3-dimensional loops on non-solvable reductive spaces

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Abstract

We treat the almost differentiable left A-loops as images of global differentiable sharply transitive sections $\sigma : G/H \to G$ for a Lie group G such that G/H is a reductive homogeneous manifold. In this paper we classify all 3-dimensional connected strongly left alternative almost differentiable left A-loops L, such that for the corresponding section $\sigma : G/H \to G$ the Lie group G is non-solvable.

Introduction

The associative law forces the identity $(ab)^{-1}ab = 1$ for all elements a and b of a group G. For loops which are structures with a binary multiplication having up to associativity the same properties as groups this behaviour changes radically. This observation led to a broader research of loops L in which the mapping $x \mapsto [(ab)^{-1}(a(bx))]$ is an automorphism of L (cf. [3], [2]). These loops have been called left A-loops.

According to [16] we treat the left A-loops L as images of global differentiable sharply transitive sections $\sigma : G/H \to G$ for a Lie group G such that the subset $\sigma(G/H)$ is invariant under the conjugation with the elements of H. Here G denotes the group topologically generated by the left translations of L and H is the stabilizer of the identity of L in G. Loops given by a differentiable section in a Lie group are called almost differentiable.

For an almost differentiable left A-loop L the tangent space $T_1\sigma(G/H)$ of the image of σ at $1 \in G$ can be provided with a binary and a ternary multiplication and yields a Lie triple algebra (cf. [11], Definition 7.1, p. 173). Since the Lie triple algebras correspond to affine reductive spaces, which are essential objects in differential geometry (cf. [13], [8]), there is

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a strong connection between the theory of differential left A-loops and the theory of affine reductive homogeneous spaces (cf. [12]). In particular the theory of connected differentiable Bruck loops (which form a subclass of the class of left A-loops) is essentially the theory of affine symmetric spaces (cf. [16], Section 11).

The smallest dimension for a connected almost differentiable non-associative left A-loop is equal 2. There exist precisely two isotopism classes of 2-dimensional left A-loops. In the one class there lies only the hyperbolic plane loop which is related to the hyperbolic symmetric plane (cf. [16], Section 22). In the other isotopism class we may choose as a representative the 2-dimensional Bruck loop L which is realized on the pseudo-euclidean affine plane E such that the group topologically generated by its left translations is the connected component of the group of pseudo-euclidean motions and the elements of L are the lines of positive slope in E (cf. [16], Section 25).

Our aim in this paper is to classify the 3-dimensional connected almost differentiable left A-loops, which have a non-solvable Lie group G as the group topologically generated by their left translations and which correspond to differentiable sections $\sigma : G/H \to G$ such that the exponential image of the tangent space $\mathbf{m} = T_1(\sigma(G/H))$ is contained in $\sigma(G/H)$. These loops are called strongly left alternative almost differentiable left A-loops.

Using the standard enveloping Lie algebra of a Lie triple algebra one sees that G is four, five or six dimensional. For the classification we determine all complements $\mathbf{m}$ of the Lie algebra $\mathbf{h}$ of H in the Lie algebra $\mathbf{g}$ of G such that $\mathbf{m}$ generates $\mathbf{g}$ and satisfies the relation $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$. The submanifold $\exp \mathbf{m}$ can be extended to a global section if and only if $\exp \mathbf{m}$ forms a system of representatives for the cosets $\{xH^g |\ x \in G\}$ in G, where $H^g = g^{-1}Hg$ with $g \in G$.

In contrast to a frequent occurance of reductive spaces and hence strongly left alternative almost differentiable local left A-loop, which can be represented as local sections in non-solvable Lie groups, the global loops in this class are rare and in strong relation to geometries on 3-dimensional manifolds as the following theorem shows.

Theorem *There are precisely two classes $\mathcal{C}_i$ $(i = 1, 2)$ of connected almost differentiable strongly left alternative simple left A-loops L having dimension 3 such that the group G generated by the left translations is a non-solvable Lie group.*

The class $\mathcal{C}_1$ consists of left A-loops having the simple Lie group $G = PSL_2(\mathbb{C})$ as the group topologically generated by their left translations, and the stabilizer H of $e \in L$ in G is the group $SO_3(\mathbb{R})$.
Any loop in the class $\mathcal{C}_1$ can be represented by a real parameter $a \in \mathbb{R}$. For

all $a \in \mathbb{R}$ the loops L_a and L_{-a} are isomorphic. These two loops form a full isotopism class. The loops L_a, $a \in \mathbb{R}$ are realized on the hyperbolic symmetric space H_3 such that the group topologically generated by their left translations is the connected component of the group of motions of H_3. The elements of all loops L_a in $\mathcal{C}_1$ are the points of H_3, but the sets of left translations differ. The hyperbolic space loop L_0, which is the unique Bruck loop in $\mathcal{C}_1$, is defined by the multiplication $x \cdot y = \tau_{e,x}(y)$, where $\tau_{e,x}$ is the hyperbolic translation moving e onto x.

The class $\mathcal{C}_2$ of simple left A-loops consists of 3-dimensional connected differentiable left A-loops such that the group $G = PSL_2(\mathbb{R}) \ltimes \mathbb{R}^3$, where the action of $PSL_2(\mathbb{R})$ on $\mathbb{R}^3$ is the adjoint action of $PSL_2(\mathbb{R})$ on its Lie algebra, is the group topologically generated by the left translations. This group is the connected component of the group of pseudo-euclidean motions and the stabilizer H of $e \in L$ in G is the stabilizer of a plane on which the euclidean metric is induced.

The loops in $\mathcal{C}_2$ can be represented by two real parameters a, b and form precisely two isomorphism classes, which coincide with the isotopism classes. The one isomorphism class consists of Bruck loops $L_{a,0}$, $a \in \mathbb{R}$, and we may choose the pseudo-euclidean space loop $L_{0,0} = \hat{L}_0$ as a representative of this isomorphism class. As a representative of the other isomorphism class which contains the loops $L_{a,b}$ with $b \neq 0$ may be chosen the loop $L_{0,1} = \hat{L}_1$. Any loop in the class $\mathcal{C}_2$ is realized on the pseudo-euclidean affine space $E(2,1)$. The elements of these loops are the planes on which the euclidean metric is induced but the sets of left translations differ.

Moreover, the 3-dimensional strongly left alternative almost differentiable non-simple left A-loops are either the products of a 1-dimensional Lie group with a 2-dimensional left A-loop isomorphic to the hyperbolic plane loop or the Scheerer extensions of the Lie group $SO_2(\mathbb{R})$ by the 2-dimensional left A-loop isomorphic to the hyperbolic plane loop and the coverings of these Scheerer extensions.

Another class of almost differentiable loops which has been thoroughly investigated is the class of differentiable Bol loops. The sections $\sigma : G/H \to G$ of Bol loops are characterized by the fact that for all $a, b \in \sigma(G/H)$ the element aba is also contained in $\sigma(G/H)$. The 3-dimensional almost differentiable Bol loops with non-solvable Lie groups have been classified in [5]; the Lie groups G topologically generated by their left translations as well as the corresponding stabilizers H are the same as in the case of 3-dimensional almost differentiable left A-loops, but the sections essentially differ. The intersection of these two classes are only the Bruck loops and the Scheerer extensions of the orthogonal group $SO_2(\mathbb{R})$ by the hyperbolic plane loop and the coverings of these Scheerer extensions.

1. Left A-loops

1.1 A set L with a binary operation $(x, y) \mapsto x \cdot y$ is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution which we denote by $y = a \backslash b$ and $x = b/a$. The left translation $\lambda_a : y \mapsto a \cdot y : L \to L$ is a bijection of L for any $a \in L$. Two loops $(L_1, \circ)$ and $(L_2, *)$ are called isotopic if there are three bijections $\alpha, \beta, \gamma : L_1 \to L_2$ such that $\alpha(x) * \beta(y) = \gamma(x \circ y)$ holds for any $x, y \in L_1$. An isotopism is an equivalence relation. If $\alpha = \beta = \gamma$ then the isotopic loops $(L_1, \circ)$ and $(L_2, *)$ are called isomorphic. Let $(L_1, \cdot)$ and $(L_2, *)$ be two loops. The direct product $L = L_1 \times L_2 = \{(a, b) \mid a \in L_1, b \in L_2\}$ with the multiplication $(a_1, b_1) \circ (a_2, b_2) = (a_1 \cdot a_2, b_1 * b_2)$ is again a loop, which is called the direct product of L_1 and L_2, and the loops $(L_1, \cdot)$, $(L_2, *)$ are subloops of $(L, \circ)$.
A loop is called a left A-loop if each mapping $\lambda_{x,y} = \lambda_{xy}^{-1} \lambda_x \lambda_y : L \to L$ is an automorphism of L.
Let G be the group generated by the left translations of L and let H be the stabilizer of $e \in L$ in the group G. The left translations of L form a subset of G acting on the cosets $\{xH; x \in G\}$ such that for any given cosets aH and bH there exists precisely one left translation λ_z with $\lambda_z aH = bH$.
Conversely let G be a group, H be a subgroup containing no normal non-trivial subgroup of G and $\sigma : G/H \to G$ be a section with $\sigma(H) = 1 \in G$ such that the set $\sigma(G/H)$ of representatives for the left cosets $\{xH, x \in G\}$ and acts sharply transitively on the space G/H of $\{xH, x \in G\}$ (cf. [16], p. 18). Such a section we call a sharply transitive section. Then the multiplication defined by $xH * yH = \sigma(xH)yH$ on the factor space G/H or by $x * y = \sigma(xyH)$ on $\sigma(G/H)$ yields a loop $L(\sigma)$. The group G is isomorphic to the group generated by the left translations of $L(\sigma)$.
If G is a Lie group and σ is a differentiable section satisfying the above conditions then the loop $L(\sigma)$ is almost differentiable. This loop is a left A-loop if and only if the subset $\sigma(G/H)$ is invariant under the conjugation with the elements of H.
Let L_1 be a loop defined on the factor space G_1/H_1 with respect to a section $\sigma_1 : G_1/H_1 \to G_1$ the image of which is the set $M_1 \subset G_1$. Let G_2 be a group let $\varphi : H_1 \to G_2$ be a homomorphism and $(H_1, \varphi(H_1)) = \{(x, \varphi(x)); x \in H_1\}$. A loop L is called a Scheerer extension of G_2 by L_1 if the loop L is defined on the factor space $(G_1 \times G_2)/(H_1, \varphi(H_1))$ with respect to the section $\sigma : (G_1 \times G_2)/(H_1, \varphi(H_1)) \to G_1 \times G_2$ the image of which is the set $M_1 \times G_2$. The loops L_1 and L_2 having the same group G of the group generated by

the left translations and the same stabilizer H of $e \in L_1, L_2$ are isomorphic if there is an automorphism of G leaving H invariant and mapping the section $\sigma_1(G/H)$ onto the section $\sigma_2(G/H)$. Moreover let L and L' be loops having the same group G generated by their left translations. Then L and L' are isotopic if and only if there is a loop L'' isomorphic to L' having G again as the group generated by its left translations such that there exists an inner automorphism τ of G mapping the section $\sigma''(G/H)$ belonging to L'' onto the section $\sigma(G/H)$ corresponding to L (cf. [16], Theorem 1.11. pp. 21-22). If L is a connected almost differentiable left A-loop, then the group G topologically generated by the left translations of L within the group of autohomeomorphisms is a connected Lie group (cf. [15]; [16], Proposition 5.20. p. 75), and we may describe L by a differentiable section.
Let L be a connected almost differentiable left A-loop. Let G be the Lie group topologically generated by the left translations of L, and let $(\mathbf{g}, [.,.])$ be the Lie algebra of G. Denote by $\mathbf{h}$ the Lie algebra of the stabilizer H of the identity $e \in L$ in G and by $\mathbf{m} = T_1\sigma(G/H)$ the tangent space at $1 \in G$ of the image of the section $\sigma : G/H \to G$ corresponding to L. Then $\mathbf{m}$ generates $\mathbf{g}$ and the homogeneous space G/H is reductive, i.e. we have $\mathbf{g} = \mathbf{m} \oplus \mathbf{h}$ and $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$. (cf. [13] Vol II, p. 190; [16], Proposition 5.20. p. 75) The subspace $\mathbf{m}$ with the operations $X \cdot Y := [X, Y]_{\mathbf{m}}$ and $[X, Y, Z] := [[X, Y]_{\mathbf{h}}, Z]$ yields a Lie triple algebra ([11], Definition 7.1, p. 173). If $X \cdot Y = 0$ for all $X, Y \in \mathbf{m}$ then $\mathbf{m}$ is a Lie triple system. In this case the factor space G/H is an affine symmetric space ([14]) and the corresponding loop L is called a Bruck loop. The Lie algebra $\mathbf{g}$ of G is isomorphic to the standard enveloping Lie algebra of the Lie triple algebra $\mathbf{m}$ generating $\mathbf{g}$. If the dimension of $\mathbf{m}$ is n then $\mathbf{g}$ has dimension at most $n(n+1)/2$.
In this paper we investigate strongly left alternative (cf. [16], Definition 5.3, p. 67) almost differentiable left A-loops L of dimension 3; these loops satisfy $\exp[T_1\sigma(L)] \subset \sigma(L)$ ([16], Proposition 5.5 p. 68). Hence every global left A-loop contains an exponential image of a complement $\mathbf{m}$ of the Lie algebra $\mathbf{h}$ of H in the Lie algebra $\mathbf{g}$ of G, such that $\mathbf{m}$ generates $\mathbf{g}$ and satisfies the relation $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$.
In this paper we often compute the images of subspaces $\mathbf{m}$ of the Lie algebras $sl_2(\mathbb{R})$, $sl_2(\mathbb{C})$, $su_2(\mathbb{C})$ under the exponential map.
1.2 The exponential function of the Lie algebras $sl_2(\mathbb{R})$, $sl_2(\mathbb{C})$, $su_2(\mathbb{C})$.
The exponential map $\exp : \mathbf{g} \to G$ is defined in the following way: For $X \in \mathbf{g}$ we have $\exp\ X = \gamma_X(1)$, where $\gamma_X(t)$ is the 1-parameter subgroup of G with the property $\frac{d}{dt}\big|_{t=0}\gamma_X(t) = X$. The matrices

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

form a real basis of $sl_2(\mathbb{R})$. The Lie algebra multiplication is given by the rules:

$$[K,T]=2U,\quad [K,U]=2T,\quad [U,T]=2K\ .$$

The normalized Cartan-Killing form $k: sl_2(\mathbb{R})\times sl_2(\mathbb{R})\to\mathbb{R}$ of $sl_2(\mathbb{R})$ is the bilinear form defined by $k(X,Y)=\dfrac{1}{8}\mathrm{trace}(\mathrm{ad}X\ \mathrm{ad}Y)$.

If $X\in sl_2(\mathbb{R})$ has the decomposition

$$X=\lambda_1\ K+\lambda_2\ T+\lambda_3\ U$$

then the Cartan-Killing form k satisfies

$$k(X)=\lambda_1^2+\lambda_2^2-\lambda_3^2.$$

According to [9] for the exponential function $\exp: sl_2(\mathbb{R})\to SL_2(\mathbb{R})$ we have

$$\exp\ X=C(k(X))\ I+S(k(X))\ X.$$

Here is

$$C(x)=\begin{cases}\cosh\sqrt{x} & \text{for}\quad 0\le x,\\ \cos\sqrt{-x} & \text{for}\quad 0>x,\end{cases}\qquad \sqrt{|x|}\ S(x)=\begin{cases}\sinh\sqrt{x} & \text{for}\quad 0\le x,\\ \sin\sqrt{-x} & \text{for}\quad 0>x.\end{cases}$$

As a natural generalization of this formula we obtain the explicite form for the exponential function of $sl_2(\mathbb{C})$. Representing the Lie algebra $\mathbf{g}=sl_2(\mathbb{C})$ as complex (2×2)-matrices we may choose as basis $\{K,T,U,\mathrm{i}K,\mathrm{i}T,\mathrm{i}U\}$, where K,T,U are the basis elements of $sl_2(\mathbb{R})$ (see in **1.2**).

The normalized complex Cartan-Killing form $k_{\mathbb{C}}: sl_2(\mathbb{C})\times sl_2(\mathbb{C})\to\mathbb{C}$ of $sl_2(\mathbb{C})$ is the bilinear form defined by: $k_{\mathbb{C}}(X,Y)=\dfrac{1}{8}\mathrm{trace}(\mathrm{ad}X\ \mathrm{ad}Y)$. If $X\in sl_2(\mathbb{C})$ has the decomposition

$$X=\lambda_1\ K+\lambda_2\ T+\lambda_3\ U+\lambda_4\ \mathrm{i}K+\lambda_5\ \mathrm{i}T+\lambda_6\ \mathrm{i}U$$

then the complex Cartan-Killing form $k_{\mathbb{C}}$ satisfies

$$k_{\mathbb{C}}(X)=\lambda_1^2+\lambda_2^2+\lambda_6^2-\lambda_3^2-\lambda_4^2-\lambda_5^2+i\ (2\ \lambda_1\lambda_4+2\ \lambda_2\lambda_5-2\ \lambda_3\lambda_6)$$

(cf. [6], Section 1, pp. 1-3). The normalized real Cartan-Killing form $k_{\mathbb{R}}: sl_2(\mathbb{C})\times sl_2(\mathbb{C})\to\mathbb{R}$ is the restriction of $k_{\mathbb{C}}$ to $\mathbb{R}$ such that

$$k_{\mathbb{R}}(X)=\lambda_1^2+\lambda_2^2+\lambda_6^2-\lambda_3^2-\lambda_4^2-\lambda_5^2.$$

For the exponential function $\exp: sl_2(\mathbb{C})\to SL_2(\mathbb{C})$ one has

$$\exp\ X=C(k_{\mathbb{C}}(X))\ I+S(k_{\mathbb{C}}(X))\ X,$$

where $C(z)=\cosh\sqrt{z}$ and $S(z)=\dfrac{\sinh\sqrt{z}}{\sqrt{z}}$, $z\in\mathbb{C}$.

The group $SU_2(\mathbb{C})$ is the 3-dimensional compact subgroup of $SL_2(\mathbb{C})$, which can be represented by (2×2)-complex matrices of the form:

$$\left\{\begin{pmatrix} a & b\\ -\bar{b} & \bar{a}\end{pmatrix}; a,b\in\mathbb{C}, a\bar{a}+b\bar{b}=1\right\}.$$

Therefore the Lie algebra $\mathbf{g}=su_2(\mathbb{C})$ is generated by the basis elements U, iK, iT. The restriction of the formula for the exponential function of $sl_2(\mathbb{C})$

to $su_2(\mathbb{C})$ gives the formula for $\exp : su_2(\mathbb{C}) \to SU_2(\mathbb{C})$.

1.3 Now we study which pairs (G,H) of Lie groups can admit differentiable sections $\sigma : G/H \to G$ corresponding to 3-dimensional almost differentiable left A-loops.
We start with a well known fact from linear algebra:

Lemma 1. *If* $\mathbf{g} = \mathbf{a} \oplus \mathbf{b}$ *with a* 3*-dimensional subspace* $\mathbf{a}$ *and the dimension of* $\mathbf{g}$ *is 4 or* 5 *then* $\mathbf{m} \cap \mathbf{a}$ *is at least* 2 *respectively* 1*-dimensional for any* 3*-dimensional subspace* $\mathbf{m}$*.*

The next fact is proved in [5], Lemma 3.

Lemma 2. *Let L be an almost differentiable global loop and denote by* $\mathbf{m}$ *the tangent space of* $T_1\sigma(G/H)$*, where* $\sigma : G/H \to G$ *is the section corresponding to L. Then* $\mathbf{m}$ *does not contain any element of* $Ad_{g^{-1}}\mathbf{h} = g\mathbf{h}g^{-1}$ *for some* $g \in G$*. Moreover every element of G can be uniquely written as a product of an element of* $\sigma(G/H)$ *with an element of H.*

Since a 1-dimensional almost differentiable left A-loop is a group an analogue of Proposition 1 in [5] is the following

Proposition 3. *Let L be a loop and let G be the group generated by the left translations of L, and denote by H the stabilizer of* $e \in L$ *in G. If G and H are direct products* $G = G_1 \times G_2$ *and* $H = H_1 \times H_2$ *with* $H_i \subset G_i$ $(i = 1, 2)$ *then L is the product of two loops* L_1 *and* L_2*, and* L_i *is isomorphic to a loop* L_i^* *having* G_i *as the group generated by the left translations of* L_i^* *and* H_i *as the corresponding stabilizer subgroup* $(i = 1, 2)$*.*
In particular there exists no 3*-dimensional left A-loop L such that L is the product of a* 1*-dimensional and a* 2*-dimensional left A-loop and L has a* 5*- or* 6*-dimensional Lie group as the group topologically generated by its left translations.*

Lemma 4. *Let* $\mathbf{g} = \mathbf{g}_1 \oplus \mathbf{g}_2$ *be the Lie algebra of the Lie group* $G = G_1 \times G_2$*, such that G is the group topologically generated by the left translations of a* 3*-dimensional almost differentiable left A-loop L. Let* $\mathbf{m}$ *be the tangent space of the manifold* Λ *of the left translations of L at* $1 \in G$*. Denote by* $\mathbf{h}$ *the Lie algebra of the stabilizer H of* $e \in L$ *in G and let* $\pi_i : \mathbf{g} \to \mathbf{g}_i$*,* $i = 1, 2$ *be the natural projection of* $\mathbf{g}$ *onto* $\mathbf{g}_i$*. We assume that* $\mathbf{g}_1$ *is isomorphic to* $sl_2(\mathbb{R})$ *and dim* $\pi_1(\mathbf{h}) = 2$*. Then:*
(i) dim $\pi_1(\mathbf{m}) = 3$*.*
(ii) If dim $\pi_2(\mathbf{m}) \geq 2$ *then the Lie algebra* $\mathbf{h}$ *has the form*
$$\mathbf{h} = \{(x, \varphi(x)) | x \in \pi_1(\mathbf{h})\},$$

with an isomorphism $\varphi : \pi_1(\mathbf{h}) \to \pi_2(\mathbf{h})$. *Moreover, one has dim* $\pi_2(\mathbf{m}) = 2$ *and dim* $\mathbf{g} = 5$.

Proof. (i) We have dim $\pi_1(\mathbf{m}) \geq 2$ since otherwise the set Λ would not generate G. If dim $\pi_1(\mathbf{h}) = 2$ then we may assume that $\pi_1(\mathbf{h})$ is generated by the elements $K, U+T$ of the Lie algebra $sl_2(\mathbb{R})$ (see **1.2** in section 1). If $\pi_1(\mathbf{m})$ were 2-dimensional then it has one of the following forms (up to conjugation)
a) $\pi_1(\mathbf{m}) = \langle U + a_1K + a_2(U+T), K + b_1K + b_2(U+T)\rangle$,
b) $\pi_1(\mathbf{m}) = \langle U + a_1K + a_2(U+T), T + b_1K + b_2(U+T)\rangle$,
c) $\pi_1(\mathbf{m}) = \langle T + a_1K + a_2(U+T), K + b_1K + b_2(U+T)\rangle$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$.
One has

$$(*) \qquad \pi_1([\mathbf{h}, \mathbf{m}]) = [\pi_1(\mathbf{h}), \pi_1(\mathbf{m})] \subseteq \pi_1(\mathbf{m}).$$

In the case a) the element

$$[K, U + a_1K + a_2(U+T)] = 2T + 2a_2(U+T)$$

is contained in $\pi_1(\mathbf{m})$ if and only if $a_1 = 0, a_2 = -\frac{1}{2}$. Moreover the element $[U+T, U-T] = -4K$ is contained in $\pi_1(\mathbf{m})$ precisely if $b_2 = 0$ and $b_1 \neq -1$. But then $\pi_1(\mathbf{m}) = \langle (1+b_1)K, U-T\rangle$ is a subalgebra of $\mathbf{g}_1$.
In the case b) the element

$$[U+T, U + a_1K + a_2(U+T)] = 2K + 2a_1(U+T)$$

is not contained in $\pi_1(\mathbf{m})$, this is a contradiction to $(*)$.
In the case c) we obtain the same contradiction in the same way as in the case a). Therefore is dim $\pi_1(\mathbf{m}) = 3$.
(ii) If dim $\pi_2(\mathbf{h}) = 3$ then one has $\pi_2(\mathbf{h}) = \mathbf{g}_2$ and $\mathbf{h} \cap (0, \mathbf{g}_2) \neq (0,0)$. Then there exists a homomorphism $\beta : \pi_2(\mathbf{h}) \to \pi_1(\mathbf{h})$ such that $\mathbf{h} \cap (0, \mathbf{g}_2) = \beta^{-1}(0)$. This is a contradiction since $\mathbf{h}$ does not contain non-trivial ideal of $\mathbf{g}$.
Since $\mathbf{m} \subseteq \pi_1(\mathbf{m}) \times \pi_2(\mathbf{m})$ and according (i) dim $\pi_1(\mathbf{m}) = 3$ there is a linear mapping $\alpha : \pi_1(\mathbf{m}) \to \pi_2(\mathbf{m})$ such that α is a linear isomorphism if dim $\pi_2(\mathbf{m}) = 3$ and dim $\alpha^{-1}(0) = 1$ for dim $\pi_2(\mathbf{m}) = 2$.
If dim $\pi_2(\mathbf{h}) \leq 2$ and (ii) does not hold then there is a homomorphism $\gamma : \pi_1(\mathbf{h}) \to \pi_2(\mathbf{h})$ with $0 \neq S = Ker\ \gamma$. If dim $S = 2$ then we have $\pi_1(\mathbf{h}) = S$ and the Proposition 3 gives a contradiction. Hence dim $S = 1$. Then $(S, 0) = \mathbf{h} \cap (\pi_1(\mathbf{m}), 0) = \mathbf{h} \cap (\mathbf{g}_1, 0)$ and $(S, 0) = \langle (U+T, 0)\rangle$ is a 1-dimensional subalgebra of $\mathbf{h}$.
First we treat the case that dim $\pi_2(\mathbf{m}) = 3$. Then one has $\mathbf{m} \cap (\mathbf{g}_1, 0) = (0,0)$ and $\pi_2(\mathbf{m}_2) = \mathbf{g}_2$. Then there is an element $m_2 \in \pi_2(\mathbf{m})$ such that

$$[(r\ (U+T), 0), (\alpha(m_2), m_2)] = ([r\ (U+T), \alpha(m_2)], 0) \neq (0,0),$$

where $r \in \mathbb{R}$. This is a contradiction.
Now we assume dim $\pi_2(\mathbf{m}) = 2$. Then we have $\mathbf{m} \cap (\mathbf{g}_1, 0) = (S', 0)$, where $S' = \alpha^{-1}(0)$. Since

$$[\mathbf{h}, \mathbf{m} \cap (\mathbf{g}_1, 0)] = [(h_1, h_2), (m_1, 0)] = ([h_1, m_1], 0) \subset \mathbf{m} \cap (\mathbf{g}_1, 0)$$

with $(h_1, h_2) \in \mathbf{h}$ and $(m_1, 0) \in \mathbf{m}$ it follows that $\pi_1(\mathbf{h})$ normalizes S' and therefore $(S', 0) = \langle (U + T, 0) \rangle = (S, 0)$, which is a contradiction. $\square$

Proposition 5. *Let $G = G_1 \times G_2$ be the group topologically generated by the left translations of a 3-dimensional connected almost differentiable proper left A-loop L. Let the group G_1 be locally isomorphic either to $SO_3(\mathbb{R})$ or to $PSL_2(\mathbb{R})$. Then for the pair (G, H), where H is the stabilizer of $e \in L$ in G, one of the following cases occurs:*
1) L is the product of the hyperbolic plane loop with a 1-dimensional Lie group and $H \cong SO_2(\mathbb{R}) \times \{1\}$.
2) G is isomorphic to $PSL_2(\mathbb{R}) \times \mathbb{R}$ and $H = \{(x, \varphi(x))\}$, where φ is a monomorphism from the 1-dimensional subgroup $\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbb{R} \right\}$ or from $\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a > 0 \right\}$ of $PSL_2(\mathbb{R})$ onto $\mathbb{R}$.
3) $G = PSL_2(\mathbb{R}) \times SO_2(\mathbb{R})$ such that $H = \{(x, x^n) | x \in SO_2(\mathbb{R}), n \in \mathbb{N}\}$.
4) $G \cong PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ and H has the form

$$H = \{(x, x) \mid x \in PSL_2(\mathbb{R})\}.$$

Proof. First we assume that dim $G = 6$. If G_1 is locally isomorphic to $PSL_2(\mathbb{R})$ then it follows from Lemma 4 and from the proof of Proposition 4 in [5] that we are in the case 4). If G_1 is locally isomorphic to $SO_3(\mathbb{R})$ then it is easy to see that G_2 is also locally isomorphic to $SO_3(\mathbb{R})$ and we may assume that $H = \{(x, x) \mid x \in G_1\}$. This case is excluded by Proposition 16.11 in [16] (p. 205).
Now we suppose that dim $G = 5$. We may assume that dim $\pi_1(\mathbf{h}) = 2$ since otherwise H would be a direct product $H = H_1 \times H_2$ which contradicts Proposition 3. Now it follows from Lemma 4 that

$$H = \{(x, \varphi(x)) | x \in \pi_1(H)\},$$

where $\pi_1(H)$ is isomorphic to the group $\mathcal{L}_2 = \{x \mapsto ax + b; a > 0, b \in \mathbb{R}\}$ and $\varphi : \pi_1(H) \to \pi_2(H)$ is an isomorphism. A real basis of the Lie algebra $\mathbf{g} = \mathbf{sl_2}(\mathbb{R}) \oplus \boldsymbol{\mathcal{L}_2}$ is

$$\mathbf{g} = \langle (K, 0),\ (T, 0),\ (U, 0),\ (0, e_1),\ (0, e_2) \rangle,$$

where K, T and U are the basis elements of $sl_2(\mathbb{R})$ (see **1.2**) and e_1, e_2 are the basis elements of $\mathcal{L}_2$ with the rule $[(0, e_1), (0, e_2)] = -(0, e_2)$. The Lie algebra $\mathbf{h}$ of H is given by

$$\mathbf{h} = \langle (K, e_1), (U + T, e_2) \rangle.$$

An arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ has as generators

$$l_1 = (U + a_1 K + a_2 (U + T), a_1 e_1 + a_2 e_2),$$
$$l_2 = (b_1 K + b_2 (U + T), e_1 + b_1 e_1 + b_2 e_2),$$
$$l_3 = (c_1 K + c_2 (U + T), e_2 + c_1 e_1 + c_2 e_2),$$

where $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$. Since one has dim $\mathbf{m} \cap (sl_2(\mathbb{R}) \oplus \{0\}) \geq 1$ and the relation $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$ holds we obtain that G cannot have dimension 5.
Finally let dim $G = 4$. Then dim $H = 1$. First we assume that G_1 is locally isomorphic to $PSL_2(\mathbb{R})$. If $\pi_2(H) = 1$ according to Proposition 3 and Theorem 27.1, Theorem 18.14 in [16] one has $H = SO_2(\mathbb{R}) \times \{1\}$ and L is a product of the hyperbolic plane loop with a 1-dimensional Lie group. This is the case 1).
Let now $\pi_2(H) \neq 1$. If G_2 is isomorphic to $\mathbb{R}$ then $H = \{(\varphi(x), x) \mid x \in \mathbb{R}\}$, where φ is a monomorphism onto G_1. The inverse of φ is again a monomorphism. Since the group $PSL_2(\mathbb{R})$ has precisely 2 conjugacy classes of 1-dimensional subgroups isomorphic to $\mathbb{R}$ we obtain the cases 2). If G_2 is isomorphic to $SO_2(\mathbb{R})$ then we may assume that

$$H = \{(x, x^n) | x \in SO_2(\mathbb{R}), n \in \mathbb{N}\}.$$

It remains to consider a group G locally isomorphic to $SO_3(\mathbb{R}) \times SO_2(\mathbb{R})$. Since H does not contain any non-trivial normal subgroup the group G is isomorphic to $SO_3(\mathbb{R}) \times SO_2(\mathbb{R})$ and H has one of the following forms:

$$H' = \{K \times \{0\}\}, \quad \text{or} \quad H = \{(k, \varphi(k)) | k \in K\},$$

where K is isomorphic to $SO_2(\mathbb{R})$ and φ is a non-trivial homomorphism.
Since in the first case the factor space G/H' is a topological product of spaces having as a factor the 2-sphere or the projective plane we have to consider only the second case ([16], Theorem 19.1, p. 249).
The Lie algebra $\mathbf{g}$ of G can be represented as $su_2(\mathbb{C}) \oplus \mathbb{R}$. Then as a basis of $\mathbf{g}$ may be chosen the following elements $i(K, 0), (U, 0), i(T, 0)$, $(0, e_1)$, where iK, U, iT is the real basis of $su_2(\mathbb{C})$ which is introduced in **1.2** and e_1 is the basis element of $\mathbb{R}$. Moreover the Lie group H has one of the following shapes $H_n = \{(x, x^n) | x \in SO_2(\mathbb{R}), n \in \mathbb{N}\}$ and for the Lie algebra $\mathbf{h}$ of H_n has the form $\mathbf{h} = \langle (U, e_1) \rangle$. An arbitrary complement $\mathbf{m}$ to the Lie algebra $\mathbf{h}$ of H_n in $\mathbf{g}$ has the shape:

$$\mathbf{m} = \langle (iK + a_1\, U, a_1\, e_1), (iT + a_2\, U, a_2\, e_1), (a_3\, U, e_1 + a_3\, e_1) \rangle,$$

where $a_1, a_2, a_3 \in \mathbb{R}$. From Lemma 1 and from the property $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$ we obtain that the unique reductive complement $\mathbf{m}$ generating $\mathbf{g}$ has the form

$$\mathbf{m} = \langle i(K, 0), i(T, 0), (a\, U, e_1 + a\, e_1) \rangle,$$

where $a \in \mathbb{R} \backslash \{-1\}$.
For $a > -\dfrac{1}{2}$ and the basis element $(U, e_1) \in \mathbf{h}$ one has

$$Ad_g(U, e_1) = -2kl(iT, 0) + (\tfrac{a}{1+a} U, e_1) \in \mathbf{m}_a$$

with $g = \left(\pm \begin{pmatrix} k - li & 0 \\ 0 & k + li \end{pmatrix}, 0 \right) \in G$, such that $k^2 - l^2 = \frac{a}{1+a}$ and $k^2 + l^2 = 1$. This contradicts Lemma 2.
For $a < -\frac{1}{2}$ the vectors

$$X_1 = \left(\tfrac{\pi}{6}U + \tfrac{\sqrt{143}}{6}\pi i K, \tfrac{\pi(1+a)}{6a}e_1\right) \quad \text{and}$$
$$X_2 = \left(\tfrac{\pi(1+a)}{6(1+a-na)}U, \tfrac{\pi(1+a)^2}{6a(1+a-na)}e_1\right)$$
are contained in $\mathbf{m}_a$. According to **1.2** we get

$$\exp X_1 = \left(\pm I, \begin{pmatrix} \cos\frac{\pi(1+a)}{6a} & \sin\frac{\pi(1+a)}{6a} \\ -\sin\frac{\pi(1+a)}{6a} & \cos\frac{\pi(1+a)}{6a} \end{pmatrix}\right),$$

$$\exp X_2 = \left(\pm \begin{pmatrix} \cos l & \sin l \\ -\sin l & \cos l \end{pmatrix}, \begin{pmatrix} \cos\frac{l(1+a)}{a} & \sin\frac{l(1+a)}{a} \\ -\sin\frac{l(1+a)}{a} & \cos\frac{l(1+a)}{a} \end{pmatrix}\right),$$

where $l = \frac{\pi(1+a)}{6(1+a-na)}$, $\pm I$ is the identity of $SO_3(\mathbb{R})$. For the element

$$g = \left(\pm I, \begin{pmatrix} \cos\frac{\pi(1+a)}{6a} & \sin\frac{\pi(1+a)}{6a} \\ -\sin\frac{\pi(1+a)}{6a} & \cos\frac{\pi(1+a)}{6a} \end{pmatrix}\right) \in G$$

one has

$$g = \exp X_1 = \exp X_2 \cdot h$$

with

$$h = \left(\pm \begin{pmatrix} \cos l & -\sin l \\ \sin l & \cos l \end{pmatrix}, \begin{pmatrix} \cos nl & -\sin nl \\ \sin nl & \cos nl \end{pmatrix}\right).$$

This is again a contradiction to Lemma 2. Therefore there is no global section $\sigma : G/H_n \to G$ satisfying $\exp \mathbf{m}_a \subseteq \sigma(G/H_n)$. □

Corollary 6. *There is no global left A-loop L homeomorphic to the compact space S^3 or P^3.*

Proof. The group G topologically generated by the left translations of an almost differentiable proper left A-loop L homeomorphic to S^3 acts transitively on L. According to 96.16 in [17] any maximal compact subgroup of G acts also transitively on S^3. Since a transitive compact subgroup of G is a non-solvable subgroup of $SO_4(\mathbb{R})$ (96.20 in [17]) the group G is non-solvable. According to Proposition 16.11 in [16] and Proposition 5 there is no almost differentiable left A-loop homeomorphic to S^3 or P^3 having a non-solvable Lie group as the group topologically generated by its left translations. □

2. Left A-loops as sections in semisimple Lie groups

In this section we classify all 3-dimensional connected strongly left alternative almost differentiable left A-loops L having a semisimple Lie group G as the group topologically generated by its left translations and describe the reductive spaces and natural geometries associated with them.
It follows from Lemma 4 and Proposition 5 that the group G must be locally isomorphic either to $PSL_2(\mathbb{C})$ or to $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$. In the second case

we may assume that the stabilizer H of $e \in L$ in G is locally isomorphic to $\{(x,x); x \in PSL_2(\mathbb{R})\}$.

Lemma 7. *For all $\lambda \in \mathbb{R}\backslash\{0,1\}$ there is a reductive complement*

$$\mathbf{m}_\lambda = \{(X, \lambda X) | X \in sl_2(\mathbb{R})\}$$

to the Lie algebra $\mathbf{h} = \{(X,X);\ X \in sl_2(\mathbb{R})\}$ of H in $\mathbf{g} = sl_2(\mathbb{R}) \oplus sl_2(\mathbb{R})$.

Proof. Let K, U and T be the real basis of $sl_2(\mathbb{R})$ induced in **1.2**. In this case a 3-dimensional complement $\mathbf{m} \subset \mathbf{g}$ has the shape

$$\{(X, \varphi(X)) | X \in sl_2(\mathbb{R})\},$$

where $\varphi : sl_2(\mathbb{R}) \to sl_2(\mathbb{R})$ is a linear map. From the relation $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$ we obtain the assertion. □

Proposition 8. *There is no global sharply transitive section $\sigma : G/H \to G$ satisfying the relation $\sigma(G/H) = \exp \mathbf{m} = \{(\exp X, (\exp X)^\lambda); X \in sl_2(\mathbb{R})\}$, where $\exp X \mapsto (\exp X)^\lambda : PSL_2(\mathbb{R}) \to PSL_2(\mathbb{R})$ is a mapping.*

Proof. Let $S_X = \{\exp tX; t \in \mathbb{R}\}$ be a 1-parameter subgroup of $PSL_2(\mathbb{R})$ isomorphic to $SO_2(\mathbb{R})$. For all $x, y \in S_X$ is satisfied $(xy)^\lambda = x^\lambda y^\lambda$ and $(S_X, S_X^\lambda) \cap H = \{(1,1)\}$. Hence the mapping $x \to x^{\lambda-1}$ is an automorphism of S_X. The only non-trivial automorphism of S_X is the mapping $x \mapsto x^{-1}$. Therefore the automorphism $x \mapsto x^{\lambda-1}$ must be the identity map and we have $\lambda = 2$. For $x_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$ and $x_2 = \begin{pmatrix} \frac{1}{2} & -9 \\ 0 & 2 \end{pmatrix}$ we have

$$(x_i, x_i^2) = (R, 1)(U_i, D^{-1} U_i D),$$

where $R = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, $D = \begin{pmatrix} \frac{\sqrt{5}}{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix}$, and $U_i = D x_i^2 D^{-1}$. This means that the coset $(R,1)H^D$ of the conjugate subgroup H^D of H contains two different elements (x_i, x_i^2) of $\sigma(G/H)$ $(i = 1, 2)$. Hence we have a contradiction to Proposition 1.6. in [16] (p. 19). □

Lemma 9. *If the group G locally isomorphic to $PSL_2(\mathbb{C})$ is the group topologically generated by the left translations of a 3-dimensional almost differentiable left A-loop, then G is isomorphic to $PSL_2(\mathbb{C})$ and H is isomorphic to $SO_3(\mathbb{R})$.*

Proof. According to [1] (pp. 273-278) there are 4 conjugacy classes of the 3-dimensional subgroups of $G = SL_2(\mathbb{C})$, which are denoted in [1] by W_r, U_0, U_1 and $SU_2(\mathbb{C})$. Since the factor spaces $SL_2(\mathbb{C})/U_i$ and $PSL_2(\mathbb{C})/(U_i/\mathbb{Z}_2)$ for $i = 0, 1$ are homeomorphic to the topological direct product having as a factor the 2-sphere or the projective plane respectively there is no differentiable loop realized on these factor spaces (cf. [5], Proposition 2).

Let now H be locally isomorphic one of the subgroups W_r or $W_r\mathbb{Z}_2/\mathbb{Z}_2$, where

$$W_r = \left\{ \begin{pmatrix} \exp((ri-1)x) & 0 \\ z & \exp(-(ri-1)x) \end{pmatrix}; x \in \mathbb{R}, z \in \mathbb{C} \right\} \quad \text{for } r \in \mathbb{R}.$$

The Lie algebra $\mathbf{h} = w_r$ of the stabilizer W_r has following basis elements:

$$\{riK - K, iT - iU, U - T\} \quad r \in \mathbb{R}.$$

A complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ contains a basis element $K + f(K)$ or $iK + f(iK)$, where $f : \mathbf{m} \to \mathbf{h}$ is a linear map. Since the element

$$[U - T, K + f(K)] = [U - T, K] + [U - T, f(K)] = 2U - 2T + [U - T, f(K)]$$

is an element of the intersection $\mathbf{h} \cap \mathbf{m} = \{0\}$, we have $[U - T, f(K)] = 2T - 2U$. This is the case precisely if $f(K) = -K$ but then $f(K)$ is not an element of $\mathbf{h}$. This is a contradiction. We obtain the same contradiction if $iK + f(iK) \in \mathbf{m}$.

Since $SU_2(\mathbb{C})$ contains central elements $\neq 1$ of $SL_2(\mathbb{C})$ the assertion follows. □

Lemma 10. *For all $a \in \mathbb{R}$ there is a reductive complement*

$$\mathbf{m} = \langle T + aiT, iU - aU, K + aiK \rangle$$

to $\mathbf{h} = so_3(\mathbb{R})$ generating $\mathbf{g} = sl_2(\mathbb{C})$.

Proof. According to **1.2** let $\{K, T, U, \mathrm{i}K, \mathrm{i}T, \mathrm{i}U\}$ be a real basis of $\mathbf{g} = sl_2(\mathbb{C})$. The Lie algebra $\mathbf{h}$ of the stabilizer $H = SO_3(\mathbb{R})$ has the form $\mathbf{h} = \langle U, \mathrm{i}T, \mathrm{i}K \rangle$. An arbitrary component $\mathbf{m}$ to $\mathbf{h}$ has the shape

$\mathbf{m} = \langle T + \mathrm{a}U + \mathrm{b}\ \mathrm{i}T + \mathrm{c}\ \mathrm{i}K, \mathrm{i}U + \mathrm{d}U + \mathrm{e}\ \mathrm{i}T + \mathrm{f}\ \mathrm{i}K, K + \mathrm{g}U + \mathrm{h}\ \mathrm{i}T + \mathrm{k}\ \mathrm{i}K \rangle$, where $a, b, c, d, e, f, g, h, k \in \mathbb{R}$. The property $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$ gives the assertion. □

Now we determine the isomorphism classes and the isotopism classes of the loops L_a, $a \in \mathbb{R}$ belonging to the complements $\mathbf{m}_a$. Two loops corresponding to $(G, H, \exp \mathbf{m}_a)$ and $(G, H, \exp \mathbf{m}_b)$ are isomorphic if and only if there exists an automorphism α of $\mathbf{g}$ such that $\alpha(\mathbf{m}_a) = \mathbf{m}_b$ and $\alpha(\mathbf{h}) = \mathbf{h}$. The automorphism group of $\mathbf{g}$ leaving $\mathbf{m}_0$ and $\mathbf{h}$ invariant is the semidirect product Θ of Ad_H and the group generated by the involutory map $\varphi : z \mapsto \bar{z}$. Since $\mathbf{m}$ is a reductive subspace the condition $\alpha(\mathbf{m}_a) = \mathbf{m}_b$, $\alpha \in \Theta$, is equivalent to $\varphi(\mathbf{m}_a) = \mathbf{m}_b$. This identity is satisfied if and only if $b = -a$. Therefore a full isomorphism class consists of the loops L_a and L_{-a} ($a \in \mathbb{R}$) and we may choose as representatives of these isomorphism classes the left A-loops $L_a,\ a \geq 0$. Since there is no $g \in G$ such that $g^{-1}\mathbf{m}_a g = \mathbf{m}_b$ for two different real numbers a, b the isotopism classes and the isomorphism classes of the left A-loops $L_a, a \in \mathbb{R}$ are the same.

The complement $\mathbf{m}_0 = \langle T, \mathrm{i}U, K \rangle$ satisfies $[\mathbf{m}_0, \mathbf{m}_0] = \mathbf{h}$, and $\mathbf{g} = \mathbf{m_0} \oplus [\mathbf{m_0}, \mathbf{m_0}]$. Hence it determines a 3-dimensional connected Riemannian symmetric space (cf. [13], Chapter VI, Theorem 2.2 (iii)) and the loop $\hat{L}_0$ corresponding to the complement $\mathbf{m}_0$ is a Bruck loop. According to [5] this is the

hyperbolic space loop.
The loops L_a for $a \in \mathbb{R}$ have elementary models in the upper half space $\mathbb{R}^{3+} = \{(x,y,z) \in \mathbb{R}^3; z > 0\}$, which may be identified with the **J**-quaternion space ([4], p. 4) such that the point j is the identity e of L_a. The elements of L_a are the points of the **J**-quaternion space. The 1-parameter subgroups through $e \in L_a$ have the same form for all loop L_a, but the sets $\exp \mathbf{m}_a$ differ.
The multiplication in the loop L_a, $a \in \mathbb{R}$ is given by

$$x * y = (\exp X)y, \text{ for all } x, y \in L_a$$

where X is the unique element of $\mathbf{m}_a$ such that $x = \exp X$.
Summarizing our discussion we obtain

Theorem 11. *If L is a connected almost differentiable strongly left alternative left A-loop with dimension 3 having a semisimple Lie group G as the group topologically generated by its left translations then G is isomorphic to $PSL_2(\mathbb{C})$ and the stabilizer H of $e \in L$ in G is isomorphic to $SO_3(\mathbb{R})$ and $L = L_a$ is characterized by a real parameter a.*
The loops L_a and L_{-a} form a full isomorphism class, which is even a full isotopism class too. Among the loops L_a only the hyperbolic space loop L_0 is a Bruck loop. This loop is realized on the hyperbolic symmetric space by the multiplication $x \cdot y = \tau_{e,x}(y)$, where $\tau_{e,x}$ is the hyperbolic translation moving e onto x. The tangent space $\mathbf{m}_0$ for the manifold of the left translations of L_0 is within the Lie algebra $\mathbf{g}$ of G orthogonal to the Lie algebra $\mathbf{h}$ of H with respect to the Cartan-Killing form of $\mathbf{g}$.

3-dimensional left A-loops corresponding to 4-dimensional non-solvable Lie groups

In this section we determine all 3-dimensional connected almost differentiable global left A-loops L having a 4-dimensional non-solvable Lie group G as the group topologically generated by their left translations. Then the stabilizer H of $e \in L$ in G has dimension 1.
In this case we have $G = PSL_2(\mathbb{R}) \times G_2$, where G_2 is one of the 1-dimensional Lie groups, and H is one of the cases 2 and 3 in the Proposition 5.
The Lie algebra $\mathbf{g}$ of G can be represented as $\mathbf{g}= sl_2(\mathbb{R}) \oplus \mathbb{R}$. Let $(K,0)$, $(T,0)$, $(U,0)$ with K,T,U defined in **1.2** be a real basis of $sl_2(\mathbb{R}) \oplus \{0\}$ and let $(0,e_1)$ be the generator of $\{0\} \oplus \mathbb{R}$.

Lemma 12. *The Lie algebra $\mathbf{g}$ is reductive with a 1-dimensional subalgebra $\mathbf{h}$ not contained in $sl_2(\mathbb{R}) \oplus \{0\}$ and a 3-dimensional complementary subspace $\mathbf{m}$ generating $\mathbf{g}$ in one of the following cases:*
1) $\mathbf{h} = \langle (K, e_1) \rangle$, $\mathbf{m}_a = \langle (U,0), (T,0), (aK, (1+a)e_1) \rangle$, *where* $a \in \mathbb{R} \backslash \{-1\}$

2) $\mathbf{h} = \langle (U+T, 2e_1) \rangle$, $\mathbf{m}_b = \langle (U+T, 0), (K, 0), (U, 2be_1) \rangle$, *where* $b \in \mathbb{R}\backslash\{0\}$
3) $\mathbf{h} = \langle (U, e_1) \rangle$, $\mathbf{m}_c = \langle (K, 0), (T, 0), (cU, (1+c)e_1) \rangle$, *where* $c \in \mathbb{R}\backslash\{-1\}$.

Proof. According to Proposition 5 we may assume that $\mathbf{h}$ has one of the shapes given in 1 till 3. An arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ has the shape in the case 1)

$$\mathbf{m} = \langle (U + a_1 K, a_1 e_1), (T + a_2 K, a_2 e_1), (a_3 K, (1+a_3) e_1) \rangle$$

in the case 2)

$$\mathbf{m} = \langle (K + a_1(U+T), 2a_1 e_1), (U + a_2(U+T), 2a_2 e_1), (a_3(U+T), e_1 + 2a_3 e_1) \rangle$$

in the case 3)

$$\mathbf{m} = \langle (K + a_1\, U, a_1\, e_1), (T + a_2\, U, a_2\, e_1), (a_3\, U, e_1 + a_3\, e_1) \rangle,$$

where $a_1, a_2, a_3 \in \mathbb{R}$. From the fact that $\dim \mathbf{m} \cap (sl_2(\mathbb{R}) \oplus \{0\}) = 2$ (Lemma 1) and from the property $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$ follows the assertion. $\square$

Proposition 13. *The complement* $\mathbf{m}_0$ *in the case 3) is the unique reductive subspace such that there are global left A-loops* L_n*,* $n \in \mathbb{N}$ *with* $\mathbf{m}_0 = T_1 L_n$*. These loops* L_n *are Scheerer extensions of the Lie group* $SO_2(\mathbb{R})$ *by the hyperbolic plane loop.*

Proof. For all $a \in \mathbb{R}\backslash\{-1\}$ the complement $\mathbf{m}_a$ contains the elements
$k_a = -(1+d+d^2)(U,0) + (-1+d+d^2)(T,0) + \left(\frac{a}{1+a}K, e_1\right)$, for which $Ad_g(k_a) = (K, e_1)$ holds with $(K, e_1) \in \mathbf{h}$ and $g = \left(\pm \begin{pmatrix} 1+d & -1 \\ -d & 1 \end{pmatrix}, 0\right)$ such that $d = -\frac{1}{2(1+a)}$. This is a contradiction to Lemma 2.
Now we deal with the complement $\mathbf{m}_b$. If $b > 0$ then for the elements $k_b = (K+U, 2be_1) \in \mathbf{m}_b$ are satisfied $Ad_g(k_b) = b(U+T, 2e_1)$, where $b(U+T, 2e_1) \in \mathbf{h}$ and $g = \left(\pm \begin{pmatrix} 1 & -\frac{2b}{\sqrt{2b}} \\ \frac{1}{\sqrt{2b}} & 0 \end{pmatrix}, 0\right) \in G$. This contradicts Lemma 2.
For $b < 0$ the subspace $\mathbf{m}_b$ contains the vectors

$$v_1 = (-3\pi b(U+T), 0), \quad v_2 = (\sqrt{5\pi^2}K + 3\pi U, 6\pi b e_1).$$

According to **1.2** the exponential images of the vectors v_1 and v_2 are

$$m_1 = \exp v_1 = \left(\begin{pmatrix} 1 & -6\pi b \\ 0 & 1 \end{pmatrix}, 0\right)$$

and

$$m_2 = \exp v_2 = (\pm I, 6\pi b),$$

where $\pm I$ is the identity of $PSL_2(\mathbb{R})$. One has $(\pm I, 6\pi b) = m_1 \cdot h_1 = m_2$, where $h_1 = \left(\begin{pmatrix} 1 & 6\pi b \\ 0 & 1 \end{pmatrix}, 6\pi b\right)$. This is a contradiction to Lemma 2.
Finally we consider the reductive complements $\mathbf{m}_c$. For $c < -1$ and for the

elements

$$k_c = \left(\frac{1-2e^4}{2e^2}\right)(T,0) + \left(\frac{c}{1+c}U, e_1\right) \in \mathbf{m}_c$$

we obtain $Ad_g(k_c) = (U, e_1)$, where $(U, e_1) \in \mathbf{h}$ and $g = \left(\pm\begin{pmatrix} \frac{1}{e} & 0 \\ 0 & e \end{pmatrix}, 0\right)$ $\in G$, choosing e such that $\frac{1+2e^4}{2e^2} = \frac{c}{1+c}$. This is a contradiction to Lemma 2.
If $c > -1$ but $c \neq 0$ the subspace $\mathbf{m}_c$ contains the vectors

$$v_1 = \left(kU, \frac{k(1+c)}{c} e_1\right)$$

and

$$v_2 = \left(\sqrt{\left(\frac{k^2(1+c-nc)^2}{(1+c)^2} - 4\pi^2\right)} T + \frac{k(1+c-nc)}{1+c} U, \frac{k(1+c-nc)}{c} e_1\right).$$

According to **1.2** the images of v_1, v_2 under the exponential map have the forms:

$$m_1 = \exp v_1 = \left(\pm\begin{pmatrix} \cos k & \sin k \\ -\sin k & \cos k \end{pmatrix}, \begin{pmatrix} \cos\frac{k(1+c)}{c} & \sin\frac{k(1+c)}{c} \\ -\sin\frac{k(1+c)}{c} & \cos\frac{k(1+c)}{c} \end{pmatrix}\right)$$

and

$$m_2 = \exp v_2 = \left(\pm I, \begin{pmatrix} \cos\frac{k(1+c-nc)}{c} & \sin\frac{k(1+c-nc)}{c} \\ -\sin\frac{k(1+c-nc)}{c} & \cos\frac{k(1+c-nc)}{c} \end{pmatrix}\right).$$

For

$$g = \left(\pm I, \begin{pmatrix} \cos\frac{k(1+c-nc)}{c} & \sin\frac{k(1+c-nc)}{c} \\ -\sin\frac{k(1+c-nc)}{c} & \cos\frac{k(1+c-nc)}{c} \end{pmatrix}\right) \in G,$$

where $k \in \mathbb{Z}$ such that $k > \sqrt{\frac{4\pi^2(1+c)^2}{(1+c-nc)^2}}$ and $\pm I$ is the identity of $PSL_2(\mathbb{R})$, one has $g = m_1 \cdot h_1 = m_2$ such that

$$h_1 = \left(\pm\begin{pmatrix} \cos k & -\sin k \\ \sin k & \cos k \end{pmatrix}, \begin{pmatrix} \cos nk & -\sin nk \\ \sin nk & \cos nk \end{pmatrix}\right).$$

This again contradicts Lemma 2.
For $c = 0$ the complement $\mathbf{m}_c$ has the shape: $\mathbf{m}_0 = \langle (K,0), (T,0), (0,e_1)\rangle$. Since $\big[[\mathbf{m}_0, \mathbf{m}_0], \mathbf{m}_0\big] \subseteq \mathbf{m}_0$ the loops L with the property $T_1 L = \mathbf{m}_0$ are global Bol loops. According to [5] we have a global Bol loop L_n for all $n \in \mathbb{N}$ having the direct product $PSL_2(\mathbb{R}) \times SO_2(\mathbb{R})$ as the group topologically generated by its left translations and as the stabilizer H of $e \in L_n$ in G the group $H_n = \{(x, x^n) | x \in SO_2(\mathbb{R}), n \in \mathbb{N}\}$. The non-isotopic loops L_n are Scheerer extensions of the Lie group $SO_2(\mathbb{R})$ by the hyperbolic plane loop (cf. [16], Section 2). $\square$

The loops L_n, $n \in \mathbb{N}$ are homeomorphic to G/H_n, which is the cylinder $\mathbb{R}^2 \times S^1$. Let $\tilde{L}$ be the universal covering of L. Since $\tilde{L}$ is homeomorphic to $\mathbb{R}^3$ the loop $\tilde{L}$ contains a central subgroup isomorphic to $\mathbb{Z}$. Moreover all other coverings of L is $\tilde{L}/n\mathbb{Z}$. The universal covering group $\tilde{G}$ of G is the group $\widetilde{PSL_2}(\mathbb{R}) \times \mathbb{R}$, which contains the central subgroup $\pi_1(G) = \mathbb{Z} \times \mathbb{Z}$.

The universal covering group $\tilde{H}$ of H is the group $\{(x, nx) | x \in \mathbb{R}, n \in \mathbb{N}\}$, which is isomorphic to $\mathbb{R}$. The group G^* topologically generated by the left translations of $\tilde{L}$ is the covering group $\tilde{G}/\{(z, nz) | z \in \mathbb{Z}, n \in \mathbb{N}\}$ and the stabilizer of the identity of $\tilde{L}$ is the group $\tilde{H}\pi_1(G)/\pi_1(G)$.
Summarizing our discussion we have:

Theorem 14. *There are precisely three classes $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ of connected strongly left alternative almost differentiable left A-loops with dimension 3 such that the group G topologically generated by their left translations is a 4-dimensional non-solvable Lie group.*
The class $\mathcal{C}_1$ consists of left A-loops L such that the group G topologically generated by their left translations is isomorphic to $PSL_2(\mathbb{R}) \times \mathbb{R}$ and the stabilizer of the identity of these loops is isomorphic to $SO_2(\mathbb{R}) \times \{0\}$. Every loop in $\mathcal{C}_1$ is a product of a 2-dimensional loop isomorphic to the hyperbolic plane loop with the Lie group $\mathbb{R}$. These loops are not isotopic. The only differentiable Bruck loop in $\mathcal{C}_1$ corresponds to the section $\sigma : G/H \to G$ such that $\sigma(G/H) = M_1 \times SO_2(\mathbb{R})$, where M_1 is the image of the section of the hyperbolic plane.
In the class $\mathcal{C}_2$ are the products of a 2-dimensional loop isomorphic to the hyperbolic plane loop with the Lie group $SO_2(\mathbb{R})$. These loops are not isotopic and the group G topologically generated by their left translations is isomorphic to $PSL_2(\mathbb{R}) \times SO_2(\mathbb{R})$ and the stabilizer of the identity of these loops is isomorphic to $SO_2(\mathbb{R}) \times \{1\}$. In $\mathcal{C}_2$ there is again precisely one differentiable Bruck loop $\hat{L}$. The image of the section of $\hat{L}$ is the direct product of the image of the hyperbolic plane loop with the Lie group $SO_2(\mathbb{R})$.
In the class $\mathcal{C}_3$ are contained the Scheerer extensions L_n, $n \in \mathbb{N}$ of the Lie group $SO_2(\mathbb{R})$ by the hyperbolic plane loop and the coverings of L_n. The group G topologically generated by the left translations of L_n is the direct product $PSL_2(\mathbb{R}) \times SO_2(\mathbb{R})$ and the stabilizer H of $e \in L_n$ in G is the group $H_n = \{(x, x^n), x \in SO_2(\mathbb{R}), n \in \mathbb{N}\}$.
The intersection of the classes $\mathcal{C}_2$ and $\mathcal{C}_3$ is the Bruck loop $\hat{L}$.

3-dimensional left A-loops belonging to 5-dimensional non-solvable Lie groups

Now we determine the 3-dimensional connected almost differentiable global left A-loops having a 5-dimensional non-solvable Lie group G as the group topologically generated by the left translations of L. In this case the stabilizer of $e \in L$ in G is a 2-dimensional closed subgroup of G containing no non-trivial normal subgroup of G. Then because of Proposition 5 we have to investigate only the following case:

G is locally isomorphic to the semi-direct product $PSL_2(\mathbb{R}) \ltimes \mathbb{R}^2$, which is the connected component of the group for area preserving affinities of $\mathbb{R}^2$.

For the Lie algebra $\mathbf{g} = sl_2(\mathbb{R}) \ltimes \mathbb{R}^2$ of G we can choose the following basis elements:

$$K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The multiplication table is given by:

$$[K, e_1] = [T, e_2] = -[U, e_2] = -2\ e_1,\ [K, e_2] = -[U, e_1] = -[T, e_1] = 2\ e_2,$$
$$[e_1, e_2] = 0,\ [K, T] = 2\ U,\ [K, U] = 2\ T,\ [U, T] = 2\ K.$$

Lemma 15. *The Lie algebra $\mathbf{g} = sl_2(\mathbb{R}) \ltimes \mathbb{R}^2$ is reductive with a subalgebra $\mathbf{h}$ which does not contain any ideal $\neq 0$ of $\mathbf{g}$ and a 3-dimensional complementary subspace $\mathbf{m}$ generating $\mathbf{g}$ in the following case $\mathbf{h} = \langle K, e_1 \rangle$ and $\mathbf{m} = \langle e_2, U + be_1, T - be_1 \rangle$, where $b \in \mathbb{R}$.*

Proof. The 2-dimensional Lie algebras $\mathbf{h}$ of $\mathbf{g}$, which does not contain any ideal $\neq 0$ of $\mathbf{g}$ (up to mapping $Ad_g, g \in G$) are $\mathbf{h_1} = \langle K, U - T \rangle$, $\mathbf{h_2} = \langle K, e_1 \rangle$, $\mathbf{h_3} = \langle U - T, e_1 \rangle$. We have for a complement $\mathbf{m}$ to $\mathbf{h_1}$ in $\mathbf{g}$ the general form: $\mathbf{m} = \langle e_1 + a_1 K + a_2(U - T), e_2 + b_1 K + b_2(U - T), U + c_1 K + c_2(U - T) \rangle$.
A complement $\mathbf{m}$ to $\mathbf{h_2}$ in $\mathbf{g}$ we can write in the following form:

$$\mathbf{m} = \langle e_2 + a_1 K + a_2 e_1, U + b_1 K + b_2 e_1, T + c_1 K + c_2 e_1 \rangle.$$

An arbitrary complement $\mathbf{m}$ to $\mathbf{h_3}$ in $\mathbf{g}$ can be given as follows:
$\mathbf{m} = \langle e_2 + a_1(U - T) + a_2 e_1,\ K + b_1(U - T) + b_2 e_1,\ U + c_1(U - T) + c_2 e_1 \rangle$.
Here $a_1, a_2, b_1, b_2, c_1, c_2$ are real parameters. The assertion follows now from the property $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$. □

Proposition 16. *There is no global left A-loop L corresponding to the reductive subspace $\mathbf{m} = \langle e_2, U + be_1, T - be_1 \rangle$, $b \in \mathbb{R}$.*

Proof. The element $e_2 \in \mathbf{m}$ is equal to $Ad_g(e_1)$, where $e_1 \in \mathbf{h}$ and $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \in G$. This is a contradiction to Lemma 2. □

This consideration yields the following

Theorem 17. *There is no 3-dimensional connected almost differentiable global left A-loop L having a 5-dimensional non-solvable Lie group as the group topologically generated by its left translations.*

3-dimensional left A-loops with 6-dimensional non-solvable Lie groups

Now we determine all 3-dimensional connected almost differentiable left A-loops such that the group G topologically generated by their left translations is a non-semisimple and non-solvable Lie group. According to Lemma 4 and Propositions 5 we have to discuss the following cases

α) G is locally isomorphic to $PSL_2(\mathbb{R}) \ltimes \mathbb{R}^3$,

β) G is the group for orientation preserving affinities of $\mathbb{R}^2$,

γ) G is locally isomorphic to $SO_3(\mathbb{R}) \ltimes \mathbb{R}^3$, which is the connected component of the euclidean motion group of $\mathbb{R}^3$.

In the case α) the group multiplication in G is given by

$$(A_1, X_1) \circ (A_2, X_2) = (A_1\ A_2,\ A_2^{-1}\ X_1\ A_2 + X_2),$$

where (A_i, X_i), $i = 1, 2$ are two elements of G such that X_i $(i = 1, 2)$ are represented by (2×2) real matrices with trace 0.

A basis of the Lie algebra $\mathbf{g} = sl_2(\mathbb{R}) \ltimes \mathbb{R}^3$ of G can be chosen as follows:

$$e_1 = \left(0, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right),\ e_2 = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0\right),\ e_3 = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0\right),$$

$$e_4 = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0\right),\ e_5 = \left(0, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right),\ e_6 = \left(0, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

According to [10] (p. 17) we obtain the following multiplication table in $\mathbf{g}$:

$$[e_1, e_2] =: e_6,\ [e_1, e_3] =: e_5,\ [e_2, e_3] =: e_4,\ [e_5, e_4] = -e_6,$$
$$[e_1, e_4] = [e_1, e_5] = [e_1, e_6] = [e_2, e_5] = [e_3, e_6] = [e_6, e_5] = 0,$$
$$[e_2, e_6] = [e_3, e_5] = -e_1,\ [e_2, e_4] = e_3,\ [e_3, e_4] = -e_2,\ [e_6, e_4] = e_5.$$

Lemma 18. *The Lie algebra* $\mathbf{g} = sl_2(\mathbb{R}) \ltimes \mathbb{R}^3$ *is reductive with a subalgebra* $\mathbf{h}$ *containing no non-zero ideal of* $\mathbf{g}$ *and a 3-dimensional complementary subspace* $\mathbf{m}$ *generating* $\mathbf{g}$ *in one of the following cases:*

(i) $\mathbf{h} = \langle e_1, e_2, e_6 \rangle$ *and* $\mathbf{m}_{b_2,b_3} = \langle e_5, e_3 - b_3 e_1 - b_2 e_6, e_4 + b_2 e_1 + b_3 e_6 \rangle$, *where* $b_2, b_3 \in \mathbb{R}$.

(ii) $\mathbf{h} = \langle e_2, e_3, e_4 \rangle$ *and* $\mathbf{m}_a = \langle e_1 + a e_4, e_6 - a e_3, e_5 + a e_2 \rangle$, *where* $a \in \mathbb{R} \backslash \{0\}$.

(iii) $\mathbf{h} = \langle e_4, e_5, e_6 \rangle$ *and* $\mathbf{m}_{b_1,b_2} = \langle e_1, e_2 + b_1 e_6 + b_2 e_5, e_3 - b_2 e_6 + b_1 e_5 \rangle$, *where* $b_1, b_2 \in \mathbb{R}$.

Proof. The 3-dimensional subalgebras $\mathbf{h}$ of $\mathbf{g}$, which does not contain any non zero-ideal are the following:

a) $\langle e_2,\ e_5,\ e_1 + e_6 \rangle$,

b) $\langle e_2 + k\ e_5,\ e_1,\ e_6 \rangle$, where $k \in \mathbb{R}$,

c) $\langle e_3 + e_4,\ e_5,\ e_1 - e_6 \rangle$,

d) $\langle e_2,\ e_3+e_4,\ e_1-e_6\rangle$,
e) $\langle e_2, e_3, e_4\rangle$,
f) $\langle e_4, e_5, e_6\rangle$.
In the case a) the basis elements of an arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ are:

$$f_1 = e_1 + a_1e_2 + a_2e_5 + a_3(e_1+e_6)\ ,\ f_2 = e_4 + b_1e_2 + b_2e_5 + b_3(e_1+e_6),$$
$$f_3 = e_3 + c_1e_2 + c_2e_5 + c_3(e_1+e_6),$$

with $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$.
If $\mathbf{h} = \langle e_2 + k\ e_5,\ e_1,\ e_6\rangle$ with $k \in \mathbb{R}$ then an arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ has as basis elements:

$$f_1 = e_5 + a_1(e_2+ke_5) + a_2e_1 + a_3e_6,\ f_2 = e_3 + c_1(e_2+ke_5) + c_2e_1 + c_3e_6,$$
$$f_3 = e_4 + b_1(e_2+ke_5) + b_2e_1 + b_3e_6,$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are real parameters.
In the case c) we can choose as basis elements of an arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ the following:

$$f_1 = e_1 + a_1(e_3+e_4) + a_2e_5 + a_3(e_1-e_6),$$
$$f_2 = e_2 + b_1(e_3+e_4) + b_2e_5 + b_3(e_1-e_6),$$
$$f_3 = e_3 + c_1(e_3+e_4) + c_2e_5 + c_3(e_1-e_6),$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$.
In the case d) the generators of an arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ are:

$$f_1 = e_1 + a_1e_2 + a_2(e_3+e_4) + a_3(e_1-e_6),$$
$$f_2 = e_5 + b_1e_2 + b_2(e_3+e_4) + b_3(e_1-e_6),$$
$$f_3 = e_3 + c_1e_2 + c_2(e_3+e_4) + c_3(e_1-e_6),$$

with the real parameters $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$.
In the case e) one has $\mathbf{h} \cong sl_2(\mathbb{R})$. An arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ has the form

$$\langle e_1 + a_1e_2 + a_2e_3 + a_3e_4, e_6 + b_1e_2 + b_2e_3 + b_3e_4, e_5 + c_1e_2 + c_2e_3 + c_3e_4\rangle,$$

with $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$.
Now we consider the last case. An arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ has the following basis elements:

$$\{e_1 + a_1e_4 + a_2e_5 + a_3e_6, e_2 + b_1e_4 + b_2e_5 + b_3e_6, e_3 + c_1e_4 + c_2e_5 + c_3e_6\},$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$. Using the relation $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$ we obtain the assertion. $\square$

Proposition 19. *There is no global left A-loop L belonging to the reductive subspaces (i) and (ii) in Lemma 18.*

Proof. The element $e_5 \in \mathbf{m}$ in the case (i) is equal to $Ad_g(e_6)$, such that $e_6 \in \mathbf{h}$ and $g = \left(\pm \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 \end{pmatrix}, 0 \right) \in G$. The element $e_6 - ae_3 + e_1 + ae_4 \in \mathbf{m}_a$ in the case (ii) for all $a \in \mathbb{R}\backslash\{0\}$ is equal to $Ad_g(a(e_4 - e_3))$ with $a(e_4 - e_3) \in \mathbf{h}$

and $g = \left(1, \begin{pmatrix} \frac{1}{2a} & 0 \\ 0 & -\frac{1}{2a} \end{pmatrix}\right) \in G$. These facts contradict Lemma 2. $\square$

Now we deal with the case (iii) in Lemma 18. Since the group $SL_2(\mathbb{R})$ has no 3-dimensional linear representation the group G is isomorphic to the semidirect product of $PSL_2(\mathbb{R}) \ltimes \mathbb{R}^3$ and H is isomorphic to the following 3-dimensional subgroup of G:

$$H = \left\{ \left(\pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \begin{pmatrix} -x & y \\ y & x \end{pmatrix} \right), t \in [0, 2\pi), x, y \in \mathbb{R} \right\}.$$

Now we determine the isomorphism classes and the isotopism classes of the left A-loops L_{b_1,b_2} having the subspaces $\mathbf{m}_{b_1,b_2}$ ($b_1, b_2 \in \mathbb{R}$) as the tangent spaces $T_1 L_{b_1,b_2}$.

We have precisely two isomorphism classes $\mathcal{C}_i$ ($i = 1, 2$) of the loops L_{b_1,b_2} belonging to the triples $(G, H, \exp \mathbf{m}_{b_1,b_2})$ for all $b_1, b_2 \in \mathbb{R}$.

The first class $\mathcal{C}_1$ consists loops belonging to $\mathbf{m}_{b_1,b_2}$ for $b_2 = 0$. Denote by $\hat{\mathbf{m}}_{b_1}$ the complement $\mathbf{m}_{b_1,0}$ for all $b_1 \in \mathbb{R}$. One has $[\hat{\mathbf{m}}_{b_1}, \hat{\mathbf{m}}_{b_1}] = \mathbf{h}$ and $\mathbf{g} = \hat{\mathbf{m}}_{b_1} \oplus [\hat{\mathbf{m}}_{b_1}, \hat{\mathbf{m}}_{b_1}]$ for all $b_1 \in \mathbb{R}$. Every loop $L_{b_1,0}$ in $\mathcal{C}_1$ is a Bruck loop and as a representative of this class we may choose the loop $\hat{L_0} = L_{0,0}$. According to [5] the loop $\hat{L_0}$ is a global differentiable Bruck loop, which is called the pseudo-euclidean space loop.

The other class $\mathcal{C}_2$ consists of loops L_{b_1,b_2} having $T_1 L_{b_1,b_2} = \mathbf{m}_{b_1,b_2}$ for $b_2 \neq 0$. Since the automorphism β of the Lie algebra $\mathbf{g}$ defined by

$\beta\ (e_1) = \sqrt{c^2 + d^2}\ e_1$,

$\beta\ (e_6) = -d\ e_5 + c\ e_6$,

$\beta\ (e_5) = c\ e_5 + d\ e_6$,

$\beta\ (e_4) = e_4$,

$$\beta\ (e_2) = \frac{c}{\sqrt{c^2 + d^2}}\ e_2 - \frac{d}{\sqrt{c^2 + d^2}}\ e_3 - c\ b_1\ e_6 + d\ b_1\ e_5,$$

$$\beta\ (e_3) = \frac{c}{\sqrt{c^2 + d^2}}\ e_3 + \frac{d}{\sqrt{c^2 + d^2}}\ e_2 - d\ b_1\ e_6\ - c\ b_1\ e_5,$$

where $\varepsilon\sqrt{c^2 + d^2} = \dfrac{1}{b_2}$ with $\varepsilon = 1$ for $b_2 > 0$ and $\varepsilon = -1$ for $b_2 < 0$, leaves the subalgebra $\mathbf{h}$ invariant and $\beta(\mathbf{m}_{b_1,b_2}) = \mathbf{m}_{0,1}$ for all $b_1 \in \mathbb{R}, b_2 \in \mathbb{R}\backslash\{0\}$ holds, we may choose the loop $\hat{L}_1 = L_{0,1}$ as a representative of the class $\mathcal{C}_2$.

Since there is no $g \in G$ such that $g^{-1}\mathbf{m}_{b_1,b_2} g = \mathbf{m}_{b_1',0}$ holds with $b_1, b_1' \in \mathbb{R}$, $b_2 \in \mathbb{R}\backslash\{0\}$ the isotopism classes of the left A-loops L_{b_1,b_2} coincide with the isomorphism classes $\mathcal{C}_1$, $\mathcal{C}_2$.

Now we prove that $\hat{L}_1 = L_{0,1}$ is a global left A-loop.

The exponential map $\exp : \mathbf{g} \to G$ is described in section 7 in [5].

The image of $\mathbf{m}_{0,1}$ under the exponential map is given as follows: The subspace $\mathbf{m}_{0,1}$ has the shape

$$\mathbf{m}_{0,1} = \left\{ \left(\begin{pmatrix} \lambda_2 & \lambda_3 \\ \lambda_3 & -\lambda_2 \end{pmatrix}, \begin{pmatrix} -\lambda_2 & -\lambda_1 - \lambda_3 \\ \lambda_1 - \lambda_3 & \lambda_2 \end{pmatrix} \right); \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \right\}.$$

According to **1.2** the first component of $\exp \mathbf{m}_{0,1}$ is

$$\left(\pm \begin{pmatrix} \cosh \sqrt{A} + \frac{\sinh \sqrt{A}}{\sqrt{A}} \lambda_2 & \frac{\sinh \sqrt{A}}{\sqrt{A}} \lambda_3 \\ \frac{\sinh \sqrt{A}}{\sqrt{A}} \lambda_3 & \cosh \sqrt{A} - \frac{\sinh \sqrt{A}}{\sqrt{A}} \lambda_2 \end{pmatrix} \right),$$

the second component of $\exp \mathbf{m}_{0,1}$ is $\begin{pmatrix} r'(1) & s'(1) \\ v'(1) & -r'(1) \end{pmatrix}$, where

$$r'(1) = \frac{\lambda_3 \, \lambda_1}{4A} \, (e^{\sqrt{A}} - e^{-\sqrt{A}})^2 - \lambda_2,$$

$$s'(1) = \frac{-\lambda_1}{4\sqrt{A}} (e^{2\sqrt{A}} - e^{-2\sqrt{A}}) - \frac{\lambda_2 \lambda_1}{4A} (e^{\sqrt{A}} - e^{-\sqrt{A}})^2 - \lambda_3,$$

$$v'(1) = \frac{\lambda_1}{4\sqrt{A}} (e^{2\sqrt{A}} - e^{-2\sqrt{A}}) - \frac{\lambda_2 \lambda_1}{4A} (e^{\sqrt{A}} - e^{-\sqrt{A}})^2 - \lambda_3,$$

and $A = \lambda_2^2 + \lambda_3^2$.

The submanifold $\exp \mathbf{m}_{0,1}$ is the image of a sharply transitive global section $\sigma : G/H \to G$ if and only if each element $g \in G$ can be uniquely written as a product $g = mh$ with $m \in \exp \mathbf{m}_{0,1}$ and $h \in H$, moreover $\exp \mathbf{m}_{0,1}$ operates sharply transitively on G/H.

In [5] section 7 we have shown that each element of $G = PSL_2(\mathbb{R})$ can be uniquely written as

$$\left(\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \right) =$$

$$\left(\begin{pmatrix} a_1 & 0 \\ b_1 & a_1^{-1} \end{pmatrix}, \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix} \right) \cdot \left(\pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \begin{pmatrix} k & l \\ l & -k \end{pmatrix} \right)$$

with $a, b, c, d \in \mathbb{R}, \; ad - bc = 1, \; x, y, z \in \mathbb{R}, a_1 > 0, b_1 \in \mathbb{R}, \; t \in [0, 2\pi)$, such that $k = x, l = \dfrac{y+z}{2}, u = \dfrac{y-z}{2}$. Therefore it is sufficient to prove that there is to each element $g \in G$ with the shape

$$\left(\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix} \right); a > 0, b, u \in \mathbb{R}$$

precisely one $m \in \exp \mathbf{m}_{0,1}$ and $h \in H$ such that $g = m \, h$ or equivalently $m = g \, h^{-1}$.

The first component of $\exp \mathbf{m}_{0,1}$ is precisely the section σ_1 of the hyperbolic plane loop given in [16] (pp. 281-282). Therefore for given $a > 0, b \in \mathbb{R}$ we have unique $\lambda_2, \lambda_3 \in \mathbb{R}, t \in [0, 2\pi)$ such that

$$\begin{pmatrix} \cosh \sqrt{A} + \frac{\sinh \sqrt{A}}{\sqrt{A}} \lambda_2 & \frac{\sinh \sqrt{A}}{\sqrt{A}} \lambda_3 \\ \frac{\sinh \sqrt{A}}{\sqrt{A}} \lambda_3 & \cosh \sqrt{A} - \frac{\sinh \sqrt{A}}{\sqrt{A}} \lambda_2 \end{pmatrix} =$$

$$\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},$$

where $A = \lambda_2^2 + \lambda_3^2$. Hence we have to consider the second component of $\exp \mathbf{m}_{0,1}$. For given u, λ_2, λ_3 we have to find unique $\lambda_1, k, l \in \mathbb{R}$ such that

$$\begin{pmatrix} r'(1) & s'(1) \\ v'(1) & -r'(1) \end{pmatrix} = \begin{pmatrix} k & l+u \\ l-u & -k \end{pmatrix},$$

where $r'(1), s'(1), v'(1)$ are values of functions, which depend on the variables $\lambda_1, \lambda_2, \lambda_3$. Since for λ_1 we obtain the equation

$$2u = \frac{-\lambda_1}{2\sqrt{\lambda_2^2+\lambda_3^2}} (e^{2\sqrt{\lambda_2^2+\lambda_3^2}} - e^{-2\sqrt{\lambda_2^2+\lambda_3^2}})$$

we have for the unique solutions $\lambda_1 = \dfrac{-4u\sqrt{\lambda_2^2+\lambda_3^2}}{e^{2\sqrt{\lambda_2^2+\lambda_3^2}} - e^{-2\sqrt{\lambda_2^2+\lambda_3^2}}}$, $k = r'(1)$, $l = \dfrac{s'(1)+v'(1)}{2}$.

Now we verify that the section σ_1 corresponding to the loop $\hat{L}_1$ is sharply transitive, this means that for given elements

$$\left(\begin{pmatrix} a_1 & 0 \\ b_1 & a_1^{-1} \end{pmatrix}, \begin{pmatrix} 0 & u_1 \\ -u_1 & 0 \end{pmatrix} \right) \text{ and } \left(\begin{pmatrix} a_2 & 0 \\ b_2 & a_2^{-1} \end{pmatrix}, \begin{pmatrix} 0 & u_2 \\ -u_2 & 0 \end{pmatrix} \right),$$

where $a_1 > 0, a_2 > 0, b_1, b_2, u_1, u_2 \in \mathbb{R}$ there exists precisely one element $z \in \exp \mathbf{m}_{0,1}$ and a $h = \left(\pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \begin{pmatrix} k & l \\ l & -k \end{pmatrix} \right) \in H$, where $t, k, l \in \mathbb{R}$ such that the equation

(I)
$$z \left(\begin{pmatrix} a_1 & 0 \\ b_1 & a_1^{-1} \end{pmatrix}, \begin{pmatrix} 0 & u_1 \\ -u_1 & 0 \end{pmatrix} \right) =$$
$$\left(\begin{pmatrix} a_2 & 0 \\ b_2 & a_2^{-1} \end{pmatrix}, \begin{pmatrix} 0 & u_2 \\ -u_2 & 0 \end{pmatrix} \right) \left(\pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \begin{pmatrix} k & l \\ l & -k \end{pmatrix} \right)$$

holds. The real variables $\lambda_1, \lambda_2, \lambda_3$ of $z \in \exp \mathbf{m}_{0,1}$ are determined by the following equations

1.
$$\frac{\sinh\sqrt{A}}{\sqrt{A}} \left(\lambda_2 \left(a_1 + \frac{a_2^2}{a_1} \right) + \lambda_3 \left(b_1 + \frac{b_2 a_2}{a_1} \right) \right) + \cosh\sqrt{A} \left(a_1 - \frac{a_2^2}{a_1} \right) = 0$$

2.
$$\frac{\sinh\sqrt{A}}{\sqrt{A}} \left(\lambda_2 \left(\frac{b_2 a_2}{a_1} - b_1 \right) + \lambda_3 \left(\frac{a_1^2 + b_2^2}{a_1} \right) \right) + \cosh\sqrt{A} \left(b_1 - \frac{b_2 a_2}{a_1} \right) = 0$$

3.
$$2(u_2 - u_1) + \lambda_3 (b_1^2 - a_1^2 + a_1^{-2}) + 2a_1 b_1 \lambda_2 =$$

$$\frac{-\lambda_1(b_1^2+a_1^2-a_1^{-2})}{4A}(e^{2\sqrt{A}}-e^{-2\sqrt{A}})+$$
$$\lambda_1\left(\frac{\lambda_2(a_1^2-b_1^2-a_1^{-2})+2\lambda_3 b_1 a_1}{4A}\right)(e^{\sqrt{A}}-e^{-\sqrt{A}})^2,$$

where $A=\lambda_2^2+\lambda_3^2$.
If z is an element of $\mathbf{m}_{0,0}$ in the equation (I) then we obtain for the variables $\lambda_1,\lambda_2,\lambda_3$ of $z\in\exp\mathbf{m}_{0,0}$ the above equations 1, 2, and the equation

3'. $$2(u_2-u_1)=\frac{-\lambda_1(b_1^2+a_1^2-a_1^{-2})}{4A}(e^{2\sqrt{A}}-e^{-2\sqrt{A}})+$$
$$\lambda_1\left(\frac{\lambda_2(a_1^2-b_1^2-a_1^{-2})+2\lambda_3 b_1 a_1}{4A}\right)(e^{\sqrt{A}}-e^{-\sqrt{A}})^2\ .$$

The equations 1, 2, 3', have unique solutions because σ_0 is a sharply transitive section. Therefore the equations 1, 2, 3, are also uniquely solvable for the variables $\lambda_1,\lambda_2,\lambda_3$. Hence the sharply transitive global section σ_1 yields also a global loop $\hat{L}_1(\sigma_1)$, which is a proper left A-loop.

An elementary model of the loop $\hat{L}_1$ may be given on the set Ψ of the euclidean planes in the pseudo-euclidean affine space (cf. [7]). The elements of the loops $\hat{L}_1$ are the same as the elements of $\hat{L}_0$ ([5]), but the sets of the left translations $\exp\mathbf{m}_{0,1}$ respectively $\exp\mathbf{m}_{0,0}$ and hence the multiplication of these two loops differ. The multiplication in the loop $\hat{L}_1$ is given by

(**) $$Q_1 * Q_2=\tau_{P,Q_1}(Q_2),\quad \text{for all } Q_1,Q_2\in\Psi,$$

where τ_{P,Q_1} is the unique element of $\exp\mathbf{m}_{0,1}$ mapping the plane P, which is the identity of $\hat{L}_1$ onto Q_1.

In the case β) the Lie algebra $\mathbf{g}$ of the group G has a real basis

$$e_1=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\ e_2=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\ e_3=\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix},$$
$$e_4=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},\ e_5=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\ e_6=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}.$$

The multiplication is given by the following rules:

$$[e_1,e_2]=[e_2,e_4]=e_2,\ [e_1,e_3]=[e_3,e_4]=-e_3,\ [e_1,e_5]=[e_3,e_6]=-e_5,$$
$$[e_1,e_6]=[e_1,e_4]=[e_2,e_6]=[e_3,e_5]=[e_4,e_5]=[e_5,e_6]=0,$$
$$[e_2,e_3]=e_1-e_4,\ [e_2,e_5]=[e_4,e_6]=-e_6.$$

Lemma 20. *The Lie algebra* $\mathbf{g}$ *is reductive with a subalgebra* $\mathbf{h}$ *containing no non-zero ideal of* $\mathbf{g}$ *and a 3-dimensional complementary subspace* $\mathbf{m}$ *generating* $\mathbf{g}$ *in the following case:* $\mathbf{h}=\langle e_1,e_4,e_5\rangle$ *and* $\mathbf{m}=\langle e_2,e_3,e_6\rangle$.

Proof. The 3-dimensional subalgebras $\mathbf{h}$, which does not contain any ideal $\neq 0$ of $\mathbf{g}$ are

a) $\mathbf{h} = \langle e_1 - e_4, e_2, e_3 \rangle$
b) $\mathbf{h} = \langle e_1, e_2, e_4 \rangle$
c) $\mathbf{h} = \langle e_1, e_4, e_5 \rangle$
d) $\mathbf{h} = \langle e_1 + e_4, e_3, e_5 \rangle$
e) $\mathbf{h} = \langle e_3, e_4, e_5 \rangle$
f) $\mathbf{h} = \langle e_1 - e_4, e_3, e_5 \rangle$.
The basis elements of an arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ in the case a) are:

$$e_1 + a_1(e_1 - e_4) + a_2 e_2 + a_3 e_3,\ e_5 + b_1(e_1 - e_4) + b_2 e_2 + b_3 e_3,$$
$$e_6 + c_1(e_1 - e_4) + c_2 e_2 + c_3 e_3,$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are real parameters.
In the case b) an arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ has the following shape:

$$\mathbf{m} = \langle e_3 + a_1 e_1 + a_2 e_2 + a_3 e_4,\ e_5 + b_1 e_1 + b_2 e_2 + b_3 e_4,$$
$$e_6 + c_1 e_1 + c_2 e_2 + c_3 e_4 \rangle,$$

with the real numbers $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$.
In the case c) an arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ has as generators:

$$\{f_1 = e_2 + a_1 e_1 + a_2 e_4 + a_3 e_5,\ f_2 = e_3 + b_1 e_1 + b_2 e_4 + b_3 e_5,$$
$$f_3 = e_6 + c_1 e_1 + c_2 e_4 + c_3 e_5\},$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$.
In the case d) the basis elements of an arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ are:

$$e_1 + a_1(e_1 + e_4) + a_2 e_3 + a_3 e_5,\ e_2 + b_1(e_1 + e_4) + b_2 e_3 + b_3 e_5,$$
$$e_6 + c_1(e_1 + e_4) + c_2 e_3 + c_3 e_5,$$

with the real parameters $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$.
In the case e) an arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ is given by:

$$\langle e_1 + a_1 e_3 + a_2 e_4 + a_3 e_5,\ e_2 + b_1 e_3 + b_2 e_4 + b_3 e_5,\ e_6 + c_1 e_3 + c_2 e_4 + c_3 e_5 \rangle,$$

with $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$.
In the last case f) for the basis elements of an arbitrary complement $\mathbf{m}$ to $\mathbf{h}$ in $\mathbf{g}$ one has:

$$e_1 + a_1(e_1 - e_4) + a_2 e_3 + a_3 e_5,\ e_2 + b_1(e_1 - e_4) + b_2 e_3 + b_3 e_5,$$
$$e_6 + c_1(e_1 - e_4) + c_2 e_3 + c_3 e_5,$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are real numbers. The assertion follows from the property $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$. □

Proposition 21. *There is no global left A-loop L having the reductive subspace $\mathbf{m} = \langle e_2, e_3, e_6 \rangle$ as the tangent space $T_1 L$.*

Proof. The subspace $\mathbf{m}$ contains the element $e_2 + e_3$, which is equal to $Ad_g(e_1 - e_4)$, where $e_1 - e_4 \in \mathbf{h}$ and $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & -1 & \frac{1}{2} \end{pmatrix} \in G$. This is a contradiction to Lemma 2. □

Now we consider the case that G is locally isomorphic to $SO_3(\mathbb{R}) \ltimes \mathbb{R}^3$. This group can be represented by the pairs of complex (2×2)-matrices

$$(A, X) = \left(\pm \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \begin{pmatrix} k & li+n \\ -li+n & -k \end{pmatrix} \right);$$

$a, b \in \mathbb{C}, a\bar{a} + b\bar{b} = 1, k, l, n \in \mathbb{R}$. Here $\bar{a}$ denotes the complex conjugate of $a \in \mathbb{C}$. The group multiplication is given by the rule

$$(A_1, X_1) \circ (A_2, X_2) = (A_1\ A_2,\ A_2^{-1}\ X_1\ A_2 + X_2).$$

Any 3-dimensional subgroup H of $G = SO_3(\mathbb{R}) \ltimes \mathbb{R}^3$, which contains no non-trivial normal subgroup of G, is locally isomorphic either to a semidirect product of a 2-dimensional translation group by a 1-dimensional rotation group $SO_2(\mathbb{R})$ or to the subgroup $\{(a, 0), a \in SO_3(\mathbb{R})\}$. The first possibility cannot occur since the factor space G/H is the topological product having as a factor the 2-sphere which is not parallelizable.
Now we deal with the second possibility for H.

Lemma 22. *For all $a \in \mathbb{R}\backslash\{0\}$ there is a reductive complement*

$$\mathbf{m}_a = \langle V_1 + aZ, V_2 + aY, V_3 - aX\rangle$$

to the Lie algebra $\mathbf{h}_2$ of H_2 generating $\mathbf{g} = so_3(\mathbb{R}) \ltimes \mathbb{R}^3$.

Proof. Denote by X, Y, Z the generators correspond to 1-dimensional rotations and let V_3, V_2, V_1 be the axes of the rotation groups corresponding to X, Y respectively Z. We can identify the basis elements of $\mathbf{g}$ with the following matrices:

$$X = \left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, 0 \right),\ Y = \left(\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, 0 \right),\ \ Z = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0 \right),$$

$$V_1 = \left(0, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right),\ V_2 = \left(0, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right),\ V_3 = \left(0, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

According to [10] (p. 17) the multiplication table of $\mathbf{g} = su_2(\mathbb{C}) \ltimes \mathbb{R}^3$ is given by:

$$[X, Y] = Z,\ [Z, X] = Y,\ [Y, Z] = X,\ [X, V_1] = [Z, V_3] = -V_2,$$
$$[X, V_2] = [Y, V_3] = V_1,\ [Z, V_2] = -[Y, V_1] = V_3,$$
$$[X, V_3] = [Y, V_2] = [V_1, V_2] = [V_1, V_3] = [V_2, V_3] = [Z, V_1] = 0.$$

The Lie algebra $\mathbf{h}_2$ of H_2 has as generators X, Y, Z. An arbitrary complement $\mathbf{m}$ to $\mathbf{h}_2$ in $\mathbf{g}$ has the following shape:

$$\mathbf{m} = \langle V_1 + aX + bY + cZ,\ V_2 + dX + eY + fZ,\ V_3 + gX + hY + iZ\rangle,$$

where $a, b, c, d, e, f, g, h, i \in \mathbb{R}$. The subspace $\mathbf{m}$ satisfies the condition $[\mathbf{h}_2, \mathbf{m}] \subseteq \mathbf{m}$ if and only if $\mathbf{m}$ has the following form:

$$\mathbf{m}_a = \langle V_1 + aZ, V_2 + aY, V_3 - aX\rangle,$$

where $a \in \mathbb{R}\backslash\{0\}$. $\square$

Using the automorphism φ of $\mathbf{g}$ having the form:
$\varphi(V_1) = \frac{1}{2a}\ V_1 + \frac{\sqrt{3}}{2a}\ V_2,$

$\varphi(V_2) = \frac{\sqrt{3}}{2a}\, V_1 - \frac{1}{2a}\, V_2,$
$\varphi(V_3) = -\frac{1}{a}\, V_3,$
$\varphi(X) = -X,$
$\varphi(Y) = -\frac{1}{2}\, Y + \frac{\sqrt{3}}{2}\, Z,$
$\varphi(Z) = \frac{\sqrt{3}}{2}\, Y + \frac{1}{2}\, Z,$
for all $a \in \mathbb{R}\backslash\{0\}$ we have $\varphi(\mathbf{h}) = \mathbf{h}$ and $\varphi(\mathbf{m}_a) = \mathbf{m}_1$. Therefore the local loops L_a having $T_1 L_a = \mathbf{m}_a$ form an isomorphism class $\mathcal{C}$ and as a representative of $\mathcal{C}$ can be chosen the local loop L_1 belonging to $\mathbf{m}_1$.

Proposition 23. *The local loop L_1 is not a global left A-loop.*

Proof. The exponential map $\exp : \mathbf{g} \to G$ is given by the following way: For $X \in \mathbf{g}$ we have $\exp X = v_X(1)$, where $v_X(t)$ is the 1-parameter subgroup of G with the property $\frac{d}{dt}\big|_{t=0} v_X(t) = X$. In the 1-parameter subgroup $\alpha(t) = (\beta(t), \gamma(t))$ of G with the conditions

$$\alpha(t=0) = (1,0) \text{ and } \tfrac{d}{dt}\big|_{t=0}\alpha(t) = (X_1, X_2) = X \in \mathbf{g}$$

the first component $\beta(t)$ is the 1-parameter subgroup of $SO_3(\mathbb{R})$ and the second component $\gamma(t)$ satisfies

$$\frac{d}{dt}\gamma(t) = \frac{d}{ds}\Big|_{s=0}\gamma(t+s) = -\frac{d}{ds}\Big|_{s=0}\beta(s)\gamma(t) + \gamma(t)\frac{d}{ds}\Big|_{s=0}\beta(s) + \frac{d}{ds}\Big|_{s=0}\gamma(s) =$$
$$-X_1\gamma(t) + \gamma(t)X_1 + X_2.$$

For $X_1 = \begin{pmatrix} \lambda_1 i & \lambda_2 i - \lambda_3 \\ -\lambda_2 i + \lambda_3 & -\lambda_1 i \end{pmatrix}$, $X_2 = \begin{pmatrix} \lambda_5 & \lambda_4 i + \lambda_6 \\ -\lambda_4 i + \lambda_6 & -\lambda_5 \end{pmatrix}$ and $\gamma(t) = \begin{pmatrix} r(t) & v(t)i + s(t) \\ -v(t)i + s(t) & -r(t) \end{pmatrix}$, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \mathbb{R}$ we get the following inhomogen system of linear differential equations:

$$\frac{d}{dt}\begin{pmatrix} r(t) \\ s(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 & -2\lambda_1 & -2\lambda_3 \\ 2\lambda_1 & 0 & 2\lambda_2 \\ 2\lambda_3 & -2\lambda_2 & 0 \end{pmatrix}\begin{pmatrix} r(t) \\ s(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} \lambda_5 \\ \lambda_6 \\ \lambda_4 \end{pmatrix}$$

with the following initial conditions:

$$r(0) = s(0) = v(0) = 0,\ \frac{d}{dt}\Big|_{t=0} r(t) = \lambda_5,\ \frac{d}{dt}\Big|_{t=0} s(t) = \lambda_6,\ \frac{d}{dt}\Big|_{t=0} v(t) = \lambda_4.$$

The solution of this inhomogeneous system is:

$$r(t) = -\frac{i[(e^{2lit} - e^{2lit})(\lambda_5\lambda_1^2 + \lambda_5\lambda_3^2 - \lambda_6\lambda_3\lambda_2 + \lambda_4\lambda_1\lambda_2)]}{4l^3}$$
$$-\frac{\left[\left(e^{lit} - e^{-lit}\right)^2(-\lambda_6\lambda_1 - \lambda_4\lambda_3) + t(4\lambda_4\lambda_1\lambda_2 - 4\lambda_6\lambda_3\lambda_2 - 4\lambda_5\lambda_2^2)\right]}{4l^2},$$

$$s(t) = -\frac{i(e^{2lit} - e^{-2lit})(\lambda_6\lambda_1^2 + \lambda_6\lambda_2^2 - \lambda_5\lambda_3\lambda_2 + \lambda_4\lambda_1\lambda_3)}{4l^3}$$
$$-\frac{\left[\left(e^{lit} - e^{-lit}\right)^2(\lambda_4\lambda_2 + \lambda_5\lambda_1) + t(4\lambda_4\lambda_1\lambda_3 - 4\lambda_5\lambda_3\lambda_2 - 4\lambda_6\lambda_3^2)\right]}{4l^2},$$
$$v(t) = -\frac{i(e^{2lit} - e^{-2lit})(\lambda_4\lambda_3^2 + \lambda_4\lambda_2^2 + \lambda_5\lambda_1\lambda_2 + \lambda_6\lambda_1\lambda_3)}{4l^3}$$
$$-\frac{\left[\left(e^{lit} - e^{-lit}\right)^2(\lambda_5\lambda_3 - \lambda_6\lambda_2) + t(4\lambda_5\lambda_1\lambda_2 + 4\lambda_6\lambda_3\lambda_1 - 4\lambda_4\lambda_1^2)\right]}{4l^2},$$
where $l = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}$. Since $\mathbf{m}_1$ has the form

$$\mathbf{m}_1 = \left\{\left(\begin{pmatrix} -ci & -a+bi \\ a+bi & ci \end{pmatrix}, \begin{pmatrix} -c & ai+b \\ -ai+b & c \end{pmatrix}\right); a,b,c \in \mathbb{R}\right\},$$

according to **1.2** (Section 1) the first component of $\exp \mathbf{m}_1$ is

$$(\exp \mathbf{m}_1)_1 = \begin{pmatrix} \cos\sqrt{k} - \frac{ci \sin\sqrt{k}}{\sqrt{k}} & \frac{(-a+bi)\sin\sqrt{k}}{\sqrt{k}} \\ \frac{(a+bi)\sin\sqrt{k}}{\sqrt{k}} & \cos\sqrt{k} + \frac{ci\sin\sqrt{k}}{\sqrt{k}} \end{pmatrix},$$

the second component of $\exp \mathbf{m}_1$ has the shape
$$(\exp \mathbf{m}_1)_2 = \begin{pmatrix} r(1) & v(1)i + s(1) \\ -v(1)i + s(1) & -r(1) \end{pmatrix},$$
where
$r(1) = -c(e^{\sqrt{k}i} - e^{-\sqrt{k}i})^2$, $s(1) = b(e^{\sqrt{k}i} - e^{-\sqrt{k}i})^2$, $v(1) = a(e^{\sqrt{k}i} - e^{-\sqrt{k}i})^2$, and $k = a^2 + b^2 + c^2$.
From the equation $g = \left(1, \begin{pmatrix} 0 & fi \\ -fi & 0 \end{pmatrix}\right) = ((\exp \mathbf{m}_1)_1, (\exp \mathbf{m}_1)_2)(h, 0)$ with $f \neq 0$ one has $h = (\exp \mathbf{m}_1)_1^{-1}$. This means that
$$\begin{pmatrix} 0 & fi \\ -fi & 0 \end{pmatrix} =$$
$$(\exp \mathbf{m}_1)_1 \begin{pmatrix} -(e^{ki} - e^{-ki})^2 c & (e^{ki} - e^{-ki})^2(ai+b) \\ (e^{ki} - e^{-ki})^2(-ai+b) & (e^{ki} - e^{-ki})^2 c \end{pmatrix} (\exp \mathbf{m}_1)_1^{-1},$$
where $k = \sqrt{a^2 + b^2 + c^2}$. Hence we obtain
$$-c(e^{\sqrt{a^2+b^2+c^2}i} - e^{-\sqrt{a^2+b^2+c^2}i})^2 = 0,\ a(e^{\sqrt{a^2+b^2+c^2}i} - e^{-\sqrt{a^2+b^2+c^2}i})^2 = f,$$
$$b(e^{\sqrt{a^2+b^2+c^2}i} - e^{-\sqrt{a^2+b^2+c^2}i})^2 = 0\ .$$
Therefore we may assume that $c = b = 0$. Then we have
$$a(e^{\sqrt{a^2}i} - e^{-\sqrt{a^2}i})^2 = f \text{ or } a(\sinh\sqrt{a^2}i)^2 = -a(\sin\sqrt{a^2})^2 = \frac{f}{4}.$$
Since the function $x \mapsto -x(\sin\sqrt{x^2})^2$ is not injective, there exist different real numbers a_1, a_2 with the properties $\sin(\sqrt{a_1^2}) \neq \sin(\sqrt{a_2^2})$ but $a_1(\sin\sqrt{a_1^2})^2 =$

$a_2(\sin\sqrt{a_2^2})^2$. Hence g can be written in different way as a product of an element in $\exp \mathbf{m}_1$ and an element of H which contradicts Lemma 2. $\square$
From the above discussion we obtain:

Theorem 24. *There is only one class $\mathcal{C}$ of the 3-dimensional connected almost differentiable left A-loops L such that the group G topologically generated by the left translations is a 6-dimensional non semisimple and non-solvable Lie group. The group G is isomorphic to the semidirect product $PSL_2(\mathbb{R}) \ltimes \mathbb{R}^3$, where the action of $PSL_2(\mathbb{R})$ on $\mathbb{R}^3$ is the adjoint action of $PSL_2(\mathbb{R})$ on its Lie algebra, and the stabilizer of the identity of the loops in $\mathcal{C}$ is the 3-dimensional subgroup of G*

$$\left\{ \left(\pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \begin{pmatrix} -x & y \\ y & x \end{pmatrix} \right); t \in [0, 2\pi), x, y \in \mathbb{R} \right\}.$$

The loops in the class $\mathcal{C}$ can be characterized by two real parameters a, b and form precisely two isomorphism classes which coincide the isotopism classes. The one isomorphism class containing the Bruck loops $L_{b_1,0}$, $b_1 \in \mathbb{R}$ has as a representative the pseudo-euclidean space loop $L_{0,0} = \hat{L}_0$. As a representative of the other isomorphism class consisting of left A-loops L_{b_1,b_2} with $b_2 \neq 0$ may be chosen the loop $L_{0,1} = \hat{L}_1$. The loops $\hat{L}_0$ and $\hat{L}_1$ are realized on the pseudo-euclidean affine space $E(2,1)$. The elements of these loops are the planes on which the euclidean metric is induced but the sets of left translations differ.

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