THE POSSIBLE NUMBER OF ISLANDS ON THE SEA

PÉTER PÁL PACH, GABRIELLA PLUHÁR, ANDRÁS PONGRÁCZ,
AND CSABA SZABÓ

[1, 2, 4]Eötvös Loránd University, Department of Algebra and Number Theory, 1117 Budapest, Pázmány Péter sétány 1/c, Hungary
ppp24@cs.elte.hu
plugab@cs.elte.hu
csaba@cs.elte.hu
[3]Central European University, Hungary
pongeee@cs.elte.hu

Abstract. Let a height function f be a real valued function on $\mathbb{R}^2$. A connected subset of $\mathbb{R}^2$ is called an island, if there is a water level such that H is an island in the classical sense. We show that an island system is always laminar. Among others, in this paper we prove that the cardinality of a maximal laminar system is either countable or continuum.

AMS Subject Classification: 05D05, 54A25

Keywords and Phrases: maximal island system, countable, continuum

1. Introduction

Let S be a subset of $\mathbb{R}^n$ and f a real valued function on S. A subset $H \subseteq S$ is called an *f-island*, if there is an open set G containing the closure of H such that $f(x) < \inf\limits_H f$ for every $x \in G \setminus H$. For a real function f, the set of all f-islands is called a *system of islands.*

For motivation, let us imagine that f is a height function on the real plane. Then H is an f-island according to our definition iff it is a "real island" in the obvious sense, where the water level is some value of f.

The concept of systems of discrete islands was introduced by G. Czédli [1]. He considered a rectangular lake whose bottom is divided into $(m+2) \times (n+2)$ cells. In other words, we identify the bottom of the lake with the table $\{0, 1, \dots, m+1\} \times \{0, 1, \dots, n+1\}$. For every square of a rectangular grid a real number a_{ij} is given, its height. The height of the bottom (above sea level) is constant on each cell but definitely less than the height of the lake shore. Now a rectangle R is called a rectangular island iff there is a possible water level such that

R is an island of the lake in the usual sense. There are other examples requiring $m \times n$ cells; for example, a_{ij} may mean a colour on a gray-scale (before we convert the picture to black and white), transparency (against X-rays) or melting temperature. His motivation came from coding theory. Földes and Singhi [3] examined "full segments" of vectors, which are just rectangular islands of $1 \times n$ tables in our terminology. According to Theorem 4 of [3], $1 \times n$ tables A whose entries form the lexicographic length sequence of some binary maximal instantaneous code are characterized by $|A|$ many equations. This makes the maximum of $\{|A|\colon A \text{ is an } 1 \times n \text{ table}\}$ important in coding theory.

For the size of maximal systems of rectangular islands, upper- and lower bounds were established in [1] and [6]. These results can be summarized as follows: If $\mathcal{H}$ is a maximal system of rectangular islands on an m by n rectangle, then

$$m + n - 1 \leq |\mathcal{H}| \leq \lfloor mn + m + n - 1 \rfloor / 2,$$

where both the upper- and the lower bounds are sharp. Several papers have been published on the subject since, investigating various extensions and generalizations (see G. Pluhár [8], E. K. Horváth, Z. Németh and G. Pluhár [4] and E. K. Horváth, G. Horváth, Z. Németh and Cs. Szabó [5]). As a possible continuous generalisation, P. P. Pach and Zs. Lengvárszky introduced the notion of continuous rectangular islands [7]. Their main result is that the size of a maximal continuous rectangular system in $\mathbb{R}^n$ is either countable or continuum. They explicitly ask what can be said about islands of different shapes, for example, circle islands.

In this paper, we examine the maximum of the number of islands of arbitrary form. We give a general condition for the shape of the islands with which the size of a maximal system of islands is either countable or continuum. We show that these properties hold for the circles, and give an example for a countable maximal circle system of islands. We prove that all island systems are laminar, but not every laminar system is a system of islands for some f.

2. Definitions

We start with the general definition of an island. Let $\mathcal{S} \subseteq \mathcal{P}(\Omega)$ for some $\Omega \subseteq \mathbb{R}^n$ and let f be a real valued function on Ω. We call a subset $H \subseteq \Omega$ an *f-island* in $\mathcal{S}$ if $H \in \mathcal{S}$ and there exists a(n) (relatively) open set $G \subseteq \Omega$ such that G contains the closure of H and for any $x \in G \setminus H$ we have $f(x) < \inf_H f$. The set of all f-islands in $\mathcal{S}$ for some function f is called an *island system* in $\mathcal{S}$.

One of the most useful properties of a system of rectangular islands is laminarity. A system of subsets $\mathcal{H}$ of a set S is called *laminar* if for any two $P, Q \in \mathcal{H}$ either $P \subseteq Q$ or $Q \subseteq P$ or $P \cap Q = \emptyset$. The case when $\Omega = [0,1]^n$ and $\mathcal{S}$ is the set of rectangles (with sides parallel to the coordinate axes) was examined in [7]. It was proved that a system of rectangular islands is laminar and that a countable maximal laminar system of closed intervals is a system of islands. In section 3 we show that similar propositions hold in general. As in the discrete case, in the continuous case a system of islands is always laminar.

The n-dimensional open (closed) disc of radius r with center x is the collection of points of $\mathbb{R}^n$ of distance less (less or equal) than r from x. We denote the open disc with center x and radius r by $B(x,r)$ and the closed disc with center x and radius r by $\overline{B}(x,r)$.

For a subset H of $\mathbb{R}^n$ we denote the interior of H by $intH$, the boundary of H by ∂H and the closure of H by clH.

Finally, let us introduce a notion having central importance in the discrete case. A system $\mathcal{S}$ of subsets of a set H, is called *weakly independent* if for any $\{S, S_1, S_2, \ldots, S_n\} \subseteq \mathcal{S}$ and $S \subseteq \bigcup_{i=1}^{n} S_i$ we have $S \subseteq S_j$ for some $1 \leq j \leq n$. This notion is due to G. Czédli, A. P. Huhn and E. T. Schmidt and was first defined in [2]. Later we show that weakly independence can be transferred to the continuous case, as well. Furthermore, note that the system $\mathcal{S}$ is a *chain* if for any two $P, Q \in \mathcal{S}$ either $P \subseteq Q$ or $Q \subseteq P$.

3. Properties of island systems

In the discrete case and in the case of continuous rectangular islands laminarity was one of the most important property of island systems. We show that in the continuous case a system of islands is laminar, as well.

Proposition 1. *Let $\Omega \subseteq \mathbb{R}^n$. Let $\mathcal{S} \subseteq \mathcal{P}(\Omega)$ be a set of connected sets in Ω. A system of f-islands in $\mathcal{S}$ on Ω is laminar.*

Proof. Let S_1 and S_2 be f-islands in $\mathcal{S}$ on Ω. According to the island property there exist open sets G_1 and G_2 such that $cl(S_1) \subseteq G_1$, $cl(S_2) \subseteq G_2$ and $\inf_{S_i} f > f(x)$ for any $x \in G_i \backslash S_i$ $(i = 1, 2)$. Without loss of generality let us assume that $\inf_{S_2} f \leq \inf_{S_1} f$. Now, $(G_2 \setminus S_2) \cap S_1 = \emptyset$ because for an x in the intersection $\inf_{S_2} f > f(x) \geq \inf_{S_1} f$ would hold. Therefore, S_1 is the disjoint union of $S_1 \cap G_2$ and $S_1 \cap (\Omega \setminus clS_2)$, since $S_2 \subseteq clS_2 \subseteq G_2$. As S_1 is connected, $S_1 \cap G_2$ or $S_1 \cap (\Omega \setminus clS_2)$ is empty.

In the first case we have $S_1 \cap S_2 = \emptyset$, and in the second case $S_1 \subseteq S_2$. Since S_1 and S_2 can be arbitrarily chosen, laminarity is obtained. $\square$

Another important property, weakly independence, plays an important role in the discrete case ([1]). A system of discrete islands is always weakly independent, but in general laminar systems are not. Observe that a laminar system of closed intervals is always weakly independent. Indeed, assume that $I, I_1, I_2, \ldots, I_n$ are intervals in the laminar system $\mathcal{H}$ such that $I \subseteq \bigcup_{i=1}^{n} I_i$. If $I \cap I_i = \emptyset$ or $I_i \subsetneq I_j$ for some j, then we can omit I_i, and the union of the others still cover I. As I cannot be covered by finitely many disjoint proper closed subintervals of I, there must be some $1 \leq j \leq n$ such that $I \subseteq I_j$. Note that in the definition of weakly independence the finiteness of the covering system of intervals is essential. If we allow infinitely many intervals, a laminar system is not necessarily weakly independent.

Example 2. Let $\mathcal{H} = \left\{[0, \frac{1}{2} - \frac{1}{n}] \,|\, n \geq 3\right\} \cup \left\{[\frac{1}{2}, 1], [0, 1]\right\}$. Then $\mathcal{H}$ is laminar. $[0, 1] \subseteq [\frac{1}{2}, 1] \cup \bigcup_{n=3}^{\infty} [0, \frac{1}{2} - \frac{1}{n}]$ and $[0, 1]$ is not contained in any of the other intervals.

It is natural to ask whether every maximal laminar system $\mathcal{H}$ is a system of islands. In many special cases one can easily construct a real valued function f such that $\mathcal{H}$ is the system of f-islands. However, this is not always the case. We present an example for an uncountable maximal laminar system which is not a system of islands for any f. Then we show that for every countable maximal laminar system there exists such a function.

Proposition 3. *Let $\mathcal{H}$ be a maximal laminar system of closed intervals on $[0, 1]$. Assume that there exists a subset of at least two disjoint intervals, $\mathcal{J} \subset \mathcal{H}$ such that*

(1) $cl(\bigcup \mathcal{J}) = [0, 1]$,
(2) for any two intervals $I_1, I_2 \in \mathcal{J}$ there is a $J \in \mathcal{J}$ between I_1 and I_2,
(3) r is the right endpoint of an interval in $\mathcal{H}$ for every $r \notin \bigcup \mathcal{J}$.

Then $\mathcal{H}$ is not a system of islands.

Proof. Assume that there exists a real valued function f such that $\mathcal{H}$ is the system of f-islands. As every interval contains a rational point, $|\mathcal{J}|$ is countable. Let $\mathcal{J} = \{J_k \mid k \in \mathbb{N}\}$, where $J_k = [x_k, y_k]$. By the island property for every J_k there is an ε_k such that $f|_{(x_k - \varepsilon_k, x_k)} < \inf_{J_k} f$.

We may assume that $\varepsilon_k \to 0$. Now, we define a sequence of intervals, $[u_k, v_k]$, such that $[u_0, v_0] = [0, 1]$ and for $i \geq 1$

(i) $[u_{i+1}, v_{i+1}] \subset [u_i, v_i]$,
(ii) $|[u_i, v_i]| < \varepsilon_i$,
(iii) $[u_i, v_i] \cap J_i = \emptyset$,
(iv) u_i is the right endpoint of some interval of $\mathcal{J}$,
(v) v_i is the left endpoint of some interval of $\mathcal{J}$.

Suppose that $[u_{k-1}, v_{k-1}]$ is already defined. Let $J_t \in \mathcal{J}$ be a subinterval of $[u_{k-1}, v_{k-1}]$. By assumption (2) such a $J_t = [x_t, y_t]$ exists. Let $\varepsilon = \min\{\varepsilon_t, \varepsilon_k,\, x_t - u_{k-1}\}$. Now, by (1) and (2) and the choice of ε, there exist $u_k, v_k \in [x_t - \varepsilon, x_t]$ such that $[u_k, v_k]$ satisfies conditions (i)-(v). Let $\bigcap[u_k, v_k] = c$. By the choice of ε we have $f(c) < f(v_k)$ for every $k \in \mathbb{N}$. By (iii) c is not contained in any interval J_k. Thus by (3) c is the right endpoint of an island. Hence, there is a $\delta > 0$ such that $f(c) > f(a)$ for every $a \in [c, c + \delta)$. As $\lim v_k = c$, there is an n such that $v_n \in [c, c + \delta)$. Then $f(c) > f(v_n)$, which is a contradiction.

$\square$

Now, we give an example of a set of intervals satisfying the conditions of Proposition 3.

Example 4. Let $\mathcal{G}$ be the set of the closures of the intervals omitted at the construction of the Cantor set, and let $\mathcal{H} = \mathcal{G} \bigcup \{[0, r] \,|\, r \notin \bigcup \mathcal{G}\}$. Moreover, let $\mathcal{H}^C$ be a maximal laminar system containing $\mathcal{H}$. Clearly, $\mathcal{H}^C$ satisfies conditions (1)-(3) of Proprosition 3, hence it is not a system of islands.

Now, we consider the countable case.

Proposition 5. *Let $\Omega \subseteq \mathbb{R}^n$. Let $\mathcal{S} \subseteq \mathcal{P}(\Omega)$ be a set of bounded connected sets in Ω. A countable maximal laminar system $\mathcal{H}$ in $\mathcal{S}$ is a system of islands if and only if the distance of any two disjoint sets in $\mathcal{H}$ is positive.*

Proof. Let us prove the "only if" direction, first. Assume that we have two disjoint sets $H_1, H_2 \in \mathcal{H}$ such that $d(H_1, H_2) = 0$. If $\mathcal{H}$ is the system of f-islands, then there exist open sets G_1 and G_2 such that $cl(H_1) \subseteq G_1$, $cl(H_2) \subseteq G_2$ and $\inf\limits_{H_i} f = h_i > f(x)$ for any $x \in G_i \setminus H_i$ $(i = 1, 2)$. By the boundedness of H_1 and H_2 there is a point $p \in cl(H_1) \cap cl(H_2)$. Thus $H_1 \cap G_2 \neq \emptyset$ and $H_2 \cap G_1 \neq \emptyset$. Let $x_1 \in H_1 \cap G_2$ and $x_2 \in H_2 \cap G_1$. Then $f(x_1) \geq h_1 > f(x_2)$ and $f(x_2) \geq h_2 > f(x_1)$ a contradiction.

For the other direction let $\mathcal{H} = \{H_n \,|\, n \in \mathbb{N}\}$ be a maximal laminar system in $\mathcal{S}$ such that the distance of any two disjoint sets in $\mathcal{H}$ is positive. We show that these are the islands corresponding to the height function:

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{3^n} \chi_{H_n}(x),$$

where $\chi_{H_n}(x)$ is the characteristic function of H_n. According to Proposition 1 a system of islands is laminar, hence it is enough to prove that each H_k is an f-island. For all $n \in \mathbb{N}$ with $H_n \cap H_k = \emptyset$ the distance of H_k and H_n is positive. Let $\varepsilon_k = \min\{d(H_k, H_n) \mid n < k, H_k \cap H_n = \emptyset\}$. Note that $\varepsilon_k > 0$. Let $G_k = \Omega \cap \left(\bigcup_{s \in cl(H_k)} B(s, \varepsilon_k) \right)$. Then G_k is open, it contains $cl(H_k)$ and for all $n < k$ with $H_k \cap H_n = \emptyset$ we have $G_k \cap H_n = \emptyset$. If $x \in H_k$, then $f(x) \geq \sum_{\{n \mid H_k \subseteq H_n\}} \frac{1}{3^n}$. If $x \in G_k \setminus H_k$, then

$$f(x) \leq \sum_{\{n \mid H_k \subsetneq H_n\}} \frac{1}{3^n} + \sum_{n=k+1}^{\infty} \frac{1}{3^n} < \sum_{\{n \mid H_k \subsetneq H_n\}} \frac{1}{3^n} + \frac{1}{3^k} = \sum_{\{n \mid H_k \subseteq H_n\}} \frac{1}{3^n},$$

hence H_k is an f-island. □

4. Maximal laminar systems

In this section we construct maximal laminar systems. At first, the countable case is described. Then a few properties of maximal laminar systems are described, and finally we manage to answer the question of Zs. Lengvárszky and P. P. Pach. We present a countable maximal set of islands consisting of closed discs.

Theorem 6. *Let $\mathcal{A}$ be a system of subsets of $\mathbb{R}^n$ with finite Lebesgue-measure such that $\mathcal{A}$ satisfies the following conditions:*

(1) $int(A) \neq \emptyset$ for every $A \in \mathcal{A}$,
(2) if $A, B \in \mathcal{A}$ and $A \nsubseteq B$, then $\lambda(A \setminus B) > 0$,
(3) if $\mathcal{C} \subseteq \mathcal{A}$ and $\mathcal{C}$ is a chain, then $\bigcap \mathcal{C} \in \mathcal{A}$ or $\lambda(\bigcap \mathcal{C}) = 0$.

The cardinality of a maximal laminar system in $\mathcal{A}$ is countable or continuum.

Proof. Let $\mathcal{H}$ be a maximal laminar system in $\mathcal{A}$. At first, we prove that every $\mathcal{C} \subseteq \mathcal{H}$ maximal chain is countable or has cardinality continuum. Let us consider the set $R = \lambda(\mathcal{C}) = \{\lambda(C) \mid C \in \mathcal{C}\} \subseteq \mathbb{R}$. We prove that if x is a right limit point of R, then $x \in R$ or $x = 0$. Suppose that x is a right limit point and $x > 0$. Let us denote the set of elements of $\mathcal{C}$ with Lebesgue-measure greater than x by $\mathcal{C}^x$. As $\lambda(\bigcap \mathcal{C}^x) = x > 0$, (3) implies that $\bigcap \mathcal{C}^x \in \mathcal{A}$, so $x \in R$. Let clR be the closure of R and

$x \in clR \setminus R$. Then either x is a left-, but not right limit point of R or $x = 0$. For every x which is not a right limit point of R there exists a $y > x$ rational number such that $(x, y) \cap R = \emptyset$. Clearly, for distinct left (but not right) limit points the corresponding rational number y is different, therefore the cardinality of the set of left-, but not right limit points is countable. It follows that $clR \setminus R$ is countable. It is well-known that the cardinality of a closed set in $\mathbb{R}$ is either countable or continuum, hence $|R| \leq \aleph_0$ or $|R| = 2^{\aleph_0}$. As $\mathcal{C}$ is a chain, by (2) we get $\lambda(C_1) \neq \lambda(C_2)$ for any $C_1 \neq C_2$. Therefore, $|\mathcal{C}| = |R|$, so the cardinality of a maximal chain in $\mathcal{H}$ is either countable or continuum.

For an $r \in \mathbb{R}^n$ let $\mathcal{C}_r$ denote the set of elements of $\mathcal{H}$ that contain r. Note that $\mathcal{C}_r$ is a chain. By (1), every element of $\mathcal{H}$ contains a rational point, thus $\mathcal{H} = \bigcup_{a \in \mathbb{Q}^n} \mathcal{C}_a$. Every chain is a subset of a maximal chain. Hence, $\mathcal{H}$ is the union of countably many maximal chains. Thus if there is a maximal chain of length continuum, then $|\mathcal{H}|$ has cardinality continuum, and $|\mathcal{H}|$ is countable, otherwise. $\square$

Note that condition (1) can substituted by

(1') $\lambda(A) > 0$ for every $A \in \mathcal{A}$,

but the proof would be too technical.

Moreover, we can omit the finitenes of the Lebesgue-measure. The proof is essentially the same, except that the role of $\lambda(C)$ will be substituted by $\gamma(C)$, where $\gamma(C)$ is defined in the following way: let $\mathbb{R}^n$ be the disjoint union of $A_1, A_2, \ldots$, where all A_i are subsets of $\mathbb{R}^n$ of Lebesgue-measure 1, and let $\gamma(C) = \sum \lambda(A_i \cap C)/2^i$.

Corollary 7. *A maximal laminar system of bounded closed convex sets (in $\mathbb{R}^n$) with nonempty interior is countable or continuum.*

Proof. We show that the system $\mathcal{A}$ of closed convex sets with nonempty interior satisfies the conditions of Theorem 6. The elements of $\mathcal{A}$ are Lebesgue-measurable, and (1) is satisfied by the conditions. Suppose that $A, B \in \mathcal{H}$ and $A \not\subseteq B$. Let $x \in A \setminus B$. As B is a closed set not containing x, there exists some $r > 0$ such that $B(x, r) \cap B = \emptyset$. The set A is convex with nonempty interior. As $x \in A$, we have $int(A \cap B(x, r)) \neq \emptyset$, and $A \cap B(x, r) \subseteq A \setminus B$. As $\lambda(A \cap B(x, r))$ is positive, we obtain that condition (2) holds. Finally, the intersection of closed convex sets is a closed convex set. Thus either its interior is nonempty or has Lebesgue-measure 0, as needed in (3). $\square$

Proposition 8. *The cardinality of a maximal laminar system of closed discs in $\mathbb{R}^n$ is either $\aleph_0$ or continuum.*

Proof. Clearly the cardinality of a maximal laminar system of closed discs is at least $\aleph_0$. We apply Theorem 6 for $\mathcal{A} = \{\text{closed discs}\}$. The closed discs are Lebesgue-measurable and the conditions (1) and (2) are satisfied. We have to prove that $\mathcal{A}$ satisfies (3), that is, the intersection of a chain $\mathcal{C}$ of closed discs is a closed disc or has Lebesgue-measure 0. Let $r(C)$ and $x(C)$ be the radius and the center of the disc C, respectively. Let $r = \inf_{C\in\mathcal{C}} r(C)$. If $r = 0$, then $\bigcap\mathcal{C}$ has Lebesgue-measure 0. Suppose that $r > 0$. If $\inf_{C\in\mathcal{C}} r(C) = r(C_0)$ for some $C_0 \in \mathcal{C}$, then $\bigcap\mathcal{C} = C_0 \in \mathcal{C}$. Suppose that $r(C) > r$ for all $C \in \mathcal{C}$. For all $n \in \mathbb{N}$ we can choose a $C_n \in \mathcal{C}$ such that $r(C_n) < r + 1/n$. The sequence $x(C_n)$ is bounded, so it has a convergent subsequence. We may assume that $x(C_n)$ itself is convergent. Let $x(C_n) \to x$. Let $B = \overline{B}(x, r)$ be the closed disc with center x and radius r. We prove that $B = \bigcap\mathcal{C}$. For every $\varepsilon > 0$ there exists an $n \in \mathbb{N}$ such that $r(C_n) < r + \varepsilon$ and $d(x(C_n), x) < \varepsilon$. Then $C_n \subseteq B(x, r + 2\varepsilon)$, hence $\bigcap\mathcal{C} \subseteq B(x, r + 2\varepsilon)$ for every $\varepsilon > 0$ and so $\bigcap\mathcal{C} \subseteq B$. Let us assume indirectly that $B \nsubseteq \bigcap\mathcal{C}$. Then there exists a $C \in \mathcal{C}$ such that $B \setminus C \neq \emptyset$. Then $B(x, r) \setminus C \neq \emptyset$, either, so we can choose a $y \notin C$ such that $d(x, y) < r$. There exists a $k \in \mathbb{N}$ such that $r(C_k) < r(C)$ and $d(x(C_k), x) < r - d(x, y)$. For this k we have that $y \in C_k$. As $\mathcal{C}$ is a chain and $r(C_k) \leq r(C)$, we obtain that $C_k \subseteq C$. Now, $y \in C_k \subseteq C$, which is a contradiction. $\square$

Proposition 9. *There exists a countable maximal laminar system of closed discs in* $\mathbb{R}^2$.

Proof. Let T_a be the equilateral triangular grid having side length a such that the origin (0,0) and $(a, 0)$ are neighbouring vertices in T_a.

$$T_a = \{i(a, 0) + j(a/2, \sqrt{3}a/2) \mid i, j \in \mathbb{Z}\}$$

If $a = bc$ for some $c \in \mathbb{N}$, then T_b is a refinement of T_a (a and b do not have to be integers). Let $\mathcal{S}_a$ be the system of closed discs with center in T_a and radius $4a/9$. The elements of $\mathcal{S}_a$ are pairwise disjoint and satisfy the following property:

(1) If C_1, C_2, $C_3 \in \mathcal{S}_a$ such that their centers form a triangle of T_a, then the convex hull of the union of any two intersects the third one. If C_1, C_2, C_3 and C form a laminar system for some disc C, and C contains C_1 and C_2, then it contains C_3, as well.

For every integer $n > 0$ we define a set of closed discs $\mathcal{A}_n$ in the following way: let $\mathcal{A}_1 = \mathcal{S}_1$. In $\mathcal{A}_1$ the minimal distance of two discs is 1/9. Let $\mathcal{A}_2$ be the set of all discs in $T_{1/90}$ that do not intersect the border of any element of $\mathcal{A}_1$. Let us assume that we have already defined $\mathcal{A}_k$ for

$k < n$ such that $\mathcal{A}_k$ consists of all elements of $\mathcal{S}_{1/r_k}$ for some integer r_k that do not intersect the border of any element of $\bigcup_{i=1}^{k-1} \mathcal{A}_i$.

At first we choose r_n such that

- r_k divides r_n for every $k < n$, hence T_{1/r_k} is a refinement of $T_{1/r_{k-1}}$
- $1/r_k$ is much smaller than the distance of any two circles in $\bigcup_{i=1}^{k-1} \mathcal{A}_i$, so small enough to "fill in the space" between any to circles from $\bigcup_{i=1}^{k-1} \mathcal{A}_i$.

For this let $\delta_k = \inf\{d(\partial C_1, \partial C_2) \mid C_1, C_2 \in \bigcup_{n=0}^{k-1} \mathcal{A}_n, C_1 \neq C_2\}$ and

$$r_k = r_{k-1}\left[\frac{10}{\delta_k}\right],$$

where ∂C is the boundary of the disc C and $[a]$ denotes the integer part of a. Note that δ_k is positive because the set $\bigcup_{n=0}^{k-1} \mathcal{S}_{1/r_n}$ is invariant under the translations by the vectors $(1,0)$ and $(0, \sqrt{3}/2)$. Let $\mathcal{A}_k$ consist of all discs of $\mathcal{S}_{1/r_k}$ that do not intersect the border of any discs in $\bigcup_{i=1}^{k-1} \mathcal{A}_i$,

$$\mathcal{A}_k = \{D \in \mathcal{S}_{1/r_k} \mid \forall C \in \bigcup_{n=0}^{k-1} \mathcal{A}_n \quad \partial D \cap \partial C = \emptyset\}.$$

Finally, let

$$\mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{A}_n.$$

By the definition of $\mathcal{A}_k$ the circles in $\{\partial C \mid C \in \bigcup_{n=0}^{k} \mathcal{A}_n\}$ are pairwise disjoint. Thus $\bigcup_{n=0}^{k} \mathcal{A}_n$ is laminar and so $\mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{A}_n$ is laminar, as well. By the construction $\mathcal{A}$ is countable. We prove that $\mathcal{A}$ is a maximal laminar system of closed discs. Suppose indirectly that $D \notin \mathcal{A}$ and $\mathcal{A} \cup \{D\}$ is laminar.

At first consider the case when $D \nsubseteq \bigcup \mathcal{A}$. Then D and $\bigcup \mathcal{A}$ are disjoint. For a sufficiently large n there exist two elements of $\mathcal{A}_n$ contained in D such that their centers are neighbours in T_{1/r_n}. If $D_1, D_2, D_3 \in \mathcal{A}_n$, $D_1, D_2 \subseteq D$ and the centers of D_1, D_2, D_3 form a triangle in T_{1/r_n}, then laminarity implies that $D_3 \subseteq D$. By repeatedly using this argument, we get that every element of the set $\{F \in \mathcal{A}_n \mid F \nsubseteq \bigcup \bigcup_{i=0}^{n-1} \mathcal{A}_i\}$ is contained in D, since the radius of discs in $\mathcal{S}_{1/r_n}$ is smaller than $d(\partial D_1, \partial D_2)/10$ for any D_1, $D_2 \in \bigcup_{i=0}^{n-1} \mathcal{A}_i$. This set is not bounded, this is a contradiction.

Now we consider the case when $D \subseteq \bigcup \mathcal{A}$. Let $D_k \in \mathcal{A}_k$, $D \subsetneq D_k$, k is maximal. For sufficiently large n there exist two elements of $\mathcal{A}_n$ contained by D such that their centers are neighbours in T_{1/r_n} and there exist $D' \in \mathcal{A}_n$ such that $D' \subseteq (D_k \setminus D) \neq \emptyset$. Similarly to the first case we get that $(\bigcup \mathcal{A}_n) \cap D_k \subseteq D$, contradiction.

$\square$

In [7] it was asked whether there exists a countable maximal laminar system of closed discs. The following theorem answers what cardinality a maximal laminar system of closed discs can have.

Theorem 10. *The size of a maximal laminar system of closed discs in $\mathbb{R}^2$ is either countable or continuum, and both cases can occur.*

Proof. The cardinality is trivially at least $\aleph_0$. The system $\{\overline{B}(0, r) \mid r > 0\}$ is a maximal laminar system of closed discs and has cardinality continuum. In Propositions 8 and 9 it is proved that the cardinality of a maximal laminar system can be $\aleph_0$ as well, but no other cardinality can occur. $\square$

In Proposition 3 we give an example of a maximal laminar system that is not a system of islands. Hence, the following problem arises naturally.

Problem 1. What conditions are necessary for a laminar system in order to be a system of islands?

Our techniques work only for maximal island systems, however it would be interesting to know what happens in the general situation.

Problem 2. Is it true that for every cardinality $\aleph_0 < \kappa < 2^{\aleph_0}$ there is a (not necessarily maximal) system of islands?

References

[1] G. Czédli, *The Number of Rectangular Islands by Means of Distributive Lattices*, European Journal of Combinatorics, **30** (2009), 208-215.

[2] G. Czédli, A. P. Huhn and E. T. Schmidt, *Weakly Independent Subsets in Lattices*, Algebra Universalis, **20** (1985), 194-196.

[3] S. Földes and N. M. Singhi, *On Instantaneous Codes*, J. of Combinatorics, Information and System Sci., **31** (2006), 317-326.

[4] E. K. Horváth, Z. Németh and G. Pluhár, *The Number of Triangular Islands on a Triangular Grid*, Periodica Mathematica Hungarica, **58** (2009), 25-34.

[5] E. K. Horváth, G. Horváth, Z. Németh and Cs. Szabó, *The Number of Square Islands on a Rectangular Sea*, Acta Sci. Math., to appear.

[6] Zs. Lengvárszky, *The minimum cardinality of maximal systems of rectangular islands*, European Journal of Combinatorics, **30** (2009), 216-219.

[7] Zs. Lengvárszky and P. P. Pach, *A Note on Systems of Rectangular Islands: The Continuous Case*, Acta Sci. Math., to appear.

[8] G. Pluhár, *The number of brick islands by means of distributive lattices*, Acta Sci. Math., **75** (2009), 3-11.