end{aligned}$$

symmetric in the first, respectively, last two variables. Consequently, the following terms are added to the second component of the expression (27) of $\mathcal{R}((x,X),(y,Y),(z,Z))$, respectively $\mathcal{S}((x,X),(y,Y),(z,Z))$:

$$\begin{aligned}&\frac{1}{2}\left(\Omega'''_{134}(x,Y,Z) + \Omega'''_{134}(y,X,Z) + \Omega'''_{233}(z,X,Y)\right), \\ &\frac{1}{2}\left(\Omega'''_{144}(x,Y,Z) + \Omega'''_{234}(z,X,Y) + \Omega'''_{234}(y,X,Z)\right).\end{aligned}$$

Therefore, the associator (29) contains the following additional terms

$$\begin{aligned}&\left(\Omega'''_{134}(x,Y,Z) + \Omega'''_{134}(y,X,Z) + \Omega'''_{233}(z,X,Y)\right) - \\ &- \left(\Omega'''_{144}(x,Y,Z) + \Omega'''_{234}(z,X,Y) + \Omega'''_{234}(y,X,Z)\right).\end{aligned}$$

Hence the maps σ, τ, ρ defined by (14) are given by

$$\begin{aligned}\sigma(x,Y,Z) &= \Omega'''_{134}(x,Y,Z) - \Omega'''_{144}(x,Y,Z), \\ \tau(x,Y,Z) &= \Omega'''_{134}(x,Z,Y) - \Omega'''_{234}(x,Z,Y), \\ \rho(x,Y,Z) &= \Omega'''_{233}(x,Y,Z) - \Omega'''_{234}(x,Y,Z).\end{aligned}$$

In this case

$$\sigma(x,Y,Z) = \tau(x,Y,Z) = \rho(x,Y,Z) = 0$$

if and only if there is a monomial $\Lambda(x,Y,Z)$ symmetric in the variables Y, Z satisfying

$$\Lambda(x,Y,Z) = \Omega'''_{134}(x,Y,Z) = \Omega'''_{144}(x,Y,Z) = \Omega'''_{234}(x,Y,Z) = \Omega'''_{233}(x,Y,Z).$$

Hence the polynomial $\Omega(x,y,X,Y)$ has the form (32), which determines a non-abelian local loop extension associated with an affine extension of an Akivis algebra by an abelian algebra. □

5 Abelian loop extensions associated with tangent algebra extensions

Let V^n and U^k be vector spaces. For the solution of equations (31) with respect to the maps Θ'''_{112} and Θ'''_{122} we discuss, that a trilinear map with vanishing alternator how can be expressed as a difference of two maps $f \in V^n_* \odot V^n_* \otimes V^n_* \otimes U^k$, $\quad g \in V^n_* \otimes V^n_* \odot V^n_* \otimes U^k$ and investigate the sum $f + g$ of these maps.

Lemma 5.1. Let $T \in V_*^n \otimes V_*^n \otimes V_*^n \otimes U^k$ be a given trilinear map such that $\mathrm{Alt}(T) = 0$. Then T can be uniquely expressed as

$$T(\xi, \eta, \zeta) = f(\xi, \eta, \zeta) - g(\xi, \eta, \zeta), \tag{34}$$

with $Sym(f + g) = 0$, where $f \in V_*^n \odot V_*^n \otimes V_*^n \otimes U^k$, $g \in V_*^n \otimes V_*^n \odot V_*^n \otimes U^k$. Any other pair $\bar{f} \in V_*^n \odot V_*^n \otimes V_*^n \otimes U^k$, $\bar{g} \in V_*^n \times V_*^n \odot V_*^n \otimes U^k$ of maps satisfying (34) is of the form

$$\bar{f}(\xi, \eta, \zeta) = f(\xi, \eta, \zeta) + \chi(\xi, \eta, \zeta), \quad \bar{g}(\xi, \eta, \zeta) = g(\xi, \eta, \zeta) + \chi(\xi, \eta, \zeta),$$

with arbitrary $\chi \in V_*^n \odot V_*^n \odot V_*^n \otimes U^k$.

Proof. Clearly, if $T \in V_*^n \otimes V_*^n \otimes V_*^n \otimes U^k$ is symmetric in two variables then $\mathrm{Alt}(T) = 0$, i.e. T belongs to the kernel $\mathrm{Ker(Alt)}$. If

$$f(\xi, \eta, \zeta) - g(\xi, \eta, \zeta) = \bar{f}(\xi, \eta, \zeta) - \bar{g}(\xi, \eta, \zeta), \quad f, \bar{f} \in V_*^n \odot V_*^n \otimes V_*^n \otimes U^k, \ g, \bar{g} \in V_*^n \otimes V_*^n \odot V_*^n \otimes U^k$$

then

$$\chi = \bar{f} - f = \bar{g} - g \in V_*^n \odot V_*^n \odot V_*^n \otimes U^k \tag{35}$$

is symmetric in all pairs of variables. Let us denote

$$\begin{aligned}
\mathcal{X} &= \{(f, g) - \mathrm{Sym}(f, g); f \in V_*^n \odot V_*^n \otimes V_*^n \otimes U^k, \ g \in V_*^n \otimes V_*^n \odot V_*^n \otimes U^k\} \subset \mathrm{Ker(Sym)},\\
\mathcal{Y} &= \{(h, -h) \in (V_*^n \odot V_*^n \otimes V_*^n \otimes U^k) \oplus (V_*^n \otimes V_*^n \odot V_*^n \otimes U^k); h \in V_*^n \odot V_*^n \odot V_*^n \otimes U^k\},\\
\mathcal{Z} &= \{(\chi, \chi) \in (V_*^n \odot V_*^n \otimes V_*^n \otimes U^k) \oplus (V_*^n \otimes V_*^n \odot V_*^n \otimes U^k); \chi \in V_*^n \odot V_*^n \odot V_*^n \otimes U^k\}.
\end{aligned}$$

It is clear that $\mathcal{X} \oplus \mathcal{Y} \oplus \mathcal{Z}$ forms an interior direct sum decomposition of the exterior direct sum $(V_*^n \odot V_*^n \otimes V_*^n \otimes U^k) \oplus (V_*^n \otimes V_*^n \odot V_*^n \otimes U^k)$. According to (35), if the maps f, g satisfy (34), then for any $\chi \in V_*^n \odot V_*^n \odot V_*^n \otimes U^k$ the maps $\bar{f} = f + \chi$, $\bar{g} = g + \chi$ also satisfy (34), hence we are looking for solutions of (34) in the subspace $\mathcal{X} \oplus \mathcal{Y}$. Considering the projections $p_1 : (f, g) \mapsto f$, respectively, $p_2 : (f, g) \mapsto g$, we define the injective linear map

$$\begin{gathered}
\varphi : ((V_*^n \odot V_*^n \otimes V_*^n \otimes U^k) \oplus (V_*^n \otimes V_*^n \odot V_*^n \otimes U^k))/\mathcal{Z} \to \mathrm{Ker(Alt)},\\
\varphi(f, g) = p_1(f, g) - p_2(f, g) = f - g.
\end{gathered} \tag{36}$$

We prove by counting dimensions, that φ maps the subspace $\mathcal{X} \oplus \mathcal{Y}$ surjectively onto $\mathrm{Ker(Alt)}$. Let $e_1, \ldots, e_n$ be a basis of V^n. The maps contained in $\mathrm{Alt}(V_*^n \otimes V_*^n \otimes V_*^n \otimes U^k) = V_*^n \wedge V_*^n \wedge V_*^n \otimes U^k$ are uniquely determined by their values on tensor products $e_i \otimes e_j \otimes e_l$, where $i < j < l$. It follows that

$$\dim(\mathrm{Alt}(V_*^n \otimes V_*^n \otimes V_*^n \otimes U^k)) = \binom{n}{3} k, \quad \text{and} \quad \dim(\mathrm{Ker(Alt)}) = (n^3 - \binom{n}{3})k.$$

The elements of $V_*^n \odot V_*^n \otimes V_*^n \otimes U^k$ are uniquely determined by their values on the tensor products $e_i \otimes e_j \otimes e_k$, where $i \leq j$, hence $\dim(V_*^n \odot V_*^n \otimes V_*^n \otimes U^k) = \frac{1}{2} n^2(n+1)k$. Similarly, we get $\dim(V_*^n \otimes V_*^n \odot V_*^n \otimes U^k) = \frac{1}{2} n^2(n+1)k$. The vector space $V_*^n \odot V_*^n \odot V_*^n \otimes U^k$ is isomorphic to the vector space of U^k-valued homogeneous polynomials of degree 3 in n variables. The $\mathbb{F}$-valued monomials are of the form x_i^3, or $x_i x_j^2$, or $x_h x_i x_j$, their number is $n + n(n-1) + \binom{n}{3} = n^2 + \binom{n}{3}$. Hence $\dim(V_*^n \odot V_*^n \odot V_*^n \otimes U^k) = \dim(\mathcal{Z}) = (n^2 + \binom{n}{3}) \dim(U^k) = (n^2 + \binom{n}{3})k$. It follows that

$$\begin{aligned}\dim\big(\mathcal{X} \oplus \mathcal{Y}\big) &= \dim\big((V_*^n \odot V_*^n \otimes V_*^n \otimes U^k) \oplus (V_*^n \otimes V_*^n \odot V_*^n \otimes U^k)\big) - \dim(\mathcal{Z}) = \\ &= n^2(n+1)k - (n^2 + \binom{n}{3})k = (n^3 - \binom{n}{3})k.\end{aligned}$$

Consequently, $\dim\big(\mathcal{X} \oplus \mathcal{Y}\big) = \dim(\mathrm{Ker}(\mathrm{Alt}))$, hence φ defined by (36) maps $\mathcal{X} \oplus \mathcal{Y}$ onto Ker(Alt) bijectively. It follows that the equation (34) has a unique solution $(f, g) \in \mathcal{X} \oplus \mathcal{Y}$. If $(b, c) \in \mathcal{X}$, $(h, -h) \in \mathcal{Y}$ and $(\chi, \chi) \in \mathcal{Z}$ then $(f, g) = (b, c) + (h, -h) + (\chi, \chi)$ belongs to $\mathcal{X} \oplus \mathcal{Y}$ if and only if $\chi = 0$, or equivalently $\mathrm{Sym}(f + g) = 0$. It follows that the equation (34) has a unique solution (f, g) satisfying $\mathrm{Sym}(f + g) = 0$. The other solutions of (34) have the form $(f + \chi, g + \chi)$ with arbitrary $\chi \in V_*^n \odot V_*^n \odot V_*^n \otimes U^k$, hence the assertion is proved. $\square$

We consider now a $\mathcal{C}^r$-differentiable local loop L of dimension n with tangent Akivis algebra $\mathrm{Ak}(L)$. Let $\mathrm{Ak}(L) \oplus_\Delta (U^k)^+$ be an affine extension of $\mathrm{Ak}(L)$ by the abelian Akivis algebra $(U^k)^+$ on the vector space U^k corresponding to a data system $\Delta = \{\alpha, \lambda, \mu, \nu, \psi, \Psi\}$. We want to determine the Taylor polynomials $\widetilde{P}(x,y)X$, $\widetilde{Q}(x,y)Y$ and $\widetilde{\Theta}(x,y)$ of order 3 of the cocycle maps $P(x,y)X$, $Q(x,y)Y$ and $\Theta(x,y)$ given by (24) and (25) of all abelian extensions of L by the vector group U^k associated with the Akivis algebra $\mathrm{Ak}(L) \oplus_\Delta (U^k)^+$. The monomial terms in $\widetilde{P}(x,y)$, $\widetilde{Q}(x,y)$, $\widetilde{\Theta}(x,y)$ satisfy the underdetermined nonlinear system of equations (30) and (31). We find the solutions of this system of equations depending on arbitrarily given tensors.

Theorem 5.2. For any given maps

$$\begin{aligned}\pi \in \mathrm{Ak}(L)_* \otimes U_*^k \otimes U^k, \quad \Pi \in \mathrm{Ak}(L)_* \odot \mathrm{Ak}(L)_* \otimes U_*^k \otimes U^k,\\ \Gamma \in \mathrm{Ak}(L)_* \odot \mathrm{Ak}(L)_* \otimes U^k, \quad \chi \in \mathrm{Ak}(L)_* \odot \mathrm{Ak}(L)_* \odot \mathrm{Ak}_*(L) \otimes U^k\end{aligned}$$

the Taylor polynomials $\widetilde{P}(x,y)X$, $\widetilde{Q}(x,y)Y$ and $\widetilde{\Theta}(x,y)$ of order 3 of cocycles of abelian extensions $\mathcal{F}(P, Q, \Theta)$ associated with $\mathrm{Ak}(L) \oplus_\Delta (U^k)^+$ are uniquely given by

$$\begin{aligned}\widetilde{P}(x,y) &= \mathrm{Id} + \pi(y) + \lambda(x,y) - \pi(y)\pi(x) + \pi(q(x,y)) + \Pi(x,y) + \frac{1}{2}\Pi(y,y),\\ \widetilde{Q}(x,y) &= \mathrm{Id} + \alpha(x) + \pi(x) + \\ &+ \frac{1}{2}\big(\lambda(x,x) - \mu(x,x) + \nu(x,x) + \alpha(x)^2 + 2\pi(x)\alpha(x) + \Pi(x,x)\big) + \\ &+ \lambda(x,y) - \mu(y,x) - \alpha(x)\pi(y) + \pi(y)\alpha(x) - \pi(x)\pi(y) + \pi(q(x,y)) + \Pi(x,y)\\ \widetilde{\Theta}(x,y) &= \frac{1}{2}\psi(x,y) + \Gamma(x,y) + \frac{1}{2}\left(f(x,x,y) + g(x,y,y)\right) + \frac{1}{2}\left(\chi(x,x,y) + \chi(x,y,y)\right).\end{aligned} \tag{37}$$

The maps $f \in \mathrm{Ak}(L)_* \odot \mathrm{Ak}(L)_* \otimes \mathrm{Ak}(L)_* \otimes U^k$ and $g \in \mathrm{Ak}(L)_* \otimes \mathrm{Ak}(L)_* \odot \mathrm{Ak}(L)_* \otimes U^k$ in the expression of $\widetilde{\Theta}(x,y)$ are the unique solution of the equations

$$\begin{aligned}&\mathrm{Sym}(f+g) = 0, \quad \text{and}\\ &f(\xi,\eta,\zeta) - g(\xi,\eta,\zeta) =\\ =&\Psi(\xi,\eta,\zeta) - \pi(\zeta)(\frac{1}{2}\psi(\xi,\eta) + \Gamma(\xi,\eta)) + (\alpha(\xi) + \pi(\xi))(\frac{1}{2}\psi(\eta,\zeta) + \Gamma(\eta,\zeta)) - \\ -&(\frac{1}{2}\psi(q(\xi,\eta),\zeta) + \Gamma(q(\xi,\eta),\zeta)) + (\frac{1}{2}\psi(\xi,q(\eta,\zeta)) + \Gamma(\xi,q(\eta,\zeta))).\end{aligned}$$

Proof. We consider the system of equations (30) for P'_2, Q'_1, P''_{12}, P''_{22}, Q''_{11}, Q''_{12}. We omit the last variable Z in the maps

$$\alpha(\xi) = \alpha(\xi, Z), \quad \lambda(\xi,\eta) = \lambda(\xi,\eta,Z), \quad \mu(\xi,\eta) = \mu(\xi,\eta,Z), \quad \nu(\xi,\eta) = \nu(\xi,\eta,Z), \tag{38}$$

and assume that $\pi(\xi) = P'_2(\xi)$ and $\Pi(\xi,\eta) = P''_{22}(\xi,\eta)$ are arbitrary linear, respectively, bilinear symmetric maps. Then we obtain from (30) the following system of equations

$$\begin{aligned}
\alpha(\xi) &= -\pi(\xi) + Q'_1(\xi),\\
\lambda(\xi,\eta) &= \pi(\eta)\pi(\xi) - \pi(q(\xi,\eta)) + P''_{12}(\xi,\eta) - \Pi(\xi,\eta),\\
\mu(\xi,\eta) &= \pi(\xi)Q'_1(\eta) - Q'_1(\eta)\pi(\xi) + P''_{12}(\eta,\xi) - Q''_{12}(\eta,\xi),\\
\nu(\xi,\eta) &= Q'_1(q(\xi,\eta)) - Q'_1(\xi)Q'_1(\eta) + Q''_{11}(\xi,\eta) - Q''_{12}(\xi,\eta)
\end{aligned}$$

for the linear Q'_1, bilinear P''_{12}, Q''_{12} and symmetric bilinear map Q''_{11}. We can express

$$\begin{aligned}
&P'_2(\xi) = \ \pi(\xi), \quad P''_{22}(\xi,\eta) = \Pi(\xi,\eta), \quad Q'_1(\xi) = \alpha(\xi) + \pi(\xi),\\
&P''_{12}(\xi,\eta) = \lambda(\xi,\eta) - \pi(\eta)\pi(\xi) + \pi(q(\xi,\eta)) + \Pi(\xi,\eta),\\
&Q''_{12}(\xi,\eta) = \lambda(\xi,\eta) - \mu(\eta,\xi) - \alpha(\xi)\pi(\eta) + \pi(\eta)\alpha(\xi) - \pi(\xi)\pi(\eta) + \pi(q(\xi,\eta)) + \Pi(\xi,\eta),\\
&Q''_{11}(\xi,\eta) = \lambda(\xi,\eta) - \mu(\eta,\xi) + \nu(\xi,\eta) -\\
&\ - \alpha(q(\xi,\eta)) + \alpha(\xi)\alpha(\eta) + \pi(\xi)\alpha(\eta) + \pi(\eta)\alpha(\xi) + \Pi(\xi,\eta).
\end{aligned}$$

The last equation gives a symmetric bilinear expression for $Q''_{11}(\xi,\eta)$, since (17) implies that

$$\begin{aligned}
Q''_{11}(\xi,\eta) - Q''_{11}(\eta,\xi) &= \lambda(\xi,\eta) - \lambda(\eta,\xi) + \mu(\xi,\eta) - \mu(\eta,\xi) + \nu(\xi,\eta) - \nu(\eta,\xi) -\\
&- 2\alpha(q(\xi,\eta)) + \alpha(\xi)\alpha(\eta) - \alpha(\eta)\alpha(\xi) = 0.
\end{aligned}$$

According to the formulas (24) we obtain the polynomial formulas $\widetilde{P}(x,y)$ and $\widetilde{Q}(x,y)$ in (37). Assuming $\Gamma(\xi,\eta) = \frac{1}{2}(\Theta''_{12}(\xi,\eta) + \Theta''_{12}(\eta,\xi))$, we get from the first of the equations (31) that

$$\Theta''_{12}(\xi,\eta) = \frac{1}{2}\psi(\xi,\eta) + \Gamma(\xi,\eta), \tag{39}$$

where $\Gamma(\xi,\eta)$ is an arbitrary symmetric bilinear map. The second one of (31) gives the equation

$$\begin{aligned}
\Theta'''_{112}(\xi,\eta,\zeta) - \Theta'''_{122}(\xi,\eta,\zeta) = \Psi(\xi,\eta,\zeta) - \pi(\zeta)\Theta''_{12}(\xi,\eta) +\\
+(\alpha(\xi) + \pi(\xi))\Theta''_{12}(\eta,\zeta) - \Theta''_{12}(q(\xi,\eta),\zeta) + \Theta''_{12}(\xi,q(\eta,\zeta))
\end{aligned} \tag{40}$$

for $\Theta'''_{112}(\xi,\eta,\zeta)$ and $\Theta'''_{122}(\xi,\eta,\zeta)$. It follows from Lemma 5.1 that (40) is solvable if the alternator map Alt annihilates the expression

$$\Psi(\xi,\eta,\zeta) - \pi(\zeta)\Theta''_{12}(\xi,\eta) + (\alpha(\xi) + \pi(\xi))\Theta''_{12}(\eta,\zeta) - \Theta''_{12}(q(\xi,\eta),\zeta) + \Theta''_{12}(\xi,q(\eta,\zeta)). \tag{41}$$

We compute the alternator of (41):

$$
\begin{aligned}
&\mathrm{Alt}(\Psi)(\xi,\eta,\zeta)-\\
&-\big(\pi(\zeta)\psi(\xi,\eta)-(\alpha(\xi)+\pi(\xi))\Theta''_{12}(\eta,\zeta)+(\alpha(\eta)+\pi(\eta))\Theta''_{12}(\xi,\zeta)+\\
&+2\Theta''_{12}(q(\xi,\eta),\zeta)-\Theta''_{12}(\xi,q(\eta,\zeta))+\Theta''_{12}(\eta,q(\xi,\zeta))+\\
&+\pi(\xi)\psi(\eta,\zeta)-(\alpha(\eta)+\pi(\eta))\Theta''_{12}(\zeta,\xi)+(\alpha(\zeta)+\pi(\zeta))\Theta''_{12}(\eta,\xi)+\\
&+2\Theta''_{12}(q(\eta,\zeta),\xi)-\Theta''_{12}(\eta,q(\zeta,\xi))+\Theta''_{12}(\zeta,q(\eta,\xi))+\\
&+\pi(\eta)\psi(\zeta,\xi)-(\alpha(\zeta)+\pi(\zeta))\Theta''_{12}(\xi,\eta)+(\alpha(\xi)+\pi(\xi))\Theta''_{12}(\zeta,\eta)+\\
&+2\Theta''_{12}(q(\zeta,\xi),\eta)-\Theta''_{12}(\zeta,q(\xi,\eta))+\Theta''_{12}(\xi,q(\zeta,\eta))\big)=\\
&=\mathrm{Alt}(\Psi)(\xi,\eta,\zeta)+\alpha(\xi)\psi(\eta,\zeta)+\alpha(\eta)\psi(\zeta,\xi)+\alpha(\zeta)\psi(\xi,\eta)-\\
&-\psi(2q(\xi,\eta),\zeta)-\psi(2q(\eta,\zeta),\xi)-\psi(2q(\zeta,\xi),\eta).
\end{aligned}
\tag{42}
$$

Since the initial data of the Akivis algebra $\mathrm{Ak}(L)\oplus_\Delta (U^k)^+$ satisfy the equation (18), we obtain from (42), remembering to the notation (38), that

$$
\begin{aligned}
&\mathrm{Alt}(\Psi)(\xi,\eta,\zeta)+\alpha(\xi,\psi(\eta,\zeta))+\alpha(\eta,\psi(\zeta,\xi))+\alpha(\zeta,\psi(\xi,\eta))-\\
&-\psi([\xi,\eta]_{\mathcal{A}},\zeta)-\psi([\eta,\zeta]_{\mathcal{A}},\xi)-\psi([\zeta,\xi]_{\mathcal{A}},\eta)=0.
\end{aligned}
$$

It follows, that the maps Θ'''_{112} and Θ'''_{122} are the solutions of the equation (34) with

$$
\begin{aligned}
&T(\xi,\eta,\zeta)=\Psi(\xi,\eta,\zeta)-\pi(\zeta)\Theta''_{12}(\xi,\eta)+(\alpha(\xi)+\pi(\xi))\Theta''_{12}(\eta,\zeta)-\\
&-\Theta''_{12}(q(\xi,\eta),\zeta)+\Theta''_{12}(\xi,q(\eta,\zeta))=\\
&=\Psi(\xi,\eta,\zeta)-\pi(\zeta)(\frac{1}{2}\psi(\xi,\eta)+\Gamma(\xi,\eta))+(\alpha(\xi)+\pi(\xi))(\frac{1}{2}\psi(\eta,\zeta)+\Gamma(\eta,\zeta))-\\
&-(\frac{1}{2}\psi(q(\xi,\eta),\zeta)+\Gamma(q(\xi,\eta),\zeta))+(\frac{1}{2}\psi(\xi,q(\eta,\zeta))+\Gamma(\xi,q(\eta,\zeta))).
\end{aligned}
$$

Using (39) and applying Lemma 5.1 we obtain from (25) that the third order Taylor polynomial $\widetilde{\Theta}(x,y)$ of the cocycle of $\mathcal{F}(P,Q,\Theta)$ can be expressed as in the form (37), where the maps $f\in\mathrm{Ak}(L)_*\odot\mathrm{Ak}(L)_*\otimes\mathrm{Ak}(L)_*\otimes U^k$, $g\in\mathrm{Ak}(L)_*\otimes\mathrm{Ak}(L)_*\odot\mathrm{Ak}(L)_*\otimes U^k$ are uniquely determined by the equations

$$T(\xi,\eta,\zeta)=f(\xi,\eta,\zeta)-g(\xi,\eta,\zeta),\quad \mathrm{Sym}(f+g)=0,$$

and the map $\chi\in\mathrm{Ak}(L)_*\odot\mathrm{Ak}(L)_*\odot\mathrm{Ak}(L)_*\otimes U^k$ can be given arbitrarily. Hence the theorem is proved. □

Corollary 5.3. The dimension of vector space formed by the arbitrarily given tensors π, Π, Γ, χ in the expressions (37) is $\frac{1}{6}(9k+3nk+6n+n^2+5)nk$.

Proof. The tensors π, Π, Γ, χ form vector spaces of dimension nk^2, $\frac{1}{2}n(n+1)k^2$, $\frac{1}{2}n(n+1)k$, $(n^2+\binom{n}{3})k$, respectively, computing their sum we get the assertion. □

6 Tangent algebras of the tangent prolongation of loops

Definition 6.1. Let L be a $\mathcal{C}^r$-differentiable local loop ($r\geq 4$) and $\alpha(t)$, $\beta(t)$ differentiable curves in L with initial data $\alpha(0)=\beta(0)=e$, $\alpha'(0)=X$, $\beta'(0)=Y$, where $X,Y\in\mathrm{T}_e(L)$. The *tangent* *prolongation* $\mathscr{T}(L\times \mathrm{T}_e(L))$ of L is the manifold $L\times \mathrm{T}_e(L)$ equipped with the $\mathcal{C}^{r-1}$-differentiable multiplication

$$(x,X)\cdot(y,Y) = \big(xy, \mathrm{d}_{xy}\lambda_{xy}^{-1}\frac{d}{dt}\big|_{t=0}(x\alpha(t)\cdot y\beta(t))\big) = (xy, \mathrm{d}_e\lambda_{xy}^{-1}\rho_y\lambda_x X + \mathrm{d}_e\lambda_{xy}^{-1}\lambda_x\lambda_y Y),$$

for all $(x,X),(y,Y)\in L\times \mathrm{T}_e(L)$.

We can see immediately, (cf. [11], Lemma 4.1.):

Lemma 6.1. The tangent prolongation $\mathscr{T}(L\times \mathrm{T}_e(L))$ of a $\mathcal{C}^r$-differentiable local loop L is a $\mathcal{C}^{r-1}$-differentiable linear abelian extension $\mathcal{F}(P,Q)$ of L determined by the $\mathcal{C}^{r-1}$-differentiable cocycle

$$P(x,y) := \mathrm{d}_e(\lambda_{xy}^{-1}\rho_y\lambda_x), \quad Q(x,y) := \mathrm{d}_e(\lambda_{xy}^{-1}\lambda_x\lambda_y). \tag{43}$$

Lemma 6.2. The monomial terms of the power series expansion (26) of

$$P(x,y)X + Q(x,y)Y = \mathrm{d}_e(\lambda_{xy}^{-1}\rho_y\lambda_x)X + \mathrm{d}_e(\lambda_{xy}^{-1}\lambda_x\lambda_y)Y$$

are expressed by

$$\begin{aligned}
&P_2'(y)X = 2q(X,y),\\
&P_{12}''(x,y)X = q(q(x,X),y) - q(q(x,y),X) - 2q(x,q(X,y)) +\\
&+ 2r(x,X,y) - 2r(x,y,X),\\
&\frac{1}{2}P_{22}''(x,y)X = -q(x,q(X,y)) - q(y,q(X,x)) - r(x,y,X) + s(X,x,y),
\end{aligned} \tag{44}$$

and

$$\begin{aligned}
&Q_1'(x) = Q_{11}''(x,y) = 0,\\
&Q_{12}''(x,y)Y = q(x,q(y,Y)) - q(q(x,y),Y) + 2s(x,y,Y) - 2r(x,y,Y).
\end{aligned} \tag{45}$$

Proof. Let us denote

$$\Sigma = x\cdot y, \quad \Gamma(z) = \rho_y\lambda_x z, \quad \Delta(z) = \lambda_x\lambda_y z.$$

The map $P(x,y) = \mathrm{d}_e(\lambda_{xy}^{-1}\rho_y\lambda_x) = \mathrm{d}_e(\Sigma\backslash\Gamma)$ is the linear part of $\Sigma\backslash\Gamma(z)$ with respect to the variable z. We have the expansions

$$\begin{aligned}
\Gamma(z) = (x\cdot z)\cdot y = x + y + z + q(x,z) + q(x+z,y) + q(q(x,z),y) +\\
+r(x,x,z) + r(x+z,x+z,y) + s(x,z,z) + s(x+z,y,y) + o(3)
\end{aligned}$$

and

$$\begin{aligned}
\Gamma(z) - \Sigma = z + q(x,z) + q(z,y) + q(q(x,z),y) + r(x,x,z) +\\
+2r(x,z,y) + r(z,z,y) + s(x,z,z) + s(z,y,y) + o(3).
\end{aligned}$$

Using

$$x\backslash y = y - x - q(x,y-x) + q(x,q(x,y-x)) - r(x,x,y-x) - s(x,y-x,y-x) + o(3)$$

we obtain

$$\begin{aligned}
&\Sigma\backslash\Gamma(z) = \Gamma(z) - \Sigma - q(\Sigma, \Gamma(z) - \Sigma) + q(\Sigma, q(\Sigma, \Gamma(z) - \Sigma)) - \\
&- r(\Sigma, \Sigma, \Gamma(z) - \Sigma) - s(\Sigma, \Gamma(z) - \Sigma, \Gamma(z) - \Sigma) + \cdots = \\
&= z + 2q(z, y) + q(q(x, z), y) - q(q(x, y), z) - \\
&- q(x + y, q(z, y) - q(y, z)) + 2r(x, z, y) - 2r(x, y, z) + \\
&+ r(z, z, y) - r(y, y, z) + s(x, z, z) + s(z, y, y) - s(x + y, z, z) + o(3).
\end{aligned}$$

The linear part of $\Sigma\backslash\Gamma$ with respect to the variable z gives

$$\begin{aligned}
P(x, y)Z = Z + 2q(Z, y) + q(q(x, Z), y) - q(q(x, y), Z) - 2q(x + y, q(Z, y)) + \\
+ 2r(x, Z, y) - 2r(x, y, Z) - r(y, y, Z) + s(Z, y, y) + o(3).
\end{aligned} \tag{46}$$

Similarly, $Q(x, y) = \mathrm{d}_e(\lambda_{xy}^{-1}\lambda_x\lambda_y) = \mathrm{d}_e(\Sigma\backslash\Delta)$ is the linear part of $\lambda_{xy}^{-1}\lambda_x\lambda_y(z) = \Sigma\backslash\Delta(z)$ with respect to the variable z. We have

$$\begin{aligned}
&\Delta(z) - \Sigma = z + q(x + y, z) + q(x, q(y, z)) + r(x, x, z) + r(y, y, z) + \\
&+ s(y, z, z) + 2s(x, y, z) + s(x, z, z) + o(3),
\end{aligned}$$

and

$$\begin{aligned}
&\Sigma\backslash\Delta(z) = \Delta(z) - \Sigma - q(\Sigma, \Delta(z) - \Sigma) + q(\Sigma, q(\Sigma, \Delta(z) - \Sigma)) - \\
&- r(\Sigma, \Sigma, \Delta(z) - \Sigma) - s(\Sigma, \Delta(z) - \Sigma, \Delta(z) - \Sigma) + \cdots = \\
&= z + q(x, q(y, z)) - q(q(x, y), z) - 2r(x, y, z) + 2s(x, y, z) + o(3).
\end{aligned}$$

It follows

$$Q(x, y)Z = Z + q(x, q(y, Z)) - q(q(x, y), Z) + 2s(x, y, Z) - 2r(x, y, Z) + o(3). \tag{47}$$

According to (24), (46) and (47) we have

$$\begin{aligned}
&P'_2(y)X + \ P''_{12}(x, y)X + \frac{1}{2}P''_{22}(y, y)X = \\
&= 2q(X, y) + q(q(x, X), y) - q(q(x, y), X) - 2q(x + y, q(X, y)) + \\
&+ 2r(x, X, y) - 2r(x, y, X) - r(y, y, X) + s(X, y, y),
\end{aligned} \tag{48}$$

and

$$\begin{aligned}
&Q'_1(x)Y + \frac{1}{2}Q''_{11}(x, x)Y + Q''_{12}(x, y)Y = \\
&= q(x, q(y, Y)) - q(q(x, y), Y) + 2s(x, y, Y) - 2r(x, y, Y).
\end{aligned} \tag{49}$$

The assertion of lemma follows from the formulas (48) and (49). □

Proposition 6.3. The commutator and the associator of the tangent Akivis algebra of the tangent prolongation $\mathscr{T}(L \times \mathrm{T}_e(L))$ are expressed by

$$[(x, X), (y, Y)] = ([x, y], [X, y] + [x, Y])$$

and

$$\langle (x, X), (y, Y), (z, Z)\rangle = \big(\langle x, y, z\rangle, \langle X, y, z\rangle + \langle x, Y, z\rangle + \langle x, y, Z\rangle\big)$$

in a distinguished coordinate chart $W^n \times V^n$.

Proof. We apply the results of Theorem 4.1 to the prolongation $\mathcal{F}(P,Q)$ of L determined by the cocycle (43). Using the formulas (28) and (29) in the case $\Theta = 0$ we get the commutator and the associator:

$$\begin{aligned}
&[(x,X),(y,Y)] = ([x,y], 2(q(X,y)-q(Y,x)) = ([x,y],[X,y]+[x,Y]),\\
&\langle (x,X),(y,Y),(z,Z)\rangle = \big(\langle x,y,z\rangle, q(q(X,y),z)-q(X,q(y,z))+2r(X,y,z)-\\
&-2s(X,y,z)+q(q(x,Y),z)-q(x,q(Y,z))+2r(x,Y,z)-2s(x,Y,z)-\\
&-q(x,q(y,Z))+q(q(x,y),Z)-2s(x,y,Z)+2r(x,y,Z)\big) =\\
&\qquad = \big(\langle x,y,z\rangle, \langle X,y,z\rangle + \langle x,Y,z\rangle + \langle x,y,Z\rangle\big).
\end{aligned}$$

Hence the assertion is proved. □

In Proposition 6.3 we identified the local loop L with the coordinate chart $W^n \subset V^n$, the tangent space $\mathrm{T}_e(L)$ with the vector space V^n, the tangent prolongation $\mathscr{T}(L \times \mathrm{T}_e(L))$ with $W^n \times V^n$, and computed the commutator and associator in the tangent space $\mathrm{T}_{(0,0)}(W^n \times V^n)$. Now we will find a coordinate-free expression for the operations of the tangent Akivis algebra of tangent prolongation $\mathscr{T}(L \times \mathrm{T}_e(L))$. Using the fact that the tangent spaces of a vector space are canonically isomorphic to the vector space we get a canonical linear isomorphism of the tangent space $\mathrm{T}_{(e,0)}(L \times \mathrm{T}_e(L))$ to the direct sum $\mathrm{T}_e(L) \oplus \mathrm{T}_e(L)$. The bilinear, respectively, trilinear forms q, r, s, the commutator and the associator of L are defined on the subspace $\mathrm{T}_e(L) \oplus \{0\} \cong \mathrm{T}_e(L)$. Let

$$\theta : \{0\} \oplus \mathrm{T}_e(L) \to \mathrm{T}_e(L) \oplus \{0\}, \quad \theta : (0,X) \mapsto (X,0)$$

be the canonical linear isomorphism induced by the identity map of $\mathrm{T}_e(L)$. In the expressions in Proposition 6.3 of the operations of the tangent Akivis algebra of $\mathscr{T}(L \times \mathrm{T}_e(L))$ we replace $x,y,z \in W^n$ with $\xi,\eta,\zeta \in \mathrm{T}_e(L) \oplus \{0\}$. Using the map $\theta : (0,X) \mapsto (X,0)$ we obtain the expression of the commutator and of the associator as follows:

Theorem 6.4. The tangent Akivis algebra $\mathrm{Ak}(\mathscr{T}(L \times \mathrm{T}_e(L)))$ of the tangent prolongation $\mathscr{T}(L \times \mathrm{T}_e(L))$ of a $\mathcal{C}^r$-differentiable local loop L is a linear semidirect sum

$$\mathrm{Ak}(\mathscr{T}(L \times \mathrm{T}_e(L))) \cong \mathrm{Ak}(L) \oplus_\Delta \mathrm{T}_e(L)^+$$

of the tangent Akivis algebra $\mathrm{Ak}(L)$ and the abelian Akivis algebra $\mathrm{T}_e(L)^+$ on the tangent space $\mathrm{T}_e(L)$, determined by the data system Δ consisting of the maps

$$\begin{gathered}
\alpha(\xi,Z) = \theta^{-1}[\xi,\theta Z],\ \lambda(\xi,\eta,Z) = \theta^{-1}\langle \theta Z,\xi,\eta\rangle,\ \mu(\xi,\eta,Z) = \theta^{-1}\langle \eta,\theta Z,\xi\rangle,\\
\nu(\xi,\eta,Z) = \theta^{-1}\langle \xi,\eta,\theta Z\rangle, \quad \text{where} \quad \theta : (0,X) \mapsto (X,0).
\end{gathered}$$

The commutator and the associator of $\mathrm{Ak}(\mathscr{T}(L \times \mathrm{T}_e(L)))$ are expressed by

$$\begin{aligned}
&[(\xi,X),(\eta,Y)] = ([\xi,\eta], \theta^{-1}([\theta(X),\eta]) + \theta^{-1}([\xi,\theta(Y)])),\\
&\langle (\xi,X),(\eta,Y),(\zeta,Z)\rangle = \big(\langle \xi,\eta,\zeta\rangle, \theta^{-1}(\langle \theta(X),\eta,\zeta\rangle) + \theta^{-1}(\langle \xi,\theta(Y),\zeta\rangle) + \theta^{-1}(\langle \xi,\eta,\theta(Z)\rangle)\big)
\end{aligned}$$

for any $(\xi,X),(\eta,Y),(\zeta,Z) \in \mathrm{T}_e(L) \oplus \mathrm{T}_e(L)$.

It follows from Lemma 2.3:

Corollary 6.5. The Sabinin algebra $\mathcal{S}(\mathscr{T}(L \times \mathrm{T}_e(L)))$ of degree 3 of the tangent prolongation $\mathscr{T}(L \times \mathrm{T}_e(L))$ of a $\mathcal{C}^r$-differentiable local loop L is defined by the operations

$$\begin{aligned}
\{(\xi, X), (\eta, Y)\} =& \big(\{\xi, \eta\}, \theta^{-1}(\{\theta(X), \eta\}) + \theta^{-1}(\{\xi, \theta(Y)\})\big), \\
\big((\xi, X), (\eta, Y), (\zeta, Z)\big) =& \big((\xi, \eta, \zeta), \theta^{-1}((\theta(X), \eta, \zeta)) + \theta^{-1}((\xi, \theta(Y), \zeta)) + \theta^{-1}((\xi, \eta, \theta(Z)))\big) \\
\Phi_{1,2}((\xi, X), (\eta, Y), (\zeta, Z)) =& \big(\Phi_{1,2}(\xi, \eta, \zeta), \theta^{-1}(\Phi_{1,2}(\theta(X), \eta, \zeta) + \theta^{-1}(\Phi_{1,2}(\xi, \theta(Y), \zeta)) + \\
& \theta^{-1}(\Phi_{1,2}(\xi, \eta, \theta(Z))\big).
\end{aligned}$$

The expressions obtained for the commutator and associator of the tangent Akivis algebra of the tangent prolongation show that this semidirect sum of Akivis algebras is constructed as described in Proposition 3.4 in the case if $\mathcal{A} = \mathcal{A}^*$ and $\theta : \{0\} \oplus \mathrm{T}_e(L) \to \mathrm{T}_e(L) \oplus \{0\}$ is induced by the identity map of $\mathrm{T}_e(L)$.

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