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Isometry classes of simply connected nilmanifolds

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Abstract

We classify the isometry equivalence classes and determine the isometry groups of connected and simply connected Riemannian nilmanifolds on filiform Lie groups of arbitrary dimension and on five dimensional nilpotent Lie groups of nilpotency class > 2. To achieve this classification we prove that up to one exceptional class the five dimensional non two-step nilmanifolds and the filiform nilmanifolds have isometry groups of the same (minimal) dimension as the nilmanifold. We give a detailed description of the exceptional case.

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1. Introduction

A connected Riemannian manifold M is said to be a *Riemannian nilmanifold* if its group of isometries contains a nilpotent Lie subgroup acting transitively on this manifold. E. Wilson proved in [10] that there is a unique nilpotent Lie subgroup N of the group of isometries acting simply transitively

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on M and hence the Riemannian nilmanifold M can be identified with the nilpotent Lie group N endowed with a left-invariant metric $\langle .,. \rangle_N$. Moreover, the group $\mathcal{I}(N)$ of all isometries of $(N, \langle .,. \rangle_N)$ is isomorphic to the semi-direct product $N \rtimes \mathcal{OA}(\mathfrak{n})$, where $\mathcal{OA}(\mathfrak{n})$ is the group of orthogonal automorphisms of the Lie algebra $\mathfrak{n}$ of N with respect to the inner product induced on $\mathfrak{n}$ by the left-invariant metric $\langle .,. \rangle_N$. It follows from this observation that the isometry equivalence classes of connected and simply connected nilmanifolds and their isometry groups can be determined by the investigation of the classes of isometrically isomorphic metric Lie algebras, i. e. Lie algebras equipped with an inner product. This procedure was applied by J. Lauret in [7] to nilpotent Lie groups of dimension 3, 4 and to a 5-dimensional two-step nilpotent group, Sz. Homolya and O. Kowalski described in [5] the isometry equivalence classes and isometry groups of all 5-dimensional simply connected two-step nilpotent Riemannian nilmanifolds. Moreover, S. Console, A. Fino, E. Samiou determined in [3] the isometry equivalence classes and isometry groups of all 6-dimensional simply connected two-step nilpotent Riemannian nilmanifolds.

Riemannian nilmanifolds of higher dimension have a rich geometry with many open questions. Among nilmanifolds the class of higher nilpotency class, particularly the filiform manifolds have a relatively rigid structure, the papers [6] of M. M. Kerr and T. L. Payne, [1], [2] of G. Cairns, A. Hinić Galić and Yu. Nikolayevsky have been devoted to the investigation of the geometry of filiform Riemannian nilmanifolds.

The aim of our paper is to investigate isometry equivalence classes and isometry groups of nilmanifolds on filiform Lie groups of arbitrary dimension and to extend the classification process of nilmanifolds to 5-dimensional nilpotent groups of nilpotency class greater than two. It turns out that the isometry groups of the investigated manifolds have minimal dimension, with the exception of one case, we give a detailed analysis of this nilmanifold.

The paper is organized as follows. In Section 2 we collect some basic definitions and notations and formulate the steps of our classification procedure. Section 3 is devoted to the study of filiform metric Lie algebras and of the isometry groups of the corresponding nilmanifolds. In Subsection 3.1 we introduce the notion of framed metric Lie algebras and show that filiform metric Lie algebras are framed, which has consequences on the structure of the connected component of the isometry group of the corresponding nilmanifolds. Subsection 3.2 deals with the classification of standard filiform metric Lie algebras and of the isometry groups of the corresponding nilmanifolds. The results are used to describe the 4- and 5-dimensional cases in detail. In Subsection 3.3 we study the non-standard filiform metric Lie algebra of smallest dimension 5. In Section 4 we complete the classification of 5-dimensional nilpotent metric Lie algebras and the corresponding isometry groups with the investigation of the two 3-step nilpotent metric Lie algebras. The such metric Lie algebras with one-dimensional center are framed metric algebras, hence we can apply in Subsection 4.1 our method of classification to this case. In contrast to the previous discussion there is a subclass of 5-dimensional 3-step nilpotent metric Lie algebras with two-dimensional center which does not have framing and hence the dimension of the isometry group of the corresponding nilmanifold is greater than 5. Subsection 4.2 is devoted to the detailed description of these metric Lie algebras and the corresponding nilmanifolds.

2. Preliminaries

In this paper we investigate on the one hand *filiform Lie algebras.* Denoting the *lower central series* of a Lie algebra $\mathfrak{n}$ by $\mathcal{C}^0\mathfrak{n} = \mathfrak{n}$ and $\mathcal{C}^{j+1}\mathfrak{n} = [\mathfrak{n}, \mathcal{C}^j\mathfrak{n}]$, $j \in \mathbb{N}$ we have the following

Definition 1. A Lie algebra $\mathfrak{n}$ is called *k-step nilpotent*, if $\mathcal{C}^k\mathfrak{n} = \{0\}$, but $\mathcal{C}^{k-1}\mathfrak{n} \neq \{0\}$ for some $k \in \mathbb{N}$.
An n-dimensional Lie algebra $\mathfrak{n}$ is called *filiform*, if it is $(n-1)$-step nilpotent. A filiform Lie algebra $\mathfrak{n}$ is *standard filiform*, if it contains a basis $\{G_1, \cdots, G_n\}$ such that the nontrivial Lie bracket relations are given by $[G_1, G_i] = G_{i+1}$, $i = 2, \dots, n-1$.

Remark 1. *For an n-dimensional filiform Lie algebra $\mathfrak{n}$ one has* $\dim(\mathcal{C}^i\mathfrak{n}) = n - i - 1$ *for* $1 \leq i \leq n-1$. *In any n-dimensional filiform Lie algebra $\mathfrak{n}$ there exists a basis $\{G_1, \cdots, G_n\}$ such that $[G_1, G_i] = G_{i+1}$, $i = 2, \dots, n-1$, (cf. M. Vergne [9], D. M. Millionschikov [8], Lemma 3.4). A general filiform Lie algebra may have more non-trivial commutation relations, the simplest examples of filiform Lie algebras are the standard filiform Lie algebras.*

On the other hand we deal with *nilpotent Lie algebras of dimension* ≤ 5 with nilpotency class > 2, which are not direct products of Lie algebras of lower dimension. According to [4], pp. 646-647, these Lie algebras are given up to isomorphism by the following non-vanishing commutators:

$$\begin{array}{lllll} \mathfrak{l}_{4,3}: & [G_1,G_2]=G_3, & [G_1,G_3]=G_4; & & \\ \mathfrak{l}_{5,5}: & [G_1,G_2]=G_4, & [G_1,G_4]=G_5, & [G_2,G_3]=G_5; & \\ \mathfrak{l}_{5,6}: & [G_1,G_2]=G_3, & [G_1,G_3]=G_4, & [G_1,G_4]=G_5, & [G_2,G_3]=G_5; \\ \mathfrak{l}_{5,7}: & [G_1,G_2]=G_3, & [G_1,G_3]=G_4, & [G_1,G_4]=G_5; & \\ \mathfrak{l}_{5,9}: & [G_1,G_2]=G_3, & [G_1,G_3]=G_4, & [G_2,G_3]=G_5, & \end{array} \tag{1}$$

with respect to a distinguished basis $\{G_1, G_2 \dots\}$, which will be called the *canonical basis* of the corresponding Lie algebra. In this list (1) of Lie algebras $\mathfrak{l}_{4,3}$ and $\mathfrak{l}_{5,7}$ are standard filiform, $\mathfrak{l}_{5,6}$ is non-standard filiform, $\mathfrak{l}_{5,5}$ and $\mathfrak{l}_{5,9}$ are 3-step nilpotent with 1-dimensional, respectively 2-dimensional center.
A Lie algebra equipped with an inner product is called *metric Lie algebra*, the automorphisms preserving the inner product are called *orthogonal automorphisms*.
In the following $\mathbb{E}^n$ denotes an n-dimensional Euclidean vector space with a distinguished orthonormal basis $\mathcal{E} = \{E_1, E_2, \dots, E_n\}$. We will use the following heuristic procedure for the classification of metric Lie algebras up to isometric isomorphisms:

1. Let $\{G_1, G_2, \dots, G_n\}$ be a fixed basis of an n-dimensional Lie algebra $\mathfrak{n}$ such that the commutation relations have a simple form (e.g. as in the list of the classification of low dimensional Lie algebras in the previous list).
2. Using the Gram-Schmidt process to the ordered basis $(G_n, G_{n-1}, \dots, G_1)$ in the metric Lie algebra $(\mathfrak{n}, \langle .,. \rangle)$ we obtain an orthonormal basis $\{F_1, F_2, \dots, F_n\}$ expressed by

$$F_i = \sum_{k=i}^{n} a_{ik} G_k, \quad a_{ik} \in \mathbb{R}, \quad \text{such that} \quad a_{ii} \geq 0.$$

Conversely, any basis $\{F_1, F_2, \dots, F_n\}$ of $\mathfrak{n}$ having the form $F_i = \sum_{k=i}^{n} a_{ik} G_k$, $a_{ik} \in \mathbb{R}$ with $a_{ii} \geq 0$ determines an inner product on $\mathfrak{n}$ as an orthonormal basis. Such bases parametrize the inner products on $\mathfrak{n}$.
3. We define a Lie bracket on $\mathbb{E}^n$ with the same structure coefficients with respect to its distinguished basis $\mathcal{E}$ as the metric Lie algebra $(\mathfrak{n}, \langle .,. \rangle)$ has with respect to its basis $\mathcal{F}$. The obtained metric Lie algebra on $\mathbb{E}^n$ is depending on real parameters, (determined by the structure coefficients), and it is isometrically isomorphic to $(\mathfrak{n}, \langle .,. \rangle)$.

4. We are looking for conditions on the real parameters of metric Lie algebras on $\mathbb{E}^n$ to get a one-to-one correspondence between the equivalence classes of isometrically isomorphic metric Lie algebras and a family of metric Lie algebras on $\mathbb{E}^n$.

This method will be applied systematically through this paper to the description of the isometry equivalence classes and the isometry groups of connected and simply connected non two-step nilpotent Riemannian nilmanifolds.

3. Filiform Lie algebras

3.1. Framing of metric Lie algebras

It turns out that every filiform metric Lie algebra and every 5-dimensional non two-step nilpotent metric Lie algebra, up to one exceptional class, can be decomposed into orthogonal direct sum of 1-dimensional subspaces. For these metric Lie algebras $(\mathfrak{n}, \langle ., . \rangle)$ this decomposition is uniquely determined by their algebraic and metric structure, or equivalently, any orthogonal automorphism of $(\mathfrak{n}, \langle ., . \rangle)$ preserves this decomposition. The metric Lie algebras satisfying this property will be called *framed*. For the determination of the group of isometries of the Riemannian nilmanifolds obtained in our classification we use that each orthogonal automorphism of a framed metric Lie algebra leaves invariant the 1-dimensional subspaces of the orthogonal direct sum decomposition (cf. Corollary 2). We will need the following

Definition 2. *An orthogonal direct sum decomposition* $\mathfrak{n} = V_1 \oplus \cdots \oplus V_n$ *on one-dimensional subspaces* $V_1, \ldots, V_n$ *of a metric Lie algebra* $(\mathfrak{n}, \langle ., . \rangle)$ *is called a* framing, *if any orthogonal automorphism of* $(\mathfrak{n}, \langle ., . \rangle)$ *preserves this decomposition.*
An orthonormal basis $\{G_1, G_2, \ldots, G_n\}$ *of* $(\mathfrak{n}, \langle ., . \rangle)$ *is* adapted *to the framing* $\mathfrak{n} = V_1 \oplus \cdots \oplus V_n$ *if* $V_i = \mathbb{R}\, G_i$ *for* $i = 1, \ldots, n$.
The metric Lie algebra $(\mathfrak{n}, \langle ., . \rangle)$ *is called* framed, *if it has a framing.*

Lemma 1. *Let* $(\mathfrak{n}, \langle ., . \rangle)$ *and* $(\mathfrak{n}^*, \langle ., . \rangle^*)$ *be isometrically isomorphic framed metric Lie algebras of dimension* n *with framings* $\mathfrak{n} = \mathbb{R}\, G_1 \oplus \cdots \oplus \mathbb{R}\, G_n$ *and* $\mathfrak{n}^* = \mathbb{R}\, G_1^* \oplus \cdots \oplus \mathbb{R}\, G_n^*$, *where* $(G_1, \ldots, G_n)$, *respectively* $(G_1^*, \ldots, G_n^*)$ *are orthonormal bases. If the commutators* $[., .]$ *of* $\mathfrak{n}$ *and* $[., .]^*$ *of* $\mathfrak{n}^*$ *are of the form*

$$[G_i, G_j] = \sum_{k=1}^{n} c_{i,j}^k G_k \quad \textit{and} \quad [G_i^*, G_j^*]^* = \sum_{k=1}^{n} {c^*}_{i,j}^k G_k^*, \quad i, j, k = 1, \ldots, n,$$

then $c_{i,j}^k = \pm {c^*}_{i,j}^k$ *for all* $i, j, k = 1, \dots, n$*. Particularly, if* $c_{i,j}^k$*,* ${c^*}_{i,j}^k \geq 0$ *then* $c_{i,j}^k = {c^*}_{i,j}^k$*.*

Proof. An isometric isomorphism $\mathfrak{n} \to \mathfrak{n}^*$ maps $G_i \mapsto \varepsilon_i G_i^*$ with $\varepsilon_i = \pm 1$. Hence $|c_{i,j}^k| = |{c^*}_{i,j}^k|$ follows from $\sum_{k=1}^n c_{i,j}^k \varepsilon_k G_k^* = \varepsilon_i \varepsilon_j \sum_{k=1}^n {c^*}_{i,j}^k G_k^*$. □

The orthogonal automorphisms of a connected and simply connected Riemannian nilmanifold $(N, \langle .,. \rangle)$ corresponding to a framed nilpotent metric Lie algebra $(\mathfrak{n}, \langle .,. \rangle)$ preserve the framing, hence we have

Corollary 2. *The group* $\mathcal{OA}(\mathfrak{n})$ *of orthogonal automorphisms of* $(\mathfrak{n}, \langle .,. \rangle)$ *is a subgroup of the group* $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$*, where the number of factors* $\leq \dim \mathfrak{n}$*. The connected component of the isometry group* $\mathcal{I}(N)$ *of* $(N, \langle .,. \rangle)$ *is isomorphic to the Lie group* N*.*

For the construction of framings we use series of ideals.

Lemma 3. *Let* $(\mathfrak{n}, \langle .,. \rangle)$ *be a metric Lie algebra of dimension* n*. If* $\mathfrak{n}$ *has a descending series of ideals* $\mathfrak{n} = \mathfrak{n}^{(1)} \supset \mathfrak{n}^{(2)} \supset \cdots \supset \mathfrak{n}^{(n)}$ *invariant under automorphisms of* $\mathfrak{n}$ *with* $\dim(\mathfrak{n}^{(j)}) = n - j + 1$*,* $j = 1, \dots, n$*, then* $(\mathfrak{n}, \langle .,. \rangle)$ *has a framing.*

Proof. Let be $V_1 = \mathfrak{n}^{(n)}$ and V_k the 1-dimensional subspace in $\mathfrak{n}^{(n-k+1)}$ which is orthogonal to $\mathfrak{n}^{(n-k+2)}$, for $k = 2, \dots, n$. Since any ideal $\mathfrak{n}^{(j)}$, $j = 1, \dots, n$, is invariant under automorphisms of $\mathfrak{n}$ the orthogonal direct sum decomposition $\mathfrak{n} = V_1 \oplus \cdots \oplus V_n$ is invariant under $\mathcal{OA}(\mathfrak{n})$. □

Theorem 4. *Any filiform metric Lie algebra* $(\mathfrak{n}, \langle .,. \rangle)$ *has a framing.*

Proof. There exists a basis $\{G_1, \cdots, G_n\}$ of $\mathfrak{n}$ satisfying $[G_1, G_i] = G_{i+1}$, $i = 2, \dots, n-1$, (cf. Remark 1). Since $\dim(\mathcal{C}^i \mathfrak{n}) = n - i - 1$ if $1 \leq i \leq n-1$, one has $\mathcal{C}^j \mathfrak{n} = \operatorname{span}(G_{j+2}, \cdots, G_n)$. We denote $\mathfrak{n}^{(j+2)} = \mathcal{C}^j \mathfrak{n}$ for $1 \leq j \leq n-2$. The factor algebra $\mathfrak{n}/\mathfrak{n}^{(5)}$ is a 4-dimensional nilpotent Lie algebra, which is isomorphic to the standard filiform Lie algebra $\mathfrak{l}_{4,3}$ according to the list (1). Hence there exists a basis $\{\bar{G}_1, \bar{G}_2, \bar{G}_3, \bar{G}_4\}$ in $\mathfrak{n}/\mathfrak{n}^{(5)}$ such that the non-vanishing Lie brackets are $[\bar{G}_1, \bar{G}_2] = \bar{G}_3$, $[\bar{G}_1, \bar{G}_3] = \bar{G}_4$. Moreover $\bar{\mathfrak{n}}^{(3)} = \operatorname{span}(\bar{G}_3, \bar{G}_4)$ is the commutator, $\bar{\mathfrak{n}}^{(4)} = \operatorname{span}(\bar{G}_4)$ is the center of $\mathfrak{n}/\mathfrak{n}^{(5)}$, and $\bar{\mathfrak{n}}^{(2)} = \operatorname{span}(\bar{G}_2, \bar{G}_3, \bar{G}_4)$ is the ideal centralizing the commutator $\bar{\mathfrak{n}}^{(3)}$. Let $\mathfrak{n}^{(h)}$ be the preimage of $\bar{\mathfrak{n}}^{(h)} \subset \mathfrak{n}/\mathfrak{n}^{(5)}$ in the algebra $\mathfrak{n}$ for $h = 2, 3, 4$. We obtain a descending series of ideals $\mathfrak{n} = \mathfrak{n}^{(1)} \supset \mathfrak{n}^{(2)} \supset \cdots \supset \mathfrak{n}^{(n)}$, which is invariant under automorphisms of $\mathfrak{n}$. Hence Lemma 3 implies the assertion. □

Corollary 5. *The connected component of the isometry group $\mathcal{I}(N)$ of a connected and simply connected Riemannian nilmanifold $(N, \langle .,. \rangle)$ corresponding to a filiform metric Lie algebra $(\mathfrak{n}, \langle .,. \rangle)$ is isomorphic to the Lie group N.*

3.2. Isometry classes of standard filiform Lie algebras

Let $\mathfrak{s}_n$ be the *standard filiform Lie algebra* of dimension n determined by the non-vanishing Lie brackets $[G_1, G_i] = G_{i+1}$, for $i = 2, \dots, n-1$ with respect to a basis $\{G_1, G_2, \dots, G_n\}$. The subspaces $\mathfrak{s}_n^{(j)} = \mathrm{span}(G_j, \cdots, G_n)$ of the Lie algebra $\mathfrak{s}_n$ satisfy $\mathfrak{s}_n^{(j)} = \mathcal{C}^{j-2}\mathfrak{n}$ for $3 \le j \le n$ and $\mathfrak{s}_n^{(2)}$ is the centralizer of $\mathcal{C}^1\mathfrak{n}$, hence the $\mathfrak{s}_n^{(j)}$, $j = 1, \dots, n$, are ideals invariant under automorphisms. It follows, that any metric standard filiform Lie algebra has a framing according to Lemma 3.

Definition 3. Let $\mathcal{C} = \{c_{j,k} \in \mathbb{R};\ 2 \le k \le j \le n-1\}$ be a lower triangular $n-2 \times n-2$ matrix with positive diagonal elements. We denote by $\mathfrak{n}_\mathcal{C}$ the Lie algebra and by $[.,.]_\mathcal{C}$ its Lie bracket defined on the Euclidean vector space $\mathbb{E}^n$ by the non-vanishing commutators

$$[E_1, E_i]_\mathcal{C} = -[E_i, E_1]_\mathcal{C} = \sum_{t=i}^{n-1} c_{t,i}\, E_{t+1}, \quad i = 2, \dots, n-1,$$

where $\{E_1, \dots, E_n\}$ is the distinguished orthonormal basis of $\mathbb{E}^n$. The metric Lie algebra $(\mathfrak{n}_\mathcal{C}, \langle .,. \rangle_\mathcal{C})$ is the Lie algebra $\mathfrak{n}_\mathcal{C}$ with the Euclidean inner product $\langle .,. \rangle_\mathcal{C}$ of $\mathbb{E}^n$.

It is easy to see that bracket operation (2) satisfies the Jacobi identity.

Lemma 6. *Any n-dimensional nilpotent Lie algebra $\mathfrak{n}_\mathcal{C}$ is isomorphic to the standard filiform Lie algebra $\mathfrak{s}_n$.*

Proof. The map $\mathfrak{n}_\mathcal{C} \to \mathfrak{s}_n$ given by $E_1 \mapsto G_1 = E_1$, $E_i \mapsto G_i = \sum_{t=i}^{n} b_{t,i}\, E_t$ with $b_{t,i} \in \mathbb{R}$, $2 \le i \le t \le n$, is an isomorphism if and only if

$$[G_1, G_i]_\mathcal{C} = \sum_{t=i}^{n-1} b_{t,i}\, [E_1, E_t]_\mathcal{C} = \sum_{t=i}^{n-1} b_{t,i} \sum_{k=t}^{n-1} c_{k,t}\, E_{k+1} = G_{i+1} = \sum_{k=i}^{n-1} b_{k+1,i+1}\, E_{k+1},$$

$i = 2, \dots, n-1$, or equivalently $\sum_{t=i}^{n-1} c_{k,t}\, b_{t,i} = b_{k+1,i+1}$. Since $c_{k,t} = 0$ if $k < t$ we obtain the recursive relation

$$\sum_{t=i}^{k} c_{k,t}\, b_{t,i} = b_{k+1,i+1}, \quad 2 \le i \le k \le n-1.$$

It follows that for any given values $b_{h,2},\ \ h = 2, \dots, n,$ there is a unique isomorphism $\mathfrak{n}_{\mathcal{C}} \to \mathfrak{s}_n$. Hence $\mathfrak{n}_{\mathcal{C}}$ is a standard filiform metric Lie algebra. $\square$

Theorem 7. *Let $\langle ., . \rangle$ be an inner product on the n-dimensional standard filiform nilpotent Lie algebra $\mathfrak{s}_n$.*

(1) There is a unique metric Lie algebra $(\mathfrak{n}_{\mathcal{C}}, \langle ., . \rangle_{\mathcal{C}})$ satisfying
  (a) $(\mathfrak{n}_{\mathcal{C}}, \langle ., . \rangle_{\mathcal{C}})$ *is isometrically isomorphic to* $(\mathfrak{s}_n, \langle ., . \rangle)$,
  (b) *if the set $\mathcal{P} = \{(k, i) :\ c_{k,i} \neq 0$ and $k - i$ is odd$\}$ is not empty then $c_{k_0,i_0} > 0$ for the minimal element (k_0, i_0) of $\mathcal{P}$ with respect to the anti-lexicographic ordering of pairs.*

(2) The group of orthogonal automorphisms of $\mathfrak{n}_{\mathcal{C}}$ is the group
  (a) *if $\{(i, k) : c_{k,i} \neq 0$ and $k - i$ is odd$\} = \emptyset$: $\mathcal{OA}(\mathfrak{n}_{\mathcal{C}}) =$*

$$\{TE_1 = \varepsilon_1 E_1, TE_h = \varepsilon_1^h \varepsilon_2 E_h, h = 2, \cdots, n, \varepsilon_1, \varepsilon_2 = \pm 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2,$$

  (b) *if $\{(i, k) :\ c_{k,i} \neq 0$ and $k - i$ is odd$\} \neq \emptyset$: $\mathcal{OA}(\mathfrak{n}_{\mathcal{C}}) =$*

$$\{TE_1 = E_1, TE_h = \varepsilon_2 E_h, h = 2, \cdots, n, \varepsilon_2 = \pm 1\} \cong \mathbb{Z}_2,$$

with respect to the basis $\{E_1, \dots, E_n\}$.

Proof. Consider the ideals $\mathfrak{s}_n^{(j)} = \operatorname{span}(G_j, \cdots, G_n),\ j = 1, \dots, n,\ \mathfrak{s}_n^{(n+1)} = \{0\}$ with the canonical basis $\{G_1, \dots, G_n\}$ of $\mathfrak{s}_n$. The Gram-Schmidt process applied to the ordered basis $(G_n, \dots, G_1)$ yields an orthonormal basis $\{F_1, \dots, F_n\}$ such that F_i is a positive multiple of G_i modulo $\mathfrak{s}_n^{(i+1)}$ and orthogonal to $\mathfrak{s}_n^{(i+1)}$ for $i = 1, \dots, n$. Hence F_i can be expressed as $F_i = \sum_{k=i}^{n} a_{k,i} G_k$, $i = 1, \dots, n$, where $\{a_{k,i}\}$ is a lower triangular $n \times n$ matrix with positive diagonal elements. The orthogonal direct sum $\mathbb{R}\, F_1 \oplus \cdots \oplus \mathbb{R}\, F_n$ is a framing of $(\mathfrak{s}_n, \langle ., . \rangle)$, since the orthogonal one-dimensional subspaces $\mathbb{R}\, F_1, \dots, \mathbb{R}\, F_n$ are determined by the inner product and by the descending series of ideals $\mathfrak{s}_n^{(j)}$, $j = 1, \dots, n$. The non-vanishing Lie brackets with respect to the new basis are of the form

$$[F_1, F_i] = -[F_i, F_1] = a_{1,1} \sum_{t=i}^{n-1} a_{t,i}\, G_{t+1} = a_{1,1} \sum_{t=i}^{n-1} a_{t,i} \sum_{j=t}^{n-1} b_{j+1,t+1}\, F_{j+1},$$

$i = 2, \dots, n-1$, where the lower triangular matrix $\{b_{j,k}\}$ is the inverse of $\{a_{j,k}\}$. It follows that $(\mathfrak{s}_n, \langle ., . \rangle)$ is isometrically isomorphic to the Lie algebra $\mathfrak{n}_{\mathcal{C}}$ corresponding to the lower triangular matrix $\mathcal{C} = \{c_{j,k}\}$ given by

$$c_{j,k} = \langle [F_1, F_k], F_{j+1} \rangle = a_{1,1} \sum_{t=k}^{n-1} a_{t,k}\, b_{j+1,t+1}, \quad 2 \leq k \leq j \leq n-1,$$

with positive diagonal elements $c_{j,j} = \langle [F_1, F_j], F_{j+1}\rangle = a_{1,1}\, a_{j,j}\, b_{j+1,j+1},\ j = 2, \dots, n-1$. Changing the orthonormal basis: $\tilde{F}_k = (-1)^k\, F_k,\ k = 1, \dots, n$, we obtain

$$\tilde{c}_{j,k} := \langle [\tilde{F}_1, \tilde{F}_k], \tilde{F}_{j+1}\rangle = \langle [-F_1, (-1)^k F_k], (-1)^{j+1} F_{j+1}\rangle = (-1)^{j-k} c_{j,k}.$$

It follows that if the set $\mathcal{P} = \{(j,k) :\ c_{j,k} \neq 0 \text{ and } j-k \text{ is odd}\}$ is not empty, then we may assume that $\{F_1, \dots, F_n\}$ is an orthonormal basis adapted to the framing of $\mathfrak{s}_n$ such that $c_{j_0,k_0} > 0$ for the minimal element (j_0, k_0) of $\mathcal{P}$ with respect to the anti-lexicographic ordering of pairs. Hence we obtained the construction of the metric Lie algebra $(\mathfrak{n}_{\mathcal{C}}, \langle .,. \rangle_{\mathcal{C}})$ corresponding to the matrix $\mathcal{C} = \{c_{j,k}\}$, which is isometrically isomorphic to $(\mathfrak{s}_n, \langle .,. \rangle)$.

Let $\Phi_{\mathcal{C}} : \mathfrak{s}_n \to \mathfrak{n}_{\mathcal{C}}$ and $\Phi_{\mathcal{D}} : \mathfrak{s}_n \to \mathfrak{n}_{\mathcal{D}}$ be isometric isomorphisms of metric Lie algebras, where $(\mathfrak{n}_{\mathcal{C}}, \langle .,. \rangle_{\mathcal{C}})$ corresponds to the matrix $\mathcal{C} = \{c_{j,k}\}$, and $(\mathfrak{n}_{\mathcal{D}}, \langle .,. \rangle_{\mathcal{D}})$ corresponds to the matrix $\mathcal{D} = \{d_{j,k}\}$. Assume the condition $c_{j_0,k_0} > 0$, respectively $d_{j_0,k_0} > 0$, if $\mathcal{P} \neq \emptyset$. Denote by $[.,.]_{\mathcal{C}}$ and $[.,.]_{\mathcal{D}}$ the Lie brackets on $\mathfrak{n}_{\mathcal{C}}$, respectively on $\mathfrak{n}_{\mathcal{D}}$. Since $(\mathfrak{n}_{\mathcal{C}}, \langle .,. \rangle_{\mathcal{C}})$ and $(\mathfrak{n}_{\mathcal{D}}, \langle .,. \rangle_{\mathcal{D}})$ are isometrically isomorphic framed metric Lie algebras, we have

$$\langle [\Phi_{\mathcal{C}}(F_1), \Phi_{\mathcal{C}}(F_i)]_{\mathcal{C}}, \Phi_{\mathcal{C}}(F_{k+1})\rangle_{\mathcal{C}} = \langle [\Phi_{\mathcal{D}}(F_1), \Phi_{\mathcal{D}}(F_i)]_{\mathcal{D}}, \Phi_{\mathcal{D}}(F_{k+1})\rangle_{\mathcal{D}}.$$

Moreover $\Phi_{\mathcal{D}}(F_i) = \varepsilon_i \Phi_{\mathcal{C}}(F_i)$, where $\varepsilon_i = \pm 1$ for $i = 1, \dots, n$, and hence

$$\langle [\Phi_{\mathcal{C}}(F_1), \Phi_{\mathcal{C}}(F_i)]_{\mathcal{C}}, \Phi_{\mathcal{C}}(F_{k+1})\rangle_{\mathcal{C}} = \varepsilon_1 \varepsilon_i \varepsilon_{k+1} \langle [\Phi_{\mathcal{C}}(F_1), \Phi_{\mathcal{C}}(F_i)]_{\mathcal{D}}, \Phi_{\mathcal{C}}(F_{k+1})\rangle_{\mathcal{D}},$$

or equivalently

$$c_{k,i} = \varepsilon_1 \varepsilon_i \varepsilon_{k+1}\, d_{k,i} \quad \text{for} \quad 2 \leq i \leq k \leq n-1.$$

Using Lemma 1 we get from the relations $c_{i,i} > 0,\ d_{i,i} > 0$ that $\varepsilon_1 \varepsilon_i = \varepsilon_{i+1}$, $i = 2, \dots, n-1$, hence we can express $\varepsilon_h = \varepsilon_1^h \varepsilon_2,\ h = 2, \dots, n$. If $\varepsilon_1 = 1$ we have $\varepsilon_2 = \dots = \varepsilon_n$ and hence $c_{k,i} = d_{k,i},\ 2 \leq i \leq k \leq n-1$. If $\varepsilon_1 = -1$ we get $\varepsilon_h = (-1)^h \varepsilon_2,\ h = 2, \dots, n$, consequently

$$c_{k,i} = (-1)^{k-i}\, d_{k,i} \quad \text{for} \quad 2 \leq i \leq k \leq n-1. \tag{2}$$

If $\mathcal{P} = \{(k,i) :\ c_{k,i} \neq 0 \text{ and } k-i \text{ is odd}\} \neq \emptyset$ then according to our assumption $c_{k_0,i_0} > 0,\ d_{k_0,i_0} > 0$, which is a contradiction, consequently $\varepsilon_1 \neq -1$. If $\mathcal{P} = \emptyset$, then equation (2) implies $c_{j,k} = d_{j,k}$ for all $2 \leq k \leq j \leq n-1$ and hence the metric Lie algebra $\mathfrak{n}_{\mathcal{C}}$ isometrically isomorphic to $\mathfrak{s}_n$ is

uniquely determined and assertion (1) is proved.
An orthogonal automorphism of $\mathfrak{n}_{\mathcal{C}}$ induces a change of the orthonormal basis: $E_i \mapsto \varepsilon_i E_i$, $\varepsilon_i = \pm 1$, $i = \{1, \ldots, n\}$, preserving the commutation relations. It follows from equation (2) that $\varepsilon_h = \varepsilon_1^h \varepsilon_2$, hence for $\varepsilon_1 = 1$ one has $\varepsilon_2 = \cdots = \varepsilon_n$, and for $\varepsilon_1 = -1$ one gets $\varepsilon_h = (-1)^h \varepsilon_2$, $h = 2, \ldots, n$. But the map $E_1 \mapsto -E_1$ and $E_h \mapsto (-1)^h \varepsilon_2 E_h$, $h = 2, \ldots, n$ is preserving the Lie bracket if and only if the set $\{(i,k) : \ c_{k,i} \neq 0 \text{ and } k - i \text{ is odd}\}$ is empty. Hence the group of orthogonal automorphisms of $\mathfrak{n}_{\mathcal{C}}$ can be represented by the group of matrices described in assertion (2). This gives the second assertion. □

Corollary 8. *Let $(N_{\mathcal{C}}, \langle .,. \rangle)$ be the connected and simply connected Riemannian nilmanifold corresponding to the metric Lie algebra $(\mathfrak{n}_{\mathcal{C}}, \langle .,. \rangle)$. The isometry group of $(N_{\mathcal{C}}, \langle .,. \rangle)$ is*

$$\mathcal{I}(N_{\mathcal{C}}) = \left\{ \begin{array}{lll} \mathbb{Z}_2 \times \mathbb{Z}_2 \ltimes N_{\mathcal{C}} & \text{if} & \{(i,k) : \ c_{k,i} \neq 0 \text{ and } k-i \text{ is odd}\} = \emptyset, \\ \mathbb{Z}_2 \ltimes N_{\mathcal{C}} & \text{if} & \{(i,k) : \ c_{k,i} \neq 0 \text{ and } k-i \text{ is odd}\} \neq \emptyset. \end{array} \right\}$$

Remark 2. *The Lie algebra $\mathfrak{l}_{4,3}$ is the 4-dimensional standard filiform Lie algebra. J. Lauret in [7] has been determined up to isometry the 4-dimensional homogeneous nilmanifolds belonging to the Lie algebra $\mathfrak{l}_{4,3}$. By Theorem 7 the metric Lie algebra $(\mathfrak{l}_{4,3}, \langle .,. \rangle)$ is isometrically isomorphic to a unique metric Lie algebra $\mathfrak{n}_{4,3}(\kappa, \lambda, \mu)$, $\kappa > 0$, $\mu > 0$, $\lambda \geq 0$, defined by the non-vanishing Lie brackets*

$$[E_1, E_2] = \kappa E_3 + \lambda E_4, \quad [E_1, E_3] = \mu E_4$$

with respect to an orthonormal basis $\{E_1, \ldots, E_4\}$ on the 4-dimensional Euclidean space $\mathbb{E}^4$. Considering the orthonormal basis $F_i = \sum_{k=i}^4 a_{ik} G_k$ with $a_{ii} > 0$, $i = 1, \ldots, 4$ obtained by the Gram-Schmidt process from the ordered canonical basis $\{G_4, G_3, G_2, G_1\}$ of $\mathfrak{l}_{4,3}$. It follows that $[F_1, F_2] = \kappa F_3 + \widetilde{\lambda} F_4$, $[F_1, F_3] = \mu F_4$, where

$$\kappa = \frac{a_{11} a_{22}}{a_{33}} > 0, \ \mu = \frac{a_{11} a_{33}}{a_{44}} > 0, \ \widetilde{\lambda} = \frac{a_{11}}{a_{44}} \left(a_{23} - \frac{a_{22} a_{34}}{a_{33}} \right). \tag{3}$$

The inner product $\langle .,. \rangle_{\mathfrak{m}}$ on the orthogonal complementary subspace $\mathfrak{m}$ to $\mathfrak{l}'_{4,3} = span(F_3, F_4)$ determines the coefficients a_{1k}, $k = 1, 2, 3, 4$ and a_{2k}, $k = 2, 3, 4$. If these coefficients and the isometric isomorphism class, given by κ, λ, μ with $\kappa, \mu > 0$, are fixed, then the equations (3) are uniquely solvable for a_{33}, a_{34} and a_{44}, determining the inner product on $\mathfrak{l}'_{4,3}$, hence:

If there is given a complementary subspace $\mathfrak{m}$ *to* $\mathfrak{l}'_{4,3}$ *in* $\mathfrak{l}_{4,3}$ *and an inner product* $\langle .,. \rangle_{\mathfrak{m}}$ *on* $\mathfrak{m}$ *than for any isometric isomorphism class of metric Lie algebras on* $\mathfrak{l}_{4,3}$ *there exists a unique inner product* $\langle .,. \rangle_{\mathfrak{l}'_{4,3}}$ *on* $\mathfrak{l}'_{4,3}$ *such that* $\mathfrak{l}_{4,3}$ *with the inner product determined by the orthogonal direct sum* $(\mathfrak{m}, \langle .,. \rangle_{\mathfrak{m}}) \oplus (\mathfrak{l}'_{4,3}, \langle .,. \rangle_{\mathfrak{l}'_{4,3}})$ *belongs to this isometric isomorphism class.*

Remark 3. *The Lie algebra* $\mathfrak{l}_{5,7}$ *is the* 5*-dimensional standard filiform Lie algebra. Let* $\mathfrak{n}_{5,7}(c_{2,2}, c_{2,3}, c_{2,4}, c_{3,3}, c_{3,4}, c_{4,4})$ *be a four-step nilpotent filiform Lie algebra defined by*

$$[E_1, E_2] = c_{2,2}E_3 + c_{2,3}E_4 + c_{2,4}E_5, \ [E_1, E_3] = c_{3,3}E_4 + c_{3,4}E_5, \ [E_1, E_4] = c_{4,4}E_5$$

with respect to the distinguished orthonormal basis $\{E_1, \dots, E_5\}$ *of the Euclidean vector space* $\mathbb{E}^5$*. The metric Lie algebra* $(\mathfrak{l}_{5,7}, \langle .,. \rangle)$ *is isometrically isomorphic to a unique metric Lie algebra* $\mathfrak{n}_{5,7}(c_{2,2}, c_{2,3}, c_{2,4}, c_{3,3}, c_{3,4}, c_{4,4})$ *with* $c_{i,j} \in \mathbb{R}$ *such that* $c_{2,2}, c_{3,3}, c_{4,4} > 0$ *and either* $c_{2,3} > 0$ *or* $c_{2,3} = 0$*,* $c_{3,4} \geq 0$ *(cf. Theorem 7).*

3.3. Non-standard filiform algebra of dimension 5

The smallest filiform but not standard filiform Lie algebra is $\mathfrak{l}_{5,6}$.

Definition 4. Let a, b, c, d, f, g, h be given real numbers with $a, d, g, h \neq 0$. The metric Lie algebra defined on the Euclidean vector space $\mathbb{E}^5$ by the non-vanishing commutators

$$[E_1, E_2] = aE_3 + bE_4 + cE_5, [E_1, E_3] = dE_4 + fE_5,$$
$$[E_1, E_4] = gE_5, [E_2, E_3] = hE_5 \tag{4}$$

is denoted by $\mathfrak{n}_{5,6}(a, b, c, d, f, g, h)$.

It easy to control that the bracket operation (4) satisfies the Jacobi identity. The Lie algebra $\mathfrak{n}_{5,6}(a, b, c, d, f, g, h)$ is four-step nilpotent and isomorphic to the filiform nilpotent algebra $\mathfrak{l}_{5,6}$ since the map

$$G_1 \mapsto G_1, \ G_2 \mapsto w(adg\, G_2 + bg\, G_3 + (c - \frac{bf}{d})G_4), \ G_3 \mapsto wdg\, G_3,$$
$$G_4 \mapsto w(g\, G_4 - \frac{f}{d}G_5), \ G_5 \mapsto wG_5, \quad \text{where} \quad w = \frac{h}{ad^2g^2},$$

is an isomorphism $\mathfrak{l}_{5,6} \to \mathfrak{n}_{5,6}(a, b, c, d, f, g, h)$.

Theorem 9. *Let $\langle .,. \rangle$ be an inner product on the 5-dimensional four-step nilpotent filiform Lie algebra $\mathfrak{l}_{5,6}$.*

(1) The metric Lie algebra $(\mathfrak{l}_{5,6}, \langle .,. \rangle)$ is isometrically isomorphic to a unique metric Lie algebra $\mathfrak{n}_{5,6}(a,b,c,d,f,g,h)$ with $a,b,c,d,f,g,h \in \mathbb{R}$ such that $a,d,g,h > 0$ and either $b > 0$ or $b = 0$, $f \geq 0$.

(2) The group of orthogonal automorphisms of $\mathfrak{n}_{5,6}(a,b,c,d,f,g,h)$ is the following group with respect to the basis $\{E_1,E_2,E_3,E_4,E_5\}$:

(i) if $b = f = 0$, then one has $\mathcal{OA}(\mathfrak{n}_{5,6}(a,b,c,d,f,g,h)) = \{TE_1 = \varepsilon_1 E_1, TE_2 = E_2, TE_3 = \varepsilon_1 E_3, TE_4 = E_4, TE_5 = \varepsilon_1 E_5, \varepsilon_1 = \pm 1\} \cong \mathbb{Z}_2$

(ii) if $b^2 + f^2 \neq 0$, then it is trivial.

Proof. In the Lie algebra $\mathfrak{l}_{5,6}$ the center is $Z(\mathfrak{l}_{5,6}) = \mathrm{span}(G_5)$, the commutator subalgebra is $\mathfrak{l}'_{5,6} = \mathrm{span}(G_3,G_4,G_5)$, the second member of the lower central series is $\mathcal{C}^2(\mathfrak{l}_{5,6}) = \mathrm{span}(G_4,G_5)$ and the centralizer of $\mathcal{C}^2(\mathfrak{l}_{5,6})$ is $\mathrm{span}(G_2,G_3,G_4,G_5)$. Hence the subspaces $\mathrm{span}(G_i,\cdots,G_n)$, $i = 1,\ldots,5$, of $\mathfrak{l}_{5,6}$ form a descending series of ideals which are invariant under all automorphisms of $\mathfrak{l}_{5,6}$. The Gram-Schmidt process applied to the ordered basis (G_5,G_4,G_3,G_2,G_1) yields an orthonormal basis $\{F_1,F_2,F_3,F_4,F_5\}$ of $\mathfrak{l}_{5,6}$, where for $i = 1,\ldots,5$ the vector F_i is a positive multiple of G_i modulo the subspace $\mathrm{span}(G_j; j > i)$ and orthogonal to $\mathrm{span}(G_j; j > i)$. It follows $F_i = \sum_{k=i}^{5} a_{ik} G_k$ with $a_{ii} > 0$ and the orthogonal direct sum $\mathbb{R}\,F_1 \oplus \cdots \oplus \mathbb{R}\,F_5$ is a framing of $(\mathfrak{l}_{5,6}, \langle .,. \rangle)$. It means that the commutators can be expressed as

$$[F_1,F_2] = aF_3 + bF_4 + cF_5, [F_1,F_3] = dF_4 + fF_5, [F_1,F_4] = gF_5, [F_2,F_3] = hF_5 \tag{5}$$

$a,d,g,h > 0$, $a,b,c,d,f,g,h \in \mathbb{R}$. Changing the orthonormal basis $F_1 \mapsto -F_1$, $F_2 \mapsto F_2$, $F_3 \mapsto -F_3$, $F_4 \mapsto F_4$, $F_5 \mapsto -F_5$ we obtain

$$[F_1,F_2] = aF_3 - bF_4 + cF_5, [F_1,F_3] = dF_4 - fF_5, [F_1,F_4] = gF_5, [F_2,F_3] = hF_5.$$

Hence there is an orthonormal basis satisfying (5) such that $a,d,g,h > 0$ and $b \geq 0$. Moreover, if $b = 0$ then we can find a basis with $a,d,g,h > 0$, $b = 0$ and $f \geq 0$. Consequently the existence of $\mathfrak{n}_{5,6}(a,b,c,d,f,g,h)$ satisfying $a,b,d,g,h > 0$, or $a,d,g,h > 0$, $b = 0$, $h \geq 0$ is proved.

Let the linear map $T : \mathfrak{n}_{5,6}(a,b,c,d,f,g,h) \to \mathfrak{n}_{5,6}(a',b',c',d',f',g',h')$ be an isometric isomorphism. The decomposition $\mathbb{R}\,E_1 \oplus \mathbb{R}\,E_2 \oplus \mathbb{R}\,E_3 \oplus \mathbb{R}\,E_4 \oplus \mathbb{R}\,E_5$

is a framing of both Lie algebras, where $a, a', d, d', g, g', h, h' > 0$ and $b, b' \geq 0$, hence by Lemma 1 we have $a = a'$, $d = d'$, $g = g'$, $h = h'$, $b = b'$, moreover $|c'| = c$, $|f'| = |f|$. Let be $T(E_i) = \varepsilon_i E_i$, $\varepsilon_i = \pm 1$, $i = 1, \ldots, 5$, then we obtain from $[TE_i, TE_j]' = T[E_i, E_j]$, $i, j = 1, \ldots, 5$, using commutation relations (5) the equations

$$\varepsilon_1\varepsilon_2(aE_3 + b'E_4 + c'E_5) = a\varepsilon_3 E_3 + b\varepsilon_4 E_4 + c\varepsilon_5 E_5,$$

$$\varepsilon_1\varepsilon_3(dE_4 + f'E_5) = d\varepsilon_4 E_4 + f\varepsilon_5 E_5,\ \varepsilon_1\varepsilon_4 gE_5 = g\varepsilon_5 E_5,\ \varepsilon_2\varepsilon_3 hE_5 = h\varepsilon_5 E_5.$$

It follows $\varepsilon_1\varepsilon_2 = \varepsilon_3$, $\varepsilon_1\varepsilon_3 = \varepsilon_4$, $\varepsilon_1\varepsilon_4 = \varepsilon_5$, $\varepsilon_2\varepsilon_3 = \varepsilon_5$ and $\varepsilon_2 = 1 = \varepsilon_4$, $\varepsilon_1 = \varepsilon_3 = \varepsilon_5$. Since $\varepsilon_1\varepsilon_2 = \varepsilon_1 = \varepsilon_5$ one has $c' = c$. If $b = b' \neq 0$ then we have in addition $\varepsilon_1\varepsilon_2 = \varepsilon_4$ and hence $\varepsilon_1 = 1 = \varepsilon_2 = \varepsilon_4 = \varepsilon_3 = \varepsilon_5$ and we obtain $f' = f$. If $b = b' = 0$ then one has $f \geq 0$, $f' \geq 0$ and Lemma 1 yields $f' = f$. This proves that the Lie algebra $\mathfrak{n}_{5,6}(a, b, c, d, f, g, h)$ is uniquely determined. If $T : \mathfrak{n}_{5,6}(a, b, c, d, f, g, h) \to \mathfrak{n}_{5,6}(a, b, c, d, f, g, h)$ is an orthogonal automorphism, then one has $TE_i = \varepsilon_i E_i$, $i = 1, \ldots, 5$, where $\varepsilon_i = \pm 1$. From Lie brackets (5) we obtain

$$\varepsilon_1\varepsilon_2\,(a\,E_3 + b\,E_4 + c\,E_5) = a\,\varepsilon_3 E_3 + b\,\varepsilon_4 E_4 + c\,\varepsilon_5 E_5, \quad \varepsilon_1\varepsilon_4 g\,E_5 = g\,\varepsilon_5 E_5,$$

$$\varepsilon_1\varepsilon_3\,(d\,E_4 + f\,E_5) = d\,\varepsilon_4 E_4 + f\,\varepsilon_5 E_5, \quad \varepsilon_2\varepsilon_3 h\,E_5 = h\,\varepsilon_5 E_5.$$

It follows

1. if $b = f = 0$, then $\varepsilon_1 = \varepsilon_3 = \varepsilon_5$ and $\varepsilon_2 = \varepsilon_4 = 1$,
2. if $b^2 + f^2 \neq 0$, then $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 1$.

Hence assertion (2) is proved. □

Corollary 10. *Let $(N_{5,6}(a, b, c, d, f, g, h), \langle ., . \rangle)$ be the connected and simply connected Riemannian nilmanifold belonging to the metric Lie algebra $(\mathfrak{n}_{5,6}(a, b, c, d, f, g, h), \langle ., . \rangle)$. The isometry group of the nilmanifold $(N_{5,6}(a, b, c, d, f, g, h), \langle ., . \rangle)$ is*

$$\mathcal{I}(N_{5,6}(a, b, c, d, f, g, h)) = \left\{ \begin{array}{lll} \mathbb{Z}_2 \ltimes N_{5,6}(a, b, c, d, f, g, h) & \text{if} & b = f = 0, \\ N_{5,6}(a, b, c, d, f, g, h) & \text{if} & b^2 + f^2 \neq 0. \end{array} \right\}$$

4. Three-step nilpotent Lie algebras of dimension 5

4.1. One-dimensional center

Now we consider the Lie algebra $\mathfrak{l}_{5,5}$.

Definition 5. Let a, b, c, d, e be real numbers with $a, d, e \neq 0$ and let $\mathfrak{n}_{5,5}(a, b, c, d, e)$ be the metric Lie algebra defined on $\mathbb{E}^5$ by the non-vanishing commutators

$$[E_1, E_2] = aE_4 + bE_5,\ [E_1, E_3] = cE_5,\ [E_1, E_4] = dE_5,\ [E_2, E_3] = eE_5. \quad (6)$$

Easy to show that bracket operation (6) satisfies the Jacobi identity and the map

$$G_1 \mapsto G_1,\ G_2 \mapsto adG_2 + bG_4,\ G_3 \mapsto \frac{e}{ad}G_3 + cG_4,\ G_4 \mapsto dG_4,\ G_5 \mapsto G_5$$

is an isomorphism $\mathfrak{l}_{5,5} \to \mathfrak{n}_{5,5}(a, b, c, d, e)$, where $\{G_1, G_2, G_3, G_4, G_5\}$ is the canonical basis of $\mathfrak{l}_{5,5}$.

Theorem 11. *Let $\langle ., . \rangle$ be an inner product on the 5-dimensional three-step nilpotent Lie algebra $\mathfrak{l}_{5,5}$.*

(1) There is a unique metric Lie algebra $\mathfrak{n}_{5,5}(a, b, c, d, e)$ with $a, d, e > 0$, $b, c \geq 0$, which is isometrically isomorphic to the metric Lie algebra $(\mathfrak{l}_{5,5}, \langle ., . \rangle)$.

(2) The group of orthogonal automorphisms of $\mathfrak{n}_{5,5}(a, b, c, d, e)$ is the group:

(a) *for $b = c = 0$:*

$$\mathcal{OA}(\mathfrak{n}_{5,5}(a, 0, 0, d, e)) = \{TE_1 = \varepsilon_1 E_1, TE_2 = \varepsilon_1\varepsilon_4 E_2, TE_3 = E_3,$$
$$TE_4 = \varepsilon_4 E_4, TE_5 = \varepsilon_1\varepsilon_4 E_5,\ \varepsilon_1, \varepsilon_4 = \pm 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2, \quad (7)$$

(b) *for $b = 0,\ c > 0$:*

$$\mathcal{OA}(\mathfrak{n}_{5,5}(a, 0, c, d, e)) = \{TE_1 = \varepsilon_1 E_1, TE_2 = \varepsilon_1 E_2, TE_3 = E_3,$$
$$TE_4 = E_4, TE_5 = \varepsilon_1 E_5,\ \varepsilon_1 = \pm 1\} \cong \mathbb{Z}_2, \quad (8)$$

(c) *for $b > 0,\ c = 0$:*

$$\mathcal{OA}(\mathfrak{n}_{5,5}(a, b, 0, d, e)) = \{TE_1 = E_1, TE_2 = \varepsilon_2 E_2, TE_3 = E_3,$$
$$TE_4 = \varepsilon_2 E_4, TE_5 = \varepsilon_2 E_5,\ \varepsilon_2 = \pm 1\} \cong \mathbb{Z}_2, \quad (9)$$

(d) *if* $b > 0,\ c > 0$*, then it is trivial*
with respect to the basis $\{E_1, E_2, E_3, E_4, E_5\}$*.*

Proof. The center $Z(\mathfrak{l}_{5,5})$ of $\mathfrak{l}_{5,5}$ is $\mathrm{span}(G_5)$, the commutator subalgebra $\mathfrak{l}'_{5,5}$ is $\mathrm{span}(G_4, G_5)$. The preimage $\pi^{-1}(Z(\mathfrak{l}_{5,5}/Z(\mathfrak{l}_{5,5})))$ of the center of the factor algebra $\mathfrak{l}_{5,5}/Z(\mathfrak{l}_{5,5})$ in $\mathfrak{l}_{5,5}$ is $\mathrm{span}(G_3, G_4, G_5)$ and the centralizer of $\mathfrak{l}'_{5,5}$ is $\mathrm{span}(G_2, G_3, G_4, G_5)$. Hence the subspaces $\mathrm{span}(G_i, \cdots, G_n)$, $i = 1, \ldots, 5$, of $\mathfrak{l}_{5,5}$ form a descending series of ideals which are invariant under all automorphisms of $\mathfrak{l}_{5,5}$. The Gram-Schmidt process applied to the ordered basis $(G_5, G_4, G_3, G_2, G_1)$ yields an orthonormal basis $\{F_1, F_2, F_3, F_4, F_5\}$ of $\mathfrak{l}_{5,5}$, where the vector F_i is a positive multiple of G_i modulo the subspace $\mathrm{span}(G_j; j > i)$ and orthogonal to $\mathrm{span}(G_j; j > i)$. According to Lemma 3 the direct sum $\mathbb{R}\, F_1 \oplus \cdots \oplus \mathbb{R}\, F_5$ is a framing of $(\mathfrak{l}_{5,5}, \langle .,. \rangle)$. Expressing the vectors of the new basis in the form $F_i = \sum_{k=i}^{5} a_{ik} G_k$ with $a_{ii} > 0$ we get

$$[F_1, F_2] = aF_4 + bF_5, [F_1, F_3] = cF_5, [F_1, F_4] = dF_5, [F_2, F_3] = eF_5, \qquad (10)$$

with suitable $a, d, e > 0$, $a, b, c, d, e \in \mathbb{R}$. Changing the orthonormal basis: $\tilde{F_1} = -F_1$, $\tilde{F_2} = -F_2$, $\tilde{F_3} = F_3$, $\tilde{F_4} = F_4$, $\tilde{F_5} = -F_5$ we obtain

$$[\tilde{F_1}, \tilde{F_2}] = a\tilde{F_4} - b\tilde{F_5},\ [\tilde{F_1}, \tilde{F_3}] = c\tilde{F_5},\ [\tilde{F_1}, \tilde{F_4}] = d\tilde{F_5},\ [\tilde{F_2}, \tilde{F_3}] = e\tilde{F_5}.$$

Similarly, the change of the basis: $\tilde{F_1} = F_1$, $\tilde{F_2} = -F_2$, $\tilde{F_3} = F_3$, $\tilde{F_4} = -F_4$, $\tilde{F_5} = -F_5$ yields

$$[\tilde{F_1}, \tilde{F_2}] = a\tilde{F_4} + b\tilde{F_5},\ [\tilde{F_1}, \tilde{F_3}] = -c\tilde{F_5},\ [\tilde{F_1}, \tilde{F_4}] = d\tilde{F_5},\ [\tilde{F_2}, \tilde{F_3}] = e\tilde{F_5}.$$

Hence there is an orthonormal basis such that in commutators (10) the coefficients b and c are non-negative, i.e. $(\mathfrak{l}_{5,5}, \langle .,. \rangle)$ is isometrically isomorphic to a metric Lie algebra $\mathfrak{n}_{5,5}(a, b, c, d, e)$ with $a, d, e > 0$, $b, c \geq 0$. Since $\{F_1, \ldots, F_5\}$ is an orthonormal basis adapted to the framing of $(\mathfrak{l}_{5,5}, \langle .,. \rangle)$ the uniqueness of the construction of $\mathfrak{n}_{5,5}(a, b, c, d, e)$ follows from Lemma 1. If the map $E_i \mapsto \varepsilon_i E_i$ is an orthogonal automorphism of $\mathfrak{n}_{5,5}(a, b, c, d, e)$ then $[\varepsilon_1 E_1, \varepsilon_2 E_2] = a\varepsilon_4 E_4 + b\varepsilon_5 E_5$, $[\varepsilon_1 E_1, \varepsilon_3 E_3] = c\varepsilon_5 E_5$, $[\varepsilon_1 E_1, \varepsilon_4 E_4] = d\varepsilon_5 E_5$, $[\varepsilon_2 E_2, \varepsilon_3 E_3] = e\varepsilon_5 E_5$
If $b = c = 0$ then we obtain $\varepsilon_4 = \varepsilon_1\varepsilon_2$, $\varepsilon_2 = \varepsilon_5 = \varepsilon_2\varepsilon_3$, hence $\varepsilon_3 = 1$ and $\varepsilon_1\varepsilon_4 = \varepsilon_2 = \varepsilon_5$. It follows that the group of orthogonal automorphisms of $\mathfrak{n}_{(a,0,0,d,e)}$ is isomorphic to the group given by (7).
For $b = 0$, $c > 0$ we get $\varepsilon_4 = \varepsilon_1\varepsilon_2$, $\varepsilon_5 = \varepsilon_1\varepsilon_3 = \varepsilon_1\varepsilon_4 = \varepsilon_2\varepsilon_3$. Then one has

$\varepsilon_3 = 1 = \varepsilon_4$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_5$. Hence the group of orthogonal automorphisms of $\mathfrak{n}_{(a,0,c,d,e)}$ is isomorphic to the group (8).
The relations $b > 0$, $c = 0$ yield $\varepsilon_4 = \varepsilon_5 = \varepsilon_1\varepsilon_2 = \varepsilon_1\varepsilon_4 = \varepsilon_2\varepsilon_3$. Hence $\varepsilon_1 = \varepsilon_3 = 1$ and $\varepsilon_4 = \varepsilon_5 = \varepsilon_2$. Consequently, the group of orthogonal automorphisms of $\mathfrak{n}_{(a,b,0,d,e)}$ is isomorphic to the group (9).
If $b, c > 0$, then we obtain $\varepsilon_4 = \varepsilon_5 = \varepsilon_1\varepsilon_2 = \varepsilon_1\varepsilon_3 = \varepsilon_1\varepsilon_4 = \varepsilon_2\varepsilon_3$. Hence one has $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 1$, i.e. the group of orthogonal automorphisms of $\mathfrak{n}_{5,5}(a,b,c,d,e)$ is trivial. This yields the second assertion. □

Corollary 12. *Let $(N_{5,5}(a,b,c,d,e), \langle .,. \rangle)$ be the connected and simply connected Riemannian nilmanifold corresponding to the metric Lie algebra $(\mathfrak{n}_{5,5}(a,b,c,d,e), \langle .,. \rangle)$. The isometry group of $(N_{5,5}(a,b,c,d,e), \langle .,. \rangle)$ is*

$$\mathcal{I}(N_{5,5}(a,b,c,d,e)) = \left\{ \begin{array}{lll} \mathbb{Z}_2 \times \mathbb{Z}_2 \ltimes N_{5,5}(a,b,c,d,e) & \textit{if} & b = c = 0, \\ \mathbb{Z}_2 \ltimes N_{5,5}(a,b,c,d,e) & \textit{if} & b > 0, c = 0 \ \textit{or} \\ & & b = 0, c > 0, \\ N_{5,5}(a,b,c,d,e) & \textit{if} & b > 0, c > 0. \end{array} \right\}$$

4.2. Two-dimensional center

We consider the Lie algebra $\mathfrak{l}_{5,9}$ with its canonical basis $\{G_1, G_2, G_3, G_4, G_5\}$. Its center and commutator subalgebra are $Z(\mathfrak{l}_{5,9}) = \text{span}(G_4, G_5)$, respectively $\mathfrak{l}'_{5,9} = \text{span}(G_3, G_4, G_5)$.

Definition 6. Let $\{E_1, E_2, E_3, E_4, E_5\}$ be an orthonormal basis in the Euclidean vector space $\mathbb{E}^5$. Denote by $\mathfrak{n}_{5,9}(k,l,m,p,q)$, $k,l,m,p,q \in \mathbb{R}$ with $k,p,q \neq 0$ the metric Lie algebra defined on $\mathbb{E}^5$ by the non-vanishing commutators

$$[E_1, E_2] = kE_3 + lE_4 + mE_5, \ [E_1, E_3] = pE_4, \ [E_2, E_3] = qE_5. \qquad (11)$$

Denote by $\tilde{\mathfrak{n}}_{5,9}(k,l,p)$ the metric Lie algebra $\mathfrak{n}_{5,9}(k,l,0,p,p)$, i.e. in the case that $m = 0$, $p = q$.

Clearly, bracket operation (11) satisfies the Jacobi identity and the map

$$E_1 \mapsto E_1 + \frac{m}{q}E_3, \ E_2 \mapsto E_2 - \frac{l}{p}E_3, \ E_3 \mapsto kE_3, \ E_4 \mapsto pkE_4, \ E_5 \mapsto qkE_5$$

is an isomorphism $\mathfrak{n}_{5,9}(k,l,m,p,q) \to \mathfrak{l}_{5,9}$.
We investigate now the existence of one-parameter subgroups in the group of orthogonal automorphisms of the metric Lie algebra $\mathfrak{n}_{5,9}(k,l,m,p,q)$.

Lemma 13. *The group $\mathcal{OA}(\mathfrak{n}_{5,9}(k,l,m,p,q))$ of orthogonal automorphisms of $\mathfrak{n}_{5,9}(k,l,m,p,q)$ contains a one-parameter subgroup if and only if $l = m = p - q = 0$. In this case the connected component of $\mathcal{OA}(\mathfrak{l}_{5,9})$ is the one-parameter group*

$$\{TE_1 = \cos tE_1 + \sin tE_2, TE_2 = -\sin tE_1 + \cos tE_2, TE_3 = E_3,$$

$$TE_4 = \cos tE_4 + \sin tE_5, TE_5 = -\sin tE_4 + \cos tE_5,\ t \in [0, 2\pi)\},$$

Otherwise, $\mathcal{OA}(\mathfrak{l}_{5,9})$ is a subgroup of the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Any one-parameter subgroup $\{\alpha_t;\ t \in \mathbb{R}\}$ of $\mathcal{OA}(\mathfrak{n}_{5,9}(k,l,m,p,q))$ preserves the center $Z(\mathfrak{l}_{5,9}) = \mathrm{span}(E_4, E_5)$, the commutator subalgebra $\mathfrak{l}'_{5,9} = \mathrm{span}(E_3, E_4, E_5)$, the orthogonal complement $\mathrm{span}(E_3)$ of $Z(\mathfrak{l}_{5,9})$ in $\mathfrak{l}'_{5,9}$ and the orthogonal complement $\mathrm{span}(E_1, E_2)$ of $\mathfrak{l}'_{5,9}$. Hence $\{\alpha_t;\ t \in \mathbb{R}\}$ induces rotations in $\mathrm{span}(E_1, E_2)$, $\mathrm{span}(E_4, E_5)$ and fixes $\mathrm{span}(E_3)$:

$$\begin{aligned} \alpha_t(E_1) = \cos t\, E_1 + \sin t\, E_2, \quad \alpha_t(E_2) = -\sin t\, E_1 + \cos t\, E_2, \quad \alpha_t(E_3) = E_3, \\ \alpha_t(E_4) = \cos\gamma\, t\, E_4 + \sin\gamma\, t\, E_5, \quad \alpha_t(E_5) = -\sin\gamma\, t\, E_4 + \cos\gamma\, t\, E_5, \end{aligned} \tag{12}$$

where $t \in \mathbb{R}$ and $\gamma \in \mathbb{R}$ is some constant. According to relations (11) we get

$$\alpha_t([E_1, E_2]) = [\alpha_t(E_1), \alpha_t(E_2)] = [E_1, E_2],$$

or equivalently

$$l(\cos\gamma\, t\, E_4 + \sin\gamma\, t\, E_5) + m(-\sin\gamma\, t\, E_4 + \cos\gamma\, t\, E_5) = lE_4 + mE_5, \quad t \in \mathbb{R},$$

from which follows either $\gamma = 0$ or $l = m = 0$. If $\gamma = 0$ then we have

$$\begin{aligned} \cos t\,[E_1, E_3] + \sin t\,[E_2, E_3] = [\alpha_t(E_1), E_3] = pE_4, \\ -\sin t\,[E_1, E_3] + \cos t\,[E_2, E_3] = [\alpha_t(E_2), E_3] = qE_5 \end{aligned}$$

for all $t \in \mathbb{R}$, giving a contradiction. Hence $l = m = 0$ and we have

$$[E_1, E_2] = k\, E_3, \quad [E_1, E_3] = p\, E_4, \quad [E_2, E_3] = q\, E_5, \quad k, p, q \neq 0.$$

Using $\alpha_t([E_1, E_3]) = [\alpha_t(E_1), E_3]$ we get

$$p(\cos\gamma\, t\, E_4 + \sin\gamma\, t\, E_5) = [\cos t\, E_1 + \sin t\, E_2, E_3] = \cos t\, p\, E_4 + \sin t\, q\, E_5.$$

It follows $\gamma = 1$ and $p = q$. Conversely, for $l = m = p - q = 0$ the maps given by (12) with $\gamma = 1$ are clearly automorphism of $\mathfrak{n}_{5,9}(k, 0, 0, p, p)$. □

Theorem 14. *Let $\langle .,. \rangle$ be an inner product on the 5-dimensional three-step nilpotent Lie algebra $\mathfrak{l}_{5,9}$.*

(1) *The metric Lie algebra $(\mathfrak{l}_{5,9}, \langle .,. \rangle)$ is isometrically isomorphic to a unique $\mathfrak{n}_{5,9}(k,l,m,p,q)$ with $k,l,m,p,q \in \mathbb{R}$ such that $k > 0$, $q > p > 0$ and $l, m \geq 0$, or to a unique $\tilde{\mathfrak{n}}_{5,9}(k,l,p)$ with $k,l,p \in \mathbb{R}$ such that $k, p > 0$ and $l \geq 0$.*

(2) *The group of orthogonal automorphisms is the following group with respect to the basis $\{E_1, E_2, E_3, E_4, E_5\}$:*

(A) *for $\mathfrak{n}_{5,9}(k,l,m,p,q)$*

(i) *if $l = m = 0$:*

$$\mathcal{OA}(\mathfrak{n}_{5,9}(k,0,0,p,q)) = \{TE_1 = \varepsilon_1 E_1, TE_2 = \varepsilon_2 E_2, TE_3 = \varepsilon_1\varepsilon_2 E_3, TE_4 = \varepsilon_2 E_4, TE_5 = \varepsilon_1 E_5,\ \varepsilon_1, \varepsilon_2 = \pm 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2,$$

(ii) *if $l = 0$, $m > 0$:*

$$\mathcal{OA}(\mathfrak{n}_{5,9}(k,0,m,p,q)) = \{TE_1 = \varepsilon_1 E_1, TE_2 = E_2, TE_3 = \varepsilon_1 E_3, TE_4 = E_4, TE_5 = \varepsilon_1 E_5, \varepsilon_1 = \pm 1\} \cong \mathbb{Z}_2,$$

(iii) *if $l > 0$, $m = 0$:*

$$\mathcal{OA}(\mathfrak{n}_{5,9}(k,l,0,p,q)) = \{TE_1 = E_1, TE_2 = \varepsilon_2 E_2, TE_3 = \varepsilon_2 E_3, TE_4 = \varepsilon_2 E_4, TE_5 = E_5, \varepsilon_2 = \pm 1\} \cong \mathbb{Z}_2, \tag{13}$$

(iv) *if $l, m > 0$, then it is trivial;*

(B) *for $\tilde{\mathfrak{n}}_{5,9}(k,l,p)$:*

(i) *if $l = 0$: $\mathcal{OA}(\tilde{\mathfrak{n}}_{5,9}(k,0,p)) =$*

$$\{TE_1 = \cos tE_1 + \varepsilon_2 \sin tE_2, TE_2 = -\sin tE_1 + \varepsilon_2 \cos tE_2, TE_3 = \varepsilon_2 E_3, TE_4 = \varepsilon_2 \cos tE_4 + \sin tE_5, TE_5 = \cos tE_5 - \varepsilon_2 \sin tE_4, \varepsilon_2 = \pm 1, t \in [0, 2\pi)\}, \tag{14}$$

(ii) *if $l > 0$, then it is group (13).*

Proof. Let be given an inner product $\langle .,. \rangle$ on $\mathfrak{l}_{5,9}$. The Gram-Schmidt process applied to $(G_5, G_4, G_3, G_2, G_1)$ gives an orthonormal basis $G_i^* = \sum_{k=i}^{5} a_{ik} G_k$, $i = 5, \dots, 1$ of $\mathfrak{l}_{5,9}$ with $a_{ii} > 0$, where G_i^* are positive multiples of G_i modulo $\mathrm{span}(G_j; j > i)$ and orthogonal to $\mathrm{span}(G_j; j > i)$ and G_5^*, $G_4^* \in Z(\mathfrak{l}_{5,9})$, $G_3^* \in \mathfrak{l}'_{5,9}$. Consequently the commutation relations of $(\mathfrak{l}_{5,9}, \langle .,. \rangle)$ with respect to the orthonormal basis $\{G_1^*, G_2^*, G_3^*, G_4^*, G_5^*\}$ are of the form

$$[G_1^*, G_2^*] = k\, G_3^* + l\, G_4^* + m\, G_5^*, \;\; [G_1^*, G_3^*] = p\, G_4^* + r\, G_5^*, \;\; [G_2^*, G_3^*] = q\, G_5^*, \quad (15)$$

with some $k, l, m, p, q, r \in \mathbb{R}$, $k, p, q > 0$.

We notice that any orthogonal automorphism of $(\mathfrak{l}_{5,9}, \langle .,. \rangle)$ preserves the subspaces $Z(\mathfrak{l}_{5,9}) = \mathrm{span}(G_4^*, G_5^*)$, $\mathfrak{l}'_{5,9} = \mathrm{span}(G_3^*, G_4^*, G_5^*)$, the orthogonal complement $\mathrm{span}(G_1^*, G_2^*)$ of $\mathfrak{l}'_{5,9}$ and the orthogonal complement $\mathrm{span}(G_3^*)$ of $Z(\mathfrak{l}_{5,9})$ in $\mathfrak{l}'_{5,9}$. We consider the one-parameter family $\{\alpha_t;\; t \in \mathbb{R}\}$ of linear maps $\alpha_t : (\mathfrak{l}_{5,9}, \langle .,. \rangle) \to (\mathfrak{l}_{5,9}, \langle .,. \rangle)$ defined by

$$\begin{aligned} &\alpha_t(G_1^*) = \cos t\, G_1^* + \sin t\, G_2^*, \quad \alpha_t(G_2^*) = -\sin t\, G_1^* + \cos t\, G_2^*, \\ &\alpha_t(G_3^*) = G_3^*, \quad \alpha_t(G_4^*) = \frac{\Phi_t^{(1)}}{\|\Phi_t^{(1)}\|}, \quad \alpha_t(G_5^*) = \frac{\Phi_t^{(2)}}{\|\Phi_t^{(2)}\|}, \end{aligned} \quad (16)$$

where

$$\begin{aligned} \Phi_t^{(1)} &= [\alpha_t(G_1^*), G_3^*] = p\, \cos t\, G_4^* + (r\, \cos t + q\, \sin t\,)\; G_5^*, \\ \Phi_t^{(2)} &= [\alpha_t(G_2^*), G_3^*] = -p\, \sin t\, G_4^* + (-r\, \sin t + q\, \cos t)\; G_5^*. \end{aligned} \quad (17)$$

Clearly, $\alpha_t(G_1^*)$, $\alpha_t(G_2^*)$ and $\alpha_t(G_3^*)$ are orthogonal unit vectors, the vectors $\alpha_t(G_4^*)$ and $\alpha_t(G_5^*)$ are linearly independent and contained in the center $Z(\mathfrak{l}_{5,9})$, moreover

$$[\alpha_t(G_1^*), \alpha_t(G_2^*)] = [G_1^*, G_2^*] = k\, G_3^*, \qquad \mathrm{mod}\;\; Z(\mathfrak{l}_{5,9}), \quad k > 0.$$

The vectors $\alpha_t(G_4^*)$ and $\alpha_t(G_5^*)$ are orthogonal if and only if $\langle \Phi_t^{(1)}, \Phi_t^{(2)} \rangle = 0$, or equivalently

$$\frac{1}{2} \left(q^2 - p^2 - r^2 \right) \sin 2t + qr\, \cos 2t = 0, \quad p,\, q > 0. \quad (18)$$

This equation shows that the vectors $\alpha_t(G_4^*)$ and $\alpha_t(G_5^*)$ are orthogonal for all $t \in \mathbb{R}$ if and only if $r = q - p = 0$, otherwise there is a unique $0 \le t_0 < \frac{\pi}{2}$ such that $\alpha_t(G_4^*)$ and $\alpha_t(G_5^*)$ are orthogonal if and only if $t = t_0 + k\frac{\pi}{2}$, $k \in \mathbb{Z}$.

Lemma 15. *The metric Lie algebras* $(\mathfrak{l}_{5,9}, \langle .,. \rangle)$ *can be classified into three types:*

(A) The family $\{\alpha_t;\ t \in \mathbb{R}\}$ *is a one-parameter group of orthogonal automorphisms of* $(\mathfrak{l}_{5,9}, \langle .,. \rangle)$ *if and only if* $(\mathfrak{l}_{5,9}, \langle .,. \rangle)$ *is isometrically isomorphic to a unique metric Lie algebra* $\tilde{\mathfrak{n}}_{5,9}(k, 0, p)$ *with* $k, p > 0$.

(B) Each element of the family $\{\alpha_t;\ t \in \mathbb{R}\}$ *is an orthogonal map, but not all are automorphisms if and only if* $(\mathfrak{l}_{5,9}, \langle .,. \rangle)$ *is isometrically isomorphic to a unique metric Lie algebra* $\tilde{\mathfrak{n}}_{5,9}(k, l, p)$ *with* $k, l, p > 0$. *In this case* $(\mathfrak{l}_{5,9}, \langle .,. \rangle)$ *has a framing.*

(C) There is only one orthogonal map in the set $\{\alpha_t;\ t \in [0, \frac{\pi}{2})\}$ *of linear maps if and only if* $(\mathfrak{l}_{5,9}, \langle .,. \rangle)$ *is isometrically isomorphic to a unique metric Lie algebra* $\mathfrak{n}_{5,9}(k, l, m, p, q)$ *with* $k > 0$, $q > p > 0$ *and* $l, m \geq 0$.

Proof. Assume that $\Phi_t^{(1)}$ and $\Phi_t^{(2)}$ are orthogonal for any $t \in \mathbb{R}$. This is the case if and only if $r = 0$ and $p = q$, according to (18), and hence

$$[G_1^*, G_2^*] = k\, G_3^* + l\, G_4^* + m\, G_5^*, \quad [G_1^*, G_3^*] = p\, G_4^*, \quad [G_2^*, G_3^*] = p\, G_5^*. \quad (19)$$

We get that $(\mathfrak{l}_{5,9}, \langle .,. \rangle)$ is isometrically isomorphic to some $\mathfrak{n}_{5,9}(k, l, m, p, p)$, in particular if $m = 0$, to some $\tilde{\mathfrak{n}}_{5,9}(k, l, p)$ with $k, l, p > 0$.
It follows from (17) that

$$G_4^* = \frac{1}{p}\left(\cos t \Phi_t^{(1)} - \sin t \Phi_t^{(2)}\right), \quad G_5^* = \frac{1}{p}\left(\sin t \Phi_t^{(1)} + \cos t \Phi_t^{(2)}\right). \quad (20)$$

Putting these expressions into the equation $[\alpha_t(G_1^*), \alpha_t(G_2^*)] = kG_3^* + lG_4^* + mG_5^*$ we get

$$[\alpha_t(G_1^*), \alpha_t(G_2^*)] = k\, G_3^* + \frac{1}{p}\left\{(l \cos t + m \sin t)\Phi_t^{(1)} + (-l \sin t + m \cos t)\Phi_t^{(2)}\right\}.$$

Assuming $l = m = 0$, or equivalently that the center $Z(\mathfrak{l}_{5,9})$ is orthogonal to the Lie bracket of any two vectors contained in $\mathrm{span}(\alpha_t(G_1^*), \alpha_t(G_2^*))$, we obtain using (16) and (17)

$$[\alpha_t(G_1^*), \alpha_t(G_2^*)] = k\, \alpha_t(G_3^*), \quad [\alpha_t(G_1^*), \alpha_t(G_3^*)] = p\, \alpha_t(G_4^*),$$

$$\alpha_t(G_2^*), \alpha_t(G_3^*)] = p\, \alpha_t(G_5^*).$$

The coefficients in these Lie brackets are independent of $t \in \mathbb{R}$, hence $\{\alpha_t;\ t \in \mathbb{R}\}$ is a one-parameter group of orthogonal automorphisms.

If $(l,m) \neq (0,0)$ we have a unique $\bar{t} \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ such that the solutions of the equation $-l\sin t + m\cos t = 0$ are $t = \bar{t} + k\,\pi$, $k \in \mathbb{Z}$ and $\alpha_{\bar{t}}(G_i^*)$, $i = 1, \ldots, 5$, satisfy

$$[\alpha_{\bar{t}}(G_1^*), \alpha_{\bar{t}}(G_2^*)] = \bar{k}\,\alpha_{\bar{t}}(G_3^*) + \bar{l}\,\alpha_{\bar{t}}(G_4^*),$$
$$[\alpha_{\bar{t}}(G_1^*), \alpha_{\bar{t}}(G_3^*)] = \bar{p}\,\alpha_{\bar{t}}(G_4^*), \quad [\alpha_{\bar{t}}(G_2^*), \alpha_{\bar{t}}(G_3^*)] = \bar{p}\,\alpha_{\bar{t}}(G_5^*) \tag{21}$$

with $\bar{k}, \bar{l}, \bar{p} \in \mathbb{R},\ \bar{k}, \bar{p} > 0,\ \bar{l} \neq 0$. The 1-dimensional subspaces $\mathrm{span}(\alpha_{\bar{t}}(G_i^*))$, $i = 1, \ldots, 5,$ are invariant with respect to any orthogonal automorphisms of $(\mathfrak{l}_{5,9}, \langle .,. \rangle)$, hence the decomposition $\mathbb{R}\,\alpha_{\bar{t}}(G_1^*) \oplus \cdots \oplus \mathbb{R}\,\alpha_{\bar{t}}(G_5^*)$ is a framing of $(\mathfrak{l}_{5,9}, \langle .,. \rangle)$. It follows that $(\mathfrak{l}_{5,9}, \langle .,. \rangle)$ has no one-parameter groups of orthogonal automorphisms. Changing the basis $\alpha_{\bar{t}}(G_i^*) \mapsto (-1)^i \alpha_{\bar{t}}(G_i^*)$, $i = 1, \ldots, 5$, we get from (21) that $[\alpha_{\bar{t}}(G_1^*), \alpha_{\bar{t}}(G_2^*)] = \bar{k}\,\alpha_{\bar{t}}(G_3^*) - \bar{l}\,\alpha_{\bar{t}}(G_4^*)$, hence we can assume $\bar{l} > 0$, consequently $(\mathfrak{l}_{5,9}, \langle .,. \rangle)$ is isometrically isomorphic to some $\tilde{\mathfrak{n}}_{5,9}(\bar{k}, \bar{l}, \bar{p})$ with $\bar{k}, \bar{l}, \bar{p} > 0$. The uniqueness of $\tilde{\mathfrak{n}}_{5,9}(\bar{k}, \bar{l}, \bar{p})$ with $\bar{k}, \bar{l}, \bar{p} > 0$, isometrically isomorphic to $(\mathfrak{l}_{5,9}, \langle .,. \rangle)$, follows from Lemma 1, hence assertion (B) is true.

In the case $l = m = 0$ for any isometric isomorphism $\Phi : \tilde{\mathfrak{n}}_{5,9}(k, 0, p) \to \tilde{\mathfrak{n}}_{5,9}(k', 0, p')$ one has

$$\Phi(G_1^*) = \pm(\cos t\, G_1^* + \sin t\, G_2^*), \Phi(G_2^*) = \pm(\cos t\, G_2^* - \sin t\, G_1^*), \Phi(G_3^*) = \pm\, G_3,$$

from which follows $k = k'$, $p = p'$. This means that $\tilde{\mathfrak{n}}_{5,9}(k, 0, p)$ is uniquely determined, giving the proof of assertion (A).

Assume now, that $\Phi_t^{(1)}$ and $\Phi_t^{(2)}$ are not orthogonal for some $t \in \mathbb{R}$. In this case there is a unique $0 \leq t_0 < \frac{\pi}{2}$ such that $\alpha_t(G_4^*)$ and $\alpha_t(G_5^*)$ are orthogonal if and only if $t = t_0 + k\frac{\pi}{2}$, $k \in \mathbb{Z}$. Moreover, the orthogonal basis $\{\alpha_t(G_1^*), \alpha_t(G_2^*), \alpha_t(G_3^*), \alpha_t(G_4^*), \alpha_t(G_5^*)\}$ satisfies

$$\begin{aligned} [\alpha_t(G_1^*), \alpha_t(G_2^*)] &= \bar{k}\,\alpha_t(G_3^*) + \bar{l}\,\alpha_t(G_4^*) + \bar{m}\,\alpha_t(G_5^*), \\ [\alpha_t(G_1^*), \alpha_t(G_3^*)] = \bar{p}\,\alpha_t(G_4^*),&\ [\alpha_t(G_2^*), \alpha_t(G_3^*)] = \bar{q}\,\alpha_t(G_5^*), \end{aligned} \tag{22}$$

with some $\bar{k}, \bar{l}, \bar{m}, \bar{p}, \bar{q} \in \mathbb{R},\ \bar{k}, \bar{p}, \bar{q} > 0$, where $\bar{p} = \|\Phi_t^{(1)}\| \neq \bar{q} = \|\Phi_t^{(2)}\|$. The change of the value from t to $t + \frac{\pi}{2}$ leads to the exchange of the subspaces $\mathrm{span}(\alpha_t(G_1^*))$ and $\mathrm{span}(\alpha_t(G_2^*))$, hence we can assume that the inequality $\|\Phi_t^{(1)}\| < \|\Phi_t^{(2)}\|$ holds. The decomposition $\mathbb{R}\,\alpha_t(G_1^*) \oplus \cdots \oplus \mathbb{R}\,\alpha_t(G_5^*)$ corresponding to this choice of $t \in \mathbb{R}$ is a framing of the metric Lie algebra $(\mathfrak{l}_{5,9}, \langle .,. \rangle)$ since the one-dimensional subspaces $\mathrm{span}(\alpha_t(G_i^*))$, $i = 1, \ldots, 5$, are invariant with respect to any orthogonal automorphisms of $(\mathfrak{l}_{5,9}, \langle .,. \rangle)$.

Changing the basis $\alpha_{t^*}(G_i^*) \mapsto (-1)^i \alpha_{t^*}(G_i^*)$, $i = 1, \ldots, 5$, we obtain instead of the first equation of (22) that $[\alpha_{t^*}(G_1^*), \alpha_{t^*}(G_2^*)] = \bar{k}\, \alpha_{t^*}(G_3^*) - \bar{l}\, \alpha_{t^*}(G_4^*) + \bar{m}\, \alpha_{t^*}(G_5^*)$. With the change $\alpha_{t^*}(G_1^*) \mapsto \alpha_{t^*}(G_1^*)$, $\alpha_{t^*}(G_2^*) \mapsto -\alpha_{t^*}(G_2^*)$, $\alpha_{t^*}(G_3^*) \mapsto -\alpha_{t^*}(G_3^*)$, $\alpha_{t^*}(G_4^*) \mapsto -\alpha_{t^*}(G_4^*)$, $\alpha_{t^*}(G_5^*) \mapsto \alpha_{t^*}(G_5^*)$ we get $[\alpha_{t^*}(G_1^*), \alpha_{t^*}(G_2^*)] = \bar{k}\, \alpha_{t^*}(G_3^*) + \bar{l}\, \alpha_{t^*}(G_4^*) - \bar{m}\, \alpha_{t^*}(G_5^*)$. Hence we can assume that the coefficients of commutators (22) satisfy $\bar{k}, \bar{p}, \bar{q} > 0$ and $\bar{l}, \bar{m} \geq 0$. It follows that there exists an isometric isomorphism $\mathfrak{l}_{5,9} \to \mathfrak{n}_{5,9}(\bar{k}, \bar{l}, \bar{m}, \bar{p}, \bar{q})$ such that $\bar{k}, \bar{p}, \bar{q} > 0$ and $\bar{l}, \bar{m} \geq 0$ and according to Lemma 1 this isometrically isomorphic Lie algebra $\mathfrak{n}_{5,9}(\bar{k}, \bar{l}, \bar{m}, \bar{p}, \bar{q})$ is uniquely determined. This proves assertion (C). Since the investigated types define disjoint classes of metric Lie algebras and their union contains all metric Lie algebras $\mathfrak{l}_{5,9}$ the lemma is proved. □

The metric Lie algebras $\mathfrak{n}_{5,9}(k, l, m, p, q)$ with $k, p, q > 0$ and $l, m \geq 0$ and $\tilde{\mathfrak{n}}_{5,9}(k, l, p)$ with $k, l, p > 0$ are framed, hence if T is an orthogonal automorphism then one has $T(E_i) = \varepsilon_i E_i$ and $[T(E_i), T(E_j)] = T([E_i, E_j])$, $i, j = 1, \ldots, 5$, where $\varepsilon_i = \pm 1$.

Let $l = m = 0$. From the Lie brackets $[\varepsilon_1 E_1, \varepsilon_2 E_2] = k\varepsilon_3 E_3$, $[\varepsilon_1 E_1, \varepsilon_3 E_3] = p\varepsilon_4 E_4$, $[\varepsilon_2 E_2, \varepsilon_3 E_3] = q\varepsilon_5 E_5$ we obtain $\varepsilon_3 = \varepsilon_1\varepsilon_2$, $\varepsilon_4 = \varepsilon_1\varepsilon_3$, $\varepsilon_5 = \varepsilon_2\varepsilon_3$, and hence $\varepsilon_2 = \varepsilon_4$, $\varepsilon_1 = \varepsilon_5$, $\varepsilon_3 = \varepsilon_1\varepsilon_2$. It follows assertion (2)(A)(i).

Let $l = 0$, $m > 0$. The Lie brackets $[\varepsilon_1 E_1, \varepsilon_2 E_2] = k\varepsilon_3 E_3 + m\varepsilon_5 E_5$, $[\varepsilon_1 E_1, \varepsilon_3 E_3] = p\varepsilon_4 E_4$, $[\varepsilon_2 E_2, \varepsilon_3 E_3] = q\varepsilon_5 E_5$ give $\varepsilon_3 = \varepsilon_1\varepsilon_2 = \varepsilon_5$, $\varepsilon_4 = \varepsilon_1\varepsilon_3$, $\varepsilon_5 = \varepsilon_2\varepsilon_3$. Then one has $\varepsilon_3 = \varepsilon_1 = \varepsilon_5$ and $\varepsilon_2 = \varepsilon_4 = 1$. Hence we obtain assertion (2)(A)(ii).

Let $l > 0$, $m = 0$. From the Lie brackets $[\varepsilon_1 E_1, \varepsilon_2 E_2] = k\varepsilon_3 E_3 + l\varepsilon_4 E_4$, $[\varepsilon_1 E_1, \varepsilon_3 E_3] = p\varepsilon_4 E_4$, $[\varepsilon_2 E_2, \varepsilon_3 E_3] = q\varepsilon_5 E_5$ one has $\varepsilon_3 = \varepsilon_1\varepsilon_2 = \varepsilon_4$, $\varepsilon_4 = \varepsilon_1\varepsilon_3$, $\varepsilon_5 = \varepsilon_2\varepsilon_3$. Hence we obtain $\varepsilon_3 = \varepsilon_4 = \varepsilon_2$ and $\varepsilon_1 = \varepsilon_5 = 1$, consequently assertions (2)(A)(iii) and (2)(B)(ii) are true.

If $m, l > 0$, then from $[\varepsilon_1 E_1, \varepsilon_2 E_2] = k\varepsilon_3 E_3 + l\varepsilon_4 E_4 + m\varepsilon_5 E_5$, $[\varepsilon_1 E_1, \varepsilon_3 E_3] = p\varepsilon_4 E_4$, $[\varepsilon_2 E_2, \varepsilon_3 E_3] = q\varepsilon_5 E_5$ follows $\varepsilon_4 = \varepsilon_5 = \varepsilon_1\varepsilon_2 = \varepsilon_3$, $\varepsilon_4 = \varepsilon_1\varepsilon_3$, $\varepsilon_5 = \varepsilon_2\varepsilon_3$. We obtain $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 1$, i.e. the group of orthogonal automorphisms of $\mathfrak{n}_{5,9}(l, m, n, p, q)$ is trivial and assertion (2)(A) is true.

If $T : \tilde{\mathfrak{n}}_{5,9}(l, 0, p) \to \tilde{\mathfrak{n}}_{5,9}(l, 0, p)$ is an orthogonal automorphism of $\tilde{\mathfrak{n}}_{5,9}(l, 0, p)$ then one has $TE_1 = \cos tE_1 - \sin tE_2$, $TE_2 = \varepsilon_2(\sin tE_1 + \cos tE_2)$, $TE_3 = \varepsilon_3 E_3$, $TE_4 = \varepsilon_4(\cos tE_4 - \sin tE_5)$, $TE_5 = \varepsilon_5(\sin tE_4 + \cos tE_5)$, $[TE_i, TE_j] =$

$T[E_i, E_j]$, for $i, j = 1, ..., 5$, where $t \in [0, 2\pi)$, $\varepsilon_i = \pm 1$. From the Lie brackets

$$[\cos tE_1 - \sin tE_2, \varepsilon_2(\sin tE_1 + \cos tE_2)] = \varepsilon_2 kE_3 = k\varepsilon_3 E_3,$$

$$[\cos tE_1 - \sin tE_2, \varepsilon_3 E_3] = \varepsilon_3 p(\cos tE_4 - \sin tE_5) = p\varepsilon_4(\cos tE_4 - \sin tE_5),$$

$$[\varepsilon_2(\sin tE_1 + \cos tE_2), \varepsilon_3 E_3] = \varepsilon_2\varepsilon_3 p(\sin tE_4 + \cos tE_5) = p\varepsilon_5(\sin tE_4 + \cos tE_5)$$

we obtain $\varepsilon_2 = \varepsilon_3$, $\varepsilon_3 = \varepsilon_4$, $\varepsilon_5 = \varepsilon_2\varepsilon_3$, and hence $\varepsilon_5 = 1$, $\varepsilon_2 = \varepsilon_3 = \varepsilon_4$. It follows that the group of orthogonal automorphisms of $\tilde{\mathfrak{n}}_{5,9}(l, 0, p)$ is isomorphic to group (14) in assertion $(2)(B)(i)$. Since the Lie algebra $\tilde{\mathfrak{n}}_{5,9}(l, m, p)$, $l > 0$, $m > 0$, $p > 0$ is framed for an orthogonal automorphism $T : \tilde{\mathfrak{n}}_{5,9}(l, m, p) \to \tilde{\mathfrak{n}}_{5,9}(l, m, p)$ of $\tilde{\mathfrak{n}}_{5,9}(l, m, p)$ we obtain the Lie brackets as for the Lie algebra $\mathfrak{n}_{5,9}(l, m, n, p, q)$ in case $(2)(A)(iii)$ taking $p = q$. Hence we have the same equations as for the Lie algebra $\mathfrak{n}_{5,9}(l, m, n, p, q)$ in case $(2)(A)(iii)$. This proves assertion $(2)(B)(ii)$. Hence Theorem 14 is proved. □

Corollary 16. *Let* $(N_{5,9}(k, l, m, p, q), \langle ., .\rangle)$, *respectively* $(\widetilde{N}_{5,9}(k, l, p), \langle ., .\rangle)$ *be the connected and simply connected Riemannian nilmanifolds belonging to the nilpotent metric Lie algebras* $(\mathfrak{n}_{5,9}(k, l, m, p, q), \langle ., .\rangle)$ *and* $(\widetilde{\mathfrak{n}}_{5,9}(k, l, p), \langle ., .\rangle)$. *Their isometry groups* $\mathcal{I}(N_{5,9}(k, l, m, p, q))$, *respectively* $\mathcal{I}(\widetilde{N}_{5,9}(k, l, p))$ *are the following groups:*

$$\mathcal{I}(N_{5,9}(k,l,m,p,q)) = \left\{ \begin{array}{lll} \mathbb{Z}_2 \times \mathbb{Z}_2 \ltimes N_{5,9}(k,l,m,p,q) & \textit{if} & m = l = 0, \\ \mathbb{Z}_2 \ltimes N_{5,9}(k,l,m,p,q) & \textit{if} & l > 0, m = 0 \ \textit{or} \\ & & l = 0, m > 0, \\ N_{5,9}(k,l,m,p,q) & \textit{if} & l > 0, m > 0. \end{array} \right\}$$

$$\mathcal{I}(\widetilde{N}_{5,9}(k,l,p)) = \left\{ \begin{array}{lll} \mathbb{Z}_2 \ltimes \widetilde{N}_{5,9}(k,l,p) & \textit{if} & l > 0 \\ \mathbb{Z}_2 \times SO(2) \ltimes \widetilde{N}_{5,9}(k,l,p) & \textit{if} & l = 0. \end{array} \right\}$$

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References

[1] G. Cairns, A. Hinić Galić, Yu. Nikolayevsky, Totally geodesic subalgebras of nilpotent Lie algebras, J. Lie Theory 23 (2013), 1023 – 1049.

[2] G. Cairns, A. Hinić Galić, Yu. Nikolayevsky, Totally geodesic subalgebras of filiform nilpotent Lie algebras, J. Lie Theory 23 (2013), 1051 – 1074.

[3] S. Console, A. Fino, E. Samiou, The Moduli Space of Six-Dimensional Two-Step Nilpotent Lie Algebras, Ann. Global Anal. Geom. 27 (2005), 17–32.

[4] W. A. de Graaf, Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2, J. Algebra 309 (2007), 640 – 653.

[5] S. Homolya, O. Kowalski, Simply connected two-step homogeneous nilmanifolds of dimension 5, Note Math. 26 (2006), 69-77.

[6] M. M. Kerr, T. L. Payne, The geometry of filiform nilpotent Lie groups, Rocky Mountain J. Math. 40 (2010), 1587–1610.

[7] J. Lauret, Homogeneous Nilmanifolds of Dimension 3 and 4, Geom. Dedicata 68 (1997), 145-155.

[8] D. V. Millionschikov, Graded Filiform Lie Algebras and Symplectic Nilmanifolds, in Geometry, Topology, and Mathematical Physics (Am. Math. Soc., Providence, RI, 2004), AMS Transl., II Ser. 212, pp. 259–279.

[9] M. Vergne, Réductibilité de la variété des algébres de Lie nilpotentes, C.R. Acad. Sci. Paris. 263 (1966), 4–6.

[10] E. Wilson, Isometry groups on homogeneous nilmanifolds, Geom. Dedicata 12 (1982), 337- 346.

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