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The multiplication groups of 2-dimensional topological loops

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Abstract. If the multiplication group $\mathrm{Mult}(L)$ of a connected simply connected 2-dimensional topological loop L is a Lie group, then $\mathrm{Mult}(L)$ is an elementary filiform Lie group $\mathscr{F}$ of dimension $n+2$ for some $n \geqslant 2$, and any such group is the multiplication group of a connected simply connected 2-dimensional topological loop L. Moreover, if the group topologically generated by the left translations of L has dimension 3, then L is uniquely determined by a real polynomial of degree n.

1 Introduction

The multiplication group $\mathrm{Mult}(L)$ and the inner mapping group $\mathrm{Inn}(L)$ of a loop L introduced in [1], [2] are important tools for studying L since they strongly reflect the structure of L. In particular, there is a strong correspondence between the normal subloops of L and certain normal subgroups of $\mathrm{Mult}(L)$. Hence, it is an interesting question which groups can be represented as multiplication groups of loops (see [9], [11], [12]). A purely group-theoretic characterization of multiplication groups is given in [10].

Topological and differentiable loops such that the groups G topologically generated by the left translations are Lie groups have been studied in [7]. There the topological loops L are treated as continuous sharply transitive sections $\sigma : G/H \to G$, where H is the stabilizer of the identity element of L in G. In [7] and [4] it is proved that essentially up to two exceptions any connected 3-dimensional Lie group occurs as the group topologically generated by the left translations of a connected 2-dimensional topological loop. These exceptions are either locally isomorphic to the connected component of the group of motions or isomorphic to the connected component of the group of dilatations of the euclidean plane. In contrast to this, if the group $\mathrm{Mult}(L)$ topologically generated by all left and right translations of a connected 2-dimensional topological loop L is a Lie group, then the isomorphism types of $\mathrm{Mult}(L)$ and of L are strongly restricted. This is shown by our theorems, in which Lie groups with filiform Lie algebras (cf. [5, pp. 626–663]) play a fundamental role.

The elementary filiform Lie group $\mathscr{F}_{n+2}$ is the simply connected Lie group of dimension $n+2 \geqslant 3$ whose Lie algebra is elementary filiform, i.e. it has a basis

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$\{e_1, \ldots, e_{n+2}\}$ such that $[e_1, e_i] = (n+2-i)e_{i+1}$ for $2 \leqslant i \leqslant n+1$ and all other Lie brackets are zero. With this notion we can formulate our theorems as follows:

Theorem 1. *Let L be a connected simply connected 2-dimensional topological loop which is not a group. The group $\mathrm{Mult}(L)$ topologically generated by all left and right translations of L is a Lie group if and only if $\mathrm{Mult}(L)$ is an elementary filiform Lie group $\mathscr{F}_{n+2}$ with $n \geqslant 2$. Moreover, the group G topologically generated by the left translations of L is an elementary filiform Lie group $\mathscr{F}_{m+2}$, where $1 \leqslant m \leqslant n$, and the inner mapping group $\mathrm{Inn}(L)$ corresponds to the abelian subalgebra $\langle e_2, e_3, \ldots, e_{n+1} \rangle$.*

The loop L of Theorem 1 is a central extension of the group $\mathbb{R}$ by the group $\mathbb{R}$ (cf. [7, Theorem 28.1]). Hence it is a centrally nilpotent loop of class 2 and can be represented in $\mathbb{R}^2$. If L is not simply connected but satisfies all other conditions of Theorem 1, then L is homeomorphic to the cylinder $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$. The multiplication group $\mathrm{Mult}(L)$ of L is a Lie group of dimension $n+2 \geqslant 4$ with elementary filiform Lie algebra such that the centre of $\mathrm{Mult}(L)$ is isomorphic to the group $\mathrm{SO}_2(\mathbb{R})$ (cf. Theorem 1 and [7, Theorem 28.1]).

Theorem 2. *Let G be the elementary filiform Lie group $\mathscr{F}_{n+2}$ with $n \geqslant 1$. Then G is isomorphic to the group topologically generated by the left translations of a connected simply connected 2-dimensional topological loop $L = (\mathbb{R}^2, *)$ with the multiplication*

$$(u_1, z_1) * (u_2, z_2) = (u_1 + u_2, z_1 + z_2 - u_2 v_1(u_1) + u_2^2 v_2(u_1) + \cdots + (-1)^n u_2^n v_n(u_1)), \tag{1}$$

where $v_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots, n$, are continuous functions with $v_i(0) = 0$ such that the function v_n is non-linear.

For $n > 1$ the group G coincides with the group $\mathrm{Mult}(L)$ topologically generated by all left and right translations of L if and only if there are continuous functions $s_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$, such that for all $x, u \in \mathbb{R}$ the equation

$$\begin{aligned} &-x(s_1(u) + v_1(u)) + x^2(s_2(u) + v_2(u)) + \cdots + (-1)^n x^n (s_n(u) + v_n(u)) \\ &\quad = -u v_1(x) + u^2 v_2(x) + \cdots + (-1)^n u^n v_n(x) \end{aligned}$$

holds.

Moreover, L is commutative if $v_1, \ldots, v_n$ satisfy the vector equation

$$\begin{pmatrix} v_1(x) \\ v_2(x) \\ \vdots \\ v_n(x) \end{pmatrix} = A \begin{pmatrix} x \\ x^2 \\ \vdots \\ x^n \end{pmatrix},$$

where $A = (a_{ij}) \in M_n(\mathbb{R})$ and $a_{ij} = (-1)^{i+j} a_{ji}$ for all $i, j \in \{1, 2, \ldots, n\}$.

Theorem 3. *If L is a 2-dimensional connected simply connected topological loop having the elementary filiform Lie group $\mathscr{F}_3$ as the group topologically generated by the left translations of L, then the multiplication of L is given by*

$$(u_1, z_1) * (u_2, z_2) = (u_1 + u_2, z_1 + z_2 - u_2 v_1(u_1)), \tag{2}$$

where $v_1 : \mathbb{R} \to \mathbb{R}$ is a non-linear continuous function with $v_1(0) = 0$.

The group $\mathrm{Mult}(L)$ topologically generated by all left and right translations of L is isomorphic to the elementary filiform Lie group $\mathscr{F}_{n+2}$ for $n \geqslant 2$ if and only if the continuous function $v_1 : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree n.

2 Preliminaries

A binary system $(L, \cdot)$ is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution, which we denote by $y = a \backslash b$ and $x = b/a$. A loop L is proper if it is not associative.

The left and right translations $\lambda_a = y \mapsto a \cdot y : L \to L$ and $\rho_a : y \mapsto y \cdot a : L \to L$, $a \in L$ are permutations of L. The permutation group $\mathrm{Mult}(L) = \langle \lambda_a, \rho_a; a \in L \rangle$ is called the multiplication group of L. The stabilizer of the identity element $e \in L$ in $\mathrm{Mult}(L)$ is denoted by $\mathrm{Inn}(L)$, and it is called the inner mapping group of L.

Let K be a group, let $S \leqslant K$, and let A and B be two left transversals to S in K (i.e. two systems of representatives for the left cosets of the subgroup S in K). We say that A and B are S-connected if $a^{-1}b^{-1}ab \in S$ for all $a \in A$ and $b \in B$. By $C_K(S)$ we denote the core of S in K (the largest normal subgroup of K contained in S). If L is a loop, then $\Lambda(L) = \{\lambda_a; a \in L\}$ and $R(L) = \{\rho_a; a \in L\}$ are $\mathrm{Inn}(L)$-connected transversals in the group $\mathrm{Mult}(L)$, and the core of $\mathrm{Inn}(L)$ in $\mathrm{Mult}(L)$ is trivial. The connection between multiplication groups of loops and transversals is given in [10, Theorem 4.1]. This theorem yields the following

Lemma 4. *Let L be a loop and $\Lambda(L)$ be the set of left translations of L. Let K be a group containing $\Lambda(L)$ and S be a subgroup of K with $C_K(S) = 1$ such that $\Lambda(L)$ is a left transversal to S in K. The group K is isomorphic to the multiplication group $\mathrm{Mult}(L)$ of L if and only if there is a left transversal T to S in K such that $\Lambda(L)$ and T are S-connected and $K = \langle \Lambda(L), T \rangle$. In this case S is isomorphic to the inner mapping group $\mathrm{Inn}(L)$ of L.*

The kernel of a homomorphism $\alpha : (L, \circ) \to (L', *)$ of a loop L into a loop L' is a normal subloop N of L. We need the following facts concerning normal subloops (cf. [3, p. 62]).

Lemma 5. *Let L be a loop with multiplication group $\mathrm{Mult}(L)$ and identity element e.*

(i) *Let α be a loop homomorphism from L with kernel N. Then α induces a surjective group homomorphism from $\mathrm{Mult}(L)$ to $\mathrm{Mult}(\alpha(L))$.*

Denote by $M(N)$ *the set* $\{m \in \mathrm{Mult}(L); xN = m(x)N \text{ for all } x \in L\}$. *Then* $M(N)$ *is a normal subgroup of* $\mathrm{Mult}(L)$, *and the multiplication group of the factor loop* L/N *is isomorphic to* $\mathrm{Mult}(L)/M(N)$.

(ii) *For every normal subgroup* $\mathscr{N}$ *of* $\mathrm{Mult}(L)$ *the orbit* $\mathscr{N}(e)$ *is a normal subloop of* L. *Moreover*, $\mathscr{N} \leqslant M(\mathscr{N}(e))$.

The theory of topological loops L is the theory of the continuous binary operations $(x, y) \mapsto x \cdot y$, $(x, y) \mapsto x/y$, $(x, y) \mapsto x\backslash y$ on the topological space L. If L is a topological loop, then the left translations λ_a as well as the right translations ρ_a, $a \in L$, are homeomorphisms of L.

Every connected topological loop having a Lie group as the group topologically generated by the left translations is realized on a homogeneous space G/H, where G is a connected Lie group, H is a closed subgroup of G with $C_G(H) = 1$ and $\sigma : G/H \to G$ is a continuous sharply transitive section with $\sigma(H) = 1 \in G$ such that the subset $\sigma(G/H)$ generates G. The multiplication of L on the manifold G/H is defined by $xH * yH = \sigma(xH)yH$, and the group G is the group topologically generated by the left translations of L. Moreover, the subgroup H is the stabilizer of the identity element $e \in L$ in the group G (cf. [7, §1.3]).

3 Proofs

If L is a connected topological loop having (with respect to the compact–open topology) a Lie group as the group $\mathrm{Mult}(L)$ topologically generated by all left and right translations, then $\mathrm{Mult}(L)$ acts transitively and effectively as a topological transformation group on L. All transitive transformation groups on a 2-dimensional manifold have been classified by Lie (cf. [6]) and Mostow in [8, §10].

Lemma 6. *If the group* $\mathrm{Mult}(L)$ *topologically generated by all left and right translations of a* 2*-dimensional connected topological loop* L *is a Lie group*, *then the group* $\mathrm{Mult}(L)$ *is solvable.*

Proof. Let L be a 2-dimensional connected topological loop such that the group $\mathrm{Mult}(L)$ topologically generated by all left and right translations is a non-solvable Lie group. We may assume that L is simply connected, since otherwise we would consider the universal covering of L. Then L is homeomorphic to $\mathbb{R}^2$ (cf. [7, Theorem 19.1]).

If the radical $\mathscr{R}$ of $\mathrm{Mult}(L)$ is trivial, then according to [8, §10] the Lie group $\mathrm{Mult}(L)$ is locally isomorphic to either $\mathrm{PSL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$. But these groups are excluded in [7, Lemma 19.5 and Theorem 19.7].

If $\dim \mathscr{R} \geqslant 1$, then $\mathrm{Mult}(L)$ has a non-trivial connected abelian normal subgroup $\mathscr{K}$. By Lemma 5, the orbit $\mathscr{K}(e)$ is a normal subloop of L. For $\mathscr{K}(e) = \{e\}$ the inner mapping group $\mathrm{Inn}(L)$ contains the non-trivial normal subgroup $\mathscr{K}$ of $\mathrm{Mult}(L)$, which is a contradiction.

If the orbit $\mathscr{K}(e)$ is the whole loop L, then $\dim \mathscr{K} = 2$, since otherwise $\mathrm{Mult}(L)$ does not act effectively on L. Moreover, $\mathscr{K}$ operates sharply transitively on L. Hence

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we have $\mathrm{Mult}(L) = \mathscr{K} \rtimes \mathrm{Inn}(L)$. If the group $\mathrm{Mult}(L)$ is non-solvable, then it is the semidirect product of the 2-dimensional abelian group $\mathscr{K}$ by a Lie group S locally isomorphic either to $\mathrm{GL}_2(\mathbb{R})$ (cf. [8, Subcases III.8 and IV.2]) or to $\mathrm{SL}_2(\mathbb{R})$ (cf. [8, Subcases III.1 and IV.1]). In all of these cases any subgroup locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ acts irreducibly on $\mathscr{K}$, which is the unique 2-dimensional sharply transitive normal subgroup of the group $\mathrm{Mult}(L)$. But then the group $\mathscr{K}$ is contained in the group topologically generated by the left translations of L, and this gives a contradiction to [7, Corollary 17.8].

Hence the orbit $\mathscr{K}(e)$ is a 1-dimensional normal subloop of L. As $L/\mathscr{K}(e)$ is a connected 1-dimensional loop such that the group topologically generated by all of its left and right translations is a factor group of $\mathrm{Mult}(L)$, we conclude that $L/\mathscr{K}(e)$ is a 1-dimensional Lie group (see [7, Theorem 18.18]). Then $\mathrm{Mult}(L)$ contains a normal subgroup M of codimension 1 such that $\mathscr{K} \leqslant M$ and each orbit $\{x\mathscr{K}(e); x \in L\}$ is invariant under M (see Lemma 5).

If $\mathrm{Mult}(L)$ is non-solvable, then according to [8, §10], it is locally isomorphic either to $\mathrm{PSL}_2(\mathbb{R}) \times \mathscr{L}_2$, where $\mathscr{L}_2$ is the 2-dimensional non-commutative Lie group (cf. [8, Subcase II.10]), or to the semidirect product of a normal subgroup $\mathscr{K} \cong \mathbb{R}^n$, for some $n \geqslant 2$, by a Lie group S locally isomorphic to $\mathrm{GL}_2(\mathbb{R})$ such that any subgroup locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ acts irreducibly on $\mathscr{K}$ (cf. Subcases III.8 and IV.2). The groups locally isomorphic to $\mathrm{PSL}_2(\mathbb{R}) \times \mathscr{L}_2$ are excluded by [7, Lemma 19.5]. If $\mathrm{Mult}(L)$ is locally isomorphic to $\mathscr{K} \rtimes \mathrm{GL}_2(\mathbb{R})$, then the subgroup M is locally isomorphic to $\mathscr{K} \rtimes \mathrm{SL}_2(\mathbb{R})$. Since $\dim \mathscr{K} \geqslant 2$, the group $\mathscr{K}$ has a subgroup $\hat{\mathscr{K}}$ of codimension 1 such that $\hat{\mathscr{K}}(e) = \{e\}$. As $\mathscr{K}$ acts transitively on the 1-dimensional Lie group $\mathscr{K}(e)$, the group $\hat{\mathscr{K}}$ fixes $\mathscr{K}(e)$ elementwise. As $\hat{\mathscr{K}} < M$ the group M does not act effectively on $\mathscr{K}(e)$. Then there exists a normal subgroup N of M which fixes every element of $\mathscr{K}(e)$ and hence $N \cap \mathscr{K} = \hat{\mathscr{K}}$. Since the abelian group $\mathscr{K}$ is the unique normal subgroup of M, we have a contradiction and the lemma is proved. □

Remark 7. If L is a 2-dimensional connected simply connected topological proper loop such that $\mathrm{Mult}(L)$ is a solvable Lie group, then $\mathrm{Mult}(L)$ is a semidirect product of the abelian group $M \cong \mathbb{R}^n$, for some $n \geqslant 2$, by a group $S \cong \mathbb{R}$, such that $M = Z \times \mathrm{Inn}(L)$, where $Z \cong \mathbb{R}$ is the centre of $\mathrm{Mult}(L)$ and $\mathrm{Inn}(L) \cong \mathbb{R}^{n-1}$ is the stabilizer of $e \in L$ in $\mathrm{Mult}(L)$ (cf. [7, Theorem 28.1]).

Proof of Theorem 1. By Lemma 6 the group $\mathrm{Mult}(L)$ is solvable. As L is not associative, $\mathrm{Mult}(L)$ has dimension at least 3. Since L is simply connected it is homeomorphic to $\mathbb{R}^2$. By the classification of Mostow (cf. [8, §10]) every solvable Lie group of dimension at least 3 acting transitively on the plane $\mathbb{R}^2$ is locally isomorphic to one of the Lie groups in the Subcases I.3, II.1, II.3, II.5, II.7, II.11, II.12, II.13, III.3, III.4, III.5 and III.7. It follows from Remark 7 that the commutator subgroup of $\mathrm{Mult}(L)$ is abelian and that $\mathrm{Mult}(L)$ has a 1-dimensional centre. Hence a direct computation shows that $\mathrm{Mult}(L)$ is isomorphic either to the 3-dimensional non-commutative nilpotent Lie group or to the direct product of a 1-dimensional Lie group with the 2-dimensional non-commutative solvable Lie group, or the Lie

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algebra of $\mathrm{Mult}(L)$ has the form

$$\mathbf{mult_1}(\mathbf{L}) = \left\langle \frac{\partial}{\partial x}, w(x)\frac{\partial}{\partial y}, \ldots, w^{(n)}(x)\frac{\partial}{\partial y} \right\rangle, \quad \text{for some } n > 1,$$

(cf. Subcase II.3). The 3-dimensional groups are excluded by [7, Theorems 23.12 and 23.7]. For every $n > 1$ the abelian normal subgroup M of codimension 1 of the $\mathrm{Mult}_1(L)$ belonging to the Lie algebra $\mathbf{mult_1}(\mathbf{L})$ is generated by the set

$$\left\{ w(x)\frac{\partial}{\partial y}, w^{(1)}(x)\frac{\partial}{\partial y}, \ldots, w^{(n)}(x)\frac{\partial}{\partial y} \right\}.$$

According to Remark 7 the group M contains a 1-parameter subgroup of central elements of the group $\mathrm{Mult}_1(L)$. This is the case precisely if the Lie algebra of M contains the generator $\partial/\partial y$ (cf. [8], Lemma 1, p. 628). Hence the function $w(x)$ has the form x^n, with $n > 1$. Then for every $n > 1$ the group $\mathrm{Mult}_1(L)$ is nilpotent with maximal nilindex. Putting

$$e_1 = \frac{\partial}{\partial x},\ e_2 = x^n\frac{\partial}{\partial y},\ e_3 = x^{n-1}\frac{\partial}{\partial y}, \ldots, e_{n+1} = x\frac{\partial}{\partial y},\ e_{n+2} = \frac{\partial}{\partial y}$$

we obtain the first assertion.

As $M = Z \times \mathrm{Inn}(L)$ (see Remark 7), where the centre Z of $\mathrm{Mult}_1(L)$ belongs to the Lie algebra $\langle e_{n+2} \rangle$, the Lie algebra of the group $\mathrm{Inn}(L)$ is given by

$$\mathbf{inn}(\mathbf{L}) = \langle e_2 + a_1 e_{n+2}, e_3 + a_2 e_{n+2}, \ldots, e_{n+1} + a_n e_{n+2} \rangle, \quad a_i \in \mathbb{R},\ i = 1, \ldots, n.$$

Using the automorphism φ of the Lie algebra $\mathbf{mult_1}(\mathbf{L}) = \langle e_1, e_2, \ldots, e_{n+2} \rangle$ defined by

$$\varphi(e_1) = e_1, \quad \varphi(e_{n+2}) = e_{n+2},$$

$$\varphi(e_i) = e_i - a_{i-1}e_{n+2} - \sum_{k=i+1}^{n+1} (n-i+2)a_{n-k+i+1}e_k \quad (2 \leqslant i \leqslant n+1)$$

we obtain the last assertion.

Since also the group G topologically generated by the left translations of L acts transitively on $\mathbb{R}^2$ and since G is an at least 3-dimensional subgroup of the elementary filiform Lie group $\mathrm{Mult}(L)$, the classification of Mostow (cf. [8, §10]) gives the second assertion. □

Lemma 8. *Let V be a non-commutative subalgebra of the elementary filiform Lie algebra of dimension $n + 2 \geqslant 3$. Then $V = V_i$ has a basis*

$$\{e_1 + t_1, e_i, e_{i+1}, \ldots, e_{n+2}\}$$

with a fixed $i \in \{2, \ldots, n+1\}$ and $t_1 \in \langle e_2, e_3, \ldots, e_{i-1} \rangle$.

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Proof. If V is not commutative, then V is a filiform Lie algebra of dimension $n+4-i$ with $2 \leqslant i \leqslant n+1$ and has a basis of the form

$$\{e_1 + t_1, e_i + t_i, e_{i+1} + t_{i+1}, \dots, e_{n+1} + t_{n+1}, e_{n+2}\}$$

with $t_1 \in \langle e_2, e_3, \dots, e_{n+2}\rangle$ and $t_{i+j} \in \langle e_{i+j+1}, \dots, e_{n+2}\rangle$, $0 \leqslant j < n+1-i$. Successive addition of suitable linear combinations $\sum_{k=0}^{j} \lambda_{n+2-k} e_{n+2-k}$ to t_{n+1-j} shows that V contains the elements $e_{n+2}, e_{n+1}, \dots, e_i$, and the assertion follows. $\square$

Proof of Theorem 2. For $n \geqslant 1$ the Lie algebra $\mathbf{g} = \langle e_1, e_2, \dots, e_{n+2}\rangle$ of the elementary filiform Lie group $G = \mathscr{F}_{n+2}$ is isomorphic to the Lie algebra of all matrices of the form

$$\begin{pmatrix} 0 & a_1 & a_2 & \cdots & a_{n-1} & a_n & b \\ 0 & 0 & 0 & \cdots & 0 & 0 & -c \\ 0 & -2c & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -3c & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -nc & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

with $a_1, \dots, a_n, b, c \in \mathbb{R}$. Hence we can represent the elements of G as the matrices

$$g(c, a_1, a_2, \dots, a_{n-1}, a_n, b) = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} & a_n & b \\ 0 & 1 & 0 & 0 & \cdots & 0 & -c \\ 0 & -2c & 1 & 0 & \cdots & 0 & c^2 \\ 0 & 3c^2 & -3c & 1 & \cdots & 0 & -c^3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & (-1)^{n-1}\binom{n}{1}c^{n-1} & (-1)^{n-2}\binom{n}{2}c^{n-2} & \cdots & (-1)\binom{n}{n-1}c^1 & 1 & (-1)^n c^n \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}, \quad (3)$$

with $a_1, \dots, a_n, b, c \in \mathbb{R}$. Since all elements of G have a unique decomposition

$$g(u, 0, \dots, 0, z) g(0, v_1, v_2, \dots, v_n, 0),$$

the continuous functions $v_1(u,z)$, $v_2(u,z), \dots, v_n(u,z)$ determine a continuous section $\sigma : G/H \to G$ defined by

$$\begin{aligned} g(u, 0, \dots, 0, z)H \mapsto\ & g(u, 0, \dots, 0, z) g(0, v_1(u,z), v_2(u,z), \dots, v_n(u,z), 0) \\ &= g(u, v_1(u,z), v_2(u,z), \dots, v_n(u,z), z), \end{aligned}$$

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where

$$H = \{g(0, v_1, v_2, \ldots, v_n, 0); v_i \in \mathbb{R} \text{ for } i = 1, \ldots, n\}.$$

The image $\sigma(G/H)$ acts sharply transitively on the factor space G/H precisely if for all $(u_1, z_1), (u_2, z_2) \in \mathbb{R}^2$ the equation

$$\begin{aligned} &g(u, v_1(u,z), v_2(u,z), \ldots, v_n(u,z), z) g(u_1, 0, \ldots, 0, z_1) \\ &\quad = g(u_2, 0, \ldots, 0, z_2) g(0, t_1, t_2, \ldots, t_n, 0) \end{aligned} \tag{4}$$

has a unique solution $(u, z) \in \mathbb{R}^2$ with a suitable element $g(0, t_1, t_2, \ldots, t_n, 0)$ in H. From (4) we obtain the equations

$$\begin{aligned} &u = u_2 - u_1 \\ &t_i = \sum_{k=i}^{n} (-1)^{k-i} \binom{k}{i} u_1^{k-i} v_k(u,z) \quad \text{for } i = 1, 2, \ldots, n, \\ &0 = z + z_1 - z_2 - u_1 v_1(u,z) + u_1^2 v_2(u,z) + \cdots + (-1)^n u_1^n v_n(u,z). \end{aligned}$$

Hence equation (4) has a unique solution if and only if for every $u_0 = u_2 - u_1$ and $u_1 \in \mathbb{R}$ the function

$$f : z \mapsto z - u_1 v_1(u_0, z) + u_1^2 v_2(u_0, z) + \cdots + (-1)^n u_1^n v_n(u_0, z)$$

is a bijective mapping from $\mathbb{R}$ to $\mathbb{R}$. Let $\psi_1 < \psi_2 \in \mathbb{R}$. Then $f(\psi_1) \neq f(\psi_2)$ and we may assume that $f(\psi_1) < f(\psi_2)$. We consider the inequality

$$\begin{aligned} 0 &< f(\psi_2) - f(\psi_1) \\ &= \psi_2 - \psi_1 - u_1(v_1(u_0, \psi_2) - v_1(u_0, \psi_1)) + u_1^2(v_2(u_0, \psi_2) - v_2(u_0, \psi_1)) \\ &\quad + \cdots + (-1)^n u_1^n (v_n(u_0, \psi_2) - v_n(u_0, \psi_1)). \end{aligned} \tag{5}$$

If for every i the function $v_i(u, z)$ is independent of z, then f is monotone and the continuous functions $v_1(u), v_2(u), \ldots, v_n(u)$ determine a 2-dimensional topological loop L. Now we represent L in the coordinate system $(u, z) \mapsto g(u, 0, \ldots, 0, z)H$. Then the product $(u_1, z_1) * (u_2, z_2)$ will be determined if we apply

$$\sigma(g(u_1, 0, \ldots, 0, z_1)H) = g(u_1, v_1(u_1), v_2(u_1), \ldots, v_n(u_1), z_1)$$

to the left coset $g(u_2, 0, \ldots, 0, z_2)H$ and find in the image coset the element of G which lies in the set $\{g(u, 0, \ldots, 0, z)H; u, z \in \mathbb{R}\}$. We obtain

$$(u_1, z_1) * (u_2, z_2) = (u_1 + u_2, z_1 + z_2 - u_2 v_1(u_1) + u_2^2 v_2(u_1) + \cdots + (-1)^n u_2^n v_n(u_1)).$$

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This loop is proper precisely if the set

$$\sigma(G/H) = \{g(u, v_1(u), v_2(u), \dots, v_n(u), z); u, z \in \mathbb{R}\}$$

generates the whole group G. The set $\sigma(G/H)$ contains the subset

$$F = \{g(u, v_1(u), v_2(u), \dots, v_n(u), 0); u \in \mathbb{R}\}$$

and the centre $Z = \{g(0, 0, \dots, 0, z); z \in \mathbb{R}\}$ of G. The group $\langle F \rangle$ topologically generated by the set F and the group Z generate G if and only if $\langle F \rangle$ is a non-commutative subgroup of codimension 1 in G. By Lemma 8 this is the case precisely if the mapping assigning to the first component of any element of $\langle F \rangle$ its $(n+1)$th component is not a homomorphism. This occurs if and only if the function v_n is non-linear; thus the first assertion follows.

For $n = 1$ the elementary filiform Lie group $\mathscr{F}_3$ cannot be the group topologically generated by all left and right translations of L (cf. Theorem 1). Now let $n > 1$. By [7, Proposition 18.16] the filiform Lie group G coincides with the group topologically generated by all translations of L given by the multiplication (1) if and only if for every $y \in L$ the map $f(y) : x \mapsto y\lambda_x\lambda_y^{-1}$ from L to L is an element of

$$H = \{g(0, t_1, \dots, t_n, 0); t_i \in \mathbb{R}, i = 1, \dots, n\}.$$

This is equivalent to the condition that the mapping

$$\begin{aligned} g(x, 0, \dots, 0, y)H \mapsto {} & (g(u, v_1(u), v_2(u), \dots, v_n(u), z))^{-1} \\ & \times (g(x, v_1(x), v_2(x), \dots, v_n(x), y))g(u, 0, \dots, 0, z)H \end{aligned}$$

has the form

$$g(x, 0, \dots, 0, y)H \mapsto g(0, s_1(u, z), s_2(u, z), \dots, s_n(u, z), 0)g(x, 0, \dots, 0, y)H$$

for suitable functions $s_1(u, z), s_2(u, z), \dots, s_n(u, z)$. This gives the relation

$$\begin{aligned} & g(x, v_1(x), \dots, v_n(x), y)g(u, 0, \dots, 0, z)H \\ & \quad = g(u, v_1(u), \dots, v_n(u), z)g(0, s_1(u, z), \dots, s_n(u, z), 0)g(x, 0, \dots, 0, y)H \end{aligned}$$

or the equation

$$\begin{aligned} & g(x, v_1(x), \dots, v_n(x), y)g(u, t_1, \dots, t_n, z) \\ & \quad = g(u, v_1(u), \dots, v_n(u), z)g(0, s_1(u, z), \dots, s_n(u, z), 0)g(x, 0, \dots, 0, y) \end{aligned} \tag{6}$$

for a suitable $g(0, t_1, \dots, t_n, 0) \in H$. Equation (6) yields

$$t_i = \sum_{k=i}^{n} (-1)^{k-i} \binom{k}{i} (x^{k-i}(s_k(u, z) + v_k(u)) - u^{k-i} v_k(x))$$

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for $i = 1, 2, \dots, n$ and

$$-x(s_1(u,z) + v_1(u)) + x^2(s_2(u,z) + v_2(u)) + \cdots + (-1)^n x^n (s_n(u,z) + v_n(u)) \\ = -uv_1(x) + u^2 v_2(x) + \cdots + (-1)^n u^n v_n(x). \tag{7}$$

Since the right-hand side of (7) is independent of z, so is the left-hand side, and we have $s_i(u,z) = s_i(u)$ for $i = 1, \dots, n$. Hence (6) is satisfied if and only if there are continuous functions $s_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, n$, depending on the variable u such that for all $x, u \in \mathbb{R}$ equation (7) holds.

Moreover, L is commutative precisely if $s_i(u) = 0$ for all $i = 1, \dots, n$ (cf. [7, Lemma 18.16]). If the functions $v_1, \dots, v_n$ are polynomials, then comparison of coefficients yields the last assertion. □

Proof of Theorem 3. For $n = 1$ equation (5) in the proof of Theorem 2 is linear in the variable u_1. Hence the function f is monotone if and only if $v_1(u,z) = v_1(u)$, and the first assertion follows from Theorem 2.

Now let K be the group of matrices (3) with $n \geqslant 2$ and let S be the subgroup

$$\{g(0, t_1, t_2, \dots, t_n, 0); t_i \in \mathbb{R}, i = 1, 2, \dots, n\}.$$

Then K is isomorphic to $\mathscr{F}_{n+2}$ (cf. proof of Theorem 2). The set

$$\Lambda_{v_1} = \{\lambda_{(u,v)}; (u,v) \in L_{v_1}\}$$

of all left translations of the loop L_{v_1} defined by (2) in the group K has the form

$$\Lambda_{v_1} = \{g(u, v_1(u), 0, 0, \dots, 0, -\tfrac{1}{2}v_1(u)u + z); u, z \in \mathbb{R}\}.$$

An arbitrary transversal T of S in K has the form

$$T = \{g(x, h_1(x,y), \dots, h_n(x,y), y); x, y \in \mathbb{R}\},$$

where $h_j : \mathbb{R}^2 \to \mathbb{R}$, $j = 1, 2, \dots, n$, are continuous functions with $h_j(0,0) = 0$. By Lemma 4, the group K is isomorphic to $\mathrm{Mult}(L_{v_1})$ precisely if the set $\{a^{-1}b^{-1}ab; a \in T, b \in \Lambda_{v_1}\}$ is contained in S and the set $\{\Lambda_{v_1}, T\}$ generates the group K. The products $a^{-1}b^{-1}ab$ with $a \in T$ and $b \in \Lambda_{v_1}$ are in S if and only if the equation

$$xv_1(u) = (-1)^{n+1} u^n h_n(x,y) + (-1)^n u^{n-1} h_{n-1}(x,y) + \cdots + (-1)^2 u h_1(x,y) \tag{8}$$

holds for all $x, y, u \in \mathbb{R}$. If $x = 0$ then equation (8) reduces to

$$0 = (-1)^{n+1} u^n h_n(0,y) + (-1)^n u^{n-1} h_{n-1}(0,y) + \cdots + (-1)^2 u h_1(0,y). \tag{9}$$

Since the polynomials $u, u^2, \dots, u^n$ are linearly independent, equation (9) is satisfied if and only if $h_i(0,y) = 0$ for all i with $1 \leqslant i \leqslant n$. As the function $v_1 : \mathbb{R} \to \mathbb{R}$ depends

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only on the variable u and the functions $h_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2, \dots, n$, are independent of u, equation (8) holds precisely if $h_i(x, y) = a_i x$ with $a_i \in \mathbb{R}$ for all i with $1 \leqslant i \leqslant n$. By Lemma 8 the set $\{\Lambda_{v_1}, T\}$ generates the group K if and only if a_n is different from 0, since then the Lie algebra of the non-commutative group generated by the set $\{\Lambda_{v_1}, T\}$ contains elements of the form $e_2 + s$ with $s \in \langle e_3, e_4, \dots, e_{n+2} \rangle$. $\square$

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