Foundations of Euclidean geometry

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1 Introduction to Euclidean geometry

1.1 Parallelograms

In what follows we suppose that we have an Euclidean space, i.e. an absolute space satisfying Euclidean parallel axiom (EPP) together with its equivalent forms [5, Theorem 17.]. Taking a plane

S in the space the points $\{A, B, C, D\} \subset S$ are the vertices of a convex quadrilateral if the segments $\bar{AC}$ and $\bar{BD}$ intersect each other at an interior point. The adjacent sides of the quadrangle are $\bar{AB}$, $\bar{BC}$, $\bar{CD}$ and $\bar{DA}$, respectively. $\bar{AB}$ and $\bar{CD}$, or $\bar{BC}$ and $\bar{DA}$ are opposite sides.

Excercise 1 Using (EPP) prove that the sum of the interior angles of a triangle is π.

Hint. Let $ABC\triangle$be a triangle and consider the line through C parallel to the opposite side. Since the sufficient conditions of parallelism are necessary it follows that we have equal alternate interior angles completing the angle $C\angle$ to π.

Definition 1 *A convex quadrilateral is a parallelogram if each side is parallel to its opposite side.*

Theorem 1 (characterization of parallelograms) *A convex quadrilateral is a parallelogram if and only if one of the following conditions is satisfied:*

(1) *each side is congruent to its opposite side*

(2) *each angle is congruent to its opposite angle*

(3) *two opposite sides are parallel and congruent*

(4) *the diagonals bisect each other*

(5) *the convex quadrilateral is central symmetric*

Proof. If a convex quadrilateral is a parallelogram then the diagonals divides the quadrilateral into congruent triangles because the sufficient conditions of parallelism are necessary (for the congruance of the triangles see (ASA)). This means that we can immediately conclude (1), (2) and (3). Using (1), statement (4) also follows. (4) and (5) are obviously equivalent. Conversely, if the opposite sides of a convex quadrilateral are congruent then the diagonals divides the quadrilateral into congruent triangles because of (SSS). Therefore the sufficient condition of parallelism implies that the opposite sides are parallel (we have equal alternate interior angles), i.e. the quadrilateral is a paralelogram. Suppose that the opposite angles are congruent. Dividing the quadrilateral into triangles we have that the sum of the angles must be 2π. Therefore we have that the the consecutive interior angles add up to π ($180°$). It is a sufficient condition of parallelism. If (3) is assumed to be true then we can divide the quadrilateral into congruent triangles by one of its diagonals (the congruence follows from the necessary conditions of parallelism and (SAS)). Using the sufficient conditions of parallelism we can conclude that the complement opposite sides are also parallel and congruent. Since (4) and (5) are obviously equivalent, it is enough to check one of them. If (4) is assumed to be true then axiom (SAS) implies that the diagonals divide the quadrilateral two pairs of congruent triangles (recall that the vertically opposite angles are congruent). The quadrilateral is a parallelogram because of the sufficient conditions of parallelism. $\square$

1.2 Theorems about parallel lines

Let a and b be lines in the plane S of a Euclidean space and consider the line $c \subset S$ intersecting both a and b. The parallel projection of a onto b sends any point $A \in a$ into $A' \in b$ such that $l_{AA'}$ is parallel to c. Especially all the lines connecting a point and its image are parallel.

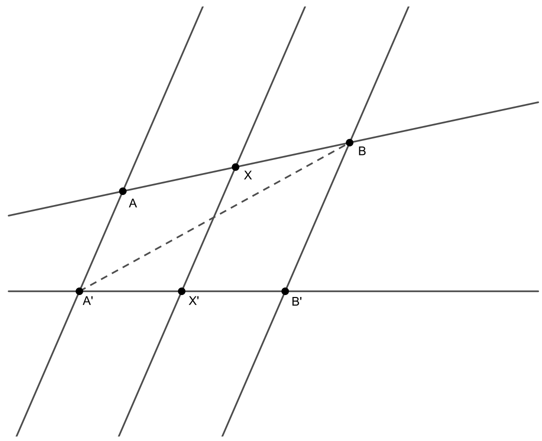

Figure 1: Parallel projection theorem I

Theorem 2 (parallel projection theorem) *The parallel projection preserves the ratio of the segments.*

Proof. In the first step of the proof we clarify that the parallel projection preserves betweeness, i.e. $A - X - B$ implies that $A' - X' - B'$. If $A = A'$, or $B = B'$ then Pasch theorem immediately implies that $A' - X' - B'$. In case of $A \neq A'$ and $B \neq B'$ let us apply Pasch theorem two times (Figure 1.) to the triangles $AA'B\triangle$ and $A'BB'\triangle$, respectively (the intersecting line is $l_{XX'}$ in both cases). The second step in the proof of the theorem is the investigation of a special case: suppose that $\bar{AB}{=}\bar{CD}$. Taking the lines parallel b through the points A and C, respectively (Figure 2.), we have congruent triangles $ABE\triangle$ and $CDF\triangle$ with equal sides $\bar{AE}{=}\bar{CF}$. Therefore Theorem 1 (characterization of parallelograms) implies that $\bar{A'B'}{=}\bar{AE}{=}\bar{CF}{=}\bar{C'D'}$. The genereal case is investigated in the thrid

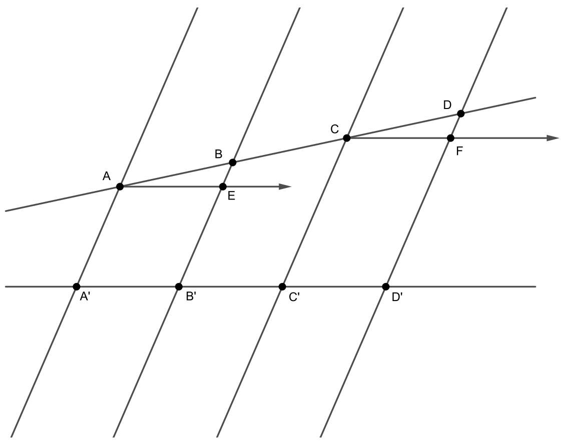

Figure 2: Parallel projection theorem I

step as follows. Dividing the segments $\bar{AB}$ into n equal parts by the points

$$A = A_1 - A_2 - A_3 - \ldots - A_{n+1} = B$$

let us put segments of length $\bar{AB}\ /n$ back-to-back along $\bar{CD}$ as many times as we can, i.e.

$$k\frac{\bar{AB}}{n} \leq \bar{CD} < (k+1)\frac{\bar{AB}}{n} \ \Rightarrow\ \frac{k}{n} \leq \bar{CD}\ /\ \bar{AB} < \frac{k+1}{n}.$$

According to the first and the second step, $A' = A'_1 - A'_2 - A'_3 - \ldots - A'_{n+1} = B'$ is the partition of $\overline{A'B'}$ into n equal parts and

$$k\frac{\overline{A'B'}}{n} \leq \overline{C'D'} < (k+1)\frac{\overline{A'B'}}{n} \;\Rightarrow\; \frac{k}{n} \leq \overline{C'D'} \;/\; \overline{A'B'} < \frac{k+1}{n}.$$

Therefore

$$\left|\left(\overline{CD} \;/\; \overline{AB}\right) - \left(\overline{C'D'} \;/\; \overline{A'B'}\right)\right| \leq \frac{1}{n}.$$

Taking the limit $n \to \infty$ we have the parallel projection theorem. $\square$

Theorem 3 (converse of the parallel projection theorem) *If we have two lines intersecting the arms of an angle with vertex O at the corresponding points A, A' and B, B' such that $\overline{OA} \;/\; \overline{OA'}=\overline{OB} \;/\; \overline{OB'}$ then the lines are parallel to each other.*

Proof. The point $B' \in \overrightarrow{OB}$ is uniquely determined by the segment construction theorem because

$$\overline{OB'}=\overline{OB} \cdot \overline{OA'} \;/\; \overline{OA}\,.$$

Using the parallel projection theorem with projecting line $l_{AA'}$, the point B' must be the parallel projected image of B as was to be proved. $\square$

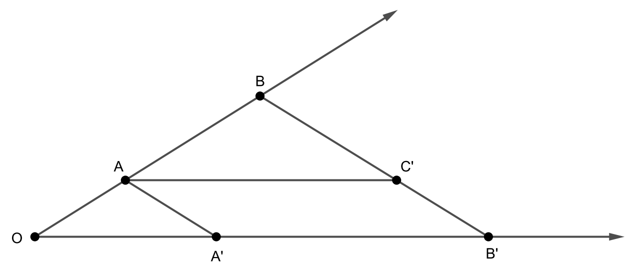

Figure 3: Application of the parallel projection theorem

Theorem 4 (application of the parallel projection theorem) *If we have two parallel lines intersecting the arms of an angle with vertex O at the corresponding points A, A' and B, B' then $\overline{OA} \;/\; \overline{OA'}=\overline{OB} \;/\; \overline{OB'}=\overline{AA'} \;/\; \overline{BB'}$, i.e. the common ratio between the segments on the same arms is equal to the ratio of the cross-segments.*

Proof. Consider the angle $AOA'\angle = BOB'\angle$ and suppose that $l_{AA'}$ and $l_{BB'}$ are parallel. Since $O' = O$, it is enough to prove that the ratio of the cross-segments is equal to the common ratio of the segments on the same arms of the angle. Applying the parallel projection theorem to the angle $OBB'\angle$ (Figure 3.) we have that

$$\frac{\overline{AO}}{\overline{BO}} = \frac{\overline{C'B'}}{\overline{BB'}} = \frac{\overline{AA'}}{\overline{BB'}}$$

because A, A', B' and C' are the vertices of a parallelogram. $\square$

1.3 Similar triangles, similarity theorems in right-angled triangles

Combining the congruence theorems and the theorems about parallel lines we have the theory of similar triangles. Therefore we do not use any kind of similarity axiom, although such an axiom can apper in different expositions of Euclidean geometry, see [1, Similarity axiom].

Definition 2 *Two triangles are similar if there is a correpondence among their vertices such that all of the corresponding angles are congruent and all of the corresponing sides are in the same ratio. It is called the ratio of the similarity.*

Remark 1 Congruent triangles are similar with similarity ratio 1.

Theorem 5 (similarity theorems) *Two triangles are similar if there is a correpondence among their vertices such that*

(S'AS') *two corresponding sides are in the same ratio and the included angles are congruent*

(S'S'S') *all of the corresponing sides are in the same ratio*

(S's'A) *two corresponding sides are in the same ratio and the non-included angles opposite to the longer sides are congruent*

(AAA) *all of the corresponding angles are congruent.*

Theorem 6 (similarity theorems in right-angled triangles) *Let a, b and c be the legs and the hypothenuse of a right-angled triangle, respectively. If the orthogonal projections of the sides a and b onto the hypotenuse are of lengths p and q, respectively, then $m^2 = pq$, $a^2 = cp$ and $b^2 = cq$, $a^2 + b^2 = c^2$, where m is the height belonging to the hypotenuse.*

Proof. Let A, B and C be the vertices of the right-angled triangle and consider the foot point T of the perpendicular line to c through the point C. Since the triangles $ABC\triangle$, $ACT\triangle$ and $BCT\triangle$ have the same angles, they are similar. The equality of the ratio of the corresponding sides gives that $m^2 = pq$ (see triangles $ACT\triangle$ and $BCT\triangle$), $a^2 = cp$ (see triangles $ABC\triangle$ and $ACT\triangle$) and $b^2 = cq$. To sum up $a^2 + b^2 = cp + cq = c(p + q) = c^2$. $\square$

Remark 2 The similarity theorem $m^2 = pq$ is called height theorem, $a^2 = cp$ and $b^2 = cq$ are leg theorems. Pythagorean theorem is $a^2 + b^2 = c^2$.

A panoramic view

Incidence geometry $\rightarrow$ absolute geometry (RP: dist., PSP, PP: angle, SAS) $\rightarrow$ hyperbolic geometry: HPP

$\downarrow$

Euclidean geometry: EPP

(parallelograms, theorems about parallel lines, similarity of triangles)

2 Isometries

2.1 Euclidean plane isometries

Definition 3 *A distance preserving one-to-one transformation* $\sigma\colon S \rightarrow S$, $A \rightarrow A' := \sigma(A)$ *of the Euclidean plane is called isometry, i.e.* $d(A', B') = d(A, B)$ *for any* $A, B \in S$.

Some properties of isometries:

- they form a group with respect to the composition of mappings.

Using the triangle inequality

- $A - B - C$ if and only if $A' - B' - C'$.

The congruence theorem (SSS) implies that

- $m(ABC\angle) = m(A'B'C'\angle)$.

Therefore an isometry sends any line into a line. The angles are also preserved and the images of parallel lines under an isometry are parallel.

Definition 4 *The reflection across the line* $l \subset S$ *is a transformation* $\sigma_l\colon S \rightarrow S$, $X \rightarrow X' := \sigma_l(X)$ *defined by the following properties:*

- $X' = X$ *for any* $X \in l$ *esetén*
- *If* $X \notin l$, *then* l *is the perpendicular bisector of* $\overline{XX'}$.

Theorem 7 *Reflection across lines are isometries.*

Proof. The distance between the points A and B is obviously preserved in case of A and $B \in l$ because of $A = A'$ and $B = B'$. Since the perpendicular bisector of a segment is the locus of points in the plane having the same distance from each of the endpoints, it follows that if $A \in l$ but $B \notin l$ then $d(A', B') = d(A, B') = d(A, B)$, because of $A' = A \in l$. Finally, suppose that $A \notin l$ and $B \notin l$. If $l_{AB} \cap l \neq \emptyset$, then the distance $d(A, B)$ can be easily expressed in terms of $d(A, C)$ and $d(B, C)$, where $C' = C$ is the intersection point of l_{AB} and l. A similar formula holds for $d(A', B')$ in terms of $d(A', C')$ and $d(B', C')$. Therefore $d(A, B) = d(A', B')$. To discuss all of the possible cases note that if l_{AB} and l are parallel then A, B, B', A' form a parallelogram and we can refer to Theorem 1. (characterization of parallelograms). $\square$

Theorem 8 (fixed points of isometries) *If an isometry σ has two fixed points then the line l of the fixed points is a pointwise fixed line and $\sigma = \sigma_l$, or σ is the identity. In case of three non-collinear fixed points σ must be the identity.*

Proof. Suppose that $A = A'$, $B = B'$ and $C = C'$ are non-collinear fixed points and let X be a point in the plane such that $X \neq X'$. Then

$$d(X, A) = d(X', A') = d(X', A), \quad d(X, B) = d(X', B') = d(X', B)$$

and

$$d(X, C) = d(X', C') = d(X', C),$$

i.e. A, B and C are on the perpendicular bisector of $\overline{XX'}$. It is a contradiction. Therefore $X' = X$ for any $X \in S$. Using the previous argument, if $A = A'$, $B = B'$ but σ is not the identical transformation then $X \neq X'$ for any $X \notin l$. On the other hand

$$d(X, A) = d(X', A') = d(X', A) \quad \text{and} \quad d(X, B) = d(X', B') = d(X', B), \tag{1}$$

i.e. both A and B are on the perpendicular bisector of $\overline{XX'}$. It is exactly the rule of the reflection across the line l for any $X \notin l$. If $X \in l$ then $X' \in l' = l$ because l is given by two fixpoints of the isometry. In particular $X = X'$ because of formula (1). $\square$

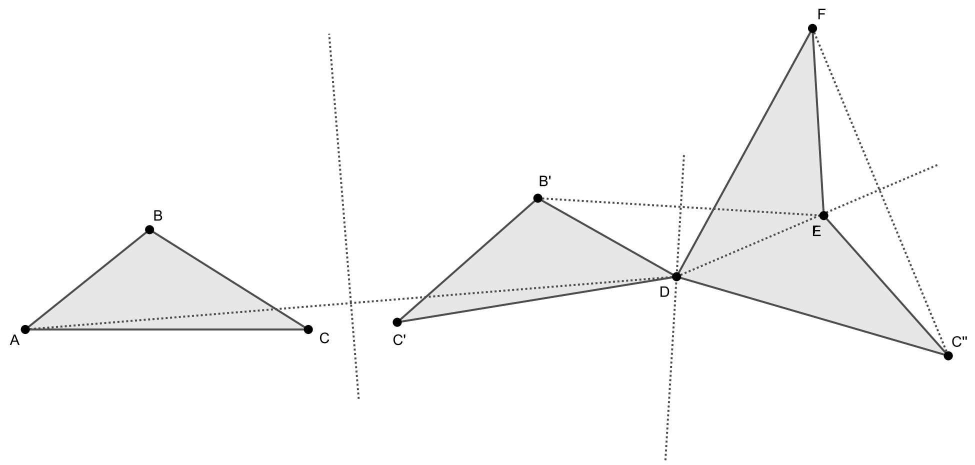

Figure 4: Fundamental theorem of plane isometries

Theorem 9 (fundamental theorem of plane isometries) *If $ABC\triangle$ and $DEF\triangle$ are congruent triangles then there exists a uniquely determine plane isometry $\sigma \colon S \to S$ such that $\sigma(A) = D$, $\sigma(B) = E$ and $\sigma(C) = F$.*

Proof. The unicity is the direct consequence of Theorem 8 because $\sigma(A) = D$, $\sigma(B) = E$ and $\sigma(C) = F$, or $\tilde{\sigma}(A) = D$, $\tilde{\sigma}(B) = E$ and $\tilde{\sigma}(C) = F$ imply that the isometry $\sigma := \sigma^{-1} \circ \tilde{\sigma}$ has three non-collinear fixed points. Therefore σ is the identity and $\sigma = \tilde{\sigma}$. The proof of the existence can be given in three steps as follows. In the first step (Figure 4.) let σ_1 be the reflection across the perpendicular bisector of $\bar{AD}$ provided that $A \neq D$; otherwise σ_1 is the identity:

$$A' = \sigma_1(A) = D,\ \sigma_1(B) = B',\ \sigma_1(C) = C'. \tag{2}$$

In the second step, let σ_2 be the reflection across the perpendicular bisector of $\bar{B'E}$ provided that $B' \neq E$; otherwise σ_2 is the identity. Since

$$d(D,E) = d(A,B) = d(A',B') = d(D,B'),$$

the point D is on the axis of reflection, i.e.

$$A'' = \sigma_2 \circ \sigma_1(A) = \sigma_2(A') = \sigma_2(D) = D,\ B'' = \sigma_2 \circ \sigma_1(B) = \sigma_2(B') = E,\ \sigma_2 \circ \sigma_1(C) = C''. \tag{3}$$

In the third step let σ_3 be the reflection across the perpendicular bisector of $\bar{C''F}$ provided that $C'' \neq F$; otherwise σ_3 is the identity. Since

$$d(D,F) = d(A,C) = d(A'',C'') = d(D,C''),$$

the point D is on the axis of reflection. In a similar way

$$d(E,F) = d(B,C) = d(B'',C'') = d(E,C''),$$

i.e. the point E is on the axis of reflection. Therefore

$$A''' = \sigma_3 \circ \sigma_2 \circ \sigma_1(A) = \sigma_3(A'') = \sigma_3(D) = D,\ B''' = \sigma_3 \circ \sigma_2 \circ \sigma_1(B) = \sigma_3(B'') = \sigma_3(E) = E,$$

$C''' = \sigma_3 \circ \sigma_2 \circ \sigma_1(C) = \sigma_3(C'') = F$. $\square$

Corollary 1 *An isometry is uniquely determined by the images of three non-collinear points.*

Corollary 2 *An isometry can be written as the composition of at most three reflections across lines.*

Figure 5. shows the essentially distinct relative positions of the axis of reflections.

Definition 5 *A translation is the composition of reflections across two parallel lines. The composition of reflections across intersecting lines is called a rotation. The intersection point of the reflection axes is the centre of the rotation. The first/second line we are reflecting across is called the interior/exterior axis.*

According to the definition of reflections across lines, a translation $\tau\colon S \to S$ keeps the distance of X and $\tau(X)$ constant for any $X \in S$. It is exactly twice the width between the axes. Furthermore, both axes are perpendicular to the line of X and $\tau(X)$ independently of the choice of X. The line of X and $\tau(X)$ represents the so-called *direction* of the translation.

Theorem 10 (three reflections theorem I) *The composition of reflections across three parallel lines is a reflection across a single line parallel to the others.*

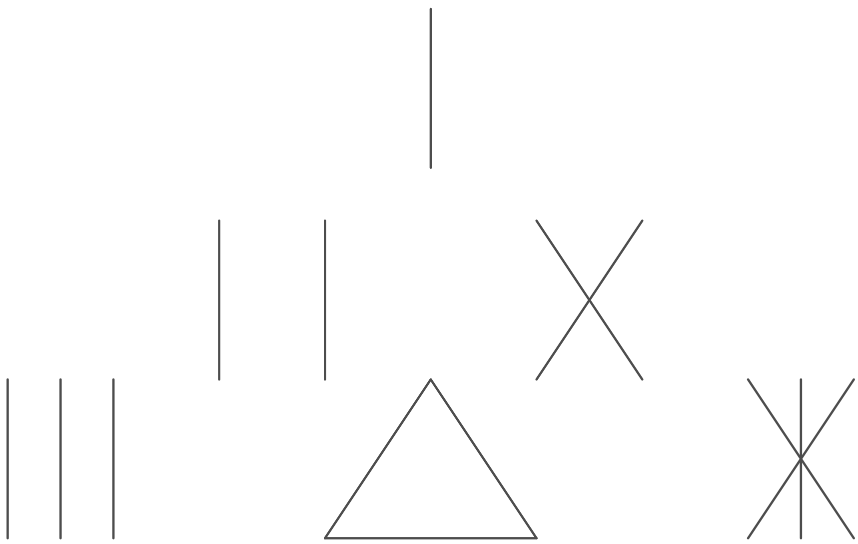

Figure 5: Essentially different relative positions of reflection axes

Proof. Let $\sigma := \sigma_3 \circ \sigma_2 \circ \sigma_1$ be the composition of reflections across three parallel lines and consider a line l such that it is perpendicular to all of them. The intersection points are denoted by A_1, A_2 and A_3, respectively. It is clear that the image of any point $X \in S$ is uniquely determined by the orthogonal projection of X onto the line l. Let $f: l \to \mathbb{R}$ be a ruler and suppose that $f(A_1) = a_1$, $f(A_2) = a_2$, $f(A_3) = a_3$. If x is the coordinate of the orthogonal projection of X onto the line l then

$$\frac{x + x_1}{2} = a_1 \;\Rightarrow\; x_1 = 2a_1 - x$$

under the reflection σ_1. A simple iteration shows that $x_3 = 2(a_3 - a_2 + a_1) - x$, i.e. $\sigma(X)$ is the image of X under the reflection across the perpendicular line to l at the point of coordinate $a_3 - a_2 + a_1$. $\square$

Corollary 3 (free choice of axis I) *To present a translation as the composition of two reflections, the interior/exterior axis perpendicular to the direction of the translation can be arbitrarily chosen but the corresponding exterior/interior axis is uniquely determined.*

Proof. Let $\tau = \sigma_2 \circ \sigma_1$ be a translation and suppose that $\tilde{\sigma}_1$ is a reflection across a line perpendicular to the direction of τ. By the three reflections theorem I, $\sigma := \sigma_2 \circ \sigma_1 \circ \tilde{\sigma}_1 = \tilde{\sigma}_2$ is a single reflection and $\tau = \sigma_2 \circ \sigma_1 = \tilde{\sigma}_2 \circ \tilde{\sigma}_1$. $\square$

According to the definition of reflections across lines, a rotation $\rho: S \to S$ keeps the measure of the central angle $XOX'\angle$ constant for any $X \in S$. It is exactly twice the measure of the angle between the axes. It is called the angle of the rotation. The direction of a rotation can be changed by changing the order of the axes.

Theorem 11 (three reflections theorems II) *The composition of reflections across three concurrent lines is a reflection across a single line through the common point.*

Proof. Let $\sigma := \sigma_3 \circ \sigma_2 \circ \sigma_1$ be the composition of reflections across three concircular lines. By the help of a circle around O let us take the points A_1, A_2 and A_3 together with the corresponding antipodal points, where the axes of reflections intersect the circle at. It is clear that the image of any point $X \in S \setminus \{O\}$ is uniquely determined by the projection onto the circle. Using a protractor let the measures be

$$m(A_1) = a_1 \pmod{\pi},\ m(A_2) = a_2 \pmod{\pi},\ m(A_3) = a_3 \pmod{\pi}.$$

If $x \pmod{2\pi}$ is the coordinate of the projection of X to the circle then

$$\frac{x + x_1}{2} \equiv a_1 \pmod{\pi} \ \Rightarrow\ x_1 \equiv 2a_1 - x \pmod{2\pi}.$$

A simple iteration shows that

$$x_3 \equiv 2(a_3 - a_2 + a_1) - x \pmod{2\pi} \ \Rightarrow\ \frac{x + x_3}{2} \equiv a_3 - a_2 + a_1 \pmod{\pi},$$

i.e. $\sigma(X)$ is the image of X under the reflection across the line determined by the antipodal points on the perimeter of the circle with coordinate $a_3 - a_2 + a_1 \pmod{\pi}$. $\square$

Corollary 4 (free choice of axis II) *To present a rotation as the composition of two reflections, the interior/exterior axis through the center of the rotation can be arbitrarily chosen but the corresponding exterior/interior axis is uniquely determined.*

Proof. Let $\rho = \sigma_2 \circ \sigma_1$ be a rotation and suppose that $\tilde{\sigma}_1$ is a reflection across a line through the center of ρ. By the three reflections theorem II, $\sigma := \sigma_2 \circ \sigma_1 \circ \tilde{\sigma}_1 = \tilde{\sigma}_2$ is a single reflection and $\rho = \sigma_2 \circ \sigma_1 = \tilde{\sigma}_2 \circ \tilde{\sigma}_1$. $\square$

Definition 6 *The composition of a reflection and a translation such that the direction of the translation is parallel to the axis of the reflection is a glide reflection.*

Remark 3 A glide reflection can be considered as a reflection followed by a translation or a translation followed by a reflection because the direction of the translation is parallel to the axis of the reflection.

Theorem 12 (classification of Euclidean plane isometries) *A Euclidean plane isometry is one of the following types: identity, reflection across a line, translation, rotation or glide reflection.*

Proof. It is enough to prove that if we have three axes of reflections in general position (not parallel or concurrent lines) then the composition $\sigma := \sigma_3 \circ \sigma_2 \circ \sigma_1$ is a glide reflection. Without loss of generality we can suppose that the interior axes l_1 and l_2 intersect each other (otherwise repeat the following process with the inverse of σ). Let O_{12} be the intersection point and present the rotation $\rho = \sigma_2 \circ \sigma_1$ by using the perpendicular line to l_3 through O_{12} instead of l_2 as the exterior axis. Roughly speaking we have to rotate the intersecting lines l_1 and l_2 about the intersection point up to the perpendicularity of the second line to the third one. Such a rigid motion of the axes does not change the center, the angle or the direction of the rotation:

$$\sigma = \sigma_3 \circ \sigma_2 \circ \sigma_1 = \sigma_3 \circ \sigma_{2'} \circ \sigma_{1'},$$

where l_2' is perpendicular to l_3. Let O_{23}' be the intersection point of the perpendicular lines and present the half-turn $\rho' = \sigma_3 \circ \sigma_{2'}$ by using the perpendicular line to l_1' through O_{23}' instead of l_3 as the exterior axis. Roughly speaking we have to rotate the intersecting lines l_3 and l_2' about the intersection point up to the perpendicularity of the third line to l_1'. Such a rigid motion of the axes does not change the center, the angle or the direction of the rotation:

$$\sigma = \sigma_3 \circ \sigma_2 \circ \sigma_1 = \sigma_3 \circ \sigma_{2'} \circ \sigma_{1'} = \sigma_{3'} \circ \sigma_{2''} \circ \sigma_{1'},$$

where l_3' is perpendicular to both l_1' and l_2'' (Figure 6.). $\square$

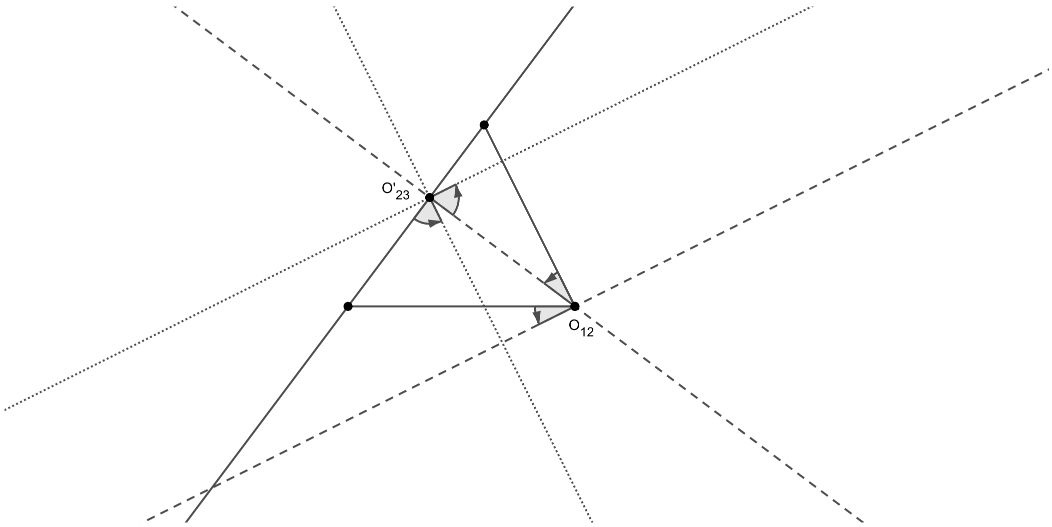

Figure 6: Classification of Euclidean plane isometries

2.2 Euclidean space isometries: a survey

Definition 7 *A distance preserving one-to-one transformation* $\sigma\colon \mathbb{E} \to \mathbb{E}, A \to A' := \sigma(A)$ *of the Euclidean space is called isometry, i.e.* $d(A', B') = d(A, B)$ *for any* $A, B \in \mathbb{E}$.

Some properties of isometries:

- they form a group with respect to the composition of mappings.

Using the triangle inequality

- $A - B - C$ if and only if $A' - B' - C'$.

The congruence theorem (SSS) implies that

- $m(ABC\angle) = m(A'B'C'\angle)$.

Therefore an isometry sends any line/plane into a line/plane. The angles are also preserved and the images of parallel lines/planes under an isometry are parallel.

Definition 8 *The reflection across the the plane* $S \subset \mathbb{E}$ *is a transformation* $\sigma_S\colon \mathbb{E} \to \mathbb{E}, X \to X' := \sigma_S(X)$ *defined by the following properties:*

- $X' = X$ *for any* $X \in S$
- *if* $X \notin S$, *then* S *is the perpendicular bisector of* $\bar{XX'}$.

Theorem 13 *Reflections across planes are isometries.*

Proof. Except the trivial case of A and $B \in S$, distance preserving goes back to the analogue property of reflections across the line $l := S \cap S_{ABA'B'}$ because A, B and the images A', B' are coplanar points. $\square$

Imitating the steps of the previous section it can be proved that any space isometry is the composition of at most four reflections across planes.

Definition 9 *A translation is the composition of reflections across two parallel planes. The composition of reflections across intersecting planes is called a rotation. The intersection of the reflection planes is the axis of the rotation. The first/second plane we are reflecting across is called the interior/exterior plane.*

According to the definition of reflections across planes, a translation $\tau\colon \mathbb{E} \to \mathbb{E}$ keeps the distance of X and $\tau(X)$ constant for any $X \in \mathbb{E}$. It is exactly twice the width between the reflection planes. Furthermore, both reflection planes are perpendicular to the line of X and $\tau(X)$ independently of the choice of X. The line of X and $\tau(X)$ represents the so-called *direction* of the translation.

Theorem 14 (three reflections theorem I) *The composition of reflections across three parallel planes is a reflection across a single plane parallel to the others.*

Proof. Let $\sigma := \sigma_3 \circ \sigma_2 \circ \sigma_1$ be the composition of reflections across three parallel planes and consider a line l such that it is perpendicular to all of them. The intersection point is denoted by A_1, A_2 and A_3, respectively. It is clear that the image of any point $X \in \mathbb{E}$ is uniquely determined by the orthogonal projection of X onto the line l. Let $f\colon l \to \mathbb{R}$ be a ruler and suppose that $f(A_1) = a_1$, $f(A_2) = a_2$, $f(A_3) = a_3$. If x is the coordinate of the orthogonal projection of X onto the line l then

$$\frac{x + x_1}{2} = a_1 \;\Rightarrow\; x_1 = 2a_1 - x$$

under the reflection σ_1. A simple iteration shows that $x_3 = 2(a_3 - a_2 + a_1) - x$, i.e. $\sigma(X)$ is the image of X under the reflection across the perpendicular plane to l at the point of coordinate $a_3 - a_2 + a_1$. $\square$

Corollary 5 (free choice of reflection plane I) *To present a translation as the composition of two reflections, the interior/exterior reflection plane perpendicular to the direction of the translation can be arbitrarily chosen but the corresponding exterior/interior reflection plane is uniquely determined.*

Proof. Let $\tau = \sigma_2 \circ \sigma_1$ be a translation and suppose that $\tilde{\sigma}_1$ is a reflection across a plane perpendicular to the direction of τ. By the three reflections theorem I, $\sigma := \sigma_2 \circ \sigma_1 \circ \tilde{\sigma}_1 = \tilde{\sigma}_2$ is a single reflection and $\tau = \sigma_2 \circ \sigma_1 = \tilde{\sigma}_2 \circ \tilde{\sigma}_1$. $\square$

According to the definition of reflections across planes, a rotation $\rho\colon \mathbb{E} \to \mathbb{E}$ keeps the measure of the central angle $XOX'\angle$ constant for any $X \in \mathbb{E}$, where O is the orthogonal projection of X to the rotational axis. It is exactly twice the measure of the angle between the planes. It is called the angle of the rotation. The direction of a rotation can be changed by changing the order of the reflection planes.

Theorem 15 (three reflections theorems II) *The composition of reflections across three planes passing through a common line is a reflection across a single plane through the common line.*

Proof. Let $\sigma := \sigma_3 \circ \sigma_2 \circ \sigma_1$ be the composition of reflections across three planes S_1, S_2 and S_3 through the common line l. If σ is restricted to a plane S perpendicular to l then it is the composition of reflections across three concurrent lines $l_1 = S \cap S_1$, $l_2 = S \cap S_2$ and $l_3 = S \cap S_3$. Applying the three reflections theorem II in the plane S we have that σ reduces to a single reflection across a line m in S. The reflection across the plane spanned by l and m substitutes the composition of reflections across S_1, S_2 and S_3 in the space. $\square$

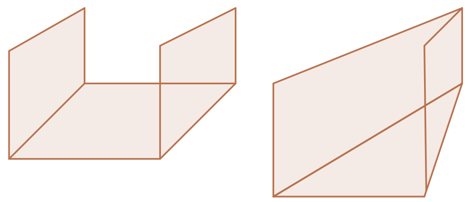

Figure 7: A glide reflection (left) and a rotoreflection (right)

Corollary 6 (free choice of axis II) *To present a rotation as the composition of two reflections, the interior/exterior reflection plane through the axis of the rotation can be arbitrarily chosen but the corresponding exterior/interior reflection plane is uniquely determined.*

Proof. Let $\rho = \sigma_2 \circ \sigma_1$ be a rotation and suppose that $\tilde{\sigma}_1$ is a reflection across a plane through the rotational axis of ρ. By the three reflections theorem II, $\sigma := \sigma_2 \circ \sigma_1 \circ \tilde{\sigma}_1 = \tilde{\sigma}_2$ is a single reflection and $\rho = \sigma_2 \circ \sigma_1 = \tilde{\sigma}_2 \circ \tilde{\sigma}_1$. $\square$

Definition 10 *The composition of a reflection and a translation such that the direction of the translation is parallel to the reflection plane is a glide reflection. The composition of a reflection and a rotation with axis perpendicular to the reflection plane is called rotoreflection.*

Remark 4 A glide reflection can be considered as a reflection followed by a translation or a translation followed by a reflection because the direction of the translation is parallel to the axis of the reflection. A rotoreflection can be considered as a reflection followed by a rotation or a rotation followed by a reflection because the rotational axis is perpendicular to the reflection plane.

Theorem 16 *The composition of three reflections across planes in general position is a glide reflection or a rotoreflection.*

Proof. The idea of the proof is the same as in 2D. Using rotations around the intersection lines we can construct an equivalent configuration of reflection planes such that two of them are perpendicular to the third one. If the planes perpendicular to the third one are parallel then we have a glide reflection. Otherwise the composition gives a rotoreflection (Figure 7.). $\square$

Definition 11 *The composition of a rotation and a translation along the rotational axis is called a rototranslation.*

Remark 5 A rototranslation be considered as a translation followed by a rotation or a rotation followed by a translation because the rotational axis is parallel to the direction of the translation.

Theorem 17 (classification of Euclidean space isometries) *A Euclidean space isometry is one of the following types: identity, reflection across a plane, translation, rotation, glide reflection, rotoreflection or rototranslation.*

2.3 The general concept of congruence

Definition 12 *Two subsets in the Euclidean plane/space are congruent if there is an isometry sending one of them to another.*

Remark 6 The fundamental theorem of plane isometries shows that the general concept of congruence is an extension of the congruence of triangles.

For the isometries of higher dimensional Euclidean spaces see [3, Chapter 11.], [6].

3 Similarities

Definition 13 *A one-to-one transformation of the Euclidean plane/space is called a similarity if* $d(A', B') = kd(A, B)$ *for any* A, B *in the plane/space, where the positive real number* $k > 0$ *is called the ratio of the similarity.*

Isometries are similarities with ratio $k = 1$.

3.1 The fixed point theorem of similarities

Theorem 18 *Any non-isometric similarity has a uniquely determined fixed point.*

Proof. The unicity is clear because two fixed points $A \neq B$ imply that

$$d(A', B') = d(A, B) = kd(A, B) \;\Rightarrow\; k = 1,$$

i.e. we have an isometry. To prove the existence suppose that $k \neq 1$ and, without loss of the generality, consider the case of $0 < k < 1$. In case of $k > 1$ we can apply the following argument to the inverse mapping with ratio $1/k$.

Choosing an arbitrary starting point A_0 let us define the sequence $A_{n+1} := A'_n$ for any natural number $n = 1, 2, \ldots$. Then we have that

$$d(A_1, A_2) = d(A'_0, A'_1) = kd(A_0, A_1),$$

$$d(A_2, A_3) = d(A'_1, A'_2) = kd(A_1, A_2) = k^2 d(A_0, A_1), \ldots$$

Using induction,

$$d(A_n, A_{n+1}) = k^n d(A_0, A_1). \tag{4}$$

By the polygonal inequality Legyen most $m > n$ és alkalmazzuk a töröttvonal-egyenlőtlenséget:

$$d(A_0, A_n) \leq d(A_0, A_1) + d(A_1, A_2) + \ldots + d(A_{n-1}, A_n) = d(A_0, A_1)\left(1 + k + k^2 + \ldots + k^{n-1}\right) =$$

$$d(A_0, A_1)\frac{1 - k^n}{1 - k} < d(A_0, A_1)\frac{1}{1 - k}.$$

Therefore the sequence A_n is bounded and we can choose a convergent subsequent A_{n_l} with limit point A. Since

$$d(A_{n_l+1}, A) \leq d(A_{n_l+1}, A_{n_l}) + d(A_{n_l}, A) \overset{(4)}{=} k^{n_l} d(A_0, A_1) + d(A_{n_l}, A)$$

and the right hand side can be arbitrarily small, it follows that the sequence A_{n_l+1} of the subsequent elements also tends to A. Therefore the subsequent element of A is equal to A because of $A' = (\lim_{l\to\infty} A_{n_l})' = \lim_{l\to\infty} A'_{n_l} = \lim_{l\to\infty} A_{n_l+1} = A$. □

3.2 Similarities in the Euclidean plane/space

Definition 14 *Let O be a given point together with the real number $\lambda \neq 0$. The central similarity $\sigma_{\lambda,O}$ is defined by the following properties: $\sigma_{\lambda,O}(O) = O$ and*

- *if $\lambda > 0$ and $X \neq O$ then $X' := \sigma_{\lambda,O}(X)$ is the uniquely determined point on $\overrightarrow{OX}$ such that $d(O,X') : d(O,X) = k$,*
- *if $\lambda < 0$ and $X \neq O$ then $X' := \sigma_{\lambda,O}(X)$ is the uniquely determined point on the complement half-line of $\overrightarrow{OX}$ such that $d(O,X') : d(O,X) = k$,*

where k is the absolute value of the so-called signed ratio λ. The point O is the center of the similarity.

Corollary 7 *If $\sigma_{\lambda,O}$ is a central similarity then $d(A',B') = kd(A,B)$, where k is the absolute value of the signed ratio λ.*

Proof. If $\lambda > 0$ then we can refer to the converse and the application of the parallel projection theorem (Theorem 3 and Theorem 4). The case of $\lambda < 0$ is also easy to discuss because the composition of the central similarity and the point reflection through O changes the sign of λ but keeps the distances unchanged because point reflections are isometries. $\square$

Using the fixed point theorem of similarities we can classify the plane similarities as follows: if we have a non-ismetric similarity φ with ratio $k \neq 1$ then it has a uniquely determined fixpoint O. It is clear that

$$\sigma := \sigma_{1/k,O} \circ \varphi$$

is an isometry (similarity with ratio 1) having a fixpoint. Therefore σ can be the identity, a reflection across a line, or a rotation and, consequemtly,

- $\varphi = \sigma_{k,O}$ is a central similarity,
- $\varphi = \sigma_{k,O} \circ \sigma_l$ is a central similarity composed with a reflection such that the reflection axis passes through the center of the central similarity (dilative reflection),
- $\varphi = \sigma_{k,O} \circ \rho$ is a central similarity composed with a rotation such that the center of the rotation is the center of the central similarity (dilative rotation).

Theorem 19 (classification of plane similarities) *A non-isometric similarity of the Euclidean plane is a central similarity, a dilative reflection or a dilative rotation.*

Central similarities with signed ratios help us to simplify the classification of space similarities. The first step is to list the space isometries with a fixed point: σ can be the identity, a reflection across a plane, a rotation or a rotoreflection. Therefore

- $\varphi = \sigma_{k,O}$ is a central similarity,
- $\varphi = \sigma_{k,O} \circ \sigma_S$ is a central similarity composed with a reflection such that the reflection plane passes through the center of the central similarity. Using a half-turn π around the line perpendicular to S at the point O, it can be easily seen that

  $$\varphi = \sigma_{k,O} \circ \sigma_S = \sigma_{-k,O} \circ \pi,$$

  i.e. we have a dilative rotation (Figure 8.),

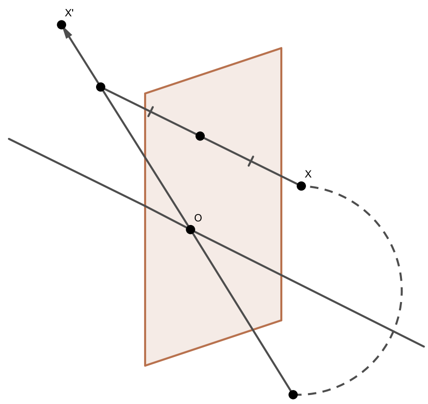

Figure 8: Classification of space similarities

- $\varphi = \sigma_{k,O} \circ \rho$ is a central similarity composed with a rotation such that the rotational axis passes through the center of the central similarity (dilative rotation).

- $\varphi = \sigma_{k,O} \circ \sigma$, where σ is a rotoreflection given by $\rho_l \circ \sigma_S$ such that the rotational axis l is perpendicular to S at the point O. As we have seen above

$$\varphi = \sigma_{k,O} \circ \sigma = \sigma_{k,O} \circ \rho_l \circ \sigma_S = \sigma_{k,O} \circ \sigma_S \circ \rho_l = \sigma_{-k,O} \circ \pi \circ \rho_l = \sigma_{-k,O} \circ \rho,$$

  where ρ is a rotation about the common rotational axis l. We have a dilative rotation again.

Theorem 20 (classification of space similarities) *A non-isometric similarity of the Euclidean space is a central similarity or a dilative rotation.*

3.3 The general concept of similarity

Definition 15 *Two subsets in the Euclidean plane/space are similar if there is a similarity sending one of them to another.*

4 Geometric measure theory

4.1 The arclength of a circle

Definition 16 *A polygonal chain inscribed in a circle means a sequence of points A_0, A_1, ..., $A_n = A_0$ on the perimeter such that*

- *A_0, A_1, ..., A_{n-1} are distinct points,*

- *$n \geq 3$ and $A'_{i-1} - A'_i - A'_{i+1}$ $(i = 2, \ldots, n-1)$ under the central projection through A_0 onto the tangent line at the diametrically opposite point to A_0.*

Lemma 1 *The set of lengths of the polygonal chains inscribed in a circle is bounded from above.*

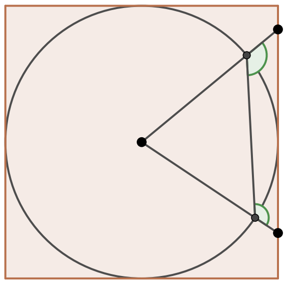

Figure 9: The projection onto the tangent square

Proof. Let a square be drawn around the circle and consider the central projection of the points of the polygonal chain onto the sides of the square through the center of the circle. By completing the polygonal chain with new points if necessary we can suppose that the projected points A'_i and A'_{i+1} are on the same side of the square. Figure 9. shows that the length of the projected segment is greater than the corresponding side of the polygonal chain. Therefore its entire length is less than the perimeter of the tangential square. □

Remark 7 Similar results can be found by using tangential regular n-gons.

As we have seen above, the perimeter of the tangential square is an upper bound of the set of lengths of the polygonal chains inscribed in a circle. The real number is called the supremum (the least upper bound) of a set in $\mathbb{R}$ if it is an upper bound and less than or equal to any upper bound of the set.

Definition 17 *The supremum of the lengths of the polygonal chains inscribed in a circle is the arclength of the circle.*

Theorem 21 *The arclength of a circle is $2r\pi$.*

Proof. By definition, the arclength K of a circle is greater than the perimeter of any cyclic regular polygon. Its side is of length $2r\sin(\alpha_n/2)$, where $\alpha_n = 2\pi/n$ is given by dividing the complete angle into n equal parts. Therefore

$$K \geq 2rn\sin(\alpha_n/2) = 2r\pi\frac{\sin(\alpha_n/2)}{(\alpha_n/2)}.$$

On the other hand, by Remark 7, K is less than any tangential regular polygon. Since its side is of length $2r\tan(\alpha_n/2)$, we have that

$$K \leq 2rn\tan(\alpha_n/2) = 2r\pi\frac{\tan(\alpha_n/2)}{(\alpha_n/2)}.$$

Taking the limit $n \to \infty$, it follows that $K = 2r\pi$ because of $\lim_{x\to 0}\frac{\sin x}{x} = 1$ and $\lim_{x\to 0}\cos(x) = 1$. □

Remark 8 Using a similar argument we have that the arclength belonging to the central angle α in a circle is $r\alpha$. Note that the angle is measured in radian.

4.2 The area of polygonal domains

In what follows we are working in a given plane of the Euclidean spce.

Definition 18 *A polygonal domain is the union of finitely many non-overlapping triangles.*

4.2.1 Axioms of area measurement of polygonal domains

Let $\mathbb{P}$ be the set of the polygonal domains in the plane. There exists a uniquely determined mapping $t: \mathbb{P} \to \mathbb{R}$ (area function) such that

(t1) $t(P) > 0$ for any $P \in \mathbb{P}$,

(t2) if P_1 and P_2 are congruent polygonal domains then $t(P_1) = t(P_2)$,

(t3) if P_1 and P_2 are non-overlapping polygonal domains, then $t(P_1 \cup P_2) = t(P_1) + t(P_2)$,

(t4) the area of a rectangle with sides of lengths a and b is ab.

The number $t(P)$ is called the area of the polygonal domain P.

Excercise 2 Using the axioms of area measurement find the formula for the area of parallelograms, triangles and trapezoid.

4.3 Jordan measure in the plane, the area of a circle

Definition 19 *The subset $K \subset S$ is called bounded if it can be covered by a polygonal domain. The outer Jordan measure of a bounded set K is defined as*

$$\overline{\mu}(K) = \inf\{t(P) \mid K \subset P \in \mathbb{P}\}.$$

The inner Jordan measure of a bounded set K is defined as

$$\underline{\mu}(K) = \sup\{t(P) \mid K \supset P \in \mathbb{P}\}.$$

If K does not contain polygonal domain, then its inner Jordan measure is zero: $\underline{\mu}(K) := 0$. The bounded set K is Jordan measurable if the inner measure is equal to the outer measure. Their common value is called the Jordan measure of the Jordan measurable set K, i.e. $\mu(K) = \underline{\mu}(K) = \overline{\mu}(K)$.

It is clear that any polygonal domain is Jordan measurable and its Jordan measure is its area. If K is Jordan measurable and L is congruent to K then L is also Jordan measurable with $\mu(L) = \mu(K)$, because any inscribed (circumscribed) polygonal domain of K is congruent to an inscribed (circumscribed) polygonal doimain of L through the isometry sending K to L. If K is Jordan measurable and L is similar to K then L is also Jordan measurable with $\mu(L) = k^2\mu(K)$, where k is the ratio of the similarity. As an example for not Jordan measurable set consider the set of points with rational coordinates in the square $[0,1] \times [0,1]$, i.e. $K := [0,1]^2 \cap \mathbb{Q}^2$. It is clear that $\overline{\mu}(K) = 1$, but $\underline{\mu}(K) = 0$. Admitting coverings with infinitely many triangles instead of the finite union in the definition of the polygonal domains we can push the outer measure of K down to be less than any positive real number $\varepsilon > 0$. The set K becomes measurable of measure zero. It is the so-called Lebesgue measure[1].

[1] C. Jordan (1838-1922), H. L. Lebesgue (1875-1941).

Theorem 22 *The area of a circle is $r^2\pi$.*

Proof. The lower bound for the inner Jordan measure can be given in terms of the area of cyclic regular polygons:

$$A_n = n\frac{r^2 \sin \alpha_n}{2} = r^2\pi\frac{\sin \alpha_n}{\alpha_n},$$

where $\alpha_n = 2\pi/n$, i.e. $\underline{\mu}(K) \geq \lim_{n\to\infty} A_n = r^2\pi$ because $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Using tangential regular polygons we have an upper bound for the outer Jordan measure:

$$\overline{\mu}(K) \leq n\frac{2r^2 \tan(\alpha_n/2)}{2} = r^2\pi\frac{\tan(\alpha_n/2)}{(\alpha_n/2)}$$

with limit $r^2\pi$. Therefore $r^2\pi \leq \underline{\mu}(K) \leq \overline{\mu}(K) \leq r^2\pi$, i.e. the circle is Jordan measurable with $\mu(K) = r^2\pi$. □

Remark 9 Using a similar argument we have that the Jordan measure belonging to the central angle α in a circle is $\frac{r^2\alpha}{2}$. Note that the angle is measured in radian.

4.4 Axioms of volume measurement, the volume of a sphere

4.4.1 Axioms of volume measurement

There exists a family $\mathbb{V}$ of bounded sets in the space as the domain of a mapping $\mu\colon \mathbb{V} \to \mathbb{R}$ (volume function) such that

(V1) $\mu(M) \geq 0$ for any $M \in \mathbb{V}$,

(V2) any bounded convex pointset is in $\mathbb{V}$,

(V3) intersection, union and difference of two sets in $\mathbb{V}$ are in $\mathbb{V}$,

(V4) μ is monotone, i.e. $\mu(M) \leq \mu(N)$ provided that M and N are in $\mathbb{V}$ and $M \subset N$,

(V5) μ is additive, i.e. $\mu(M\cup N) = \mu(M)+\mu(N)$ provided that M and N are in $\mathbb{V}$ and $\mu(M\cap N) = 0$,

(V6) the volume of a parallelepiped is the area of one of its faces times the corresponding height,

(V7) if M is in $\mathbb{V}$ and N is congruent to M, then N is in $\mathbb{V}$ and $\mu(M) = \mu(N)$.

(V8) (Cavalieri's principle) Let M and N be in $\mathbb{V}$ and suppose that they are between parallel planes. If any plane parallel to the bounding planes intersects both M and N in cross sections of equal area, then $\mu(M) = \mu(N)$, i.e. M and N are of equal volume.

The elements in $\mathbb{V}$ are called sets having volume. The volume of $M \in \mathbb{V}$ is $\mu(V)$.

Lemma 2 *If M is a bounded planar set in $\mathbb{V}$, then $\mu(M) = 0$.*

Proof. Since M is a bounded planar set we can construct a parallelepiped such that M is contained in one of its faces and the corresponding height can be arbitrarily small. It follows by (V6), that the volume of such a parallelepiped can also be arbitrarily small. Using the monotonicity of the volume function, we have that $\mu(M) = 0$. □

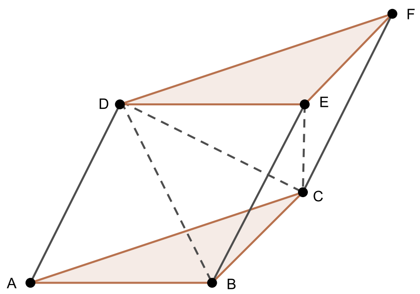

Figure 10: The volume of a pyramid

Lemma 3 *The volume of a prism is the area of the base times the height.*

Proof. Using the additivity of the volume function it is enough to prove the statement for triangular based prisms. Completing the base to a parallelogram we can complete the prism to a parallelepiped. It is the union of two congruent triangular based prisms. The additivity of the volume function and (V6) implies the statement. □

Corollary 8 *The volume of a cylinder is the area of the base times the height.*

Proof. The proof is based on Cavalieri's principle by constructing a prism such that both the base and the height are of the same measures as the measures of the base and the height of the cylinder. □

Lemma 4 *The volume of a pyramid is one-third the area of the base times the height.*

Proof. Using the additivity of the volume function it is enough to prove the statement for triangular based pyramids (tetrahedrons). First of all note that if two tetrahedrons have bases of equal area and heights of equal length then they are of equal volume. It follows by Cavalieri's principle because central similarities with respect to the opposite vertices to the bases of equal area produce cross sections of equal area in the tetrahedrons. Completing a tetrahedron to a triangular based prism as Figure 10. shows, let us apply the additivity of the volume function to finish the proof. Note that we have three tetrahedrons with pairwise common bases such that corresponding heights are of equal length. □

Corollary 9 *The volume of a cone is one-third the area of the base times the height.*

Proof. The proof is similar to the proof of Corollary 8. □

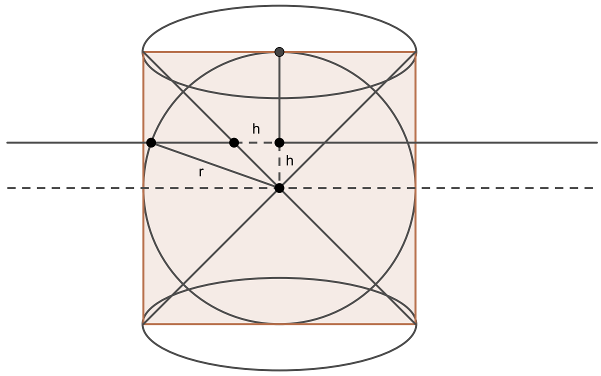

Figure 11: The volume of the sphere

4.4.2 The volume of a sphere

Theorem 23 *The volume of a sphere is* $\frac{4r^3\pi}{3}$.

Proof. Let G be a sphere. As Figure 11 shows, consider a cylinder H around the sphere. There is a double cone Kwithin the cylinder such that the vertex is at the center of the sphere. We are going to prove that G and $H \setminus K$ have cross sections of equal area parallel to the base of the cylinder. It is enough to invstigate the upper half of the cone. By Pythagorean theorem, the horizontal plane located at the heigth h above the equator intersects the sphere in a circle of area $\left(r^2 - h^2\right)\pi$. It is the same as the area of the ring that is the plane's intersection within the cylinder but outside the cone located at the same height (Figure 11.). Using Cavalieri's principle

$$V(G) = V(H \setminus K) = V(H) - V(K) = 2r^3\pi - 2\frac{r^3\pi}{3} = \frac{4r^3\pi}{3}$$

as was to be proved. $\square$

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