Foundations of absolute geometry

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Contents

1 The axioms of absolute geometry

The rest of Euclidean geometry that holds without the parallel postulate is called absolute geometry.

1.1 Incidence axioms, Incidence structures of Hilbert types

We have the following undefinied terms: points, lines and planes. The sets of points, lines and planes are denoted by $\mathbb{E}$, $\mathbb{L}$ and $\mathbb{P}$, respectively. We also have a so-called incidence relation between points and lines or planes. A line is incident with a plane if all points that are incident with the line are incident with the plane. If the incidence of points with lines or planes satisfies the following axioms (incidence axioms) then $(\mathbb{E}, \mathbb{L}, \mathbb{P})$ is called an incidence structure (space) of Hilbert type:

(I1) any two points are incident with exactly one line

(I2) any line is incident with at least two points

(I3) there are three points such that they are not incident with a line (non-collinear points)

(I4) any three non-collinear points are incident with exactly one plane

(I5) any plane is incident with at least one point

(I6) if two points are incident with a plane then the (uniquely determined) line that is incident with the points is incident with the plane

(I7) if two planes are incident with a point then there is another point that is incident with the planes

(I8) there are four points such that they are not incident with a plane (non-coplanar points)

The pair $(\mathbb{E}, \mathbb{L})$ is called an incidence plane of Hilbert type if axioms (I1)-(I3) are satisfied. To simplify the context we identify each line or plane with the set of points that are incident with them. Therefore the standard terminology and notations of the set theory can be used: intersection, union, empty set ... It is also usual to say that a point is lying on a line or a plane, a line (or a plane) contains a point ...

Proposition 1 *The intersection of two not disjoint planes is a line.*

Proof. We can refer to the following axioms:
I7. If there is a common point of two planes then there is another common point of the planes
I1. There is exactly one line passing through two points
I6. If two points of a line are lying on a plane then the line is lying on the plane
Therefore we have a line in the intersection of two not disjoint planes. Further points in the intersection contradict to
I4. There is exactly one plane passing through three non-collinear points. □

Definition 1 (relative positions of lines) *Two lines intersect each other if there is a common point of the lines but they are not coincident. Two lines are skew if they are not in the same plane. Two lines are parallel if they are in the same plane without intersection or they are coincident.*

Excercise 1 Prove that parallelism is the logical complement of intersection or skew.

Definition 2 (relative positions of planes) *Two planes intersect each other if there is a common point of the planes but they are not coincident. Two planes are parallel if they have no common points or they are coincident.*

Remark 1 In the sense of Proposition 1, the intersection of two intersecting planes is a line.

Definition 3 (relative positions of lines and planes) *A line and a plane intersect each other if there is a common point of them but the plane does not contain the line. They are parallel if they have no common points or the plane contains the line.*

Proposition 2 *A plane in an incidence space is an incidence plane.*

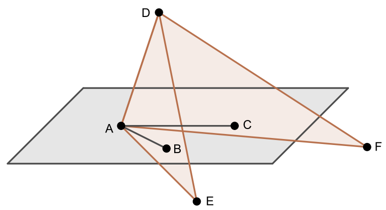

Figure 1: The proof of Proposition 2.

Proof. Let S be a plane in an incidence space. Axiom (I1) is satisfied in S because of (I6). The plane S automatically satisfies (I2). What about (I3)? To satisfy (I3) we have to construct three non-collinear points A, B and $C \in S$. The construction is started with (I5) (see Figure 1.): let $A \in S$ be a point in S. In the sense of (I8) there must be a point $D \notin S$ in the space. Consider the uniquely determined line l_{AD} passing through the points A and D (see axiom (I1)). Using (I3) we can find a point $E \notin l_{AD}$ in the space. Taking the uniquely determined plane S_{ADE} (see axiom (I4)), Proposition 1 implies that the intersection of the planes S and S_{ADE} is a line e. In the sense of (I2) we can choose a point $B \in e \subset S$ such that $A \neq B$. Using (I8) again there must be a point $F \notin S_{ADE}$ in the space. In a similar way as above, the intersection of the planes S and S_{ADF} is a line f and $C \in f$ such that $A \neq C$. To finish the proof we clarify that A, B and C are non-collinear points. First of all note that $S_{ADE} = S_{ABD}$ and $S_{ADF} = S_{ACD}$. If A, B and C are collinear points then we have that

$$S_{ADE} = S_{ABD} = S_{ACD} = S_{ADF}.$$

It is a contradiction because of the choice of the point F. $\square$

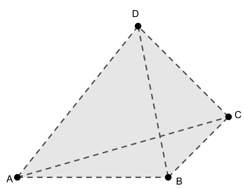

Figure 2: The four-point model.

Example 1 (the four-point model, Figure 2.) This model is minimal in the sense of (I8):

$$\mathbb{E} = \{A, B, C, D\},\ \mathbb{L} = \{\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\},$$

$$\mathbb{P} = \{\{A, B, C\}, \{A, B, D\}, \{A, C, D\}, \{B, C, D\}\},$$

i.e. the lines are the subsets containing exactly two elements of $\mathbb{E}$ and the planes are the subsets containing exactly three elements of $\mathbb{E}$. The three-point model of the incidence plane is the plane of the four-point model of the incidence space because of the previous proposition.

Excercise 2 Discuss the relative positions of the lines and (or) the planes in the four-point model.

1.2 Birkhoff's ruler axiom (the abstract description of distance measurement)

An incidence space satisfies Birkhoff's **ruler axiom** (RP, ruler postulate) if for any line l there is a one-to-one correspondence $f\colon l \to \mathbb{R}$ such that

$$|f(A) - f(B)| = d(A, B) \quad (A, B \in l),$$

where $d(A, B)$ is the distance between the points A and B (the distance is an undefinied term). The mapping f is called a coordinatization or a ruler of the line l.

Remark 2 Observe that the axioms of the real numbers are implicitly included into the formulation of Birkhoff's ruler axiom. The axiom states that any line can be mapped isometrically onto the real line. Therefore there is no need to define the distance between the points of the space. It is an undefinied term.

Ruler's comparison Let $f_1\colon l \to \mathbb{R}$ and $f_2\colon l \to \mathbb{R}$ be the rulers of the same line. Since they are distance-preserving mappings, it follows that $h\colon \mathbb{R} \to \mathbb{R}$, $h(x) := f_2 \circ f_1^{-1}(x)$ is also a distance-preserving mapping of the real line, i.e.

$$|h(x) - h(y)| = |x - y| \quad (x, y \in \mathbb{R}).$$

Suppose that $h(0) = 0$. Taking the square of both sides,

$$h^2(x) - 2h(x)h(y) + h^2(y) = x^2 - 2xy + y^2,$$

where $h^2(x) = x^2$ and $h^2(y) = y^2$ because of

$$|h(x) - h(0)| = |x - 0| = |x| \quad (x \in \mathbb{R})$$

and $h(0) = 0$. Therefore

$$h(x)h(y) = xy \quad (x, y \in \mathbb{R}).$$

Substituting $x = y = 1$, it follows that $h(1) = \pm 1$. Taking x as a free parametre, the substitution $y = 1$ gives that $h(x) = \varepsilon x$, where $\varepsilon = \pm 1$. The case of $h(0) \neq 0$ is easy to discuss by applying the previous argument to the mapping $g(x) := h(x) - h(0)$ instead of the original one. To sum up, the general form of distance-preserving mappings of the real line is $h(x) = \varepsilon x + h(0)$, where $\varepsilon = \pm 1$. Finally, we have the ruler's comparison formula

$$f_2(A) = \varepsilon f_1(A) + c \quad (A \in l), \tag{1}$$

where $c := f_2 \circ f_1^{-1}(0)$ is a constant: a ruler can be translated ($c \neq 0$) or reflected about the origin ($\varepsilon = -1$). Formula (1) allows us to pull the ordering among the reals back for the points of the line the ruler acting on. It is independent of the choice of the ruler and we can define segments and half-lines corresponding to the intervals of the real line.

Definition 4 *Let A, B and X be three collinear points. The point X is between A and B if the coordinate $f(X)$ is between $f(A)$ and $f(B)$ for a ruler $f\colon l \to \mathbb{R}$ of the common line of the points. The ordering is denoted by $A - X - B$.*

Excercise 3 Let A, B and X be three collinear points. Prove that X is between the points A and B if and only if $d(A,X) + d(X,B) = d(A,B)$.

Definition 5 *The segment with endpoints A and B is defined as*

$$\overline{AB} = \{X \in l_{AB} \mid A - X - B\} \cup \{A, B\}.$$

Its interior is $\overline{AB} \setminus \{A, B\}$. The length of the segment is $d(A,B)$. Line segments are congruent if they have the same length.

Definition 6 *The half-line starting from A and going through B is defined as*

$$\overrightarrow{AB} = \{X \in l_{AB} \mid \neg(X - A - B)\} \cup \{A\}.$$

Its interior is $\overrightarrow{AB} \setminus \{A\}$.

Theorem 1 (half-line coordinatization theorem) *For any half-line $\overrightarrow{AB}$ there is a uniquely determined ruler $f\colon l \to \mathbb{R}$ of the supporting line such that*

$$f(A) = 0 \quad \text{and} \quad \overrightarrow{AB} = \{X \in l \mid f(X) \geq 0\}.$$

Proof. Using translation and reflection about the origin if necessary, we can suppose that $f(A) = 0$ and $f(B) > 0$. Therefore $X - A - B$ if and only if the coordinate $f(A) = 0$ is between $f(X)$ and $f(B)$, i.e. $f(X) < 0$. Its logical complement is $f(X) \geq 0$. Further translations and/or reflections contradict to the conditions $f(A) = 0$ and/or $f(B) > 0$. This means that the ruler is uniquely determined. □

Theorem 2 (segment construction theorem) *The congruent copy of a segment can be given at any point into any direction.*

Proof. Consider the segment $\overline{PQ}$ of length $r \geq 0$. Let a half-line $\overrightarrow{AB}$ be given by the uniquely determined ruler $f\colon l \to \mathbb{R}$ of the supporting line such that $f(A) = 0$ and $f(B) > 0$ (see the half-line coordinatization theorem). To find the congruent copy of $\overline{PQ}$ along the half-line $\overrightarrow{AB}$ at the point A we have to solve equation $d(A,X) = r$, where $X \in \overrightarrow{AB}$. It can be written into the form

$$r = |f(X) - f(A)| = |f(X)| = f(X),$$

i.e. the uniquely determined solution is $X = f^{-1}(r)$. □

The terms "segment" and "half-line" allow us to introduce the terms "polygon" and "angle" (the union of two half-lines with the same starting point):

$$AOB\angle := \overrightarrow{OA} \cup \overrightarrow{OB}\ .$$

Moreover we can introduce the convexity of subsets in the space.

Definition 7 *A pointset $H \subset \mathbb{E}$ is convex if for any two, not necessarily distinct points A and B in H, the segment $\overline{AB}$ is contained in H.*

The following axiom supports angle measurement as we shall see in section 1.4.

1.3 Plane separation axiom

Let S be a plane in an incidence space satisfying (RP). **Plane separation axiom** (PSP, plane separation postulate) states that any line $l \subset S$ divides the plane into uniquely determined non-empty, disjoint convex subsets H_1 and H_2 such that

(PSP1) $S \setminus l = H_1 \cup H_2$

(PSP2) if $A \in H_1$ and $B \in H_2$ then $\overline{AB} \cap l \neq \emptyset$.

H_1 and H_2 are called open half-planes bounded by l. Taking the union of the open half-planes with the common bounding line we have the so-called closed half-spaces.

Theorem 3 (Pasch theorem) *If a line l in the plane of the triangle $ABC\triangle$ does not meet any vertices but l intersects one of the sides then l intersects exactly one of the other sides of the triangle.*

Proof. Suppose (for example) that l intersects the side $\overline{AB}$ at the interior point X. It is clear that A and B must be in distinct open half-planes bounded by l (it follows by the convexity of the open half-planes). In the sense of (PSP1), the vertex C must be in the opposite open half-plane to some but exactly one of them. If A and C (or B and C) are in opposite open half-planes then, by (PSP2), l intersects the side $\overline{AC}$ (or $\overline{BC}$) but exactly one of them. □

Definition 8 *Let $AOB\angle$ be a non-degenerate angle, i.e. not a line or a half-line. In the plane of $AOB\angle$ we have a closed half-plane containing B and bounded by l_{OA} and another one containing A and bounded by l_{OB}. The intersection of these closed half-planes is called the convex domain of the angle. Its interior is the intersection of the corresponding open half-planes. If $AOB\angle$ degenerates to a line l then the convex domain means an arbitrary half-plane bounded by l.*

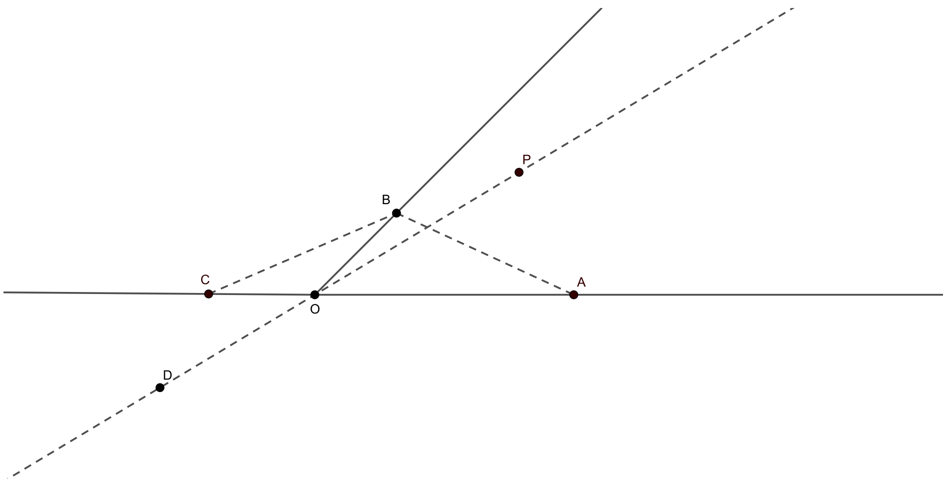

Figure 3: Crossbar theorem.

Theorem 4 (crossbar theorem) *Let $AOB\angle$ be a non-degenerate angle. The half-line $\overrightarrow{OP}$ intersects the interior of the cross-segment $\overline{AB}$ if and only if P is in the interior of the convex domain of the angle.*

Proof. Suppose that P is in the interior of the convex domain of the angle and consider two auxiliary points C and D such that $C - O - A$ and $D - O - P$. Applying Pasch theorem to the triangle $ABC\triangle$ we have that l_{OP} intersects the side $\overline{AB}$ or $\overline{BC}$ but exactly one of them at an interior point. Obsreve that the half-line $\overrightarrow{OD}$ is strictly spearated from the interiors of the segments by the line l_{OA}. Therefore the opposite half-line $\overrightarrow{OP}$ intersects the side $\overline{AB}$ or $\overline{BC}$ but exactly one of them at an interior point. Since the interior of the segment $\overline{BC}$ is strictly separated from the interior of $\overrightarrow{OP}$ by the line l_{OB}, it follows that $\overrightarrow{OP}$ intersects the cross-segment $\overline{AB}$ at an interior point. Conversely, if the half-line $\overrightarrow{OP}$ intersects the cross-segment $\overline{AB}$ at an interior point X then P and X must be in the same open half-planes with respect to both supporting lines of the arms of the angle, respectively. Therefore P is in the interior of the convex domain of the angle. □

Definition 9 *An incidence space satisfying the ruler and the plane separation axioms is called a continuously ordered incidence space*

Excercise 4 Prove the space separation property: in a continuously ordered incidence space, any plane S divides the space into uniquely determined non-empty, disjoint convex subsets K_1 and K_2 such that

(SSP1) $\mathbb{E} \setminus S = K_1 \cup K_2$

(SSP2) if $A \in K_1$ and $B \in K_2$, then $\overline{AB} \cap S \neq \emptyset$.

Hint. Taking a point $A \notin S$ let us define the "half-spaces" as follows:

$$K_1 := \{P \in \mathbb{E} \setminus S \mid \overline{AP} \cap S = \emptyset\} \cup \{A\},\ K_2 := \{P \in \mathbb{E} \setminus S \mid \overline{AP} \cap S \neq \emptyset\}.$$

1.4 Protractor axiom (the abstract descripiton of angle measurement)

A continuously ordered incidence space satisfies the **protractor axiom** (PP, protractor postulate), if the angle measurement is additive, i.e.

(PP1) if $\overrightarrow{OA} \neq \overrightarrow{OB}$ and P is in the convex domain of the angle $AOB\angle$ then

$$m(AOB\angle) = m(AOP\angle) + m(POB\angle),$$

where $m(AOB\angle) \in [0, \pi]$ is the measure of $AOB\angle$

and the following angle construction axiom[1] holds:

(PP2) if H is a closed half-plane bounded by the supporting line of $\overrightarrow{OA}$ then for any $t \in [0, \pi]$ there is a uniquely determined half-line $\overrightarrow{OB} \subset H$ such that $m(AOB\angle) = t$.

For the protractor axiom see Figure 4.: PP1 (left), PP2 (right).

[1] Although the angle construction axiom is the analogue of the segment construction theorem in some sense recall that the angle construction is formulated as an axiom assumed to be true but the segment construction is formulated as a theorem proved to be true. It is an essential difference.

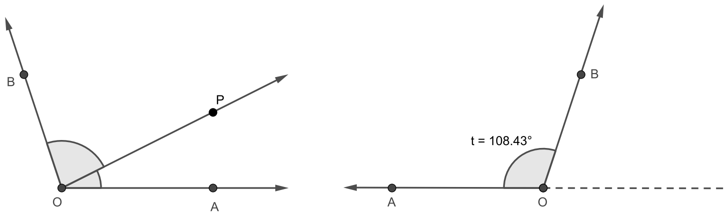

Figure 4: Protractor axiom

Definition 10 *Two angles of the same measure are called congruent.*

Using (PP) we have that

- if an angle degenerates to a line then it is of mesure π,
- if an angle degenerates to a half-line then it is of measure 0,
- vertically opposite angles are congruent, the sum of linear pair angles[2] is π.

We can easily introduce the following concepts: right angle, perpendicular bisector of a segment, bisector of an angle...

The copy of triangles in the space is based on the following ruler and compass (protractor) construction: using the notations in Figure 5, $\overline{AB}=\overline{DE}$, $m(A\angle) = m(D\angle)$ and $\overline{AC}=\overline{DF}$ due to the segment construction theorem and the angle construction axiom. These segments and angles are related by direct measurement. What about the missing sides and angles (indirect measurement)? The answer is given by the congruence axiom (see the following subsection).

Definition 11 *Two triangles are congruent if there is a correpondence among their vertices such that all of the corresponding sides and angles are congruent.*

1.5 Congruence axiom

The **congruence axiom** (SAS, side-angle-side) states that if there is a correspondence among the vertices of two triangles such that two sides and the included angle are congruent to the corresponding sides and angle then the triangles are congruent. The congruence axiom completes the system of axioms of absolute geometry. Undifined terms: $\mathbb{E}$ (points), $\mathbb{L}$ (lines), $\mathbb{P}$ (planes), d (distance), m (measure of angles). Axioms: incidence axioms, ruler axiom (RP), plane separation axiom (PSP), protractor axiom (PP) and congruence axiom (SAS).

Definition 12 *An absolute space is an incidence space satisfying the ruler axiom, the plane separation axiom, the protractor axiom and the congruence axiom.*

[2] Vertically opposite angles are formed by pairwise opposite half-lines at the same point (the common vertex of the angles). Two adjacent angles are called linear pair angles if their non-common arms are opposite half-lines.

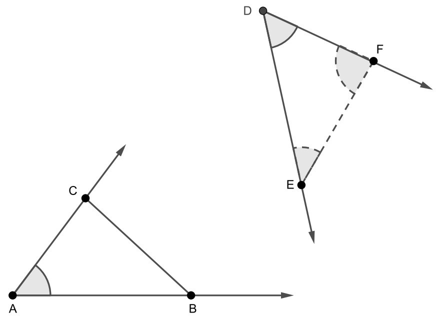

Figure 5: The copy of a triangle by ruler and compass.

2 Selected topics in absolute geometry

2.1 Congruence theorems

Theorem 5 (Pons Asinorum) *In an isosceles triangle we have congruent angles opposite to the congruent sides.*

Proof. Let $ABC\triangle$ be an isosceles triangle such that $\bar{AC}=\bar{BC}$. The crossbar theorem provides that the internal angle bisector at the vertex C intersects $\bar{AB}$ at the interior point X. Using (SAS) we have two congruent triangles $AXC\triangle$ and $BXC\triangle$ with equal angles at the vertices A and B. $\square$

Theorem 6 (exterior angle inequality, Figure 6.) *An exterior angle of a triangle is greater than any of the non-adjacent interior angles.*

Proof. Let $ABC\triangle$ be a triangle and pick the point D up as the image of B under the reflection about the midpoint F of the side $\bar{AC}$ (the vertices A and C obviously interchange). Since the vertically opposite angles are congruent it follows, by (SAS) that the triangle $DFC\triangle$ is congruent to $BFA\triangle$. Taking an auxiliary point G on the half-line $\vec{BC}$ such that $B-C-G$, we have that $ACG\angle$ is (one of) the exterior angle at vertex C. Since D is in the interior of the convex domain of the angle $ACG\angle$, it follows that $m(A\angle) = m(ACD\angle) < m(ACG\angle)$ as was to be stated. $\square$

Corollary 1 *In a right-angled/obtuse-angled triangle we have exactly one right/obtuse angle and two acute angles.*

Theorem 7 (Congruence theorems, Figure 7.) *If there is a correspondence among the vertices of two triangles such that*

(ASA, angle-side-angle) *two angles and the included side are congruent to the corresponding angles and side,*

(SAA, side-angle-angle) *two angles and one of the non-included side are congruent to the corresponding angles and side,*

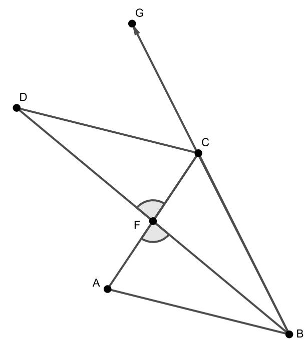

Figure 6: Exterior angle inequality.

(SSS, side-side-side) *all three sides are congruent to the corresponding sides, then the triangles are congruent.*

Proof. We prove the congruence theorems (ASA) and (SAA) simultaneously. Let $ABC\triangle$ and $DEF\triangle$ be two triangles such that $\bar{AB}{=}\bar{DE}$ and $A\angle = D\angle$. If $\bar{AC}{=}\bar{DF}$ then we are done because of (SAS). Suppose, in contrary, that $\bar{AC}$ is not congruent to $\bar{DF}$, i.e. changing the role of the triangles if necessary, $\bar{AC}{>}\bar{DF}$. In the sense of the segment construction theorem we can choose a point G in the interior of $\bar{AC}$ such that $\bar{AG}{=}\bar{DF}$. Using (SAS), the triangles $ABG\triangle$ and $DEF\triangle$ are congruent. In case of (ASA)

$$m(DEF\angle) = m(ABG\angle) < m(ABC\angle) = m(DEF\angle)$$

is a contradiction because of (PP1). In case of (SAA)

$$m(DFE\angle) = m(AGB\angle) > m(ACB\angle) = m(DFE\angle)$$

is a contradiction because of the exterior angle inequalty. Therefore $\bar{AC}{=}\bar{DF}$ and we are done because of (SAS). For the proof of (SSS) see [2] and [4]. □

Theorem 8 (the converse of Pons Asinorum) *In a triangle we have equal sides opposite to equal interior angles.*

Proof. Imitate the steps of the original proof by using the congruence therorem (SAA). □

Remark 3 There is two further possible congruence statements: (AAA), i.e. all three angles are congruent to the corresponding angles and (SSA), i.e. the congruence of two sides and one of the non-included angles to the corresponding sides and angle. There is no way to prove or disprove (AAA) in absolute gometry. It is an independent statement of its axioms: it is false in Euclidean geometry (similarity) but true in non-Euclidean (hyperbolic) geometry. The congruence statement (SSA) can be proved in absolute geometry by the additional condition for the angle to be opposite to the greater side. The proof will be presented in section 2.3 (inequalities).

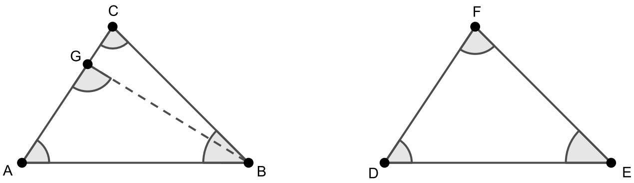

Figure 7: (ASA) and (SAA)

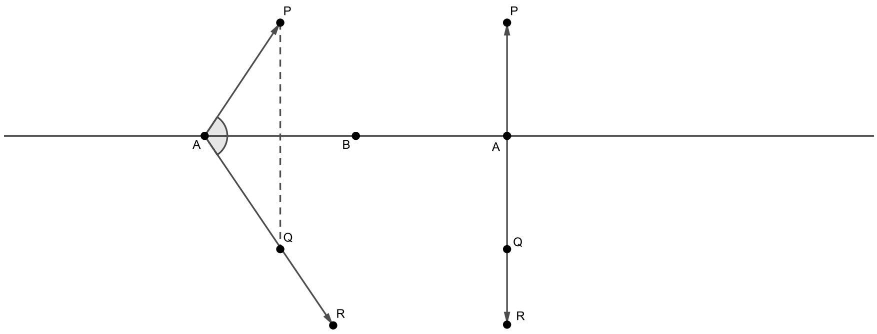

Figure 8: Existence and unicity theorem of the perpendicular line.

2.2 Perpendicular and parallel lines in the absolute plane

Theorem 9 (Existence and unicity theorem of the perpendicular line, Figure 8.) *In an absolute plane if a line $l \subset S$ and a point $P \in S$ are given then there is a uniquely determined perpendicular line to l through the point P.*

Proof. If $P \in l$, then we can refer to the angle construction axiom (P2) and the congruence of vertically opposite angles. If $P \notin l$, then the unicity of the perpendicular line follows from the exterior angle inequality. What about the existence? Consider the points A and $B \in l$; using the angle construction axiom let the half-line $\overrightarrow{AR}$ be given such that

$$m(BAP\angle) = m(BAR\angle) \tag{2}$$

but the points P and R are in opposite half-planes bounded by l. The next step is to use the segment construction theorem to choose the point Q on the half-line $\overrightarrow{AR}$ such that $\bar{AP}=\bar{AQ}$. Finally, let us join the points P and Q by a line. In the sense of (PSP2), $\bar{PQ}$ intersects l at a point T. If $T = A$,

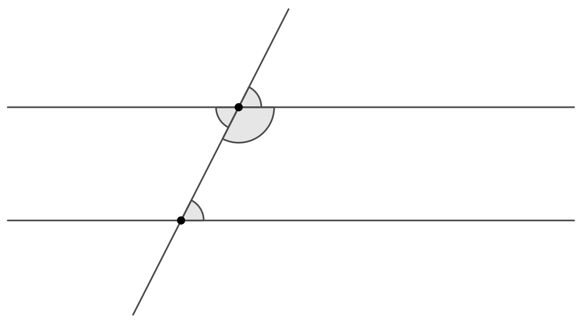

Figure 9: A sufficient condition of parallelism I.

then the angle $PAR\angle$ degenerates to a line, i.e. $\pi = m(PAR\angle) = m(BAP\angle) + m(BAR\angle)$ and (2) implies the right angle between the lines l_{PQ} and l. Otherwise the congruence of the triangles $PTA\triangle$ and $QTA\triangle$ shows that l_{PQ} is perpendicular to l because

$$\pi = m(PTQ\angle) = m(PTA\angle) + m(QTA\angle) \quad \text{and} \quad m(PTA\angle) = m(QTA\angle).$$

□

Theorem 10 (a sufficient condition of parallelism I, Figure 9.) *Consider two coplanar lines e and f together with a third one intersecting both of them. If there are equal alternate interior angles at the intersection points then e and f are parallel.*

Proof. It is a direct consequence of the exterior angle inequality. □

Theorem 11 (existence theorem of the parallel line) *If a line l and a point $P \notin l$ are given then there is a parallel line to l through the point P.*

Proof. Let S be the absolute plane determined by the given line l and the point $P \notin l$. We are going to use the existence and unicity theorem of the perpendicular line two times: let m be the line perpendicular to l through the point P and consider the line e perpendicular to m through the point P. Since the alternate interior angles are of measure $90°$, e and l are parallel because a sufficient condition of parallelism I. is satisfied. □

Remark 4 (a sufficient condition of parallelism II. Figure 9.) It is clear that the alternate interior angles are equal if and only if

- the corresponding angles are equal
- the alternate exterior angles are equal
- the consecutive interior angles add up to $180°$.

Each of them is sufficient for the parallelism but they are **not necessary**. In fact, it is obvious that if they are necessary conditions of the parallelism then we can prove not only the existence but also the **unicity** of the parallel line because of the angle construction axiom (PP2). There is no way to prove or disprove the unicity of the parallel line in absolute gometry. It is an independent statement of its axioms: it is true in Euclidean geometry but false in non-Euclidean (hyperbolic) geometry.

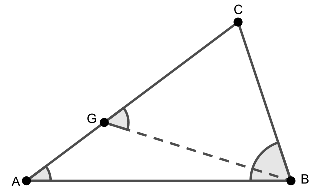

Figure 10: The proof of Theorem 12.

2.3 Inequalities

Theorem 12 *In a triangle, the angle opposite to the longer side is greater.*

Proof. Let $ABC\triangle$ be a triangle and suppose that $\overline{AC}>\overline{BC}$. If G is a point (see Figure 10.) in the interior of the side $\overline{AC}$ such that $\overline{GC}=\overline{BC}$ then we have an isosceles triangle $GBC\triangle$. Pons Asinorum together with (PP1) and the exterior angle inequality say that $m(B\angle) > m(GBC\angle) = m(BGC\angle) > m(A\angle)$ as was to be stated. □

Corollary 2 *In a triangle, the side opposite to the greater angle is longer.*

Proof. The shorter side opposite to the greater angle contradicts to the previous theorem. Equal sides contradicts to Pons Asinorum. □

Corollary 3 *In a right-angled triangle, the hypotenuse is the longest side.*

Theorem 13 (classical triangle inequality, Figure 11.) *The sum of the lengths of any two sides in a triangle is greater than the third one.*

Proof. Let $ABC\triangle$ be a triangle and consider a point G on the supporting line of the side $\overline{AB}$ such that $A-B-G$ and $\overline{BC}=\overline{BG}$. Since we have an isosceles triangle $BCG\triangle$, it follows that

$$m(AGC\angle) = m(BGC\angle) = m(BCG\angle) < m(ACG\angle).$$

Therefore, in the triangle $AGC\triangle$, the length of the side $\overline{AC}$ opposite to the less angle $AGC\angle$ is shorter then the length of the side $\overline{AG}$. It is the sum of the lengths of the sides $\overline{AB}$ and $\overline{BC}$. □

Corollary 4 *For any points A, B and C*

$$d(A,B) + d(B,C) \geq d(A,C)$$

and equality occurs if and only if $A-B-C$.

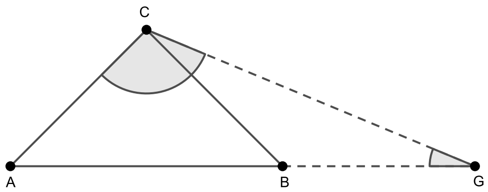

Figure 11: Classical triangle inequality.

Corollary 5 (polygonal inequality) *If* $n \geq 3$ *and* $A_1, \ldots, A_n \in \mathbb{E}$, *then*

$$d(A_1, A_2) + \ldots + d(A_{n-1}, A_n) \geq d(A_1, A_n).$$

Proof. Induction on the number of the sides. □

Theorem 14 (the congruence theorem (SsA), Figure 12.) *If there is a correspondence among the vertices of two triangles such that two sides and the non-included angle opposite to the longer one are congruent to the corresponding sides and angle then the triangles are congruent.*

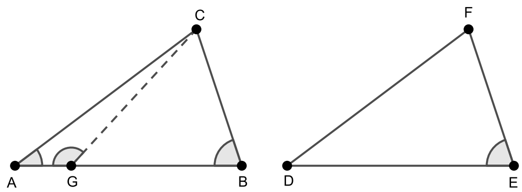

Figure 12: (SsA).

Proof. Let $ABC\triangle$ and $DEF\triangle$ be triangles such that

$$\bar{AC} = \bar{DF}, \ \bar{BC} = \bar{EF}, \ m(B\angle) = m(E\angle)$$

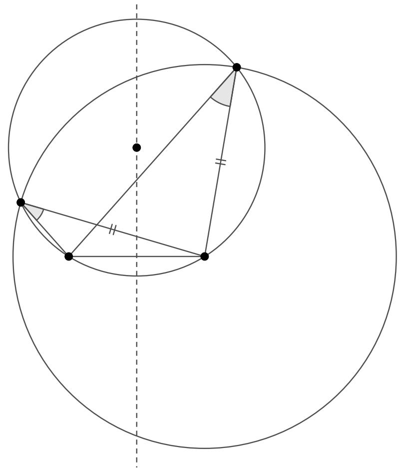

Figure 13: Non-congruent triangles having two equal corresponding sides and corresponding angles opposite to the shorter sides.

and $\overline{AC}>\overline{BC}$. Using (SAS) it is enough to prove that $\overline{AB}=\overline{DE}$. Changing the role of the triangles if necessary suppose, in contrary, that $\overline{AB}>\overline{DE}$ and let us choose a point G in the interior of the segment $\overline{AB}$ such that $\overline{GB}=\overline{DE}$. In the sense of (SAS), the triangles $GBC\triangle$ and $DEF\triangle$ are congruent. At the same time, the interior angles in the triangle $AGC\triangle$ satisfies the inequalities $m(A\angle) < m(B\angle) < m(AGC\angle)$ because of the exterior angle inequality. Therefore $\overline{AC}>\overline{GC}=\overline{DF}$ which is a contradiction. □

Remark 5 Choosing the angle opposite to the longer side is essential in (SsA) as Figure 13. shows in the context of Euclidean geometry.

Lemma 1 *In a triangle, the length of the side $\overline{AB}$ is a strictly monotone increasing function of the opposite angle $C\angle$ provided that the lengths of the sides $\overline{AC}$ and $\overline{BC}$ are constant.*

Proof. Without loss of generality we can suppose that $\overline{BC}\leq\overline{AC}$. According to Pons Asinorum, it is clear that the measure of the angle at A' in the triangle $AA'B\triangle$ (see Figure 14.) is less than the common measure of the angles on the base of the isosceles triangle $ACA'\triangle$. At the same time, the measure of the angle $A'AB\angle$ is greater. Therefore we have the inequality $\overline{AB}<\overline{A'B}$ in the triangle $AA'B\triangle$. □

Excercise 5 Using the notation in Figure 14. prove that B is in the interior of the convex domain of the angle $CA'A\angle$ and C is in the interior of the convex domain of the angle $A'AB\angle$. This completes the proof of the previous lemma.

Hint. Since A is in the convex domain of $BCA'\angle$, it follows that B and A are in the same open half-plane bounded by $l_{A'C}$. Suppose that B and C are different open half-planes bounded by $l_{AA'}$. By (PSP2), the segment $\overline{BC}$ intersects $l_{AA'}$ at the point X. Therefore

$$m(BCA\angle) = m(XCA\angle) \quad \text{and} \quad m(BCA'\angle) = m(XCA'\angle),$$

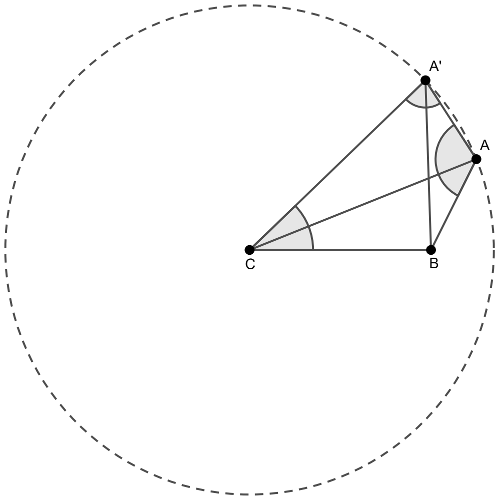

Figure 14: The proof of Lemma 1.

i.e. inequality $m(BCA\angle) < m(BCA'\angle)$ implies that $m(XCA\angle) < m(XCA'\angle)$ and, consequently, $\neg(X - A' - A)$. The collinearity of C, X and B contradicts to $A' - X - A$ as well because A and A' is on the same side of l_{BC}. We have that $A' - A - X$. Using Pons Asinorum and Corollary 2, it follows that $\overline{CX}$ is of greater length than the common length of the segments $\overline{AC}$ and $\overline{A'C}$. The same holds for the segment $\overline{CB}=\overline{CX} + \overline{XB}$ which is a contradiction because of $\overline{BC}\leq\overline{AC}$. Therefore B and C are in the same open half-plane bounded by $l_{AA'}$. This means that B is in the interior of the convex domain of the angle $CA'A\angle$. By the crossbar theorem, the half-line $\overrightarrow{A'B}$ intersects the cross-segment $\overline{AC}$ at an interior point. In other words, the half-line $\overrightarrow{AC}$ intersects the cross-segment $\overline{A'B}$ at the (same) interior point and the crossbar theorem implies that C is in the interior of the convex domain of the angle $A'AB\angle$.

Theorem 15 (Legendre theorem I., Sacchieri-Legendre theorem) *In a triangle of an absolute space the sum of the interior angles is at most* π (180°).

Proof. Suppose, in contrary, that the sum of the interior angles in the triangle $A_1B_1C_1\triangle$ is greater than π and put the congruent copies of the triangle back-to-back n-times as Figure 15. shows. The polygonal inequality says that

$$n\,\overline{A_1A_2}<\overline{A_1C_1}+(n-1)\,\overline{C_1C_2}+\overline{C_nB_n} \Rightarrow \overline{A_1A_2}-\overline{C_1C_2}<\frac{\overline{A_1C_1}-\overline{C_1C_2}+\overline{C_nB_n}}{n}.$$

Taking the limit $n \to \infty$, $\overline{A_1A_2} - \overline{C_1C_2}\leq 0$ and $\overline{A_1A_2}\leq\overline{C_1C_2}$. On the other hand, the angle δ of the gap between consecutive triangles is less than γ, i.e. $\gamma > \delta$ because $\alpha+\delta+\beta=\pi$ but $\alpha+\beta+\gamma>\pi$. By the previous lemma, $\overline{C_1C_2}<\overline{A_1A_2}$ which is a contradiction. Therefore the sum of the interior angles is at most π. □

Corollary 6 (absolute exterior angle theorem) *In a triangle, an exterior angle is greater or equal than the sum of the non-adjacent interior angles.*

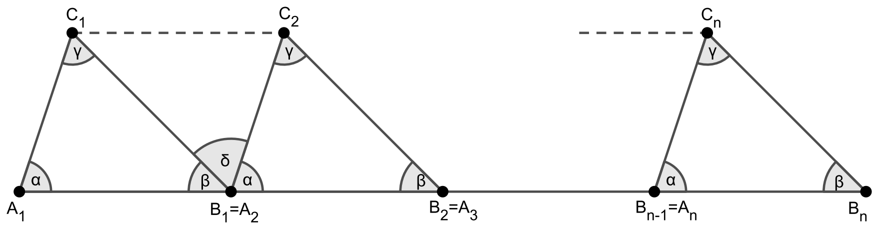

Figure 15: The proof of Legendre theorem I.

Excercise 6 Prove absolute Thales-theorem: the visibility angle of a diagonal is at most $\pi/2$ along the perimeter of a circle.

Theorem 16 (Legendre theorem II.) *In an absolute space, the sum of the interior angles is either less than π for any triangle or π for any triangle.*

Proof. For the proof we can refer to [2] and [4]. □

3 Euclidean parallel axiom and its equivalent forms

As we have seen above (existence theorem of the parallel line) there are parallel lines in absolute geometry. We have two logically complement cases:

(EPP) There is at most one line parallel to a given one through an external point (Euclidean parallel axiom).

(HPP) There are at least two lines parallel to a given one through an external point (hyperbolic parallel axiom).

Note that (HPP)=$\neg$ (EPP) because of "at least two=$\neg$ (at most one)".

Definition 13 *An absolute space satisfying Euclidean/ hyperbolic parallel axiom is called a Euclidean/hyperbolic space.*

Comparing the existence theorem of the parallel line and (EPP) we have that *in a Euclidean space, there is exactly one line parallel to a given one through an external point.*

Theorem 17 *In an absolute space the following statements are equivalent:*

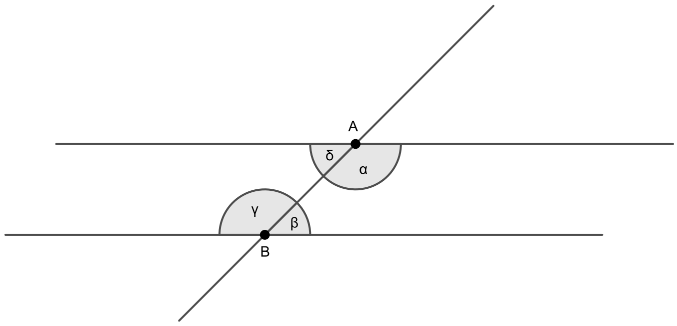

Figure 16: The proof of $(A) \Rightarrow (B)$.

(A) (Euclid's V. axiom) *Let S be a plane and consider a line $l \subset S$. Taking the points A and $B \in l$, suppose that C and D are in the same open side of l such that $m(CAB\angle) + m(DBA\angle) < \pi$. Then the half-lines $\overrightarrow{AC}$ and $\overrightarrow{BD}$ intersects each other*

(B) *The sufficient conditions of parallelism are necessary*

(E) (EPP)

(H) *The perpendicular bisectors of a triangle meet a common point*

(I) (Farkas Bolyai's theorem) *Given three non-collinear points, there is a uniquely determined circle passing through them*

(J) *In a triangle, the sum of the interior angles is π*

(K) *In a triangle, an exterior angle is the sum of the non-adjacent interior angles.*

Proof. The labelling of the statements follows [2]; as an illustration consider the implications $(A) \Rightarrow (B) \Leftrightarrow (E) \Rightarrow (A)$. Suppose that the parallel lines e and f are cut by a transversal at the points $A \in e$ and $B \in f$. Following the notations in Figure 16. let α and β, or γ and δ be the pairs of the consecutive interior angles on the same side of the transversal, respectively. We have that $\alpha + \delta = \pi$ and $\beta + \gamma = \pi$. By contraposition, (A) implies that $\alpha + \beta \geq \pi$ and $\gamma + \delta \geq \pi$ because there are no intersection points of the half-lines in the common side of the transversal. Adding these relations, it follows that $2\pi \leq \alpha + \beta + \gamma + \delta = 2\pi$, i.e. we have no strict inequalities: $\alpha + \beta = \pi$ and $\gamma + \delta = \pi$. Especially, $\alpha = \gamma$ and $\beta = \delta$. This means that we have equal alternate interior angles and we are done. Suppose that (B) holds: if the sufficient conditions of parallelism are necessary, then the parallel line is uniquely determined by any of the angles enclosed with a transversal line. The angle construction axiom (P2) provides that there are no alternative parallel lines and (EPP) follows. The converse $(E) \Rightarrow (B)$ is also clear. Finally, suppose that (EPP) holds and consider the configuration of Euclid's V. axiom (Figure 17.). Since the sufficient conditions of parallelism are necessary, it follows that l_{AC} and l_{BD} intersect each other because $m(CAB\angle) + m(DBA\angle) < \pi$ means that the sufficient conditions are taking to fail. We have four pairwise complement cases: $\overrightarrow{AC} \cap \overrightarrow{BD} \neq \emptyset$, $\overrightarrow{AC} \cap \overrightarrow{BF} \neq \emptyset$,

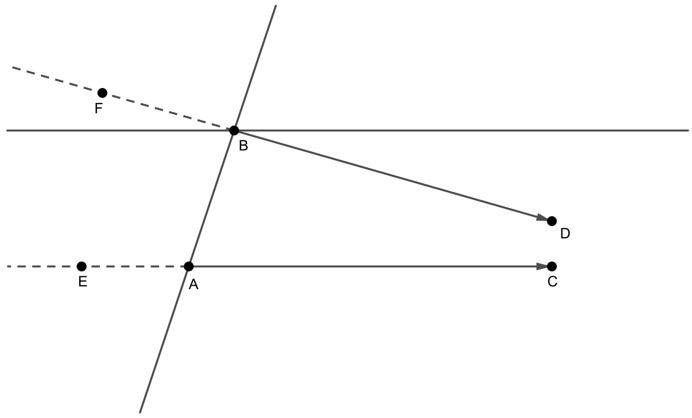

Figure 17: The proof of $(E) \Rightarrow (A)$.

$\overrightarrow{AE} \cap \overrightarrow{BF} \neq \emptyset$, $\overrightarrow{AE} \cap \overrightarrow{BD} \neq \emptyset$. The first one is the only possible choice. Otherwise the line l_{AB}, or the line parallel to l_{AC} through the point B separate the half-lines. $\square$

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[4] Szilasi József: Geometria I., KLTE TTK, Debrecen, 1990.