Loops as sections in compact Lie groups

Á. Figula and K. Strambach

Abstract

We show that there does not exist any connected topological proper loop homeomorphic to a quasi-simple Lie group and having a compact Lie group as the group topologically generated by its left translations. Moreover, any connected topological loop homeomorphic to the 7-sphere and having a compact Lie group as the group of its left translations is classical. We give a particular simple general construction for proper loops such that the compact group of their left translations is direct product of at least 3 factors.

Key words and phrases: compact connected loops, compact Lie groups, sharply transitive sections in compact Lie groups.

2010 Mathematics Subject Classification: 22C05, 20N05, 57T10, 57T15

1 Introduction

H. Scheerer has clarified in [10] for which compact connected Lie groups G and for which closed subgroups H the natural projection $G \to G/H$ has a continuous section σ. For a semisimple compact Lie group G the image $\sigma(G/H)$ is not homeomorphic to a Lie group precisely if G contains a factor locally isomorphic to $PSO_8(\mathbb{R})$. This is due to the fact that the group topologically generated by the left translations of the octonions of norm 1 is the group $SO_8(\mathbb{R})$. Hence any compact connected topological loop whose group topologically generated by the left translations is a compact Lie group containing no factor locally isomorphic to $PSO_8(\mathbb{R})$, is itself homeomorphic to a compact Lie group.

But it remained an open problem for which σ the image $\sigma(G/H)$ determines a loop. This is the case if $\sigma(G/H)$ acts sharply transitively on G/H what means that for given cosets g_1H, g_2H there exists precisely one $z \in \sigma(G/H)$ such that the equation $zg_1H = g_2H$ holds. Continuous sections σ with this property (they are called sharply transitive sections) correspond to topological loops $(L, *)$ (cf. [9], Proposition 1.21, p. 29) realized on G/H

with respect to the multiplication $xH \cdot yH = \sigma(xH)yH$. Here H plays the role of the identity element $e \in L$. If the image $\sigma(G/H)$ generates the group G, then G is the group topologically generated by the left translations of $(L, \cdot)$.

There are many examples of compact connected loops having a non-simple compact connected Lie group as the group topologically generated by their left translations (cf. [9], Theorem 16.7, p. 198 and Section 14.3, pp. 170-173). A particular simple general construction for proper loops such that the group generated by their left translations is the direct product of at least three factors is given in Section 3.

In contrast to this in this paper we prove that any connected topological loop L homeomorphic to a quasi-simple Lie group G and having a compact Lie group as the group topologically generated by its left translations must coincide with G (cf. Theorem 8). Similarly, any connected topological loop L homeomorphic to the 7-sphere and having a compact Lie group as the group topological generated by its left translations is either the Moufang loop $\mathcal{O}$ of octonions of norm 1 or the factor loop $\mathcal{O}/Z$, where Z is the centre of $\mathcal{O}$ (cf. Theorem 6). With these results the Scheerer's question concerning loops is answered (cf. [10], Section 2, p. 152).

2 Prerequisites

A set L with a binary operation $(x, y) \mapsto x \cdot y$ is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution which we denote by $y = a \backslash b$ and $x = b/a$. The left translation $\lambda_a : y \mapsto a \cdot y : L \to L$ is a bijection of L for any $a \in L$.

The kernel of a homomorphism $\alpha : (L, \circ) \to (L', *)$ of a loop L into a loop L' is a normal subloop N of L, i.e. a subloop of L such that

$$x \circ N = N \circ x, \quad (x \circ N) \circ y = x \circ (N \circ y), \quad x \circ (y \circ N) = (x \circ y) \circ N$$

holds for all $x, y \in L$. A loop $(L, \cdot)$ is a product of two subloops L_1 and L_2 if any element x of L has a representation $x = a \cdot b$, $a \in L_1$ and $b \in L_2$. A loop $(L, \cdot)$ is called a Moufang loop if for all $x, y, z \in L$ the identity $(x \cdot y) \cdot (z \cdot x) = [x \cdot (y \cdot z)] \cdot x$ holds.

Let L be a topological space. Then $(L, \cdot)$ is a topological loop if the maps $(x, y) \mapsto x \cdot y$, $(x, y) \mapsto x \backslash y$, $(x, y) \mapsto y/x : L^2 \to L$ are continuous. If only the multiplication and the left division are continuous, then the loop L is called almost topological. An almost topological loop is topological if the group generated by its left translations is a connected Lie group (see

[9], Corollary 1.22). A loop L is almost differentiable if L is a differentiable manifold and the multiplication as well as the left division are differentiable.

Let G be a compact connected Lie group, H be a connected Lie subgroup of G containing no non-trivial normal subgroup of G and $G/H = \{xH, x \in G\}$. Let $\sigma : G/H \to G$ be a continuous map with $\sigma(H) = 1 \in G$ such that the set $\sigma(G/H)$ is a system of representatives for G/H which generates G and acts sharply transitively on G/H. The last property means that to any given left cosets xH and yH there exists precisely one $z \in \sigma(G/H)$ with $zxH = yH$. Then the multiplication $xH \cdot yH = \sigma(xH)yH$ on the factor space G/H, respectively $x \cdot y = \sigma(xyH)$ on the set $\sigma(G/H)$ yields a compact topological loop having G as the group topologically generated by the left translations $\lambda_{xH} : yH \mapsto \sigma(xH)yH$, respectively $\lambda_x : y \mapsto \sigma(xyH)$.

If L is a compact topological loop such that the group topologically generated by all left translations of L is a compact connected Lie group, then the set $\{\lambda_a, a \in L\}$ forms a sharply transitive continuous section $\sigma : G/G_e \to G$ with $\sigma(\lambda_a G_e) = \lambda_a$, where G_e is the stabilizer of $e \in L$ in G.

If the section $\sigma : G/H \to G$ is differentiable, then the loop L is almost differentiable.

A quasi-simple compact Lie group is a compact Lie group G containing a normal finite central subgroup N such that the factor group G/N is simple. A semisimple connected compact group G is a Lie group containing a normal finite central subgroup N such that the factor group G/N is a direct product of simple Lie groups. A connected compact Lie group is an almost direct product of compact semisimple Lie groups if its universal covering (cf. [2], Appendix A) is a direct product of simply connected quasi-simple Lie groups.

For connected and locally simply connected topological loops there exist universal covering loops (cf. [3], [4], [6], IX.1). This yields the following

Lemma 1. *The universal covering loop $\tilde{L}$ of a connected and locally simply connected topological loop L is simply connected and L is isomorphic to a factor loop $\tilde{L}/N$, where N is a central subgroup of $\tilde{L}$.*

3 Loops corresponding to products of groups

Let $G = K \times P \times S$ be a group, where K is a group, P is a non-abelian group. Let $g : K \to S$ be a map which is not a homomorphism but $g(1) = 1$, the set $\{(k, 1, g(k)); k \in K\}$ generates the group $K \times \{1\} \times S$ and S is isomorphic to a subgroup of P having with the centre of P trivial intersection. Hence there is a monomorphism from S into P and we may assume $H = \{(1, x, x);\ x \in S\}$. Moreover, we put $M = \{(k, lg(k), g(k)); k \in K,\ l \in P\}$. Every element $(a, b, c) \in G$ may be uniquely decomposed as $(a, b, c) = (a, bc^{-1}, 1)(1, c, c)$

with $(1,c,c) \in H$. For all $a \in K, b \in P$ there are unique elements $m = (a, bg(a), g(a)) \in M$ and $h = (1, g(a)^{-1}, g(a)^{-1}) \in H$ such that $(a,b,1) = mh$. Hence the set M determines the section $\sigma : G/H \to G; (x,y,1)H \mapsto \sigma((x,y,1)H) = (x, yg(x), g(x))$. The set M acts sharply transitively on the left cosets $\{(a,b,1)H;\ a \in K, b \in P\}$ since for given $a_1, a_2 \in K, b_1, b_2 \in P$ the equation

$$(k, lg(k), g(k))(a_1, b_1, 1) = (a_2, b_2, 1)(1, d, d)$$

has the unique solution $k = a_2 a_1^{-1}$, $l = b_2 g(a_2 a_1^{-1}) b_1^{-1} g(a_2 a_1^{-1})^{-1}$ with $d = g(a_2 a_1^{-1}) \in P$. As the group H contains no normal subgroup of G the map σ corresponds to a loop L having the group $G = K \times P \times S$ as the group generated by its left translations, the subgroup H as the stabilizer of $e \in L$ and the set M as the set of all left translations of L. The multiplication of $L = (L, *)$ can be defined on the set $\{(a,b,1)H;\ a \in K, b \in P\}$ by

$$(a_1, b_1, 1)H * (a_2, b_2, 1)H = (a_1 a_2, b_1 g(a_1) b_2 g(a_1)^{-1}, 1)H. \tag{1}$$

Since $(1, l_1, 1)H * (1, l_2, 1)H = (1, l_1 l_2, 1)H$ for all $l_1, l_2 \in P$ holds $(N, *) = (\{(1, l, 1)H; l \in P\}, *)$ is a subgroup of $(L, *)$ isomorphic to P. As G is the direct product $G = G_1 \times G_2$, where $G_1 = K \times \{1\} \times S$ and $G_2 = \{1\} \times P \times \{1\}$, and $\sigma(G/H) = M = M_1 \times G_2$ with $M_1 = \{(k, 1, g(k)); k \in K\} \subset G_1$ Proposition 2.4 in [9], p. 44, yields that the group $(N, *)$ is normal in the loop $(L, *)$. Moreover, for all $k_1, k_2 \in K$ one has $(k_1, 1, 1)H * (k_2, 1, 1)H = (k_1 k_2, 1, 1)H$. Hence $(K, *) = (\{(k, 1, 1)H; k \in K\}, *)$ is a subgroup of $(L, *)$ isomorphic to K. Therefore the loop $(L, *)$ defined by (1) is a semidirect product of the normal subgroup $(N, *)$ by the subgroup $(K, *)$.

The loop L is a group if the multiplication (1) is associative, i.e.

$$((a_1, b_1, 1)H * (a_2, b_2, 1)H) * (a_3, b_3, 1)H =$$

$$(a_1, b_1, 1)H * ((a_2, b_2, 1)H * (a_3, b_3, 1)H).$$

This identity holds if and only if for all $a_i \in K$ and $b_i \in P$ one has

$$(a_1 a_2 a_3, b_1 g(a_1) b_2 g(a_1)^{-1} g(a_1 a_2) b_3 g(a_1 a_2)^{-1}, 1)H =$$

$$(a_1 a_2 a_3, b_1 g(a_1) b_2 g(a_2) b_3 g(a_2)^{-1} g(a_1)^{-1}, 1)H$$

or equivalently $g(a_1) g(a_2) b_3 g(a_2)^{-1} g(a_1)^{-1} = g(a_1 a_2) b_3 g(a_1 a_2)^{-1}$. This is a contradiction since g is not a homomorphism. Therefore L is a proper loop.

If K, P and S are connected Lie groups and the function g is continuous, then L has continuous multiplication and left division (cf. [9], p. 29). Hence L is a connected locally compact topological proper loop. If g is differentiable, then L is a connected almost differentiable proper loop (cf. [9], p. 32).

The constructed examples show the following

Remark 1. There exist proper loops with normal connected subgroups having a compact connected Lie group G as the group topologically generated by the left translations if $G = G_1 \times G_2 \times G_3$, where G_3 is isomorphic to a subgroup of G_2 having with the centre of G_2 trivial intersection.

The aim of the paper is to demonstrate that this is a typical situation for connected compact Lie groups G being groups generated by the left translations of a proper loop.

4 Results

Lemma 2. *Any one-dimensional connected topological loop having a compact Lie group as the group topologically generated by its left translations is the orthogonal group $SO_2(\mathbb{R})$.*

Proof. See in [9], Proposition 18.2, p. 235. □

Lemma 3. *Let G be a connected semisimple compact Lie group. Assume that G is the group topologically generated by the left translations of a compact simply connected loop L homeomorphic to a semisimple Lie group K_1. Then G has the form $K_1 \times \rho(H_1)$, where H_1 is a subgroup of K_1 and ρ is a monomorphism. For the stabilizer H of $e \in L$ one has $H = (H_1, \rho(H_1)) = \{(x, \rho(x)), x \in H_1\}$ and H_1 has with the centre Z_1 of K_1 a trivial intersection.*

Proof. Since G/H is homeomorphic to K_1 the group G is homeomorphic to $K_1 \times H$. According to [10] or to Theorem 16.1 in [9], p. 195, the group G has the form $G = K_1 \times K_2$ and the stabilizer H of $e \in L$ is $H = (H_1, \rho(H_1))$, where K_2 is a Lie group isomorphic to H and ρ is a monomorphism. From this it follows that $G = K_1 \times \rho(H_1)$. If $Z_1 \cap H_1 \neq 1$, then H has with the centre Z of G a non-trivial intersection. But this is a contradiction to the fact that H has no normal subgroup $\neq 1$ of G. □

Lemma 3 yields

Corollary 4. *Let $G = K_1 \times K_2$ be a connected compact semisimple Lie group such that K_1 is semisimple and H_1 is a subgroup of K_1 isomorphic to K_2. Let H be the subgroup $(H_1, \rho(H_1))$ of G, where ρ is a monomorphism $H_1 \to K_2$. If H_1 intersects the centre of K_1 non trivially, then there does not exist any proper loop L homeomorphic to K_1 having G as the group topologically generated by the left translations of L and H as the stabilizer of $e \in L$.*

Corollary 5. *Let G be a connected semisimple compact Lie group having a covering K_1 of a product of the groups $SO_3(\mathbb{R})$ as a proper direct factor. Then there does not exist any connected topological proper loop L homeomorphic to K_1 and having G as the group topologically generated by the left translations of L.*

Proof. We may assume that L is simply connected. Hence K_1 is a direct product of groups isomorphic to $Spin_3(\mathbb{R})$. Then the group G topologically generated by the left translations of L has the form $K_1 \times \rho(H_1)$, where H_1 is a subgroup of K_1, the map ρ is a monomorphism, and the stabilizer H of $e \in L$ is the subgroup $(H_1, \rho(H_1))$ of G (cf. Lemma 3). Since any subgroup H_1 intersects the centre of K_1 non trivially Corollary 4 yields the assertion. □

Theorem 6. *Let L be a topological loop homeomorphic to the 7-sphere or to the 7-dimensional real projective space such that the group G topologically generated by the left translations of L is a compact Lie group. Then L is one of the two 7-dimensional compact Moufang loops, G is locally isomorphic to $PSO_8(\mathbb{R})$ and the stabilizer H of $e \in L$ is isomorphic to $SO_7(\mathbb{R})$.*

Proof. We may assume that L is simply connected. Since G is a compact Lie group, using Proposition 2.4 in [7] and Ascoli's Theorem, from IX.2.9 Theorem of [6] it follows that the loop L has a left invariant uniformity. Therefore IX.3.14 Theorem in [6] yields that L is the multiplicative loop $\mathcal{O}$ of octonions having norm 1. Then G is isomorphic to $SO_8(\mathbb{R})$ and the stabilizer H of $e \in L$ is isomorphic to $SO_7(\mathbb{R})$ (cf. [10]).

If L is homeomorphic to the 7-dimensional real projective space, then the universal covering $\tilde{L}$ of L is a Moufang loop homeomorphic to the 7-sphere. It follows from [6], p. 216, that the loop L is a factor loop $\tilde{L}/N$, where N is a central subgroup of $\tilde{L}$ of order 2. Lemma 1.33 in [9] yields that L is the Moufang loop $\mathcal{O}/Z$, where Z is the centre of the multiplicative loop of octonions having norm 1. □

If L is a differentiable loop homeomorphic to the 7-sphere and if we assume that the group G topologically generated by the left translations of L is a quasi-simple compact Lie group, then G is isomorphic to $SO_8(\mathbb{R})$ and the stabilizer H of $e \in L$ is isomorphic to $SO_7(\mathbb{R})$. This allows us to obtain the assertion of Theorem 6 also in the following way. We identify the set G/H of the left cosets with the set $\mathcal{S}$ of the left translations of the Moufang loop $\mathcal{O}$. The section $\sigma : G/H \to G$ belonging to L has the form $\sigma(xH) = x\phi(x)$, where $x \in \mathcal{S}$ and ϕ is a differentiable map from $\mathcal{S}$ to H. Since any two elements of $\mathcal{S}$ are contained in a subgroup D isomorphic to $Spin_3(\mathbb{R})$ the restriction of ϕ to D is a homomorphism (Corollary 5). Hence L is a diassociative Lie loop ([6], IX.6.42) and Theorem 16.10 in [9] yields the assertion of Theorem 6.

Theorem 7. *Let G be a compact Lie group which is the group topologically generated by the left translations of a proper topological loop L homeomorphic to a connected semisimple compact Lie group. Then G is a connected semisimple Lie group.*

Proof. Since L is connected also G is connected. By Hofmann-Scheerer Splitting Theorem (cf. [2], p. 474) the group G is isomorphic to a semidirect product $G = G' \rtimes T$, where G' is the semisimple commutator subgroup of G and T is a torus. The group G' is isomorphic to an almost direct product $G' = K_1 \cdots K_m$ of quasi-simple compact Lie groups K_i, $i = 1, \cdots, m$. The loop L is homeomorphic to a connected semisimple compact Lie group $K = K_1 \cdots K_s$ with $s \leq m$. We may assume that L and hence also K are simply connected. Since the universal covering $\tilde{G}'$ of G' is the direct product of K and the universal covering $\tilde{S}$ of $S = K_{s+1} \cdots K_m$, the group G' is the direct product of K and S. As L is homeomorphic to the image of the section $\sigma : G/H \to G$, where H is the stabilizer of $e \in L$, the set $\sigma(G/H)$ has the form $\{(x, \alpha(x))\}$, where $x \in K$ and α is a continuous mapping from K into $S \rtimes T$. The group $T = T_1 \times \cdots \times T_h$ is the direct product of one-dimensional tori T_i.

Let π be the projection from $S \rtimes T$ into T along S and ι_i be the projection from T into T_i along the complement $\prod_{j \neq i} T_j$. As $\sigma(G/H)$ is a compact connected homogeneous space and T_i is a 1-sphere for all i any $\iota_i \pi \alpha(K)$ is either constant or surjective. Since $\sigma(G/H)$ generates G there exists one i such that $\iota_i \pi \alpha(K)$ is different from $\{1\}$. As the group T is the direct product of 1-dimensional tori T_i the Bruschlinsky group B (cf. [8], p. 47) of K is not trivial. By Theorem 7.1 in [8], p. 49, B is isomorphic to the first cohomology group $H^1(K)$. The graded cohomology algebra of the compact Lie group K is the tensor product of the cohomology algebras $H^1(K_i)$ of the quasi-simple factors K_i of K, $i = 1, \cdots, s$. Since $H^1(K_i)$ has no generators of degree 1 and 2 ([1], pp. 126-127) also the cohomology algebra $H^1(K)$ has no generators of degree 1 and 2. Hence the Poincare polynomial $\psi(K)$ has no linear and quadratic monomials, which is a contradiction. □

Remark 2. If a topological loop L is homeomorphic to a non semisimple compact connected Lie group, then in contrast to Theorem 7 the group topologically generated by the left translations of L may be non semisimple.

Let T_1 be a torus of dimension $m \geq 1$, P be a connected semisimple compact Lie group and T_2 be a torus of dimension s with $1 \leq s \leq m$ such that there exists a monomorphism $\varphi : T_2 \to P$ with $\varphi(T_2) \cap Z(P) = \{1\}$, where $Z(P)$ is the centre of P. Further let $g : T_1 \to T_2$ be a continuous surjective mapping which is not a homomorphism but $g(1) = 1$. Then according to

Section 3 there exists a proper connected loop L homeomorphic to $T_1 \times P$ having the direct product $G = T_1 \times P \times T_2$ as the group topologically generated by the left translations of L and $H = \{(1, \varphi(x), x); x \in T_2\} < G$ as the stabilizer of $e \in L$.

Theorem 8. *There does not exist any proper topological loop which is homeomorphic to a connected quasi-simple Lie group and has a compact Lie group as the group topologically generated by its left translations.*

Proof. By Lemma 1 we may assume that L is a proper loop homeomorphic to a simply connected quasi-simple compact Lie group K_1. Then the stabilizer H of $e \in L$ has the form $H = (H_1, \rho(H_1)) = \{(x, \rho(x)); x \in H_1\}$, where ρ is a monomorphism and the group G topologically generated by the left translations of L has the form $G = K_1 \times \rho(H_1)$ (cf. Lemma 3). Identifying the space G/H with K_1 one has that the image $\sigma(K_1)$ of the section $\sigma : K_1 \to G$ intersects H trivially. As ρ is a monomorphism we may assume that $H = \{(x, x); x \in H_1\}$. The restriction of σ to a one-dimensional torus subgroup A of H_1 yields $\sigma(A) = \{(u, f(u))\}$, where f is a continuous function. Since the compact loop $\sigma(A)$ is a group (cf. Lemma 2) the map f is a homomorphism. It follows that $\sigma(A)$ has the form $\{(u, u^n)\}$ with fixed $n \in \mathbb{Z}$. Since $\{(u, u^n); u \in A \cong SO_2(\mathbb{R})\} \cap H = \{1\}$ the equation $x^n = x$, $x \in A$, can be satisfied only for $x = 1$. Equivalently, $x \mapsto x^{n-1}$ is an automorphism of $SO_2(\mathbb{R})$. Besides the identity the only automorphism of the group $SO_2(\mathbb{R})$ is the map $x \mapsto x^{-1}$. Therefore we obtain $n \in \{0, 2\}$. Let C_1 be a 3-dimensional subgroup of H_1. By Corollary 5 any 3-dimensional compact loop havig a compact Lie group as the group topologically generated by its left translations is a group. Hence the set $C = \sigma(C_1) = (C_1, \psi(C_1))$ is locally isomorphic to $SO_3(\mathbb{R})$ and ψ is a homomorphism of C_1. Besides a homomorphism with finite kernel any continuous homomorphism of $SO_3(\mathbb{R})$ is an automorphism induced by a conjugation with elements of the orthogonal group $O_3(\mathbb{R})$. Hence none of the 1-dimensional subgroups A of C_1 satisfies that $\sigma(A) = \{(x, x^2), x \in A\}$. As in compact groups the exponential map is surjective the compact group C is the union of the one-dimensional connected subgroups $\sigma(A) = \{(x, 1), x \in A\}$. Therefore C has the form $(C_1, 1)$. Since the 3-dimensional subgroups of H_1 covers H_1 (cf. [5], Propositions 6.45 and 6.46) for the continuous section σ one has $\sigma(H_1) = (H_1, 1)$.

Let B_i be a one-dimensional torus subgroup of K_1 such that $\sigma(B_i) = (B_i, 1)$. The union $B = \bigcup B_i$ of the one-dimensional subgroups B_i of K_1 is a subgroup of K_1 containing H_1.

Let F_i be a 1-dimensional torus subgroup of K_1 such that $\sigma(F_i) \neq (F_i, 1)$. Then one has $\sigma(F_i) = \{(x, x^n); x \in F_i\}$, where $n \in \mathbb{Z} \setminus \{0\}$. Since any 1-dimensional subgroup of K_1 is contained in a 3-dimensional subgroup of K_1

(cf. [5], Propositions 6.45 and 6.46) and any 3-dimensional loop homeomorphic to a cover of $SO_3(\mathbb{R})$ is a group (cf. Corollary 5), besides a homomorphism with finite kernel we obtain that $x \mapsto x^n$ is either an isomorphism or an anti-isomorphism of $SO_3(\mathbb{R})$. Hence one has $n = 1$ or -1. It follows that either $\sigma(F_i) = \{(x, x); x \in F_i\}$ or $\sigma(F_i) = \{(x, x^{-1}); x \in F_i\}$ for any 1-dimensional torus F_i of K_1 with $\sigma(F_i) \neq (F_i, 1)$. The union F of such 1-dimensional tori F_i of K_1 is isomorphic to the group $\rho(H_1) \cong H_1$.

The subgroups F and B yield a factorization of K_1 such that the intersection $F \cap B$ is discrete. This contradicts the fact that K_1 is quasi-simple (cf. Theorem 4.6 in [1], p. 145). □

Corollary 9. *Let L be a proper topological loop homeomorphic to a product of quasi-simple compact Lie groups and having a compact connected Lie group G as the group topologically generated by its left translations. Then the minimal dimension of G is 14. In this case G is locally isomorphic to $Spin_3(\mathbb{R}) \times SU_3(\mathbb{C}) \times Spin_3(\mathbb{R})$ and L is homeomorphic to a group locally isomorphic to $Spin_3(\mathbb{R}) \times SU_3(\mathbb{C})$.*

Proof. We assume that the loop L is simply connected. Then L is homeomorphic to the direct product K_1 of at least two quasi-simple simply connected factors (cf. Theorem 8). According to Theorem 7 the connected group G is semisimple. By Lemma 3 the stabilizer H of $e \in L$ has the form $(H_1, \rho(H_1)) = \{(x, \rho(x)), x \in H_1\}$ and $G = K_1 \times \rho(H_1)$, where H_1 is a subgroup of K_1 and ρ is a monomorphism. Therefore G has at least three quasi-simple factors such that one of these is different from $Spin_3(\mathbb{R})$ (cf. Corollary 5). Since any quasi-simple compact Lie group different from $Spin_3(\mathbb{R})$ is at least 8-dimensional (cf. [11]), the group G has dimension at least 14. If $\dim(G) = 14$, then by Section 3 the group K_1 coincides with $K \times P = Spin_3(\mathbb{R}) \times SU_3(\mathbb{C})$ and $G = K \times P \times S$ has the form as in the assertion. The loop L exists since for the function $g : K \to S$ in Section 3 one can choose the function $x \mapsto x^n$ with $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$. □

Remark 3. Euclidean and hyperbolic symmetric spaces correspond to global differentiable loops (cf. [9], Theorem 11.8, p. 135). In contrast to this, compact simple symmetric spaces which are not Lie groups yield only local Bol loops L since for L the exponential map is not a diffeomorphism (cf. [9], Proposition 9.19, p. 115).

This paper was supported by the János Bolyai Research Fellowship and the European Union's Seventh Framework Programme (FP7/2007-2013) under grant agreement no. 317721, no. 318202.

References

[1] V. V. Gorbatsevich, A. L. Onishchik, *Lie transformation groups*, in Lie Groups and Lie Algebras I (A. L. Onishchik, ed.); Encyklopedia of Mathematical Sciences **20**, Springer Verlag, Berlin, 1993, 95-229.

[2] J. Hilgert, K.-H. Neeb, *Structure and Geometry of Lie Groups*, Springer Monographs in Mathematics **3733**, New York, 2012.

[3] K. H. Hofmann, *Topologische Loops*, Math. Z. **70** (1958), 13-37.

[4] K. H. Hofmann, *Non-associative Topological Algebra*, Tulane University Lecture Notes, New Orleans, 1961.

[5] K. H. Hofmann, S. A. Morris, *Structure of Compact Groups*, de Gruyter Expositions in Mathematics **25**, Berlin, New York, 1998.

[6] K. H. Hofmann, K. Strambach, *Topological and analytical loops*, in Quasigroups and Loops: Theory and Applications (O. Chein, H. O. Pflugfelder, J. D. H. Smith, ed.); Sigma Series in Pure Mathematics **8**, Heldermann Verlag, Berlin, 1990, 205-262.

[7] S. N. Hudson, *Lie loops with invariant uniformities*, Trans. Amer. Math. Soc. **115** (1965), 417-432.

[8] Sz.-T. Hu, *Homotopy Theory*, Academic Press, New York, London, 1959.

[9] P. T. Nagy, K. Strambach, it Loops in Group Theory and Lie Theory, de Gruyter Expositions in Mathematics **35**, Berlin, New York, 2002.

[10] H. Scheerer, *Restklassenräume kompakter zusammenhängender Gruppen mit Schnitt*, Math. Ann. **206** (1973), 149-155.

[11] Tits, *Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen*, Lectures Notes in Mathematics, Springer, Berlin, 1967.

Á. Figula, Institute of Mathematics, University of Debrecen, Debrecen, Hungary, figula@science.unideb.hu and

K. Strambach, Department Mathematik, Universität Erlangen-Nürnberg, Erlangen, Germany, stramba@math.fau.de