Bol loops as sections in semi-simple Lie groups of small dimension

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Abstract

Using the relations between the theory of differentiable Bol loops and the theory of affine symmetric spaces we classify all connected differentiable Bol loops having an at most 9-dimensional semi-simple Lie group as the group topologically generated by their left translations. We show that all these Bol loops are isotopic to direct products of Bruck loops of hyperbolic type or to Scheerer extensions of Lie groups by Bruck loops of hyperbolic type.

1   Introduction

In [15] and [17] the authors have thoroughly studied the relations between smooth Bol loops and homogeneous spaces of Lie groups. Using their point of view we treat the connected differentiable Bol loops L as images of global differentiable sections $\sigma : G/H \to G$, where G is a connected Lie group, H is a closed subgroup containing no non-trivial normal subgroup of G and for all $r, s \in \sigma(G/H)$ the element rsr lies in $\sigma(G/H)$. The differentiable Bruck loops, these are Bol loops L satisfying the identity $(xy)^{-1} = x^{-1}y^{-1}$, $x, y \in L$, have realizations on differentiable affine symmetric spaces G/H, where H is the set of fixed elements of an involutory automorphism of G and $\sigma(G/H)$ is the exponential image of the (-1)-eigenspace of the corresponding automorphism of the Lie algebra $\mathbf{g}$ of G. An impotant subclass of Bruck loops are the Bruck loops of hyperbolic type which correspond to Lie groups G and involutions τ fixing elementwise a maximal compact subgroup of G (cf. [4], 64.9, 64.10). With this notions our main result reads as follows.

Main Theorem *Let L be a connected differentiable Bol loop having an at most 9-dimensional semi-simple Lie group G as the group topologically generated by its left translations. Then L is isotopic to a direct product of Bruck loops of hyperbolic type or to a Scheerer extension of a Lie group by a Bruck loop of hyperbolic type.*
If $\dim G \leq 5$ then L is isotopic to the hyperbolic plane loop $\mathbb{H}_2$ and G is isomorphic to $PSL_2(\mathbb{R})$ (cf. [17], Sec. 22).
If $\dim G = 6$ then L is isotopic either to the direct product $\mathbb{H}_2 \times \mathbb{H}_2$ or to the 3-dimensional hyperbolic space loop $\mathbb{H}_3$ (cf. [3], p. 446) or to a Scheerer extension of a 3-dimensional simple Lie group G_1 by $\mathbb{H}_2$. In the first case G is isomorphic to $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$, in the second case to $PSL_2(\mathbb{C})$, in the third case to $PSL_2(\mathbb{R}) \times G_1$.

If $7 \leq \dim G \leq 8$ then either L is isotopic to the complex hyperbolic plane loop having $PSU_3(\mathbb{C}, 1)$ as the group topologically generated by its left translations (cf. [14], Prop. 5.1, p.152) or L is isotopic to the 5-dimensional Bruck loop of hyperbolic type having the group $PSL_3(\mathbb{R})$ as the group topologically generated by its left translations.

If $\dim G = 9$ then L is isotopic either to $\mathbb{H}_2 \times \mathbb{H}_2 \times \mathbb{H}_2$ or to $\mathbb{H}_2 \times \mathbb{H}_3$ or to a Scheerer extension of a 6-dimensional semi-simple Lie group G_2 by $\mathbb{H}_2$ or to a Scheerer extension of a 3-dimensional simple Lie group G_1 by $\mathbb{H}_2 \times \mathbb{H}_2$ respectively by $\mathbb{H}_3$. In the first case G is isomorphic to $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$, in the second case to $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{C})$, in the third case to $PSL_2(\mathbb{R}) \times G_2$ and in the fourth case to $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R}) \times G_1$ respectively to $PSL_2(\mathbb{C}) \times G_1$.

All known differentiable connected Bol loops having a semi-simple Lie group as the group topologically generated by their left translations are isotopic to direct products of Bruck loops of hyperbolic type or to Scheerer extensions of Lie groups by Bruck loops of hyperbolic type since they are constructed using Cartan involutions.

For the classification of differentiable Bol loops L up to isotopisms we proceed in the following way ([15], pp. 424-425, [17], pp. 78-79 and Proposition 1.6, p. 18). In the Lie algebra $\mathbf{g}$ of the group G topologically generated by the left translations of L we determine the (-1)-eigenspaces $\mathbf{m}$ for all involutory automorphisms of $\mathbf{g}$. After this we seek for any $\mathbf{m}$ a system of representatives $\mathbf{h}^*$ of the sets $\{Ad_g\mathbf{h}; g \in G\}$ which consists of subalgebras with $\mathbf{g} = \mathbf{m} \oplus Ad_g\mathbf{h} = \mathbf{m} \oplus \mathbf{h}^*$. Any triple $(G, \exp \mathbf{h}^*, \exp \mathbf{m})$ determines a local Bol loop. Global differentiable Bol loops L correspond precisely to those exponential images of $\mathbf{m}$, which form a system of representatives for the cosets of $\exp \mathbf{h}$ in G. To show which local Bol loop is extendible to a global one we need a futher analytic treatment since there are much more local than global differentiable Bol loops.

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2   Some basic notions of the theory of Bol loops

A binary system $(L, \cdot)$ is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution which we denote by $y = a \backslash b$ and $x = b/a$. Two loops $(L_1, \circ)$ and $(L_2, *)$ are called isotopic if there are three bijections $\alpha, \beta, \gamma : L_1 \to L_2$ such that $\alpha(x) * \beta(y) = \gamma(x \circ y)$ holds for any $x, y \in L_1$. Isotopism of loops is an equivalence relation. Let $(L_1, \cdot)$ and $(L_2, *)$ be two loops. The set $L = L_1 \times L_2 = \{(a, b) \mid a \in L_1, b \in L_2\}$ with the componentwise multiplication is again a loop, which is called the direct product of L_1 and L_2, and the loops $(L_1, \cdot)$, $(L_2, *)$ are subloops of L.

A loop L is called a Bol loop if for any two left translations λ_a, λ_b the product $\lambda_a\lambda_b\lambda_a$ is again a left translation of L. If L_1 and L_2 are Bol loops, then the direct product $L_1 \times L_2$ is again a Bol loop. Every subloop of a Bol loop satisfies the Bol identity.

The theory of differentiable loops L is essentially the theory of the smooth binary operations $(x, y) \mapsto x \cdot y$, $(x, y) \mapsto x/y$, $(x, y) \mapsto x\backslash y$ on the connected differentiable manifold L. If L is a connected differentiable Bol loop then the left translations $\lambda_a = y \mapsto a \cdot y : L \to L$, $a \in L$, which are diffeomorphisms of L, topologically generate a connected Lie group G (cf. [17], p. 33; [15], pp. 414-416). Moreover the manifold L is parallelizable since the set of the left translations is sharply transitive.

Every connected differentiable Bol loop is isomorphic to a loop L realized on the homogeneous space G/H, where G is a connected Lie group, H is a connected closed subgroup which is not allowed to contain a non-trivial normal subgroup of G and $\sigma : G/H \to G$ is a differentiable section with $\sigma(H) = 1 \in G$ such that the subset $\sigma(G/H)$ generates G

and for all $r, s \in \sigma(G/H)$ the element rsr is contained in $\sigma(G/H)$ (cf. [17], p. 18 and Lemma 1.3, p. 17, [9], Corollary 3.11, p. 51). The multiplication of L on the space G/H is defined by $xH * yH = \sigma(xH)yH$ and the group G is the group topologically generated by the left translations of L.

Let $\mathbf{m} = T_1\sigma(G/H)$ be the tangent space of $\sigma(G/H)$ at $1 \in G$. If $(\mathbf{g}, [.,.])$ respectively $\mathbf{h}$ denotes the Lie algebra of G respectively of H then one has $\mathbf{g} = \mathbf{m} \oplus \mathbf{h}$, $\big[[\mathbf{m},\mathbf{m}],\mathbf{m}\big] \subseteq \mathbf{m}$ and $\mathbf{m}$ generates $\mathbf{g}$. ¿From every Bol triple $((\mathbf{g}, [.,.]), \mathbf{h}, \mathbf{m})$ we can obtain with a canonical construction a triple $((\mathbf{g}^*, [.,.]^*), \mathbf{h}^*, \mathbf{m}^*)$, such that $\mathbf{m}^*$ is the (-1)-eigenspace of an involutory automorphism of $\mathbf{g}^*$ and the subalgebra $\mathbf{h}^*$ complements $\mathbf{m}^*$ in $\mathbf{g}^*$ (cf. [15], pp. 424-425). If the Lie algebra $\mathbf{g}$ is semi-simple then it is isomorphic to $\mathbf{g}^*$ and it is the Lie algebra of the displacement group for the symmetric space belonging to the Lie triple system $A := \big(\mathbf{m}, (.,.,.)\big)$, with $(.,.,.) = [[.,.],.]$ (cf. [13], pp. 78-79). Hence one has $\mathbf{g} = \mathbf{m} \oplus [\mathbf{m},\mathbf{m}]$ (cf. [17], Section 6).

Remark. Let $\mathbf{g}$ be the Lie algebra of the group topologically generated by the left translations of a connected differentiable Bol loop L such that $\mathbf{g}$ is the direct sum $\mathbf{g}_1 \oplus \mathbf{g}_2$ of Lie algebras $\mathbf{g}_i$, $i = 1,2$. If an involutory automorphism τ of $\mathbf{g}$ has the shape (τ_1, id), where τ_1 is an involutory automorphism of $\mathbf{g}_1$ and $id : \mathbf{g}_2 \to \mathbf{g}_2$ is the identity map, then one has $\mathbf{m} = \mathbf{m}_1 \oplus \mathbf{g}_2$ and $\mathbf{h} = \mathbf{h}_1 \oplus \{0\}$, where $\mathbf{m}_1$ is the (-1)-eigenspace and $\mathbf{h}_1$ the $(+1)$-eigenspace of τ_1. The Bol loop L corresponding to such a triple $(\mathbf{g}, \mathbf{h}, \mathbf{m})$ is the direct product of a Bol loop $\tilde{L}$ isotopic to the Bruck loop realized on the symmetric space $\exp \mathbf{g}_1 / \exp \mathbf{h}_1$ and the Lie group $\exp \mathbf{g}_2$.

Let L_1 be a loop defined on the factor space G_1/H_1 with respect to a section $\sigma_1 : G_1/H_1 \to G_1$ the image of which is the set $M_1 \subset G_1$. Let G_2 be a group, let $\varphi : H_1 \to G_2$ be a homomorphism and $(H_1, \varphi(H_1)) = \{(x, \varphi(x)) \mid x \in H_1\}$. A loop L is called a Scheerer extension of G_2 by L_1 if the loop L is defined on the factor space $(G_1 \times G_2)/(H_1, \varphi(H_1))$ with respect to the section $\sigma : (G_1 \times G_2)/(H_1, \varphi(H_1)) \to G_1 \times G_2$ the image of which is the set $M_1 \times G_2$.

Lemma 1. *Let $\mathbf{g}$ be the Lie algebra of the group topologically generated by the left translations of a connected differentiable Bol loop L such that $\dim L \geq 4$ and the tangent space $T_1 L$ does not contain a Lie algebra as a direct factor. Let $\mathbf{g}$ be the direct product of simple Lie algebras with $\dim \mathbf{g} \leq 9$.*
i) If $\dim \mathbf{g} = 6$ then $\dim L = 4$.
ii) If $\dim \mathbf{g} = 9$ then $\dim L \in \{5, 6\}$.

Proof. Let τ be the involutory automorphism corresponding to L. Neither the (-1)-eigenspace nor the $(+1)$-eigenspace of τ can contain a simple direct factor of $\mathbf{g}$. The (-1)-eigenspace of each involutory automorphism of a 3-dimensional simple Lie algebra is 2-dimensional (cf. [5], pp. 44-45). Hence if $\mathbf{g} = \mathbf{g}_1 \oplus \mathbf{g}_2$ with simple 3-dimensional Lie algebras $\mathbf{g}_i$ then $\dim L = 4$. If $\mathbf{g} = \mathbf{g}_1 \oplus \mathbf{g}_2 \oplus \mathbf{g}_3$ with 3-dimensional simple Lie algebras $\mathbf{g}_i$, $(i = 1,2,3)$ then the $(+1)$-eigenspace of an involutory automorphism of $\mathbf{g}$ has dimension either 3 or 4. Let $\mathbf{g} = \mathbf{g}_1 \oplus \mathbf{g}_2$, where $\mathbf{g}_i$ are simple Lie algebras with $\dim \mathbf{g}_1 = 6$ and $\dim \mathbf{g}_2 = 3$. The dimension of a (-1)-eigenspace of an involutory automorphism of $\mathbf{g}_1$ is either 3 or 4 (cf. [14], p. 153). Hence $\dim L \notin \{4, 7, 8\}$. $\qquad\square$

¿From topological reasons we obtain

Lemma 2. *Let G be isomorphic to the Lie group $G_1 \times G_2$, such that $G_2 \cong SO_3(\mathbb{R})$ and for the subgroup H of G one has $H = H_1 \times H_2$ with $1 \neq H_2 \leq G_2$. Then G cannot be the group topologically generated by the left translations of a topological loop.*

Proof. The factor space G/H is a topological product of spaces having as a factor the 2-sphere or the projective plane, which are not parallelizable. $\qquad\square$

Lemma 3. *Let G be a Lie group isomorphic to $K \times SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$ and let H be a subgroup of G such that $H = H_1 \times \{(a,a) \mid a \in SO_3(\mathbb{R})\}$ with $H_1 \leq K$. Then there is no Bol loop L corresponding to the pair (G, H).*

Proof. The loop L would be a product of a loop L_1 corresponding to the factor space K/H_1 with a compact proper loop L_2 having $SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$ as the group topologically generated by its left translations (cf. [17], Proposition 1.18, p. 26). But such a loop L_2 does not exist (cf. [17], Corollary 16.9, p. 204). □

Proposition 4. *There is no connected differentiable Bol loop such that the group G topologically generated by its left translations is a compact Lie group G with dim $G \leq 9$.*

Proof. If G is a quasi-simple Lie group then it admits a continuous section if and only if G is locally isomorphic to $SO_8(\mathbb{R})$ (cf. [18], pp. 149-150). Since dim $L \geq 4$ ([17], Corollary 16.8, p. 204) it follows from [17], Theorem 16.1 that G is locally isomorphic to $SO_3(\mathbb{R}) \times SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$. This is excluded by Theorem 16.7 in [17], p. 198 and Lemmata 2 and 3. □

An important tool to eliminate certain stabilizers H is the fundamental group π_1 of a connected topological space.

Lemma 5. *Denote by G a connected Lie group and by H a connected subgroup of G. Let $\sigma : G/H \to G$ be a global section. Then $\pi_1(K) \cong \pi_1(\sigma(G/H)) \times \pi_1(K_1)$, where K respectively K_1 is a maximal compact subgroup of G respectively of H.*

Proof. Since G respectively H is homeomorphic to a topological product $K \times \mathbb{R}^n$ respectively to $K_1 \times \mathbb{R}^n$ (cf. [16], p. 178) one has $\pi_1(G) = \pi_1(K)$ and $\pi_1(H) = \pi_1(K_1)$. The group G is homeomorphic to the topological product $\sigma(G/H) \times H$. Hence $\pi_1(G) = \pi_1(K) \cong \pi_1(\sigma(G/H)) \times H) \cong \pi_1(\sigma(G/H)) \times \pi_1(K_1)$ (cf. [7], Theorem 2.1, p. 144). □

Since the Bol loops are strongly left alternative (Definition 5.3. in [17]) every global Bol loop L contains the exponential image of the tangent space $\mathbf{m}$ at $e \in L$. In the discussion which submanifolds $\exp \mathbf{m}$ can be extended to a global section we use the following Lemma of [3].

Lemma 6. *Let L be a differentiable loop and denote by $\mathbf{m}$ the tangent space $T_1\sigma(G/H)$, where $\sigma : G/H \to G$ is the section corresponding to L. Then $\mathbf{m}$ does not contain any element of $Ad_g\mathbf{h}$ for some $g \in G$. Moreover, every element of G can be uniquely written as a product of an element of $\sigma(G/H)$ with an element of H.*

Lemma 7. *Let $\sigma : G/H \to G$ be a continuous section, where $G \cong PSL_2(\mathbb{R})$ and H is the group $\left\{ \begin{pmatrix} l & b \\ 0 & l^{-1} \end{pmatrix} ; b \in \mathbb{R} \right\}$ such that either $l = 1$ for all $b \in \mathbb{R}$ or $0 < l \in \mathbb{R}$. Then the image $\sigma(G/H)$ cannot contain the manifold $M = \left\{ \begin{pmatrix} x+y & z \\ z & x-y \end{pmatrix} ; y, z \in \mathbb{R}, x \geq 1, x^2 - y^2 - z^2 = 1 \right\}$.*

Proof. The coset $g(c)H = \begin{pmatrix} 1+c & 1 \\ c & 1 \end{pmatrix} H$, $c > -1$, contains the element $s(c) = \begin{pmatrix} 1+c & c \\ c & \frac{c^2+1}{1+c} \end{pmatrix} \in M$. But $\lim_{c \to -1} \sigma(g(c)H)$ should be $\lim_{c \to -1} s(c)$ which is a contradiction. □

Remark. If the Lie group G is the group topologically generated by the left translations of two Bol loops L_1 and L_2 such that the corresponding symmetric spaces are isomorphic then L_1 and L_2 are isotopic (cf. [17], Theorem 1.11, pp. 21-22).

In our computations we often use the following facts.

4

As a real basis of $\mathfrak{sl}_2(\mathbb{R})$ we choose

$$(1) \qquad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \ e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \ e_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(cf. [6], pp. 19-20).

Then the Lie algebra multiplication is given by

$$[e_1, e_2] = 2e_3, \ \ [e_1, e_3] = 2e_2, \ \ [e_3, e_2] = 2e_1.$$

An element $X = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in \mathfrak{sl}_2(\mathbb{R})$ is elliptic, parabolic or hyperbolic according to whether

$$k(X, X) = \lambda_1^2 + \lambda_2^2 - \lambda_3^2 \ \text{ is smaller, equal, or greater } 0.$$

1.1 The basis elements e_1, e_2 are hyperbolic, e_3 is elliptic and the elements $e_2 + e_3$, $e_1 + e_3$ are both parabolic. All elliptic elements are conjugate under $Ad\ G$ to e_3, all hyperbolic elements to e_1 and all parabolic elements to $e_2 + e_3$ (cf. [6], p. 23). There is precisely one conjugacy class $\mathcal{C}$ of the two dimensional subgroups of $PSL_2(\mathbb{R})$; as a representative of $\mathcal{C}$ we choose

$$\mathcal{L}_2 = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}; \ a > 0, b \in \mathbb{R} \right\}.$$

The Lie algebra of $\mathcal{L}_2$ is generated by the elements e_1, $e_2 + e_3$.

1.2 As a real basis of the Lie algebra $\mathfrak{so}_3(\mathbb{R}) \cong \mathfrak{su}_2(\mathbb{C})$ we may choose the basis elements $\{ie_1, ie_2, e_3\}$, where $i^2 = -1$. Every elements of $\mathfrak{so}_3(\mathbb{R})$ is conjugate to e_3.

3 Bruck loops of hyperbolic type

Now we give a procedure to construct many Bruck loops having a non-compact Lie group as the group topologically generated by the left translations.

Theorem 8. *Let G be a simple non-compact Lie group, let H be a maximal compact subgroup of G and let τ be the involutory automorphism of the Lie algebra $\mathbf{g}$ of G such that the Lie algebra $\mathbf{h}$ of H is the $(+1)$-eigenspace of τ.*

a) The factor space G/H is a Riemannian symmetric space diffeomorphic to the manifold $\exp \mathbf{m}$, where $\mathbf{m}$ is the (-1)-eigenspace of τ. The group G is the group of displacements of G/H and $\exp \mathbf{m} = \{\sigma_{xH}\sigma_H\}$, where σ_{xH} is the reflection at the point xH. Any coset xH contains precisely one element of $\exp \mathbf{m}$.

b) The section $\sigma : G/H \to G$ assigning to the coset xH the element of $\exp \mathbf{m}$ contained in xH defines a global Bruck loop L on G/H by $(xH) * (yH) = \sigma(G/H)yH$.

Proof. According to [13], Proposition 1.7, p. 148, the factor space G/H is a Riemannian symmetric space having the group G as its group of displacements, such that $\exp \mathbf{m}$ consists of the products of the reflection at H and a reflection at arbitrary point (cf. [13], p. 64 and Proposition 3.2, p. 95 and Theorem 1.3, p. 73). It follows from [13], Theorem 3.2, p. 165, that G/H is diffeomorphic to $\exp \mathbf{m}$, any coset xH contains precisely one element of $\exp \mathbf{m}$ and the section $\sigma : G/H \to G$ is differentiable. For elements $r = \sigma_{xH}\sigma_H$, $s = \sigma_{yH}\sigma_H$ of $\exp \mathbf{m}$ one has $rsr = (\sigma_{xH}\sigma_H)(\sigma_{yH}\sigma_H)(\sigma_{xH}\sigma_H) = (\sigma_{xH}(\sigma_H\sigma_{yH}\sigma_H)\sigma_{xH})\sigma_H \in \exp \mathbf{m}$. Hence the multiplication $* : G/H \times G/H \to G/H$ defines on G/H a global differentiable Bruck loop L (cf. [17], Proposition 9.25, p. 118). $\qquad \square$

Another proof of this theorem is given in [10], Theorem 3.3, p. 319.

Any simple non-compact Lie group G admits an involutory automorphism τ the centralizer of which is a maximal compact subgroup of G. This Cartan involution τ determines a symmetric space $\mathcal{S}(\tau)$ of hyperbolic type (cf. [4], 64.9, p. 375). For this reason we call the Bruck loop realized on $\mathcal{S}(\tau)$ a differentiable Bruck loop of hyperbolic type.

4   Bol loops corresponding to simple Lie groups

First we investigate the Lie groups locally isomorphic to $PSL_2(\mathbb{C})$. A real basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is given by $\{e_1, e_2, e_3, ie_1, ie_2, ie_3\}$, where $\{e_1, e_2, e_3\}$ is the basis of $\mathfrak{sl}_2(\mathbb{R})$ described by (1).

According to [14], p. 153, there is only one conjugacy class of involutory automorphisms of $G = PSL_2(\mathbb{C})$ leaving an at most 2-dimensional subgroup H of G elementwise fixed. A representative of this class is the map $\tau : PSL_2(\mathbb{C}) \to PSL_2(\mathbb{C}) : X \mapsto A^{-1}XA$ with $A = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The centralizer of τ is the group $H = \left\{ \pm \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \; a, b \in \mathbb{C}, a^2 + b^2 = 1 \right\}$.
The Lie algebra $\mathbf{h}$ of H is generated by e_3, ie_3 and the tangent space $\mathbf{m}$ of the corresponding 4-dimensional symmetric space has as basis elements e_1, e_2, ie_1, ie_2.

Lemma 9. *Let $\mathbf{g}$ be the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Any 2-dimensional subalgebra $\mathbf{h}$ of $\mathbf{g}$ has (up to conjugation) one of the following forms:*

$$\mathbf{h}_1 = \langle e_3, ie_3 \rangle, \quad \mathbf{h}_2 = \langle e_1, e_2 + e_3 \rangle, \quad \mathbf{h}_3 = \langle i(e_2 + e_3), e_2 + e_3 \rangle.$$

The element $e_2 + e_3 \in \mathbf{h}_2$ as well as $i(e_2 + e_3) \in \mathbf{h}_3$ is conjugate under $Ad\,G$ to $e_1 + ie_2 \in \mathbf{m}$. The element $e_3 \in \mathbf{h}_1$ respectively $ie_3 \in \mathbf{h}_1$ is conjugate to $ie_1 \in \mathbf{m}$ respectively to $-e_1 \in \mathbf{m}$.

Proof. The first assertion follows from Theorem 15 in [12], p. 129.
Futhermore one has $Ad_{g_1}(e_2 + e_3) = e_1 + ie_2$, $Ad_{g_2}(i(e_2 + e_3)) = e_1 + ie_2$, $Ad_{g_3}(e_3) = ie_1$, $Ad_{g_3}(ie_3) = -e_1$, where $g_1 = \pm \begin{pmatrix} -1 + i & 0 \\ -\frac{1}{2} + \frac{1}{2}i & -\frac{1}{2} - \frac{1}{2}i \end{pmatrix}$, $g_2 = \pm \begin{pmatrix} -\frac{2}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}} \end{pmatrix}$ and $g_3 = \pm \begin{pmatrix} 1 & \frac{1}{2}i \\ i & \frac{1}{2} \end{pmatrix}$. $\qquad\square$

Lemma 9 yields the

Proposition 10. *There is no at least 4-dimensional Bol loop having a group locally isomorphic to $PSL_2(\mathbb{C})$ as the group topologically generated by its left translations.*

Now let G be locally isomorphic to the non-compact Lie group $PSU_3(\mathbb{C}, 1)$. The Lie algebra $\mathbf{g} = \mathfrak{su}_3(\mathbb{C}, 1)$ of G can be treated as the Lie algebra of matrices

$$(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + \lambda_5 e_5 + \lambda_6 e_6 + \lambda_7 e_7 + \lambda_8 e_8) \mapsto$$

$$\begin{pmatrix} -\lambda_1 i & -\lambda_2 - \lambda_3 i & \lambda_4 + \lambda_5 i \\ \lambda_2 - \lambda_3 i & \lambda_1 i + \lambda_6 i & \lambda_7 + \lambda_8 i \\ \lambda_4 - \lambda_5 i & \lambda_7 - \lambda_8 i & -\lambda_6 i \end{pmatrix}; \lambda_j \in \mathbb{R}, j = 1, \cdots, 8.$$

Then the multiplication in $\mathbf{g}$ is given by the following:

$$[e_1, e_6] = 0, \; [e_3, e_2] = 2e_1, \; [e_4, e_5] = 2(e_1 - e_6), \; [e_8, e_7] = 2e_6,$$

$$[e_6, e_3] = [e_7, e_4] = [e_8, e_5] = \frac{1}{2}[e_1, e_3] = e_2,$$

$$[e_2, e_6] = [e_4, e_8] = [e_7, e_5] = \frac{1}{2}[e_2, e_1] = e_3,$$

$$[e_7, e_2] = [e_3, e_8] = [e_5, e_6] = [e_1, e_5] = e_4,$$

$$[e_8, e_2] = [e_7, e_3] = [e_6, e_4] = [e_4, e_1] = e_5,$$

$$[e_2, e_4] = [e_3, e_5] = [e_8, e_1] = \frac{1}{2}[e_8, e_6] = e_7,$$

$$[e_2, e_5] = [e_4, e_3] = [e_1, e_7] = \frac{1}{2}[e_6, e_7] = e_8.$$

There are two conjugacy classes of involutory automorphism of G with 4-dimensional centralizers (see [14] (p. 155)). The centralizers of suitable representatives of these classes are isomorphic either to $H_1 = Spin_3 \times SO_2(\mathbb{R})/\langle(-1,-1)\rangle$ or to $H_2 = SL_2(\mathbb{R}) \times SO_2(\mathbb{R})/\langle(-1,-1)\rangle$. Moreover, the (-1)-eigenspaces $\mathbf{m}_i$, $i = 1, 2$ of these representatives are

$$\mathbf{m}_1 = \langle e_4, e_5, e_7, e_8\rangle, \quad \mathbf{m}_2 = \langle e_1, e_3, e_4, e_5\rangle.$$

An Iwasawa decomposition (cf. [8], Theorem 6, p. 530) of the Lie algebra $\mathfrak{su}_3(\mathbb{C}, 1)$ is given by

$$\mathfrak{su}_3(\mathbb{C}, 1) = \mathfrak{k} + \mathfrak{a} + \mathfrak{n},$$

where $\mathfrak{k} = \langle e_1, e_2, e_3, e_6\rangle$ is compact, $\mathfrak{n} = \langle e_4 - e_3, e_5 + e_2, e_6 + e_7\rangle$ is nilpotent and $\mathfrak{a} = \langle e_8\rangle$. Using this decomposition and the classification in [2], Chap. 5, p. 276, we obtain that the conjugacy classes of the 4-dimensional subalgebras of $\mathfrak{su}_3(\mathbb{C}, 1)$ are the following

$$\mathbf{h}_1 = \mathfrak{so}_3(\mathbb{R}) \oplus \mathfrak{so}_2(\mathbb{R}) = \langle e_1, e_2, e_3, e_6\rangle, \quad \mathbf{h}_2 = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{so}_2(\mathbb{R}) = \langle e_2, e_6, e_7, e_8\rangle,$$

$$\mathbf{h}_3 = \langle e_4 - e_3, e_2 + e_5, e_6 + e_7, e_8\rangle.$$

The intersection of the subspace $\mathbf{m}_2$ with the subalgebras $\mathbf{h}_i$, $i = 1, 3$ as well as the intersection of $\mathbf{m}_1$ with the subalgebras $\mathbf{h}_j$, $j = 2, 3$ is not trivial. Moreover, the subgroup H_2 can not be the stabilizer of the identity of a 4-dimensional differentiable loop L (see Lemma 5). Hence it remains to prove the triple $(G, H_1, \exp \mathbf{m}_1)$.

The group H_1 is a maximal compact subgroup of G hence there is a global differentiable Bruck loop L_0 of hyperbolic type having $G \cong PSU_3(\mathbb{C}, 1)$ as the group topologically generated by its left translations. The loop L_0 is realized on the complex hyperbolic plane geometry (cf. [14], p. 152). Since only groups conjugate to H_1 can be complements of $\exp \mathbf{m}_1 = \exp\{\lambda_4 e_4 + \lambda_5 e_5 + \lambda_7 e_7 + \lambda_8 e_8; \ \lambda_i \in \mathbb{R}\}$ there is precisely one isotopism class $\mathcal{C}$ of differentiable Bol loops which are sections in $PSU_3(\mathbb{C}, 1)$. As a representative of $\mathcal{C}$ we can choose the loop L_0 which we call the complex hyperbolic plane loop.

The adjoint map $\tau : X \mapsto (\bar{X})^t$ can be chosen as a representative of involutions of $\mathbf{g} = \mathfrak{su}_3(\mathbb{C}, 1)$ fixing elementwise a 3-dimensional subalgebra. The centralizer of τ is the subalgebra $\mathbf{h} = \langle e_2, e_4, e_7\rangle \cong \mathfrak{sl}_2(\mathbb{R})$. The tangent space $\mathbf{m}$ of the 5-dimensional symmetric space corresponding to τ is generated by the basis elements e_1, e_3, e_5, e_6, e_8.

Using the Iwasawa decomposition of $\mathfrak{su}_3(\mathbb{C}, 1)$ and the classification in [2], Chap. 5, p. 276, we see that every 3-dimensional subalgebra $\mathbf{h}$ of $\mathbf{g}$ has one of the following shapes:

$$\mathbf{h_1} \cong \mathfrak{su}_2(\mathbb{C}), \quad \mathbf{h_2} \cong \mathfrak{sl}_2(\mathbb{R}),$$

$$\mathbf{h_3} = \langle e_5 + e_2, e_6 + e_7, e_8\rangle, \quad \mathbf{h_4} = \langle e_4 - e_3 + be_8, e_5 + e_2, e_6 + e_7\rangle,$$

$$\mathbf{h_5} = \langle e_4 - e_3 + b(e_5 + e_2), e_6 + e_7, e_8 + c(e_5 + e_2)\rangle,$$

$\mathbf{h_6} = \langle e_1 - \frac{1}{2}e_6 + \frac{3}{2}c(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2), e_8 + b(e_4 - e_3) + c(e_5 + e_2), e_6 + e_7\rangle$, where $b, c \in \mathbb{R}$.

Since the subalgebras of $\mathfrak{su}_3(\mathbb{C}, 1)$ isomorphic to $\mathfrak{so}_3(\mathbb{R})$ are conjugate under $Ad\, G$ (cf. [2], Chap. 5, p. 276) we may assume that $\mathbf{h_1} = \langle e_1, e_2, e_3\rangle$. The subalgebras of $\mathfrak{su}_3(\mathbb{C}, 1)$ isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ form two conjugacy classes. As representatives of these conjugacy classes we can choose the subalgebras $\mathbf{h_{2,1}} = \langle e_2, e_4, e_7\rangle$ and $\mathbf{h_{2,2}} = \langle e_6, e_7, e_8\rangle$. The subalgebras $\mathbf{h_1}$ and $\mathbf{h_{2,1}}$ contain the compact element e_2 which is conjugate to $e_1 \in \mathbf{m}$. The element $e_6 + e_7 \in \mathbf{h_{2,2}} \cap \mathbf{h_3} \cap \mathbf{h_4} \cap \mathbf{h_5} \cap \mathbf{h_6}$ is conjugate to $e_6 + e_8 \in \mathbf{m}$ since both are hyperbolic elements in the same Lie subalgebras isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.

The above considerations yield the

Theorem 11. *All Bol loops having a group locally isomorphic to $PSU_3(\mathbb{C}, 1)$ as the group topologically generated by its left translations are isotopic to the complex hyperbolic plane loop.*

Let now G be isomorphic to $SL_3(\mathbb{R})$. According to [14], p. 155, all involutory automorphisms of $SL_3(\mathbb{R})$ are induced by reflections or polarities of the real projective plane $\mathbb{P}_2(\mathbb{R})$. The Lie algebra $\mathbf{g} = \mathfrak{sl}_3(\mathbb{R})$ is isomorphic to the Lie algebra of matrices

$$(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + \lambda_5 e_5 + \lambda_6 e_6 + \lambda_7 e_7 + \lambda_8 e_8) \mapsto$$

$$\begin{pmatrix} -\lambda_5 - \lambda_8 & \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_5 & \lambda_6 \\ \lambda_4 & \lambda_7 & \lambda_8 \end{pmatrix} ; \lambda_i \in \mathbb{R}, i = 1, \cdots, 8.$$

Then the Lie multiplication of $\mathbf{g}$ is given by

$$[e_1, e_2] = [e_1, e_7] = [e_2, e_6] = [e_3, e_4] = [e_3, e_6] = [e_4, e_7] = [e_5, e_8] = 0,$$

$$[e_1, e_6] = [e_2, e_5] = \frac{1}{2}[e_2, e_8] = e_2, \ [e_1, e_8] = [e_2, e_7] = \frac{1}{2}[e_1, e_5] = e_1,$$

$$[e_4, e_6] = [e_3, e_8] = \frac{1}{2}[e_3, e_5] = -e_3, \ [e_3, e_7] = [e_4, e_5] = \frac{1}{2}[e_4, e_8] = -e_4,$$

$$[e_6, e_8] = [e_5, e_6] = [e_3, e_2] = e_6, \ [e_1, e_4] = [e_5, e_7] = [e_7, e_8] = -e_7,$$

$$[e_1, e_3] = -e_5, \ [e_2, e_4] = -e_8, \ [e_6, e_7] = e_5 - e_8.$$

We choose an elliptic polarity in $\mathbb{P}_2(\mathbb{R})$ such that the corresponding involution τ_1 induces on $\mathfrak{sl}_3(\mathbb{R})$ the automorphism $\tau_1^* : \mathfrak{sl}_3(\mathbb{R}) \to \mathfrak{sl}_3(\mathbb{R}); X \mapsto -X^t$. The Lie algebra $\mathfrak{so}_3(\mathbb{R}) = \langle e_1 - e_3, e_2 - e_4, e_7 - e_6 \rangle$ is the $(+1)$-eigenspace of τ_1^*. The tangent space $\mathbf{m_1} \subset \mathfrak{sl}_3(\mathbb{R})$ of the 5-dimensional symmetric space belonging to τ_1^* has as generators $e_5, e_8, e_1 + e_3$, $e_2 + e_4, e_6 + e_7$. We can choose a hyperbolic polarity in $\mathbb{P}_2(\mathbb{R})$ such that the corresponding involution τ_2^* of $\mathfrak{sl}_3(\mathbb{R})$ is given by $\tau_2^* : X \mapsto -\mathrm{diag}(1, 1, -1)X^t \, \mathrm{diag}(1, 1, -1)$. The Lie algebra fixed by τ_2^* elementwise is $\langle e_1 - e_3, e_2 + e_4, e_6 + e_7 \rangle$. The tangent space $\mathbf{m_2}$ of the corresponding symmetric space is generated by $\{e_1 + e_3, e_2 - e_4, e_6 - e_7, e_5, e_8\}$.

According to Lemma 5 any maximal compact subgroup of the stabilizer H is trivial or locally isomorphic to $SO_3(\mathbb{R})$. Now using the classification of Lie, who has determined all subalgebras of $\mathfrak{sl}_3(\mathbb{R})$ (cf. [12], pp. 288-289 and [11], p. 384) we obtain that the 3-dimensional subalgebras $\mathbf{h}$ of the stabilizer H have one of the following shapes:

$\mathbf{h_1} = \mathfrak{so}_3(\mathbb{R}), \ \mathbf{h_2} = \langle a(e_5 + e_8) + e_6 - e_7, e_1, e_2 \rangle, a > 0, \ \mathbf{h_3} = \langle e_1, e_2, e_6 \rangle,$

$\mathbf{h_4} = \langle e_5 - e_8, e_2 + e_3, e_6, \rangle, \ \mathbf{h_5} = \langle e_3, e_6, e_8 + e_2 \rangle, \ \mathbf{h_6} = \langle e_2, e_6, e_5 + e_8 - e_3 \rangle,$

$\mathbf{h_7} = \langle e_5, e_8, e_6 \rangle, \ \mathbf{h_8} = \langle e_2, e_5 + e_8, e_6 \rangle, \ \mathbf{h_9} = \langle e_3, e_6, e_8 \rangle,$

$\mathbf{h_{10}} = \langle e_2, e_6, (b-1)e_5 + be_8, \rangle, b \in \mathbb{R}, \ \mathbf{h_{11}} = \langle e_3, e_6, e_5 + ce_8 \rangle, c \in \mathbb{R}.$

The element $e_8 + e_2 \in \mathbf{h_5}$ is conjugate to $e_8 \in \mathbf{m_1} \cap \mathbf{m_2}$ under the element $g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & \frac{1}{2} \end{pmatrix}$. Futhermore the element $e_5 + e_8 - e_3 \in \mathbf{h_6}$ is conjugate to $e_8 - 2e_5 \in \mathbf{m_1} \cap$ $\mathbf{m_2}$ under $g = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \end{pmatrix}$ and $e_2 \in \mathbf{h_2} \cap \mathbf{h_3}$ is conjugate to $e_8 - e_5 - (e_6 - e_7) \in \mathbf{m_2}$ under $g = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. The intersection of the subspaces $\mathbf{m_1}$ and $\mathbf{m_2}$ with the subalgebras $\mathbf{h_i}$, $i = 4, 7, 8, 9, 10, 11$, as well as the intersection of $\mathbf{m_2}$ with the subalgebra $\mathbf{h_1}$

is not trivial. Hence we may suppose that the Bol loop L is realized on $\exp \mathbf{m_1}$ and the stabilizer of the identity of L has one of the following shapes:

a) $H_1 = SO_3(\mathbb{R})$,

b) $H_2 = \left\{ \begin{pmatrix} d^{-2t} & a & b \\ 0 & d^t \cos t & d^t \sin t \\ 0 & -d^t \sin t & d^t \cos t \end{pmatrix} ; t \in [0, 2\pi), a, b \in \mathbb{R} \right\}$, with $d > 1$,

c) $H_3 = \left\{ \begin{pmatrix} 1 & k & l \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} ; k, l, m \in \mathbb{R} \right\}$.

Proposition 12. *There is no differentiable Bol loop L such that the stabilizer of the identity $e \in L$ is one of the Lie groups H_i, $i = 2, 3$ given in b) and c).*

Proof. The exponential image of the subspace $\mathbf{m_1}$ consists of all positive definite matrices of the shape

$$\left\{ A = \begin{pmatrix} a & b & c \\ b & e & d \\ c & d & f \end{pmatrix} ; \det A = 1 \right\}.$$

A differentiable Bol loop L exists if and only if every coset gH_i, $g \in G$, $i = 2, 3$ contains precisely one element $m \in \exp \mathbf{m_1}$ (see Lemma 6). For $c \in \mathbb{R}$ denote by $g_c H_i$ the coset $\begin{pmatrix} 1+c & 1 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} H_i$, $i = 2, 3$. The coset $g_1 H_2$ contains different elements

$m_1 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $m_2 = \begin{pmatrix} 2d^{-4\pi} & d^{-4\pi} & 0 \\ d^{-4\pi} & \frac{d^{-4\pi}+d^{2\pi}}{2} & 0 \\ 0 & 0 & d^{2\pi} \end{pmatrix}$ of $\exp \mathbf{m_1}$. If $c > -1$ then any coset $g_c H_3$ contains precisely the element

$$s_3(c) = \begin{pmatrix} (1+c) & c & 0 \\ c & \frac{c^2+1}{c+1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \sigma(G/H_3).$$

But $\lim_{c \to -1} \sigma(g_c H_3)$ should be $\lim_{c \to -1} s_3(c)$ which is a contradiction. $\qquad \square$

A reflection τ_3 in $\mathbb{P}_2(\mathbb{R})$ can be chosen in such a way that it induces on $\mathfrak{sl}_3(\mathbb{R})$ the involution $\tau_3^* : \mathfrak{sl}_3(\mathbb{R}) \to \mathfrak{sl}_3(\mathbb{R}); X \mapsto A^{-1}XA$, $A = \operatorname{diag}(1, -1, -1)$, which fixes elementwise the subalgebra $\langle e_5, e_6, e_7, e_8 \rangle$ isomorphic to $\mathfrak{gl}_2(\mathbb{R})$. The tangent space $\mathbf{m_3}$ of the corresponding 4-dimensional symmetric space has as generators e_1, e_2, e_3, e_4. The group associated with $\langle e_5, e_6, e_7, e_8 \rangle$ is excluded by Lemma 5 and hence the classification of Lie (cf. [12], pp. 288-289 and [11], p. 384) yields that the Lie algebra $\mathbf{h}$ of the stabilizer of the identity of a 4-dimensional Bol loop has (up to conjugation) one of the following forms:

$\mathbf{h_1} = \langle e_1, e_2, e_6, e_5 + c e_8 \rangle$, $\quad \mathbf{h_2} = \langle e_3, e_5, e_6, e_8 \rangle$, $\quad \mathbf{h_3} = \langle e_1, e_2, e_6, e_8 \rangle$,

$\mathbf{h_4} = \langle e_2, e_5, e_6, e_8 \rangle$, where $c \in \mathbb{R}$.

The intersection of all these subalgebras $\mathbf{h_i}$, $i = 1, \cdots, 4$ with the subspace $\mathbf{m_3}$ is not trivial.

This contradiction to Lemma 6 and the above considerations yield the main part of the following

Theorem 13. *Every Bol loop with the group $SL_3(\mathbb{R})$ as the group topologically generated by the left translations is isotopic to the 5-dimensional Bruck loop L_0 of hyperbolic type having the group $SO_3(\mathbb{R})$ as the stabilizer of $e \in L_0$.*

Proof. The group $SO_3(\mathbb{R})$ is a maximal compact subgroup of $SL_3(\mathbb{R})$. According to Theorem 8 there is a 5-dimensional Bruck loop L_0 of hyperbolic type realized on the differentiable manifold $\exp \mathbf{m_1}$. Since up to isomorphisms there is only one symmetric space $\mathcal{S}$ having $SL_3(\mathbb{R})$ as the group of displacements and $SO_3(\mathbb{R})$ as the centralizer of the involutory automorphism belonging to $\mathcal{S}$ there exists precisely one isotopism class $\mathcal{C}$ of differentiable Bol loops corresponding to $SL_3(\mathbb{R})$ and L_0 is a representative of $\mathcal{C}$. $\qquad\square$

5 Bol loops corresponding to semi-simple Lie groups

Let $G = G_1 \times G_2$ be the group topologically generated by the left translations of a connected differentiable Bol loop L such that $M = M_1 \times M_2$ holds, where M is the symmetric space belonging to L. If for the stabilizer H of $e \in L$ one has $H = H_1 \times H_2$ with $1 \neq H_i < G_i$, $i = 1, 2$, then L is a direct product of two proper Bol loops L_1 and L_2 such that L_i is realized on M_i, has G_i as the group topologically generated by the left translations and H_i as the stabilizer of $e \in L_i$, $i = 1, 2$, (cf. [17], Proposition 1.19, p.28). If $M_2 \cong G_2$ and the stabilizer H of $e \in L$ has the shape $H = (H_1, \varphi(H_1))$, where $\varphi : H_1 \to G_2$ is a homomorphism, then L is a Scheerer extension of the Lie group G_2 by the proper Bol loop L_1 (cf. [17], Proposition 2.4, p. 44). If G has dimension ≤ 9 and the direct factors are simple then G_i is isomorphic either to $PSL_2(\mathbb{R})$ or to $PSL_2(\mathbb{C})$ or to $SO_3(\mathbb{R})$. There is no proper Bol loop corresponding to $SO_3(\mathbb{R})$ (cf. [17], Corollary 16.8, p. 204) and every proper Bol loop having $PSL_2(\mathbb{R})$ respectively $PSL_2(\mathbb{C})$ as the group topologically generated by the left translations is isotopic to the hyperbolic plane loop (cf. [17], Section 22) respectively to the hyperbolic space loop (cf. [3], Theorem 5, p. 446 and Proposition 10). This discussion yields

Theorem 14. *Let L be a connected differentiable Bol loop having an at most 9-dimensional semi-simple Lie group G as the group topologically generated by its left translations.*
i) If the stabilizer H is a direct product of subgroups $1 \neq H_i$ contained in the simple factors G_i of G then L is a direct product of proper Bol loops L_i isotopic to the hyperbolic plane loop $\mathbb{H}_2$ respectively to the hyperbolic space loop $\mathbb{H}_3$. Furthermore, G is isomorphic either to $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ or to $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ respectively to $PSL_2(\mathbb{C}) \times PSL_2(\mathbb{R})$.

ii) If the stabilizer H is not a direct product of subgroups $1 \neq H_i$ contained in the simple factors G_i of G then one has $G = G_1 \times S$, where G_1 is isomorphic either to $PSL_2(\mathbb{R})$ or to $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ respectively to $PSL_2(\mathbb{C})$ and S is the complement of G_1 in G. Moreover, the stabilizer H has the shape $\{(x, \varphi(x)) \mid x \in H_1\}$, where H_1 is isomorphic either to $SO_2(\mathbb{R})$ or to $SO_2(\mathbb{R}) \times SO_2(\mathbb{R})$ respectively to $SO_3(\mathbb{R})$ and the loop L is a Scheerer extension of S by $\mathbb{H}_2$ in the first case, by $\mathbb{H}_2 \times \mathbb{H}_2$ in the second case and by $\mathbb{H}_3$ in the third case.

From now we assume that the subgroup H is not decomposable into a direct product. Denote by $p_i : G \to G_i$ the projection of G onto the i-th components G_i of G.

Lemma 15. *There is one conjugacy class $\mathcal{C}_1$ of involutory automorphisms of $\mathfrak{so}_3(\mathbb{R})$ and two conjugacy classes $\mathcal{C}_2$ and $\mathcal{C}_3$ of involutory automorphisms of $\mathfrak{sl}_2(\mathbb{R})$. As a representative of $\mathcal{C}_1$ we can choose one which fixes the 1-dimensional subalgebra $\langle e_3 \rangle$ elementwise. As a representative of $\mathcal{C}_2$ respectively $\mathcal{C}_3$ we can choose one fixing $\langle e_3 \rangle$ respectively $\langle e_2 \rangle$ elementwise.*

Proof. The assertion follows from [5], pp. 44-45. $\qquad\square$

Proposition 16. *Every connected differentiable Bol loop having a group locally isomorphic to $PSL_2(\mathbb{R}) \times SO_3(\mathbb{R})$ as the group topologically generated by its left translations is isotopic to a Scheerer extension of $SO_3(\mathbb{R})$ by $\mathbb{H}_2$.*

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Proof. Assume that L is not a Scheerer extension in the assertion. The automorphism group Γ of the Lie algebra $\mathbf{g}$ of G is the direct product of the automorphism group of $\mathfrak{sl}_2(\mathbb{R})$ and the automorphism group of $\mathfrak{so}_3(\mathbb{R})$. According to Lemma 15 there are two conjugacy classes of involutory automorphisms of $\mathbf{g}$ fixing elementwise a 2-dimensional subalgebra. The (-1)-eigenspaces of suitable representatives of these classes are

$$\mathbf{m}_1 = \langle (e_1, 0), (e_2, 0), (0, ie_1), (0, ie_2) \rangle, \ \mathbf{m}_2 = \langle (e_1, 0), (e_3, 0), (0, ie_1), (0, ie_2) \rangle.$$

Since $SO_3(\mathbb{R})$ has no 2-dimensional subgroup we have to investigate the case that $\dim p_1(H) = 2$ and $\dim p_2(H) = 1$. Then we may assume that $p_1(\mathbf{h}) = \langle e_2 + e_3, e_1 \rangle$, $p_2(\mathbf{h}) = \langle e_3 \rangle$ (see **1.1** and **1.2**) and the Lie algebra $\mathbf{h}$ has the shape $\mathbf{h} = \langle (e_2 + e_3, 0), (e_1, e_3) \rangle$. The element $(e_2 + e_3, 0) \in \mathbf{h}$ is conjugate to $(e_1 + e_3, 0) \in \mathbf{m}_2$ (see **1.1**). Hence there is no differentiable Bol loop L with $T_e L = \mathbf{m}_2$. Since $p_1(\exp \mathbf{m}_1)$ and $p_1(\exp \mathbf{h})$ satisfy the conditions of Lemma 7 we have also here a contradiction. $\qquad \square$

Proposition 17. *If the Lie group $G' = G_1 \times G_2$ or $G'' = G_1 \times G_2 \times G_3$, where G_i $(i = 1, 2, 3)$ is locally isomorphic to $PSL_2(\mathbb{R})$, is the group topologically generated by the left translations of a connected differentiable Bol loop L then L is either a direct product of proper Bol loops isotopic to the hyperbolic plane loop $\mathbb{H}_2$ or a Scheerer extension of $PSL_2(\mathbb{R})$ by $\mathbb{H}_2$ respectively by $\mathbb{H}_2 \times \mathbb{H}_2$ or a Scheerer extension of $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ by $\mathbb{H}_2$.*

Proof. Assume that L has not a form as in the assertion. The automorphism group Γ of the Lie algebra $\mathbf{g}' = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$ of G' is the semidirect product of the normal automorphism group $\Gamma_1 \times \Gamma_1$, where Γ_1 is the automorphism group of $\mathfrak{sl}_2(\mathbb{R})$, by the group generated by $\sigma : \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{sl}_2(\mathbb{R})$; $(u, v) \mapsto (v, u)$. To the symmetric space determined by σ there corresponds a 3-dimensional Bol loop having $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ as the group topologically generated by the left translations, but such a Bol loop does not exit (cf. [3], pp. 442-444). Hence there exist up to automorphisms of G' precisely three 4-dimensional symmetric spaces of G' the tangent spaces of which are given by

$$\mathbf{m}_1 = \langle (e_1, 0), (e_2, 0), (0, e_1), (0, e_2) \rangle, \ \mathbf{m}_2 = \langle (e_1, 0), (e_2, 0), (0, e_1), (0, e_3) \rangle,$$
$$\mathbf{m}_3 = \langle (e_1, 0), (e_3, 0), (0, e_1), (0, e_3) \rangle.$$

Moreover, there are up to automorphisms of G'' four 6-dimensional symmetric spaces in G'' the tangent spaces of which are

$$\widetilde{\mathbf{m}}_1 = \langle (e_1, 0, 0), (e_2, 0, 0), (0, e_1, 0), (0, e_2, 0), (0, 0, e_1), (0, 0, e_2) \rangle,$$
$$\widetilde{\mathbf{m}}_2 = \langle (e_1, 0, 0), (e_2, 0, 0), (0, e_1, 0), (0, e_2, 0), (0, 0, e_1), (0, 0, e_3) \rangle,$$
$$\widetilde{\mathbf{m}}_3 = \langle (e_1, 0, 0), (e_2, 0, 0), (0, e_1, 0), (0, e_3, 0), (0, 0, e_1), (0, 0, e_3) \rangle,$$
$$\widetilde{\mathbf{m}}_4 = \langle (e_1, 0, 0), (e_3, 0, 0), (0, e_1, 0), (0, e_3, 0), (0, 0, e_1), (0, 0, e_3) \rangle.$$

If L corresponds to G' then $\dim H = 2$ and $H = (\varphi(\mathcal{L}_2), \mathcal{L}_2)$, where $\varphi \neq 1$ is a homomorphism of $\mathcal{L}_2$ into $PSL_2(\mathbb{R})$. If φ is injective then the Lie algebra of H has the shape $\mathbf{h} = \langle (e_1, e_1), (e_2 + e_3, e_2 + e_3) \rangle$ and the intersection of $\mathbf{h}$ with $\mathbf{m}_i$ for $i = 1, 2, 3$, is not trivial. If φ has 1-dimensional kernel then $\mathbf{h}$ contains the element $(0, e_2 + e_3)$ which is conjugate to $(0, e_1 + e_3) \in \mathbf{m}_2 \cap \mathbf{m}_3$ (see **1.1**). Hence the subspaces $\mathbf{m}_2$ and $\mathbf{m}_3$ cannot determine a Bol loop. Since $p_2(\exp \mathbf{m}_1)$ and $p_2(\exp \mathbf{h})$ have a shape as in Lemma 7 we obtain also a contradiction.

If L corresponds to G'' then $\dim H \in \{3, 4\}$ (see Lemma 1). The dimension of H cannot be 4 since every 4-dimensional subgroup of G'' contains a direct factor and there is no 3-dimensional Bol loop corresponding to $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$.

Let now $\dim H = 3$ and $p_i(H) \neq (H \cap G_i)$ for $i = 1, 2, 3$. If $p_1(H)$ is isomorphic to $PSL_2(\mathbb{R})$ then one has $p_j(H) \cong PSL_2(\mathbb{R})$ for $j = 2, 3$ and the Lie algebra $\mathbf{h}$ of H contains (e_1, e_1, e_1) up to conjugacy. If $p_i(H)$ is isomorphic to $\mathcal{L}_2$ for $i = 2, 3$ then the projection of H onto

the first component of G'' is a non-trivial homomorphic image of $\mathcal{L}_2$. In this case the Lie algebra $\mathbf{h}$ contains either the element (e_1, e_1, e_1) or $(0, 0, e_2 + e_3)$ up to conjugacy. The first of them lies in $\widehat{\mathbf{m}_i}$ for $i = 1, 2, 3, 4$ and the second is conjugate to $(0, 0, e_1 + e_3) \in \widehat{\mathbf{m}_i}$, $i = 2, 3, 4$ (see **1.1**). Since $p_3(\exp \widehat{\mathbf{m}_1})$ and $p_3(\exp \mathbf{h})$ satisfies the conditions of Lemma 7 we have a contradiction and the assertion follows. $\qquad\square$

Proposition 18. *Every connected differentiable Bol loop L having a group locally isomorphic either to $G' = PSL_2(\mathbb{R}) \times SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$ or to $G'' = PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R}) \times SO_3(\mathbb{R})$ as the group topologically generated by its left translations is a Scheerer extension of $SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$ respectively of $PSL_2(\mathbb{R}) \times SO_3(\mathbb{R})$ by the hyperbolic plane loop $\mathbb{H}_2$ or a Scheerer extension of $SO_3(\mathbb{R})$ by $\mathbb{H}_2 \times \mathbb{H}_2$.*

Proof. Assume that L is not a Scheerer extension in the assertion. Lemmata 1, 2 and 3 exclude the group G'.

According to Proposition 16 and Lemma 2 we may assume that $p_i(H) \neq (H \cap G_i)$, $i = 1, 2, 3$. Hence the stabilizer H of the identity of L in G'' has dimension 3. Moreover, the Lie algebra of H has the form

$$\mathbf{h} = \langle (e_1, e_1, e_3), (e_2 + e_3, 0, 0), (0, e_2 + e_3, 0) \rangle.$$

According to Lemma 15 we have four conjugacy classes of involutory automorphisms of $\mathbf{g}'' = sl_2(\mathbb{R}) \oplus sl_2(\mathbb{R}) \oplus so_3(\mathbb{R})$ which fix elementwise a 3-dimensional subalgebra of $\mathbf{g}''$. The (-1)-eigenspaces of suitable representatives of these classes are given by

$$\mathbf{m}_1 = \langle (e_1, 0, 0), (e_2, 0, 0), (0, e_1, 0), (0, e_2, 0), (0, 0, \mathrm{i}e_1), (0, 0, \mathrm{i}e_2) \rangle,$$

$$\mathbf{m}_2 = \langle (e_1, 0, 0), (e_2, 0, 0), (0, e_1, 0), (0, e_3, 0), (0, 0, \mathrm{i}e_1), (0, 0, \mathrm{i}e_2) \rangle,$$

$$\mathbf{m}_3 = \langle (e_1, 0, 0), (e_3, 0, 0), (0, e_1, 0), (0, e_2, 0), (0, 0, \mathrm{i}e_1), (0, 0, \mathrm{i}e_2) \rangle,$$

$$\mathbf{m}_4 = \langle (e_1, 0, 0), (e_3, 0, 0), (0, e_1, 0), (0, e_3, 0), (0, 0, \mathrm{i}e_1), (0, 0, \mathrm{i}e_2) \rangle.$$

As the elements e_3 and $\mathrm{i}e_1$ are conjugate in $SO_3(\mathbb{R})$ (see **1.2**) the subalgebras $\mathbf{m}_i$, $i = 1, 2, 3, 4$, cannot be the tangent spaces of Bol loops corresponding to G''. $\qquad\square$

Proposition 19. *Let the group G be locally isomorphic to a direct product $G = G_1 \times G_2$, where $G_1 \cong PSL_2(\mathbb{C})$ and G_2 is either $PSL_2(\mathbb{R})$ or $SO_3(\mathbb{R})$. Every connected differentiable Bol loop having G as the group topologically generated by its left translations is either a direct product of the hyperbolic space loop $\mathbb{H}_3$ and the hyperbolic plane loop $\mathbb{H}_2$ or a Scheerer extension of $PSL_2(\mathbb{C})$ by $\mathbb{H}_2$ or a Scheerer extension of a 3-dimensional simple Lie group by $\mathbb{H}_3$.*

Proof. There is precisely one conjugacy class of 4-dimensional subgroups of $SL_2(\mathbb{C})$ (see [1], p. 277). This class can be represented by a subgroup V, the Lie algebra of which is

$$v = \langle e_1, \mathrm{i}\, e_1, e_2 + e_3, \mathrm{i}(e_2 + e_3) \rangle.$$

The maximal compact subgroups of V are isomorphic to $SO_2(\mathbb{R})$. Representatives of the 3-dimensional subalgebras of $SL_2(\mathbb{C})$ (cf. [1], p. 277-278) are given by

$$\mathfrak{so}_3(\mathbb{R}) = \langle \mathrm{i}e_1, \mathrm{i}e_2, e_3 \rangle, \quad \mathfrak{sl}_2(\mathbb{R}) = \langle e_1, e_2, e_3 \rangle,$$

$$w_r = \langle (r\mathrm{i} - 1)e_1, e_2 + e_3, \mathrm{i}(e_2 + e_3) \rangle, \quad u_1 = \langle \mathrm{i}e_1, e_2 + e_3, \mathrm{i}(e_2 + e_3) \rangle.$$

Any 2-dimensional ideal of the subalgebras w_r, $r \in \mathbb{R}$, and u_1 is isomorphic to $\langle e_2 + e_3, \mathrm{i}(e_2 + e_3) \rangle$. The maximal compact subgroups of the Lie group corresponding to w_r are trivial for any $r \in \mathbb{R}$, whereas the maximal compact subgroups of the Lie group corresponding to u_1 are isomorphic to $SO_2(\mathbb{R})$.

According to Lemma 1 we have $\dim L \in \{5, 6\}$. If G is locally isomorphic to $PSL_2(\mathbb{C}) \times SO_3(\mathbb{R})$, then $\dim L \neq 5$. Otherwise $\dim H$ would be 4 and $p_1(H) = V$ which contradicts

Lemma 5.

There are precisely three classes of involutions of $\mathfrak{sl}_2(\mathbb{C})$ (cf. [14], pp. 152-153). The $(+1)$-eigenspaces of suitable representatives of these involutions are $\mathbf{h} = \langle e_3, ie_3 \rangle$ or $\mathbf{h} = \langle e_1, e_2, e_3 \rangle$ respectively $\mathbf{h} = \langle ie_1, ie_2, e_3 \rangle$. The automorphism group Γ of the Lie algebra $\mathbf{g}' = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{R})$ respectively of $\mathbf{g}'' = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{so}_3(\mathbb{R})$ is the direct product of the automorphism group of $\mathfrak{sl}_2(\mathbb{C})$ and the automorphism group of $\mathfrak{sl}_2(\mathbb{R})$ respectively of $\mathfrak{so}_3(\mathbb{R})$. According to Lemma 15 there exist precisely two conjugacy classes of involutions of $\mathbf{g}'$ having 3-dimensional subalgebras as their $(+1)$-eigenspaces. The (-1)-eigenspaces of suitable representatives of these classes are given by

$$\mathbf{m}_1 = \langle (e_1, 0), (e_2, 0), (ie_1, 0), (ie_2, 0), (0, e_1), (0, e_2) \rangle,$$

$$\mathbf{m}_2 = \langle (e_1, 0), (e_2, 0), (ie_1, 0), (ie_2, 0), (0, e_1), (0, e_3) \rangle.$$

Moreover, there are precisely four conjugacy classes of involutions of $\mathbf{g}'$ leaving 4-dimensional subalgebras elementwise fix. The tangent spaces $\mathbf{m}_i$, $i = 3, 4, 5, 6$, of the corresponding symmetric spaces are:

$$\mathbf{m}_3 = \langle (ie_1, 0), (ie_2, 0), (ie_3, 0), (0, e_1), (0, e_2) \rangle,$$

$$\mathbf{m}_4 = \langle (ie_1, 0), (ie_2, 0), (ie_3, 0), (0, e_1), (0, e_3) \rangle,$$

$$\mathbf{m}_5 = \langle (e_1, 0), (e_2, 0), (ie_3, 0), (0, e_1), (0, e_2) \rangle,$$

$$\mathbf{m}_6 = \langle (e_1, 0), (e_2, 0), (ie_3, 0), (0, e_1), (0, e_3) \rangle.$$

For $\mathbf{g}''$ there is one conjugacy class of involutory automorphisms which fix 3-dimensional subalgebras elementwise. The (-1)-eigenspace of a representative of this class is the subspace

$$\widetilde{\mathbf{m}}_1 = \langle (e_1, 0), (e_2, 0), (ie_1, 0), (ie_2, 0), (0, ie_1), (0, ie_2) \rangle.$$

First we consider the case that G is locally isomorphic to $PSL_2(\mathbb{C}) \times SO_3(\mathbb{R})$ and $\dim H = 3$. According to Lemma 5 the projection $p_1(\mathbf{h})$ of the Lie algebra $\mathbf{h}$ of H onto the first component is isomorphic either to $\mathfrak{so}_3(\mathbb{R})$ or to a Lie algebra w_r, $r \in \mathbb{R}$. If $p_1(\mathbf{h}) \cong p_2(\mathbf{h}) \cong \mathfrak{so}_3(\mathbb{R})$ then we may assume that $\mathbf{h}$ has the shape $\{(x, x);\ x \in \mathfrak{so}_3(\mathbb{R})\}$. But then $\mathbf{h} \cap \widetilde{\mathbf{m}}_1$ is not trivial. If $p_1(\mathbf{h})$ is isomorphic to w_r, $r \in \mathbb{R}$, then up to conjugation $\mathbf{h}$ has the form $\mathbf{h} = (w_r, \phi(w_r))$, where ϕ is a homomorphism of w_r onto $\mathfrak{so}_2(\mathbb{R})$ and $\phi^{-1}(0) = \langle (e_2 + e_3, 0), (i(e_2 + e_3, 0)) \rangle$. Since $(e_2 + e_3, 0)$ is conjugate to $(e_1 + ie_2, 0) \in \widetilde{\mathbf{m}}_1$ the subspace $\widetilde{\mathbf{m}}_1$ cannot be the tangent space of a differentiable Bol loop.

Now we assume that G is locally isomorphic to $PSL_2(\mathbb{C}) \times PSL_2(\mathbb{R})$ and $\dim H = 3$. If $\dim p_2(\mathbf{h}) = 3$ then the subgroup H is locally isomorphic to $\{(x, x);\ x \in PSL_2(\mathbb{R})\}$ and the Lie algebra $\mathbf{h}$ of H is generated by (e_1, e_1), $(e_2 + e_3, e_2 + e_3)$, (e_3, e_3). But then $\mathbf{h} \cap \mathbf{m}_i \neq \{0\}$ for $i = 1, 2$.

If $\dim p_2(\mathbf{h}) = 2$ then $\dim p_1(\mathbf{h}) \in \{2, 3\}$. If $p_1(H) \cong p_2(H) \cong \mathcal{L}_2$ then the Lie algebra $\mathbf{h}$ has the shape $\langle (e_1, e_1), (e_2 + e_3, 0), (0, e_2 + e_3) \rangle$. If $p_1(\mathbf{h})$ is a 2-dimensional abelian subalgebra of $\mathfrak{sl}_2(\mathbb{C})$ then $\mathbf{h}$ has the shape $\mathbf{h} = (K, 0) \oplus (\varphi(p_2(\mathbf{h})), p_2(\mathbf{h}))$, where φ is a homomorphism with the nucleus $(0, e_2 + e_3)$ and K is a complement of $\varphi(p_2(\mathbf{h}))$ in $p_1(\mathbf{h})$. According to Lemma 9 the homomorphism φ may be chosen in such a way that $\varphi(p_2(\mathbf{h}))$ has one of the following shapes: $\langle e_3 \rangle$, $\langle ie_3 \rangle$, $\langle e_2 + e_3 \rangle$, $\langle i(e_2 + e_3) \rangle$. Hence $\mathbf{h}$ contains one of the following 1-dimensional algebras: $\langle (e_3, e_1) \rangle$, $\langle (ie_3, e_1) \rangle$, $\langle (e_2 + e_3, e_1) \rangle$, $\langle (i(e_2 + e_3), e_1) \rangle$. But $(e_1, e_1) \in \mathbf{h} \cap \mathbf{m}_1 \cap \mathbf{m}_2$ and the element $(0, e_2 + e_3) \in \mathbf{h}$ is conjugate to $(0, e_1 + e_3) \in \mathbf{m}_2$ (see **1.1**). Since $p_1(\mathbf{m}_1)$ as well as $p_1(\mathbf{h})$ have a form as in Lemma 9 these subalgebras $\mathbf{h}$ are excluded.

Let now $\dim p_1(\mathbf{h}) = 3$ and $\dim p_2(\mathbf{h}) \in \{2, 1\}$. Then $\mathbf{h}$ has the shape $\mathbf{h} = (p_1(\mathbf{h}), \varphi(p_1(\mathbf{h})))$, where φ is a homomorphism of a 3-dimensional subalgebra of $\mathfrak{sl}_2(\mathbb{C})$ onto $p_2(\mathbf{h})$ with $\dim \varphi^{-1}(0) \in \{2, 1\}$. Then $p_2(\mathbf{h})$ contains the element $(e_2 + e_3, 0)$ or $(i(e_2 + e_3), 0)$. However, both of them are conjugate to $(e_1 + ie_2, 0) \in \mathbf{m}_1 \cap \mathbf{m}_2$ (see Lemma 9).

Finally let G be locally isomorphic to $PSL_2(\mathbb{C}) \times PSL_2(\mathbb{R})$ and let $\dim H = 4$. Since H does not decompose onto a direct product $H_1 \times H_2$ with $H_i < G_i$ we have $\dim p_1(\mathbf{h}) + \dim p_2(\mathbf{h}) \geq 5$ and hence $\dim p_1(\mathbf{h}) \geq 3$.

We consider first the case $\dim p_1(\mathbf{h}) = 3$. The subalgebra $p_2(\mathbf{h})$ cannot be $\mathfrak{sl}_2(\mathbb{R})$ since $\mathfrak{sl}_2(\mathbb{R})$ is simple.

If $\dim p_2(\mathbf{h}) = 2$ then we may assume that $p_2(\mathbf{h}) = \langle e_1, e_2 + e_3 \rangle$ (see **1.1**). Then $\mathbf{h}$ has the form $(p_1(\mathbf{h}), \varphi(p_1(\mathbf{h})) \oplus \langle (0, e_2 + e_3) \rangle$, where φ is a homomorphism of $p_1(\mathbf{h})$ into $p_2(\mathbf{h})$ with $\dim \varphi^{-1}(0) = 2$ and $p_1(\mathbf{h})$ is either the subalgebra u_1 or w_r, $r \in \mathbb{R}$. It follows that $\mathbf{h}$ contains the elements $(e_2 + e_3, 0)$ and $(\mathrm{i}(e_2 + e_3), 0)$. The element $(\mathrm{i}(e_2 + e_3), 0)$ is contained in $\mathbf{m}_i$, $i = 3, 4$, and $(0, e_1 + e_3) \in \mathbf{m}_6$ is conjugate to $(0, e_2 + e_3) \in \mathbf{h}$ (see **1.1**).

Therefore it remains to investigate the triples $(G, \exp \mathbf{h}_i, \exp \mathbf{m}_5)$, $i = 1, 2$, with

$$\mathbf{h}_1 = \langle (e_2 + e_3, 0), (\mathrm{i}(e_2 + e_3), 0), ((r\mathrm{i} - 1)e_1, e_1), (0, e_2 + e_3) \rangle$$

$$\mathbf{h}_2 = \langle (e_2 + e_3, 0), (\mathrm{i}(e_2 + e_3), 0), (\mathrm{i}e_1, e_1), (0, e_2 + e_3) \rangle.$$

Let $\dim p_1(\mathbf{h}) = 4$. Then up to conjugation we have $p_1(\mathbf{h}) = v$. Since v is solvable and $\mathfrak{sl}_2(\mathbb{R})$ is simple it follows $\dim p_2(\mathbf{h}) < 3$.

If $\dim p_2(\mathbf{h}) = 2$ then we may assume that $p_2(\mathbf{h}) = \langle e_1, e_2 + e_3 \rangle$ (see **1.1**) and the subalgebra $\mathbf{h}$ has the form $\mathbf{h} = (v, \psi(v))$, where ψ is a homomorphism from v onto $p_2(\mathbf{h})$ such that $\psi^{-1}(0) = \langle e_2 + e_3, \mathrm{i}(e_2 + e_3) \rangle$. Since the image $\psi(w_r)$, $r \in \mathbb{R}$, as well as $\psi(u_1)$ is the 1-dimensional ideal $\langle e_2 + e_3 \rangle$ of $p_2(\mathbf{h})$ the subalgebra $\mathbf{h}$ would contain $((r\mathrm{i} - 1)e_1, e_2 + e_3)$ for all $r \in \mathbb{R}$ and $(\mathrm{i}e_1, e_2 + e_3)$. This contradicts $\dim \mathbf{h} = 4$.

Let now $\dim p_2(\mathbf{h}) = 1$. Then one has $\mathbf{h} = (v, \varphi(v))$, where φ is a homomorphism from v onto a 1-dimensional subalgebras of $\mathfrak{sl}_2(\mathbb{R})$. Since $\varphi^{-1}(0)$ is 3-dimensional the commutator subalgebra $\langle (e_2 + e_3, 0), (\mathrm{i}(e_2 + e_3), 0) \rangle$ of v is contained in $\varphi^{-1}(0)$. If the element $((r\mathrm{i} - 1)e_1, 0) \in w_r$ lies in $\mathbf{h}$ then for the fourth generator of $\mathbf{h}$ we have up to conjugation the following possibilities: (e_1, e_1), $(e_1, e_2 + e_3)$, (e_1, e_3), $(\mathrm{i}e_1, e_1)$, $(\mathrm{i}e_1, e_2 + e_3)$, $(\mathrm{i}e_1, e_3)$. If $\mathbf{h}$ contains the element $(\mathrm{i}e_1, 0)$ then for the fourth basis element of $\mathbf{h}$ we may choose one of the following: (e_1, e_1), $(e_1, e_2 + e_3)$, (e_1, e_3). The element $(\mathrm{i}(e_2 + e_3), 0) \in \mathbf{h}$ lies in $\mathbf{m}_3 \cap \mathbf{m}_4$, the elements (e_1, e_1) and $((r\mathrm{i} - 1)e_1, 0) - r(\mathrm{i}e_1, e_1)$ of $\mathbf{h}$ are contained in $\mathbf{m}_5 \cap \mathbf{m}_6$ and (e_1, e_3), $((r\mathrm{i} - 1)e_1, 0) - r(\mathrm{i}e_1, e_3)$ are elements of $\mathbf{m}_6$. Moreover, $(e_1, e_2 + e_3)$ respectively $((r\mathrm{i} - 1)e_1, 0) - r(\mathrm{i}e_1, e_2 + e_3)$ is conjugate to $(e_1, e_1 + e_3)$ respectively to $(-e_1, -r(e_1 + e_3))$ which are elements of $\mathbf{m}_6$ (see **1.1**).

It remains to investigate the triples $(G, \exp \mathbf{h}_i, \exp \mathbf{m}_5)$, where $\mathbf{h}_i$ has one of the following shapes:

$\mathbf{h}_3 = \langle (e_2 + e_3, 0), (\mathrm{i}(e_2 + e_3), 0), ((r\mathrm{i} - 1)e_1, 0), (e_1, e_2 + e_3) \rangle$,

$\mathbf{h}_4 = \langle (e_2 + e_3, 0), (\mathrm{i}(e_2 + e_3), 0), ((r\mathrm{i} - 1)e_1, 0), (\mathrm{i}e_1, e_2 + e_3) \rangle$,

$\mathbf{h}_5 = \langle (e_2 + e_3, 0), (\mathrm{i}(e_2 + e_3), 0), ((r\mathrm{i} - 1)e_1, 0), (e_1, e_3) \rangle$,

$\mathbf{h}_6 = \langle (e_2 + e_3, 0), (\mathrm{i}(e_2 + e_3), 0), ((r\mathrm{i} - 1)e_1, 0), (\mathrm{i}e_1, e_3) \rangle$,

$\mathbf{h}_7 = \langle (e_2 + e_3, 0), (\mathrm{i}(e_2 + e_3), 0), (\mathrm{i}e_1, 0), (e_1, e_2 + e_3) \rangle$,

$\mathbf{h}_8 = \langle (e_2 + e_3, 0), (\mathrm{i}(e_2 + e_3), 0), (\mathrm{i}e_1, 0), (e_1, e_3) \rangle$.

The exponential image of the subspace $\mathbf{m}_5$ consists of elements

$$\left(\begin{pmatrix} a_1 + a_2 & b_1 + b_2\mathrm{i} \\ b_1 - b_2\mathrm{i} & a_1 - a_2 \end{pmatrix}, \begin{pmatrix} c + d & f \\ f & c - d \end{pmatrix} \right); a_1 \geq 1, c \geq 1, a_2, b_1, b_2, d, f \in \mathbb{R}$$

with $a_1^2 - a_2^2 - b_1^2 - b_2^2 = 1 = c^2 - d^2 - f^2$. Since $p_2(H_i)$, $i = 1, 2, 3, 4, 7$, and $p_2(\exp \mathbf{m}_5)$ satisfy the conditions of Lemma 7 the subalgebras H_i for $i = 1, 2, 3, 4, 7$ cannot occur as the stabilizer of identity for a Bol loop.

The first component of $\exp \mathbf{h}_i$, $i = 5, 6, 8$, has the form $\begin{pmatrix} \exp v & z \\ 0 & \exp -v \end{pmatrix}$, where $z \in \mathbb{C}$,

and $v = (r\mathrm{i} - 1)x + \varepsilon y$, $x, y \in \mathbb{R}$ with $\varepsilon = 1$ for $j = 5$ and $\varepsilon = i$ in the case $j = 6$, whereas $v = \mathrm{i}x + y$ for $j = 8$.

The cosets $\left(\begin{pmatrix} 5 & 1 \\ 4 & 1 \end{pmatrix}, 1 \right) H_i$, $i = 5, 6, 8$, contain the different elements $m_1 = \left(\begin{pmatrix} \frac{5}{4} & 1 \\ 1 & \frac{8}{5} \end{pmatrix}, 1 \right)$
and $m_2 = \left(\begin{pmatrix} 5 & 4 \\ 4 & \frac{17}{5} \end{pmatrix}, 1 \right)$ of $\exp \mathbf{m}_5$ which is a contradiction to Lemma 7. $\qquad\square$

References

[1] Asoh, T.: On smooth SL(2, C) actions on 3-manifolds. Osaka J. Math. **24**, 271-298 (1987)

[2] Chen, S.: On subgroups of the Noncompact real exceptional Lie group F_4^*. Math. Ann. **204**, 271-284 (1973)

[3] Figula, Á.: 3-dimensional Bol loops as sections in non-solvable Lie groups. Forum Math. **17**(3), 431-460 (2005)

[4] Freudenthal, H., de Vries, H.: *Linear Lie Groups*. Academic Press, New York, London, 1969

[5] Furness, P. M. D.: Locally symmetric structures on surfaces. Bull. London Math. Soc. **8**, 44-48 (1976)

[6] Hilgert, J., Hofmann, K. H.: Old and new on $SL(2)$. Manuscr. Math. **54**, 17-52 (1985)

[7] Hu, Sze-Tsen: *Homotopy Theory*. Academic Press, New York, London, 1959

[8] Iwasawa, K.: On some types of topological groups. Ann. Math. **50**, 507-558 (1949)

[9] Kiechle, H.: *Theory of K-Loops*. Lecture Notes in Mathematics, 1778, Springer-Verlag, Berlin, 2002

[10] Krammer, W., Urbantke, H. K.: K-Loops, gyrogroups and symmetric spaces. Results Math. **33**, 310-327 (1998)

[11] Lie, S., Engel, F.: *Theorie der Transformationsgruppen*. 3. Abschnitt. Verlag von B.G. Teubner, Leipzig, 1893

[12] Lie, S.: *Vorlesungen über Continuierliche Gruppen*. Bearbeitet und herausgegeben von G. Scheffers. Chelsea Publishing Company, Bronx, New York, 1971

[13] Loos, O.: *Symmetric Spaces*. Vol. I, Benjamin, New York, 1969

[14] Löwen, R.: Classification of 4-dimensional symmetric planes. Math. Z. **167**, 137-159 (1979)

[15] Miheev, P. O., Sabinin, L. V.: *Quasigroups and Differential Geometry*. In: Chein, O., Pflugfelder, H. O., Smith, J. D. H.: *Quasigroups and Loops: Theory and Applications*. Sigma Series in Pure Math. 8, Heldermann-Verlag, Berlin, 1990, pp. 357-430

[16] Montgomery, D., Zippin, L.: *Topological Transformation Groups*. Wiley Interscience Publishers, New York, 1955

[17] Nagy, P. T., Strambach, K.: *Loops in Group Theory and Lie Theory*. de Gruyter Expositions in Mathematics, 35, Berlin, New York, 2002

[18] Scheerer, H.: Restklassenräume kompakter zusammenhängender Gruppen mit Schnitt. Math. Ann. **206**, 149-155 (1973)

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