manuscripta math. 130, 183–199 (2009)

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Subloop incompatible Bol loops

Received: 10 December 2008 / Revised: 10 April 2009
Published online: 27 May 2009

Abstract. We give a necessary modification of Proposition 1.18 in Nagy and Strambach (Loops in Group Theory and Lie Theory. de Gruyter Expositions in Mathematics Berlin, New York, 2002) and close the gap in the classification of differentiable Bol loops given in Figula (Manuscrp Math 121:367–385, 2006). Moreover, using the factorization of Lie groups we determine the simple differentiable proper Bol loops L having the direct product $G_1 \times G_2$ of two groups with simple Lie algebras as the group topologically generated by their left translations such that the stabilizer of the identity element of L is the direct product $H_1 \times H_2$ with $H_i < G_i$. Also if $G_1 = G_2 = G$ is a simple permutation group containing a sharply transitive subgroup A, then an analogous construction yields a simple proper Bol loop. If A is cyclic and G is finite and primitive, then all such loops are classified.

1. Introduction

In [12] the loops L are consistently considered as sharply transitive sections $\sigma : G/H \to G$, where G is the group generated by the left translations of L and H is the stabilizer of the identity element e of L in G.

This point of view is applied there for a classification of differentiable loops of low dimension. Using the methods of [12] in [3] a classification of differentiable Bol loops having an at most nine-dimensional semi-simple Lie group as the group topologically generated by their left translations is given.

A useful tool proving this classification was Proposition 1.18 in [12]: If the group G generated by the left translations of a loop L is the direct product $G = G_1 \times G_2$ and for the stabilizer H of $e \in L$ in G one has $H = H_1 \times H_2$ with $H_i < G_i$, then L is a product of two loops L_1 and L_2. But the further claim of this proposition that the loop L_i, $i = 1, 2$, is isomorphic to a loop having G_i as the group generated by its left translations and H_i as the stabilizer of the identity needs a modification (see Proposition 1). Namely, there are loops $L = L_1 L_2$, which we call subloop incompatible loops such that at least one of the subgroups generated by the left translations of L_i, $i = 1, 2$, is a proper subgroup of G_i.

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Mathematics Subject Classification (2000): 53C30, 53C35, 20N05, 22E15

DOI: 10.1007/s00229-009-0279-y

Due to this fact the classification of differentiable Bol loops L given in *Main Theorem* in [3, pp. 367–368], is complete under the following additional assumption:
$(*)$ If L is a connected differentiable Bol loop having a semi-simple Lie group $G = G_1 \times \cdots \times G_n$ with the non-trivial simple direct factors G_i as the group topologically generated by its left translations and $\sigma : G/H \to G$ as the corresponding section, then $\sigma(G/H) = \sigma_1(G/H) \times \cdots \times \sigma_n(G/H)$, where σ_i is the projection of $\sigma(G/H)$ into G_i, and $\sigma_i(G/H)$ generates G_i for all $i = 1, \ldots, n$.

One aim of this paper is to classify connected differentiable Bol loops which do not satisfy the condition $(*)$ and have an at most nine-dimensional semi-simple Lie group G as the group topologically generated by their left translations (cf. Theorems 5, 12 and 13). These loops are subloop incompatible.

Our investigation shows that subloop incompatible differentiable Bol loops L which have a semi-simple Lie group as the group topologically generated by the left translations occur only if the section corresponding to L has as direct factor a simple symmetric space generating a non-simple group of displacements. This allows to determine all simple differentiable proper Bol loops L having the direct product $G_1 \times G_2$ of two groups with simple Lie algebras as the group topologically generated by their left translations such that the stabilizer of $e \in L$ in G is the direct product $H_1 \times H_2$ with $H_i < G_i$. These loops are products of two Lie groups (cf. Theorem 4). To classify these loops we essentially use results on factorizations of simple Lie groups (cf. [14, 13]).

Proposition 2 and Lemma 2 in [11] are, as the author there shows, powerful tools for a general construction of simple proper Bol loops. We use this construction for simple permutation groups G acting on a set Ω and having a sharply transitive subgroup C. Let S be the stabilizer of a point $p \in \Omega$ in G. Then there is a simple proper Bol loop L having $G \times G$ as the group generated by its left translations. The stabilizer H of the identity $e \in L$ has the form $H = C \times S$ and L is a product of the groups S and C. If G is a finite primitive permutation group and C is cyclic, then using [7] we obtain that G must be one of the following groups: the alternating group A_{2k+1}, $k \geq 2$, the group $PSL_2(11)$, the Mathieu group M_{11} or M_{23} and the group $PSL_d(q)$ with $d \geq 2$, $(d, q) \notin \{(2, 2), (2, 4)\}$ such that the greatest common divisor of d and $q - 1$ equals 1 (cf. Corollary 9). Moreover, for every such G we determine the stabilizer H and the corresponding loop L.

2. Notation

Let G be a connected semi-simple Lie group with trivial centre. A decomposition $G = G_1 \cdot G_2$, where G_i, $i = 1, 2$, are closed connected subgroups is called an Iwasawa decomposition if G_1 is a maximal compact subgroup of G and G_2 has only trivial compact subgroups. One has $G_1 \cap G_2 = \{1\}$. If $\widetilde{G}$ is a covering group of G, then an Iwasawa decomposition is given by $\widetilde{G} = \widetilde{G_1} \cdot \widetilde{G_2}$, where $\widetilde{G_1}$ is a covering group of G_1 and $\widetilde{G_2}$ is isomorphic to G_2.

Let G_1 and G_2 be groups and let $\varphi : G_1 \to G_2$ be a homomorphism. Then we distinguish between the following two subgroups of $G_1 \times G_2$:

$$G_1 \times \varphi(G_1) = \{(x_1, \varphi(x_2));\ x_1, x_2 \in G_1\}$$

and

$$(G_1, \varphi(G_1)) = \{(x, \varphi(x));\ x \in G_1\}.$$

Let G_2 be a group, H_2 be a subgroup of G_2 and let L_2 be a loop realized on the factor space G_2/H_2 with respect to the section $\sigma_2 : G_2/H_2 \to G_2$ the image of which is the set $M \subset G_2$. A loop L is a Scheerer extension of the group G_1 by the loop L_2 if L is realized on G/H, where G is the direct product $G_1 \times G_2$, H is the subgroup $(\rho(H_2), H_2)$ with a homomorphism $\rho : H_2 \to G_1$ and L corresponds to the sharply transitive section $\sigma : G/H \to G$ with $\sigma(G/H) = G_1 \times M$.

2.1 If the group G is locally isomorphic to $PSL_2(\mathbb{R})$, then we choose as a real basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ always

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(cf. [4, pp. 19–20]). An element $X = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in \mathfrak{sl}_2(\mathbb{R})$ is elliptic, parabolic or hyperbolic depending on whether

$$k(X, X) = \lambda_1^2 + \lambda_2^2 - \lambda_3^2 \text{ is smaller, equal, or greater } 0,$$

(cf. [4, p. 20], where k is called the normalized Cartan-Killing form of $\mathfrak{sl}_2(\mathbb{R})$). The basis elements e_1, e_2 are hyperbolic, e_3 is elliptic and the elements $e_2 + e_3$, $e_1 + e_3$ are both parabolic. The group G contains 3 conjugate classes of 1-parameter subgroups; the parabolic 1-parameter subgroups corresponding to the subalgebra $\mathbb{R}(e_2 + e_3)$, the hyperbolic 1-parameter subgroups corresponding to $\mathbb{R}e_1$ and the elliptic 1-parameter subgroups belonging to $\mathbb{R}e_3$. An element $g \in G$ is called parabolic, hyperbolic or elliptic depending on whether g is contained in a parabolic, hyperbolic or elliptic subgroup. Moreover, the group G contains precisely one conjugacy class $\mathcal{C}$ of 2-dimensional subgroups; as a representative of $\mathcal{C}$ we choose

$$\mathcal{L}_2 = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix};\ a > 0, b \in \mathbb{R} \right\}.$$

The Lie algebra of $\mathcal{L}_2$ is generated by the elements $e_1, e_2 + e_3$.

2.2 In the direct product $\mathcal{L}_2 \times \mathcal{L}_2$ there is precisely one conjugacy class of 3-dimensional connected subgroups having no 1-dimensional direct factor, namely the direct product $\mathcal{L}_2 \times \mathcal{L}_2$ with amalgamated commutator factor subgroups. This subgroup has the form

$$\left\{ \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix} \right);\ a > 0, b, c \in \mathbb{R} \right\}$$

(see [6, 9.11 Satz, p. 50]).

Other basic notions used in this paper are contained in Sect. 2 in [3, pp. 368–372].

For Lie groups having a simple Lie algebra we use the notation of [16].

3. Bol loops corresponding to the direct product of two groups

First we give the necessary modification of Proposition 1.18 in [12].

Proposition 1. (a) *Let L be a loop, G be the group generated by the left translations of L and let H be the stabilizer of the identity $e \in L$ in G. If G and H are the direct products $G = G_1 \times G_2$ and $H = H_1 \times H_2$ with $H_i < G_i$ $(i = 1, 2)$, then L is the product of two loops L_1 and L_2. Let $\sigma : G/H \to G_1 \times G_2$ be the section corresponding to L and let $\sigma_i = p_i \circ \sigma$, where p_i is the natural projection $G \to G_i$ $(i = 1, 2)$. The loop L_1, respectively the loop L_2 is a normal subloop of L if and only if*

$$\sigma_2(g_1 H_1, g_2 H_2) = \sigma_2(H_1, g_2 H_2),$$

respectively

$$\sigma_1(g_1 H_1, g_2 H_2) = \sigma_1(g_1 H_1, H_2)$$

for all $g_1 \in G_1$, $g_2 \in G_2$.

(b) *The image of the section belonging to the subloop L_1 is the set $S_1 = \{\sigma_1((g_1, 1)(H_1, H_2));\ g_1 \in G_1\}$ and the image of the section corresponding to the subloop L_2 is the set $S_2 = \{\sigma_2((1, g_2)(H_1, H_2));\ g_2 \in G_2\}$. Moreover, one has $L_1 \cap L_2 = \{e \in L\}$.*

(c) *The loop L_i is isomorphic to a loop $\widehat{L_i}(i = 1, 2)$ having the group $\widehat{G_i}/N_i$ as the group generated by the left translations of $\widehat{L_i}$, where $\widehat{G_i}$ is the subgroup of G_i generated by S_i and N_i is a normal subgroup of $\widehat{G_i}$ with $N_i < H_i$ which is maximal with respect to this property. The stabilizer of $e \in \widehat{L_i}$ in $\widehat{G_i}$ is isomorphic to the group $\widehat{H_i} = (H_i \cap \widehat{G_i})/N_i$ $(i = 1, 2)$.*

Proof. The restriction of σ_1, respectively σ_2 to the subsets $(G_1 \times H_2)/(H_1 \times H_2)$, respectively $(H_1 \times G_2)/(H_1 \times H_2)$ of $(G_1 \times G_2)/(H_1 \times H_2)$ yields that $L_1 = \{(g_1, 1)(H_1, H_2);\ g_1 \in G_1\}$ with the multiplication

$$(g_1, 1)(H_1, H_2) * (k_1, 1)(H_1, H_2) = (\sigma_1((g_1, 1)(H_1, H_2))k_1 H_1, H_2)$$

and $L_2 = \{(1, g_2)(H_1, H_2);\ g_2 \in G_2\}$ with the multiplication

$$(1, g_2)(H_1, H_2) * (1, k_2)(H_1, H_2) = (H_1, \sigma_2((1, g_2)(H_1, H_2))k_2 H_2)$$

are subloops of $L(\sigma)$ (see Proposition 1.18 in [12, p. 27]). This proves (a) and (b).

Because of

$$(G_1 \times H_2)/H_2 \cong G_1, \quad (H_1 \times H_2)/H_2 \cong H_1,$$
$$(H_1 \times G_2)/H_1 \cong G_2, \quad (H_1 \times H_2)/H_1 \cong H_2$$

one can define the sharply transitive sections

$$\widehat{\rho} : G_1/H_1 \to G_1 \text{ by } \widehat{\rho}(g_1 H_1) = \sigma_1((g_1, 1)(H_1, H_2)); \quad g_1 \in G_1$$

and

$$\widehat{\tau} : G_2/H_2 \to G_2 \, by \, \widehat{\tau}(g_2H_2) = \sigma_2((1, g_2)(H_1, H_2)); \quad g_2 \in G_2.$$

The section $\widehat{\rho}$ determines a loop $\widehat{L_1}$ on the factor space G_1/H_1 by the rule $g_1H_1 \circ k_1H_1 = \widehat{\rho}(g_1H_1)k_1H_1$ and the section $\widehat{\tau}$ determines a loop $\widehat{L_2}$ on the factor space G_2/H_2 by the rule $g_2H_2 \circ k_2H_2 = \widehat{\tau}(g_2H_2)k_2H_2$. A direct computation shows that the mapping $\varphi_1 : (g_1H_1, H_2) \mapsto g_1H_1$ with $g_1 \in G_1$ is an isomorphism of the loop $(L_1, *)$ onto the loop $(\widehat{L_1}, \circ)$ and the mapping $\varphi_2 : (H_1, g_2H_2) \mapsto g_2H_2$ with $g_2 \in G_2$ is an isomorphism of the loop $(L_2, *)$ onto the loop $(\widehat{L_2}, \circ)$. Let $\widehat{G_1}$ be the subgroup of G_1 generated by $\{\widehat{\rho}(g_1H_1);\ g_1 \in G_1\} = S_1$ and $\widehat{G_2}$ be the subgroup of G_2 generated by $\{\widehat{\tau}(g_2H_2);\ g_2 \in G_2\} = S_2$. It follows from Proposition 1.13 in [12, p. 25], that the group generated by the left translations of the loop $\widehat{L_i}$ and the stabilizer of the identity of $\widehat{L_i}$ $(i = 1, 2)$ has the form as in the assertion (c). □

Proposition 1(c) differs from Proposition 1.18 in [12] only in the conclusion that the groups $\widehat{G_i}$ and $\widehat{H_i}$, $i = 1, 2$, can be proper subgroups of G_i, respectively H_i. The group $\widehat{G_i}$ coincides with G_i for $i = 1, 2$, if $\sigma(G/H) = M_1 \times M_2$ with $M_i \subset G_i$ (cf. Proposition 1.19 in [12, p. 28]), but not in general. This is already the case for many examples contained in [12] (see [12, pp. 50–51, pp. 190–193 and Theorem 16.7, p. 198]). More precisely we have the following

Remark 2. Let the group G be the direct product $K_1 \times K_2 \times K_2$ such that there is a non-trivial homomorphism $\varphi : K_2 \to K_1$. Then there is a Scheerer extension L of the group K_1 by the group K_2. This extension L is defined on the factor space G/H, where $H = \{(\varphi(k_2), 1, k_2);\ k_2 \in K_2\}$, and belongs to the section $\sigma : G/H \to G$ such that $\sigma(G/H)$ is the set $\{(k_1, k_2, k_2^{-1});\ k_1 \in K_1, k_2 \in K_2\}$ (see Proposition 15.15 in [12, p. 190]).

Since $G = G_1 \times G_2$ with $G_1 = K_1 \times \{1\} \times K_2$, $G_2 = \{1\} \times K_2 \times \{1\}$ and $H = H_1 \times H_2$, where $H_1 = H$, $H_2 = \{(1, 1, 1)\}$, the loop L is a product of a normal subgroup L_1 isomorphic to K_1 with a group L_2 isomorphic to K_2 and $L_1 \cap L_2 = \{1\}$. The subgroup L_1 corresponds to the section $\sigma_1 : G_1/H_1 \to G_1$ the image of which is the set $\{(k_1, 1, 1);\ k_1 \in K_1\}$ and the subgroup L_2 belongs to the section $\sigma_2 : G_2/H_2 \to G_2$ the image of which is the set $\{(1, k_2, 1);\ k_2 \in K_2\}$ (see Proposition 1 and Proposition 2.14 in [12, p. 51]). Hence the group L_1 cannot generate the group G_1.

Let L be a connected topological loop belonging to the section $\sigma : G/H \to G$ and let $M := \sigma(G/H)$. Let $\widetilde{L}$ be the universal covering of L corresponding to the section $\sigma^* : G^*/H^* \to G^*$, where G^* is the group topologically generated by the left translations of $\widetilde{L}$ and H^* be the stabilizer of $e \in \widetilde{L}$ in G^*. Then G^* is a covering group of G such that for the covering map $\rho : G^* \to G$ one has $\rho(\sigma^*(G^*/H^*)) = M$, $\rho(H^*) = H$ and the kernel of ρ is the subgroup Z^* of $\sigma^*(G^*/H^*)$ which is isomorphic to the fundamental group Z of L. Moreover $H^* \cap Z^* = \{1\}$ and hence H^* is isomorphic to H (cf. Lemma 1.34 in [12, p. 34]).

Let L be a connected differentiable Bol loop having a semi-simple Lie group G with trivial centre as the group topologically generated by its left translations. Then the image $M = \sigma(G/H)$ of the section $\sigma : G/H \to G$ corresponding to L

has the form $M = K \times V_1 \times \cdots \times V_s$, where K is a direct factor of G and V_i are submanifolds of G corresponding to simple symmetric spaces. Moreover, G is the direct product $K \times I(V_1) \times \cdots \times I(V_s)$, where $I(V_i)$ is the group of displacements of the symmetric space V_i ([12, Proposition 6.6 and Lemma 6.7, p. 85], [10, pp. 424–425], and [12, Theorem 13.14, p. 163]). By Proposition 1.2, in [9, p. 141], the group $I(V_i)$ of the simple symmetric space V_i is either simple or it is the direct product $S \times S$ of two simple isomorphic direct factors and the symmetric space V_i has the form $\{(x, x^{-1});\ x \in S\}$.

If each group $I(V_i)$ is simple, then for the projection $\sigma_i = p_i \circ \sigma$ of M into $I(V_i)$ one has $\sigma_i(G/H) = V_i$ and $M = K \times \sigma_1(G/H) \times \cdots \times \sigma_s(G/H)$. In this case the loop L corresponding to this section σ satisfies $(*)$.

If $\widetilde{G}$ is the universal covering of the group G, then $\widetilde{G} = \widetilde{K} \times \widetilde{I(V_1)} \times \cdots \times \widetilde{I(V_s)}$, where $\widetilde{K}$ is the universal covering of K and $\widetilde{I(V_i)}$ is the universal covering group of $I(V_i)$. The preimage $\rho^{-1}(\sigma_i(G/H)) = \sigma_i\widetilde{(G/H)}$ of $\sigma_i(G/H)$ with respect to the covering homomorphism $\rho : \widetilde{G} \to G$ generates $\widetilde{I(V_i)}$. If G' is the group topologically generated by the left translations of a covering loop L' of the loop L corresponding to σ, then there exists a covering homomorphism $\rho' : \widetilde{G} \to G'$ such that $\rho'(\sigma_i\widetilde{(G/H)})$ generates $\rho'(\widetilde{I(V_i)})$ for all $i = 1, \ldots, s$. Hence with L also L' satisfies the condition $(*)$.

Using Proposition 1.2 in [9, p. 141], the previous discussion gives

Lemma 3. *Let L be a connected differentiable Bol loop such that for the Lie algebra $\mathbf{g}$ of the group G topologically generated by the left translations of L one has $\mathbf{k} \oplus \mathbf{g_1} \oplus \cdots \oplus \mathbf{g_s}$, where $\mathbf{k}$ is semi-simple and each $\mathbf{g_i}$ is a simple Lie algebra. Let $\mathbf{m} = \mathbf{k} \oplus \mathbf{v_1} \oplus \cdots \oplus \mathbf{v_l}$ be the tangent space of the image of the section belonging to L, where $\mathbf{v_i}$ is the tangent space of a simple symmetric space. If L is subloop incompatible, then one has $l < s$ and there exists an involutory automorphism α of $\mathbf{g}$ and two isomorphic subalgebras $\mathbf{g}_i$ and $\mathbf{g}_j$, $i, j \in \{1, 2, \ldots, s\}$ such that $\alpha(\mathbf{g}_i) = \mathbf{g}_j$ and $\mathbf{m}_L = \{X - \alpha(X);\ X \in \mathbf{g}_i\}$.*

Now we describe all subloop incompatible connected differentiable Bol loops for which the group G topologically generated by their left translations has the form $G = G_1 \times G_2$, where the Lie algebra $\mathbf{g_i}$ of G_i is simple and the stabilizer H of $e \in L$ in G has the form $H_1 \times H_2$ with $H_i < G_i$, $i = 1, 2$.

Theorem 4. *Let L be a subloop incompatible simply connected differentiable Bol loop. Assume that the group G topologically generated by its left translations is the direct product $G_1 \times G_2$, where the Lie algebra $\mathbf{g_i}$ of G_i is simple, and the stabilizer H of $e \in L$ in G is the direct product $H = H_1 \times H_2$ with $H_i < G_i$, $i = 1, 2$. Then:*

(a) *The group G_1 is simply connected and there is a covering map $\rho : G_1 \to G_2$ such that G_2 is abstract simple. The preimage $\rho^{-1}(H_2)$ is a connected subgroup of G_1 and $G_1 = \rho^{-1}(H_2) \cdot H_1$ forms a factorization of G_1 such that $\rho^{-1}(H_2) \cap H_1 = \{1\}$ and no element of $\rho^{-1}(H_2) \setminus \{1\}$ is conjugate to an element of H_1. Moreover, the group G is not compact.*
(b) *The simply connected loop L is the product of a Lie group isomorphic to $\rho^{-1}(H_2)$ with a Lie group isomorphic to H_1.*

(c) *Then the loop L/Z, where Z is the centre of L, is a simple Bol loop which is the product of a Lie group isomorphic to H_2 with a Lie group isomorphic to H_1.*

Proof. According to Lemma 3 the tangent space of the image M of the section belonging to L has the form $\mathbf{m} = \{(X, -X);\ X \in \mathbf{g_1}\}$, where $\mathbf{g_1}$ is the Lie algebra of G_1 and the Lie algebra $\mathbf{g_2}$ of G_2 is isomorphic to $\mathbf{g_1}$. Since $\dim M = \dim G_1 = n$ the stabilizer H of $e \in L$ has also dimension n. Every element of the Lie algebra $\mathbf{g} = \mathbf{g_1} \oplus \mathbf{g_2}$ of G has a unique decomposition as the direct sum $m + h$ with $m \in \mathbf{m}$ and $h \in \mathbf{h} = \mathbf{h_1} \oplus \mathbf{h_2}$, where $\mathbf{h_i}$ is the Lie algebra of H_i. In particular for $(0, a)$ one has $(x, -x) + (h_1, h_2)$, where $(x, -x) \in \mathbf{m}$ and $(h_1, h_2) \in \mathbf{h}$. It follows that $x = -h_1$ and $a = h_1 + h_2$. Therefore the Lie algebra $\mathbf{g_2}$ is the direct sum of the vector subspaces $\mathbf{h_1}$ and $\mathbf{h_2}$. The group $G = G_1 \times G_2$ is homeomorphic to $L \times H_1 \times H_2$. As $H_1 \times H_2$ is homeomorphic to G_2 the simply connected loop L is homeomorphic to G_1 and hence G_1 is simply connected. Therefore there exists a covering map $\rho : G_1 \to G_2$. The intersection $H_1 \cap \rho^{-1}(H_2)$ is trivial; otherwise $M = \{(x, \rho(x)^{-1});\ x \in G_1\}$ would contain an element of H. Moreover, $H = H_1 \times H_2$ contains no element $h \neq 1$ which is conjugate to an element of M. Therefore, no element of $\rho^{-1}(H_2)\backslash\{1\}$ is conjugate to an element of H_1. Every element of $G = G_1 \times G_2$ has a unique decomposition as a product $m \cdot h$, where $m \in M$ and $h \in H$. Then for every $a \in G_1$ there exist elements $x \in G_1$ and $h_i \in H_i$, $i = 1, 2$, such that $(a, 1) = (x, \rho(x)^{-1}) \cdot (h_1, h_2)$. This yields that $a = \rho^{-1}(h_2)h_1$ and hence we have $G_1 = \rho^{-1}(H_2) \cdot H_1$. As $H = H_1 \times H_2$ does not contain any non-trivial normal subgroup of G one has $H_2 \cap Z^* = H_1 \cap Z^* = \{1\}$, where Z^* is the centre of G. Therefore the group $G_2 = \rho(G_1) = \rho(\rho^{-1}(H_2) \cdot H_1) = H_2 \cdot \rho(H_1) \cong H_2 \cdot H_1$ is abstract simple.

Any factorization $S = S_1 \cdot S_2$ with $S_1 \cap S_2 = \{1\}$ of a compact connected semi-simple Lie group S is isomorphic to the direct product $S = S_1 \times S_2$ (cf. [8,15], also Theorem 4.4 in [14], p. 531). This is a contradiction to the condition that the Lie algebra $\mathbf{g_i}$ of G_i, $i = 1, 2$, is simple. Hence G is not compact (which follows also from Theorem 16.7 and Corollary 16.9 in [12]). With this the proof of the assertion (a) is complete.

According to Proposition 1 the loop L is the product of two loops L_1 and L_2. As every element $g_1 \in G_1$ has the form $g_1 = \rho^{-1}(h_2)h_1$, where $h_i \in H_i$, $i = 1, 2$, the unique element $(x, \rho(x)^{-1}) \in M$ containing in the left coset $(g_1, 1)(H_1, H_2) = (\rho^{-1}(h_2), 1)(H_1, H_2)$ has the form $(\rho^{-1}(h_2), h_2^{-1})$. Hence the subloop L_1 which belongs to $\{\sigma_1(g_1 H_1, H_2);\ g_1 \in G_1\}$ is isomorphic to the Lie group $\rho^{-1}(H_2)$. A similar consideration yields that the subloop L_2 belonging to $\{\sigma_2(H_1, g_2 H_2);\ g_2 \in G_2\}$ is isomorphic to the Lie group $\rho(H_1) \cong H_1$. Hence the assertion (b) is proved.

The centre Z^* of G has the form $\{(z, 1);\ z \in Z_1\}$, where Z_1 is the centre of G_1. One can define the subgroup Z^* and the section σ corresponding to L in the factor group G/Z^* in a natural way, which determine a Bol loop L^* with a surjective homomorphism $L \to L^*$. The kernel of this homomorphism is central and isomorphic to Z^* (cf. [12, Lemma 1.34, p. 34]). Hence the subgroup Z^* corresponds to a central subgroup Z of L. The loop L/Z has the group $G/Z^* = G_1/Z_1 \times G_2$ as the group topologically generated by the left translations. The group G_1/Z_1 can be identified with the group G_2. Therefore the mapping $(a, b) \mapsto (b, a)$; $a \in G_1/Z_1, b \in G_2$, may be seen as an involutory automorphism τ of G/Z^* which leaves the image of the section corresponding to L/Z invariant. Since the group G/Z^* has no proper τ-invariant normal subgroups the loop L/Z is simple (cf. Lemma 2 in [11, p. 83]) and the assertion (c) follows. □

Now we determine all subloop incompatible connected differentiable Bol loops L having an at most nine-dimensional Lie group $G = G_1 \times G_2$ with simple Lie algebra $\mathbf{g}_i$ of $G_i, i = 1, 2$, as the group topologically generated by their left translations. These loops shape up as minimal examples of subloop incompatible differentiable Bol loops of Theorem 4.

Theorem 5. *Let L be a connected differentiable Bol loop such that for the at most 9-dimensional Lie group G topologically generated by its left translations one has $G = G_1 \times G_2$, where both factors G_i, $i = 1, 2$, have simple Lie algebras. If L is subloop incompatible, then G_1 is isomorphic to a covering of $PSL_2(\mathbb{R})$, the group G_2 is isomorphic to $PSL_2(\mathbb{R})$, the stabilizer H is the direct product $\mathcal{L}_2 \times SO_2(\mathbb{R})$ and the loop L is the product of a covering of $SO_2(\mathbb{R})$ with a group isomorphic to $\mathcal{L}_2$.*

Proof. By Lemma 3 the Lie groups G_1 and G_2 have isomorphic Lie algebras and the tangent space $\mathbf{m}$ of the image of the section belonging to L is $\{(X, -X);\ X \in \mathbf{g}_1\}$. It follows from Proposition 2 d) in [2, p. 435] that G_1 and G_2 are locally isomorphic to $PSL_2(\mathbb{R})$. According to [2, pp. 442–444] the 3-dimensional stabilizer H has the form $H_1 \times H_2$ with $1 \neq H_i < G_i$. Up to automorphisms of G we may assume that H_1 is the 2-dimensional Lie group $\mathcal{L}_2$ and H_2 is isomorphic to a 1-dimensional subgroup of G_2. By Theorem 4 the subgroup H_2 cannot be conjugate to a subgroup of $\mathcal{L}_2$. Hence the group H_2 is locally isomorphic to the group $SO_2(\mathbb{R})$ and in view of Theorem 4 the assertion follows. □

Let S be a connected non-compact Lie group having a simple Lie algebra. Following [13, Sect. 2], we call a factorization $S = S_1 S_2$ into closed subgroups S_1 and S_2 intersection-free factorization if $S_1 \cap S_2 = \{1\}$. An intersection-free factorization of S yields a subloop incompatible loop L of Theorem 4 (cf. [11, Proposition 2, p. 85]).

Proposition 6. *Let G be a connected non-compact Lie group with a simple Lie algebra. Then*:

(a) *Any Iwasawa decomposition of G gives a loop of Theorem 4.*
(b) *The Iwasawa decompositions are the only intersection-free factorizations of G if G is either a complex Lie group or it is locally isomorphic to one of the following groups: $SL_{n+1}(\mathbb{R})$ $(n \neq 3)$, $SL_{m+1}(\mathbb{H})$ $(m \geq 1)$, $SO_{2n+1}(\mathbb{R}, 1)$ $(n > 2)$, $SO_{2n}(\mathbb{R}, 1)$ $(n > 3)$, the exceptional group F_4 with maximal compact subgroups of type B_4, the exceptional group E_6 with maximal compact subgroups of type either F_4 or C_4, the exceptional group E_7 with maximal compact subgroups of type A_7, the exceptional group E_8 with maximal compact subgroups of type D_8.*

Proof. The claim (a) is clear. The maximal compact subgroups K of the groups listed in (b) have simple Lie algebras (cf. [16]). Let $G = G_1 \cdot G_2$ be an intersection-free factorization of G different from an Iwasawa decomposition. Then a maximal compact subgroup K of G has a factorization $K = K_1 \cdot K_2$ such that $K_1 \cap K_2 = \{1\}$, where $1 \neq K_i \leq G_i$, $i = 1, 2$, is a maximal compact subgroup of G_i (cf. Lemma 1.2 in [14, p. 520]). Since K has simple Lie algebra we obtain a contradiction to Theorem 4.4 in [14, p. 531]. □

In Sect. 2 of [13] the intersection-free decompositions of classical Lie groups are determined. In particular, if $G = G_1 \cdot G_2$ is such a factorization, then either G_1 or G_2, say G_2, is compact. More precisely, for intersection-free factorizations which are not Iwasawa decompositions we have:

(1) Let G be locally isomorphic to $SU_{n+1}(\mathbb{C}, h)$, where $n > 1$ and the hermitian form h has index i with $1 \leq i \leq [\frac{n+1}{2}]$. Then there are intersection-free factorizations $G = G_1 \cdot G_2$ such that:
   (i) G_2 is locally isomorphic to $SU_{n+1-i}(\mathbb{C}) \times SO_2(\mathbb{R})$, where $i > 1$ and the maximal compact subgroups K_1 of G_1 are locally isomorphic to $SU_i(\mathbb{C})$.
   (ii) G_2 is locally isomorphic to $SU_{n+1-i}(\mathbb{C})$, $i > 1$ and the maximal compact subgroups K_1 of G_1 are locally isomorphic to $SU_i(\mathbb{C}) \times SO_2(\mathbb{R})$.
   (iii) G_2 is locally isomorphic to $SU_{n+1-i}(\mathbb{C}) \times SU_i(\mathbb{C})$ and the maximal compact subgroups K_1 of G_1 are locally isomorphic to $SO_2(\mathbb{R})$.

(2) Let G be locally isomorphic to $SO_n(\mathbb{R}, h)$, where $n > 4$ and h is a quadratic form of index $2 \leq i \leq [\frac{n}{2}]$. Then there exist intersection-free factorizations $G = G_1 \cdot G_2$ such that G_2 is locally isomorphic to $SO_{n-i}(\mathbb{R})$ and the maximal compact subgroups K_1 of G_1 are locally isomorphic to $SO_i(\mathbb{R})$.
   If $i = 3$ and $n = 7$ as well as if $i = 4$, then G_2 can be also locally isomorphic to $SO_3(\mathbb{R}) \times SO_{n-4}(\mathbb{R})$ and the maximal compact subgroups of G_1 are locally isomorphic to $SO_3(\mathbb{R})$.

(3) Let G be locally isomorphic to $S_\alpha U_n(\mathbb{H}, h)$, where h is a quaternional antihermitean form of index $[\frac{n}{2}]$. Then there are intersection-free factorizations $G = G_1 \cdot G_2$ such that G_2 is locally isomorphic to $SU_n(\mathbb{C})$ and the maximal compact subgroups K_1 of G_1 are locally isomorphic to $SO_2(\mathbb{R})$.

(4) Let G be locally isomorphic to $Sp_{2n}(\mathbb{R})$, $n \geq 3$. Then there exist intersection-free factorizations $G = G_1 \cdot G_2$ such that G_2 is locally isomorphic to $SU_n(\mathbb{R})$ and the maximal compact subgroups K_1 of G_1 are locally isomorphic to $SO_2(\mathbb{R})$.

(5) Let G be locally isomorphic to $SU_n(\mathbb{H}, h)$, where h is a quaternional hermitean form of index $i \in \{3, 4, \ldots, [\frac{n}{2}]\}$. Then there are intersection-free factorizations $G = G_1 \cdot G_2$ such that G_2 is locally isomorphic to $SU_{n-i}(\mathbb{H})$ and the maximal compact subgroups K_1 of G_1 are locally isomorphic to $SU_i(\mathbb{H})$.

For more information about the structure of the groups G_1 consult [13, Theorems 2.1, 2.2].

The following proposition is a consequence for Lie groups of Proposition 2 in [11, p. 85].

Proposition 7. *Any intersection-free factorization of a classical Lie group G yields a subloop incompatible Bol loop of Theorem 4.*

Proposition 8. *Let G be a simple group which has a representation on a set Ω as a permutation group containing a sharply transitive subgroup C. Let S be the stabilizer of a point of Ω in G. Then there exists a simple proper Bol loop L having the group $\tilde{G} = G_1 \times G_2$ with $G_1 = G_2 = G$ as the group generated by its left translations, the group $H = C \times S$ with $C \subset G_1$ and $S \subset G_2$ as the stabilizer of the identity of L in $\tilde{G}$ and corresponding to the section with the image $M = \{(x, x^{-1}), x \in G\}$. The loop L is a product of the groups S and C.*

Proof. Proposition 2 in [11] yields that the loop L is a proper Bol loop. Since the only non-trivial proper normal subgroups of $\tilde{G}$ are $G_1 \times \{1\}$ and $\{1\} \times G_2$ it follows from Lemma 4 in [11] that $\tilde{G}$ is the group generated by the left translations of L. The map $\tau : (a, b) \mapsto (b, a), a \in G_1, b \in G_2$ is an involutory automorphim of $\tilde{G}$ leaving M invariant. Since $\tilde{G}$ does not contain a non-trivial normal subgroup invariant under τ the loop L is simple (cf. Lemma 2 in [11]). Moreover, it follows from Proposition 1 that L is a product of the groups S and C. □

Using the classification of finite primitive groups G containing a sharply transitive cyclic subgroup (cf. [7]) and some informations on $PSL_d(q)$ as well as on the Mathieu groups (cf. [1] and [6], Satz 6.14, p. 183 and Satz 8.28, p. 214) we obtain as a consequence of the preceding proposition the following

Corollary 9. *Let G be a finite simple group which has a representation on a set Ω as a primitive permutation group containing a sharply transitive cyclic subgroup C. Let L be a simple Bol loop constructed as in preceding proposition. Then precisely one of the following cases occurs*:

(1) *G is the alternating group A_{2k+1}, $k \geq 2$, the loop L is a product of the alternating group A_{2k} with the cyclic group of order $2k + 1$ and has order $\frac{(2k+1)!}{2}$.*
(2) *G is the group $PSL_d(q)$ with $d \geq 2$, $(d, q) \notin \{(2, 2), (2, 4)\}$ and the greatest common divisor of d and $q - 1$ equals 1, the loop L is a product of the group of affinities of the $(d - 1)$-dimensional affine space over $GF(q)$ with the cyclic group of order $\frac{q^d - 1}{q - 1}$ and has order $q^{\frac{1}{2}d(d-1)} \prod_{i=1}^{d-1} (q^{i+1} - 1)$.*
(3) *G is the group $PSL_2(11)$, the loop L is a product of the alternating group A_5 with the cyclic group of order 11 and has order 660.*
(4) *G is the Mathieu group M_{11}, the loop L is a product of the Mathieu group M_{10} with the cyclic group of order 11 and has order 7920.*
(5) *G is the Mathieu group M_{23}, the loop L is a product of the Mathieu group M_{22} with the cyclic group of order 23 and has order 10200960.*

In the cases (2) till (5) in Corollary 9 the condition that C is cyclic can be replaced by the condition that C is abelian since any abelian transitive subgroup in these groups is cyclic (for the case (2) cf. [7, Theorem 1]). In contrast to this in suitable alternating groups there are transitive abelian subgroups which are not cyclic and which can be taken as the subgroup C. This yields further simple Bol loops which are products of groups. The simplest examples of this type can be realized in alternating groups A_{2k+1}, where $2k+1=n^2$. In such a group let α respectively β be the element with the following cycle representation:

$$\alpha=(12\dots n)((n+1)(n+2)\dots(2n))\dots(((n-1)n+1)((n-1)n+2)\dots(n^2))$$
$$\beta=(1\ (n+1)\ \dots\ ((n-1)n+1))(2(n+2)\ \dots\ ((n-1)n+2))\dots(n(2n)\dots(n^2)).$$

Since $\alpha\beta=\beta\alpha$ the group C generated by α and β is the direct product of two cyclic groups of order n and acts transitively.

4. Bol loops having a Lie group with three simple factors as the left translation group

We remark that a differentiable Bol loop L having the direct product $K_1\times K_2\times K_3$ with simple Lie algebras of K_i, $i=1,2,3$, as the group G topologically generated by its left translations is subloop incompatible precisely if the direct product $K_i\times K_j$ of two factors of G is the group of displacements of a simple symmetric space (cf. Lemma 3). A construction of sections which define subloop incompatible differentiable Bol loops is difficult already for the case that $K_1=K_2=K_3=K$. But if K has dimension 3, then only the following example yields a new phenomenon with respect to the Main Theorem of [3].

Example. Let G be the group $PSL_2(\mathbb{R})\times PSL_2(\mathbb{R})\times PSL_2(\mathbb{R})$ and let $H<G$ be the direct product $H_1\times H_2$, where $H_1=\{(k,1,k);\ k\in PSL_2(\mathbb{R})\}$ and $H_2=\{(1,s,1);\ s\in SO_2(\mathbb{R})\}$. Moreover, let the image of the section $\sigma: G/H\to G$ be the set $\sigma(G/H)=\{(x,x^{-1},y);\ x\in PSL_2(\mathbb{R}), y\in F_1\}$, where $F_1=\left\{\begin{pmatrix} m+n & z\\ z & m-n\end{pmatrix}; m\geq 1, n,z\in\mathbb{R}, m^2-n^2-z^2=1\right\}$. The set $\sigma(G/H)$ is the symmetric space of G corresponding to the involutory automorphism $\tau:(u,v,z)\mapsto(v,u,\alpha(z)),\ u,v,z\in PSL_2(\mathbb{R})$, where α is the involutory automorphism of $PSL_2(\mathbb{R})$ fixing the subgroup $SO_2(\mathbb{R})$ elementwise. The factor space $G/H=\{gH;\ g\in G\}$ has the form

$$\{(1,l,g_3)H;\ l\in\mathcal{L}_2, g_3\in PSL_2(\mathbb{R})\}.$$

The section σ determines a global differentiable Bol loop $\check{L}$ if and only if every left coset $(1,l,g_3)H$ contains precisely one element of the set $\sigma(G/H)$ (cf. Proof of Lemma 1.3 in [12, p. 17]). This happens precisely if for every given $p,q,r,s\in\mathbb{R}$ with $ps-qr=1$ and $a>0$, $b\in\mathbb{R}$ the equation

$$\left(x,x^{-1},y\right)=\left(1,\begin{pmatrix} a & b\\ 0 & a^{-1}\end{pmatrix},\pm\begin{pmatrix} p & q\\ r & s\end{pmatrix}\right)\left(k,\pm\begin{pmatrix} c & d\\ -d & c\end{pmatrix},k\right)\tag{1}$$

has a unique solution $x\in PSL_2(\mathbb{R})$, $y\in F_1$ for suitable $k\in PSL_2(\mathbb{R})$, $c,d\in\mathbb{R}$ with $c^2+d^2=1$. We obtain that $x=k=\pm\begin{pmatrix} c & -d\\ d & c\end{pmatrix}\begin{pmatrix} a^{-1} & -b\\ 0 & a\end{pmatrix}$ and

$$y=\begin{pmatrix} m+n & z\\ z & m-n\end{pmatrix}=\pm\begin{pmatrix} p & q\\ r & s\end{pmatrix}\left[\pm\begin{pmatrix} c & -d\\ d & c\end{pmatrix}\right]\begin{pmatrix} a^{-1} & -b\\ 0 & a\end{pmatrix}.\tag{2}$$

Comparing the (1, 2)- and (2, 1)-entries of both sides of Eq. (2) we have

$$z = -p(cb + da) + q(-db + ca) = rca^{-1} + sda^{-1}. \qquad (3)$$

From Eq. (3) we get the equation

$$c(qa^2 - pab - r) = d(pa^2 + qab + s). \qquad (4)$$

If $qa^2 - pab - r = 0$, then $ps - qr = 1$ yields $pa^2 + qab + s \neq 0$. In this case from equation (4) it follows that $d = 0$ and $c = \pm 1$. If $qa^2 - pab - r \neq 0$, then we have $c = \frac{d[pa^2+qab+s]}{qa^2-pab-r}$. Using the relation $c^2 + d^2 = 1$ we obtain that

$$|c| = \frac{pa^2 + qab + s}{\sqrt{(pa^2 + qab + s)^2 + (qa^2 - pab - r)^2}},$$

$$|d| = \frac{qa^2 - pab - r}{\sqrt{(pa^2 + qab + s)^2 + (qa^2 - pab - r)^2}}.$$

The Eq. (4) holds precisely if sign c = sign d. The values c and d determine the elements $y \in F_1$, $x \in PSL_2(\mathbb{R})$ in a unique way since m and n of Eq. (2) as well as x can be computed knowing c and d. Hence the loop $\check{L}$ belonging to the triple $(G, H, \sigma(G/H))$ is a proper differentiable Bol loop. This loop is a product of a normal 3-dimensional Bol loop L_1 having $SO_2(\mathbb{R}) \times \{1\} \times PSL_2(\mathbb{R})$ as the group topologically generated by the left translations and $\{(s, 1, s);\ s \in SO_2(\mathbb{R})\}$ as the stabilizer of the identity of L_1 and a Bol loop L_2 isotopic to the hyperbolic plane loop $\mathbb{H}_2$ (see Proposition 1 and Sect. 22 in [12]). According to Theorem 6 in [2, p. 448] the loop L_1 is a Scheerer extension of the Lie group $SO_2(\mathbb{R})$ by $\mathbb{H}_2$.

Remark 10. Let $\widetilde{L}$ be the universal covering of the loop $\check{L}$ in Example. If G^* is the group topologically generated by the left translations of $\widetilde{L}$, then $G^* = (\widetilde{G_1} \times G_2 \times \widetilde{G_3})/N$ such that $\widetilde{G_1} = \widetilde{G_3}$ is the universal covering of $PSL_2(\mathbb{R})$ and $N = \{(z, 1, z);\ z \in Z\}$, where Z is the centre of $\widetilde{G_1}$. The stabilizer H^* of $e \in \widetilde{L}$ is the subgroup $\{(k, s, k);\ k \in \widetilde{G_1}, s \in SO_2(\mathbb{R})\}/N$ of G^*.

Now we classify subloop incompatible connected differentiable Bol loops such that the Lie algebra of the group G topologically generated by their left translations is the direct sum of three 3-dimensional simple Lie algebras.

Lemma 11. *Let $G = G_1 \times G_2$, where G_1, G_2 are locally isomorphic to the simple Lie groups $PSL_2(\mathbb{R})$ or $SO_3(\mathbb{R})$. Let H be a 4-dimensional connected subgroup of G. Then one has $H = H_1 \times H_2$ with $H_i \leq G_i$.*

Proof. Let $p_i : G \to G_i$ be the natural projections of G onto the i-th components G_i of G and write $H_i = p_i(H)$. If $\dim(H_1) \leq 2$ and $\dim(H_2) \leq 2$, then we are done. Hence we may assume that $\dim(H_1) = 3$ which yields $H_1 = G_1$. In this case if $\dim(H_2) = 1$, then we obtain the assertion. If $\dim(H_2) = 2$, then H_2 is isomorphic to $\mathcal{L}_2$ and G_2 is locally isomorphic to $PSL_2(\mathbb{R})$. Since there is only trivial homomorphism from a 3-dimensional simple Lie group G_1 into $\mathcal{L}_2$, we have a contradiction. The case that $\dim(H_2) = 3$ is impossible since in this case the dimension of H is equal 6 or 3. □

Theorem 12. *Let L be a connected differentiable Bol loop such that for the Lie algebra $\mathbf{g}$ of the group $G = G_1 \times G_2 \times G_3$ topologically generated by its left translations one has $\mathbf{g} = \mathbf{g_1} \oplus \mathbf{g_2} \oplus \mathbf{g_3}$, where $\mathbf{g_i}$, $i = 1, 2, 3$, are 3-dimensional simple Lie algebras corresponding to G_i and $\mathbf{g_1}$ is isomorphic to $\mathbf{g_2}$, but $\mathbf{g_3}$ is not isomorphic to $\mathbf{g_1}$. If L is subloop incompatible, then L is a Scheerer extension of the group G_3 by a Bol loop $\widehat{L}$ of Theorem 5. In this case G_1 is isomorphic to a covering of $PSL_2(\mathbb{R})$, the group G_2 is isomorphic to $PSL_2(\mathbb{R})$ and G_3 is isomorphic either to $SO_3(\mathbb{R})$ or to $Spin_3(\mathbb{R})$.*

Proof. As the Lie algebra $\mathbf{g_1}$ is isomorphic to $\mathbf{g_2}$ by Lemma 3 the tangent space $\mathbf{m}$ of the image M of the section belonging to L has the form $\mathbf{m} = \mathbf{m_1} \oplus \mathbf{m_2}$ with $\mathbf{m_1} = \{(X, -X);\ X \in \mathbf{g_1}\}$ and $\mathbf{m_2} \subset \mathbf{g_3}$. Moreover, dim $\mathbf{m_2} \geq 2$ (see Lemma 15 in [3, p. 379]).

First we treat the case that $\mathbf{g_1} = \mathbf{g_2}$ is the Lie algebra $\mathfrak{so}_3(\mathbb{R})$ and $\mathbf{g_3}$ is the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. If $\dim(\mathbf{m}) = 6$, then one has $\dim(\mathbf{h}) = 3$. If $\dim(\mathbf{m}) = 5$, then we have $\dim(\mathbf{h}) = 4$. In the first case we obtain $\mathbf{h} = \mathbf{h_1} \oplus \mathbf{h_2}$ with $\mathbf{h_1} \subset \mathbf{g_1} \oplus \mathbf{g_1}$ and $\mathbf{h_2} \subset \mathbf{g_3}$, since any homomorphism of a 3-dimensional subgroup of $G_1 \times G_1$ into G_3 is trivial and no element of $\mathbf{m}$ is conjugate to an element of $\mathbf{h}$. In the second case if $\mathbf{h}$ is not the direct product $\mathbf{h_1} \oplus \mathbf{h_2}$ with $\mathbf{h_1} \subset \mathbf{g_1} \oplus \mathbf{g_1}$ and $\mathbf{h_2} \subset \mathbf{g_3}$, then $\mathbf{h}$ would have the form $((\mathbf{g_1} \oplus \mathbf{a}), \varphi(\mathbf{g_1} \oplus \mathbf{a}))$, where the homomorphism φ is not trivial and $\mathbf{a}$ is a 1-dimensional Lie algebra of $\mathfrak{so}_3(\mathbb{R})$ (see Lemma 11). Since the kernel of φ contains as direct factor $\mathbf{g_1}$, the Lie algebra of $\mathbf{h}$ must contain an ideal of $\mathbf{g}$. This is a contradiction. By Proposition 1.19 in [12, p. 28] the Bol loop L corresponding to the triple (G, H, M) is the direct product of Bol loops L_1 and L_2. The loop L_1 is realized on the manifold S/H_1, where S is the direct product of two groups locally isomorphic to $SO_3(\mathbb{R})$. But such a Bol loop does not exist (see Theorem 16.7 in [12, p. 198]).

Now we consider the case that $\mathbf{g_1} = \mathbf{g_2}$ is the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ and $\mathbf{g_3}$ is the Lie algebra $\mathfrak{so}_3(\mathbb{R})$. First we assume that the tangent space $\mathbf{m}$ has the form $\mathbf{m} = \mathbf{m_1} \oplus \mathbf{m_2}$ such that $\mathbf{m_2} \subset \mathfrak{so}_3(\mathbb{R})$ and dim $\mathbf{m_2} = 2$. Then the Lie algebra $\mathbf{h}$ has dimension 4. Since all 1-dimensional subalgebras of $\mathfrak{so}_3(\mathbb{R})$ are conjugate and $\mathbf{m_2}$ contains 1-dimensional subalgebras one has $\mathbf{h} \cap \mathbf{g_3} = 0$. As $\mathbf{h}$ does not contain any non-trivial ideal of $\mathbf{g}$ it follows that $\mathbf{h} = (\mathfrak{l}_2 \oplus \mathfrak{l}_2, \varphi(\mathfrak{l}_2 \oplus \mathfrak{l}_2))$, where $\mathfrak{l}_2$ is the Lie algebra of $\mathcal{L}_2$ and φ is a homomorphism from $\mathfrak{l}_2 \oplus \mathfrak{l}_2$ into $\mathfrak{so}_3(\mathbb{R})$. Since the kernel of φ contains the Lie algebra of the commutator subgroup of $\mathcal{L}_2 \times \mathcal{L}_2$, the intersection $\mathbf{h} \cap \mathbf{m_1}$ is not trivial.

Finally let $\mathbf{m} = \mathbf{m_1} \oplus \mathbf{g_3}$. Then the stabilizer H of $e \in L$ in G is 3-dimensional. Since $H \cap G_3 = \{1\}$ the stabilizer H has the form $(H_1, \varphi(H_1))$, where H_1 is a 3-dimensional subgroup of $G_1 \times G_2$ and $\varphi : H_1 \to G_3$ is a homomorphism. The subgroup H_1 is either the direct product $\mathcal{L}_2 \times \mathcal{L}_2$ with amalgamated commutator factor subgroups $\mathcal{F}$ (see **2.2**) or $H_1 = \mathcal{L}_2 \times A$, where A is a 1-dimensional subgroup of G_2. If H_1 contains $\mathcal{F}$ or A is not elliptic, then H_1 contains a subgroup $K \times K$, where K is a 1-dimensional parabolic or hyperbolic subgroup of G_1. But then there is an element of M which is conjugate to an element of H.

If A is a 1-dimensional elliptic subgroup of G_2 and $\varphi : \mathcal{L}_2 \times A \to G_3$ is a homomorphism, then L is a Scheerer extension of the Lie group G_3 by a Bol loop

$\widehat{L}$ of Theorem 5 (cf. Proposition 2.4, and 2.5 in [12, pp. 44–45]). In particular, if the homomorphism φ is trivial, then by Proposition 1.19 in [12, p. 28], the loop L is the direct product of $\widehat{L}$ with the group $SO_3(\mathbb{R})$, respectively $Spin_3(\mathbb{R})$. If $\widetilde{L}$ is the universal covering of a loop L, then the group G^* topologically generated by the left translations of $\widetilde{L}$ is isomorphic to the group $\widetilde{G_1} \times G_2 \times \widetilde{G_3}$, where $\widetilde{G_1}$ is the universal covering of $PSL_2(\mathbb{R})$, G_2 is the group $PSL_2(\mathbb{R})$ and $\widetilde{G_3}$ is the group $Spin_3(\mathbb{R})$. □

Theorem 13. *Let L be a connected differentiable Bol loop such that for the Lie algebra* $\mathbf{g}$ *of the group G topologically generated by its left translations one has* $\mathbf{g} = \mathbf{g_1} \oplus \mathbf{g_2} \oplus \mathbf{g_3}$, *where* $\mathbf{g_1} = \mathbf{g_2} = \mathbf{g_3} = \mathbf{g}^*$ *is a 3-dimensional simple Lie algebra. If L is subloop incompatible, then one of the following holds*:

(i) *L is either isomorphic to a Scheerer extension of a Lie group G' by a Lie group G'' both belonging to the Lie algebra* $\mathbf{g}^*$ *or a Scheerer extension of a Lie group G_3 corresponding to the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ by a Bol loop $\widehat{L}$ of Theorem 5.*

(ii) *L is isomorphic to the direct product of a Bol loop $\widehat{L}$ in Theorem 5 with the hyperbolic plane loop $\mathbb{H}_2$.*

(iii) *L is a covering of the loop $\check{L}$ in Example.*

Proof. We may assume that the tangent space $\mathbf{m}$ of the image M of the section belonging to the loop L has the form $\mathbf{m_1} \oplus \mathbf{m_2}$ such that $\mathbf{m_1} = \{(X, -X);\ X \in \mathbf{g_1}\}$ (cf. Lemma 3) and $\mathbf{m_2} \subseteq \mathbf{g_3}$. By Lemma 15 in [3] one has $\dim \mathbf{m_2} \geq 2$.

If $\mathbf{m_2} = \mathbf{g_3}$, then for the Lie algebra $\mathbf{h}$ of the stabilizer H of $e \in L$ one has $\mathbf{h} \cap \mathbf{g_3} = 0$. Therefore $\mathbf{h}$ has the form $(\mathbf{h_1}, \varphi(\mathbf{h_1}))$, where $\mathbf{h_1}$ is a 3-dimensional subalgebra of $\mathbf{g_1} \oplus \mathbf{g_2}$ and $\varphi : \mathbf{h_1} \to \mathbf{g_3}$ is a homomorphism. The loop L belonging to the triple (G, H, M) is a Scheerer extension of a group G_3 belonging to the Lie algebra $\mathbf{g_3}$ by a Bol loop L'. The loop L' corresponds to a sharply transitive section $\sigma_1 : (G_1 \times G_2)/H_1 \to G_1 \times G_2$ the image of which is $M_1 = \exp \mathbf{m_1}$, where $G_1 \times G_2$, respectively H_1 belongs to $\mathbf{g_1} \oplus \mathbf{g_2}$, respectively $\mathbf{h_1}$ (see Propositions 2.4 and 2.5 in [12]). The group H_1 has either the form $G_1 \times \{1\}$ and φ is an isomorphism or it is the direct product $\mathcal{L}_2 \times A$ with a 1-dimensional elliptic subgroup A of G_2. In the first case L is isomorphic to a Scheerer extension of a group G_3 by a group G_1 both having the Lie algebra $\mathbf{g}^*$ (see Remark 2). In the latter case L is a Scheerer extension of G_3 belonging to the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ by a Bol loop $\widehat{L}$ in Theorem 5. This is the case (i) of the assertion.

Now we consider the case that $\dim \mathbf{m_2} = 2$. Then one has $\dim \mathbf{h} = 4$.

If $\mathbf{g}^* = \mathfrak{so}_3(\mathbb{R})$, then the fundamental group π_1 of the group G is finite. Every 4-dimensional core-free subgroup H of G is a direct product $SO_2(\mathbb{R}) \times \{(x, \varphi(x)); x \in G^*\}$, where φ is a non-trivial homomorphism of a 3-dimensional Lie group G^* which Lie algebra is $\mathfrak{so}_3(\mathbb{R})$. As the group G is homeomorphic to the topological product $\sigma(G/H) \times H$ one has $\pi_1(G) = \pi_1(K) \cong \pi_1(\sigma(G/H) \times H) \cong \pi_1(\sigma(G/H)) \times \pi_1(K_1)$, where K respectively K_1 is a maximal compact subgroup of G respectively of H (cf. [5, Theorem 2.1, p. 144]). Since $\pi_1(SO_2(\mathbb{R}))$ is isomorphic to $\mathbb{Z}$ we obtain a contradiction. Hence one has $\mathbf{g}^* = \mathfrak{sl}_2(\mathbb{R})$.

If $\mathbf{m_2}$ is the tangent space of the simple symmetric space M_2 corresponding to an involutory automorphism fixing a 1-dimensional hyperbolic subgroup elementwise (cf. Lemma 15 in [3, p. 379]), then $\mathbf{m_2}$ contains 1-dimensional subalgebra of any conjugate class of $\mathfrak{sl}_2(\mathbb{R})$ (see **2.1**). It follows that the intersection $\mathbf{h} \cap \mathbf{g_3}$ is trivial since otherwise $\mathbf{m_2}$ contains an element which is conjugate to an element of $\mathbf{h}$. Therefore the Lie algebra $\mathbf{h}$ has the form $(\mathbf{h_1}, \varphi(\mathbf{h_1}))$, where $\mathbf{h_1}$ is a 4-dimensional subalgebra of $\mathbf{g_1} \oplus \mathbf{g_2}$ and $\varphi : \mathbf{h_1} \to \mathbf{g_3}$ is a homomorphism. The subgroup H_1 is isomorphic either to $\mathcal{L}_2 \times \mathcal{L}_2$ or to $G_1 \times A$, where A is a 1-dimensional subgroup of G_2 (cf. Lemma 11). Since H does not contain any normal subgroup of G, the group H contains in both cases elements of type $(a^{-1}, a, \varphi(a^{-1}, a))$; $(a^{-1}, a) \in H_1$ which are conjugate to an element of M.

Finally we consider the case that $\mathbf{m_2}$ is the tangent space of the simple symmetric space belonging to an involutory automorphism fixing a 1-dimensional elliptic subgroup elementwise (cf. Lemma 15 in [3, p. 379]). Then $\mathbf{m_2}$ contains only hyperbolic elements. We have dim $(H \cap G_3) \leq 1$ since otherwise $\mathbf{m_2}$ would contain an element which is conjugate to an element of $\mathbf{h}$.

In this part of the proof we use the natural projections $p_i : G \to G_i, i = 1, 2, 3,$ of G onto G_i.

First we assume that $H \cap G_3$ is a 1-dimensional subgroup of G_3. Since $H \cap G_3$ is normal in $p_3(H)$ one has dim $p_3(H) \leq 2$.

For dim $p_3(H) = 1$ we have $p_3(H) = H \cap G_3$ and $H = H_1 \times (H \cap G_3)$ with $H_1 < G_1 \times G_2$. In this case one has $L = L_1 \times L_2$ (cf. [12], Lemma 1.19, p. 28), where L_1 is a loop $\widehat{L}$ of Theorem 5 with $G_1 \times G_2$ as the group topologically generated by the left translations and L_2 is a loop isotopic to the hyperbolic plane loop $\mathbb{H}_2$ with G_3 as the group generated by the left translations (cf. [12, Section 22]). This yields the case (ii) of the assertion.

If dim $p_3(H) = 2$, then $p_3(H)$ is isomorphic to $\mathcal{L}_2$ and at least one of the groups $p_i(H)$, $i = 1, 2$, has dimension 2. If both $p_1(H)$ and $p_2(H)$ are isomorphic to $\mathcal{L}_2$, then H is isomorphic to the direct product of three groups $\mathcal{L}_2$ with amalgamated commutator factor subgroups (cf. **2.2**). In this case H contains an element of M. If only one of the projections $p_i(H), i = 1, 2,$ has dimension 2, then we may assume that $p_1(H)$ is isomorphic to $\mathcal{L}_2$ and $p_2(H)$ has dimension 1. It follows that H is isomorphic either to $H' = (A, 1, B) \times (1, C, 1)$, where $A \times B$ is isomorphic to $\mathcal{L}_2 \times \mathcal{L}_2$ with amalgamated commutator factor subgroups (see **2.2**) and $C = p_2(H)$, or $H'' = (F, 1, 1) \times (1, \psi(S), S)$, where F and S are isomorphic to $\mathcal{L}_2$ and $\psi(S) = p_2(H)$ is the image of a homomorphism $\psi : p_3(H) \to G_2$ having the commutator subgroup of $\mathcal{L}_2$ as the kernel.

If $p_2(H)$ is a parabolic or hyperbolic subgroup of G_2, then H contains an element which is conjugate to an element of M. If $p_2(H)$ is an elliptic subgroup of G_2, then using the real basis $\{e_1, e_2, e_3\}$ of $\mathfrak{sl}_2(\mathbb{R})$ given by **2.1** the Lie algebra $\mathbf{h}'$ of the stabilizer H' has the form

$$\langle V_1 = (e_1, 0, e_1), V_2 = (e_2 + e_3, 0, 0), V_3 = (0, e_3, 0), V_4 = (0, 0, e_2 + e_3)\rangle$$

and the Lie algebra $\mathbf{h}''$ of the stabilizer H'' has the form

$$\langle Z_1 = (e_1, 0, 0), Z_2 = (e_2 + e_3, 0, 0), Z_3 = (0, e_3, e_1), Z_4 = (0, 0, e_2 + e_3)\rangle.$$

The tangent space $\mathbf{m}$ of M is defined by

$$(**)\quad \mathbf{m} = \langle X_1 = (e_1, -e_1, 0), X_2 = (e_2, -e_2, 0), X_3 = (e_3, -e_3, 0), \\ X_4 = (0, 0, e_1), X_5 = (0, 0, e_2)\rangle.$$

Since $\mathbf{g} = \mathbf{h}' \oplus \mathbf{m} = \mathbf{h}'' \oplus \mathbf{m}$ any element of $\mathbf{g}$ has a representation of the form $Y = \sum_{i=1}^{4} \lambda_i V_i + \sum_{j=1}^{5} \nu_j X_j$ as well as $W = \sum_{i=1}^{4} \lambda_i Z_i + \sum_{j=1}^{5} \nu_j X_j$ with suitable $\lambda_i, \nu_j \in \mathbb{R}$. The elements $Y = (0, 0, Y_3)$ and $W = (0, 0, W_3)$ contained in $\mathbf{g_3}$ have both the form

$$(0, 0, \lambda_4(e_2 + e_3) + \nu_4 e_1 + \nu_5 e_2), \text{ with suitable } \lambda_4, \nu_4, \nu_5 \in \mathbb{R}.$$

The Cartan-Killing form k of $\mathfrak{sl}_2(\mathbb{R})$ yields $k(Y_3) = k(W_3) = \nu_4^2 + \nu_5^2 \geq 0$. It follows that $\mathbf{g_3}$ does not contain elliptic elements which is a contradiction.

Let now $H \cap G_3 = \{1\}$. Then the loop L is realized on the factor space $G_1 \times G_2 \times G_3/(H_1, \varphi(H_1))$, where H_1 is a 4-dimensional subgroup of $G_1 \times G_2$ and $\varphi : H_1 \to G_3$ is a homomorphism. In this case H_1 is isomorphic either to the direct product $\mathcal{L}_2 \times \mathcal{L}_2$ or to the direct product $G_1 \times A$, where A is a 1-dimensional subgroup of G_2 (cf. Lemma 11).

If H_1 contains $\mathcal{L}_2 \times \mathcal{L}_2$ or A is hyperbolic, then there is an element of H which is conjugate to an element of M.

If A is a parabolic subgroup of G_2, then using the real basis of $\mathfrak{sl}_2(\mathbb{R})$ given in **2.1** the Lie algebra $\mathbf{h}$ of H has the form

$$\mathbf{h} = \langle U_1 = (e_1, 0, e_1), U_2 = (e_2, 0, e_2), U_3 = (e_3, 0, e_3), U_4 = (0, e_2 + e_3, 0)\rangle.$$

According to $(**)$ any element of $\mathbf{g} = \mathbf{h} \oplus \mathbf{m}$ can be represented as $K = \sum_{i=1}^{4} \lambda_i U_i + \sum_{j=1}^{5} \nu_j X_j$, where $\lambda_i, \nu_j \in \mathbb{R}$. The elements $K = (0, K_2, 0)$ contained in $\mathbf{g_2}$ have the form $(0, \lambda_1 e_1 + \lambda_2 e_2 + \lambda_4(e_2 + e_3), 0)$ with suitable $\lambda_1, \lambda_2, \lambda_4 \in \mathbb{R}$. The Cartan-Killing form k of K_2 satisfies $k(K_2) = \lambda_1^2 + \lambda_2^2 \geq 0$. It follows that $\mathbf{g_2}$ does not contain elliptic element which is a contradiction.

If A is a 1-dimensional elliptic subgroup of G_2 and G_1 is isomorphic to $PSL_2(\mathbb{R})$, then $H = (G_1 \times A, \varphi(G_1 \times A))$, where $\varphi(G_1 \times A) \cong PSL_2(\mathbb{R})$ and the corresponding Bol loop L is $\check{L}$ given in Example. The structure of the coverings of $\check{L}$ and their groups generated by the left translations are given in Remark 10. This is the case (iii) of the assertion. □

If L falls into (ii) in Theorem 13, then the group G^* topologically generated by the left translations of the universal covering $\widetilde{L}$ of L is isomorphic to $\widetilde{G_1} \times G_1 \times G_1$, where G_1 is the group $PSL_2(\mathbb{R})$, the group $\widetilde{G_1}$ is the universal covering of G_1 and the stabilizer H^* of $e \in \widetilde{L}$ has the form $(\mathcal{L}_2 \times SO_2(\mathbb{R})) \times SO_2(\mathbb{R})$.

Let L be a Scheerer extension of a connected simple Lie group S by S (cf. the case (i) in Theorem 13). The image of the section belonging to L has the form $\mathcal{Q} = \{(y, \varphi(y^{-1}), x);\ x, y \in S\}$, where φ is a covering homomorphism. Then $\mathcal{Q}$ generates the group $S \times \varphi(S) \times S$. The group G^* topologically generated by the left translations of the universal covering $\widetilde{L}$ of L is isomorphic to $(\widetilde{S} \times \varphi(S) \times \widetilde{S})/N$, where $\widetilde{S}$ is the universal covering of S. The central normal subgroup N has the form $N = \{(z, 1, z);\ z \in Z\}$, where Z is the centre of $\widetilde{S}$. Moreover, the stabilizer H^* of $e \in \widetilde{L}$ is the group $H^* = \{(x, 1, x);\ x \in \widetilde{S}\}/N$.

Let L be a Scheerer extension of a covering group G_3 of $PSL_2(\mathbb{R})$ by a Bol loop $\widehat{L}$ of Theorem 5 falling into (i) in Theorem 13. If G^* is the group topologically generated by the left translations of the universal covering $\widetilde{L}$ of L, then G^* has the shape $G^* = \widetilde{G_3} \times G_2 \times \widetilde{G_3}$, where $\widetilde{G_3}$ is the universal covering of the group $PSL_2(\mathbb{R})$ and G_2 is isomorphic to $PSL_2(\mathbb{R})$.

Sincere thanks to the referee for the careful reading of our manuscript and for many improvements.

The first author was supported by the Hungarian Scientific Research Fund (OTKA) Grant PD 77392.

References

[1] Conway, J.H., Curtis, R.T., Norton, S.P., Parker, R.A., Wilson, R.A.: Atlas of Finite Groups. Clarendon Press, Oxford (1985)
[2] Figula, Á.: 3-dimensional Bol loops as sections in non-solvable Lie groups. Forum Math. **17**(3), 431–460 (2005)
[3] Figula, Á.: Bol loops as sections in semi-simple Lie groups of small dimension. Manuscr. Math. **121**, 367–384 (2006)
[4] Hilgert, J., Hofmann, K.H.: Old and new on $SL_2(\mathbb{R})$. Manuscr. Math. **54**, 17–52 (1985)
[5] Hu, S.-T.: Homotopy Theory. Academic Press, New York (1959)
[6] Huppert, B.: Endliche Gruppen I. Springer, Berlin (1967)
[7] Jones, G.A.: Cyclic regular subgroups of primitive permutation groups. J. Group Theory **5**, 403–407 (2002)
[8] Koszul, J.L.: Variante d'un théoréme de H. Ozeki. Osaka J. Math. **15**, 547–551 (1978)
[9] Loos, O.: Symmetric Spaces, vol. I. Benjamin, New York (1969)
[10] Miheev, P.O., Sabinin, L.V.: Quasigroups and differential geometry. In: Chein, O., Pflugfelder, H.O., Smith, J.D.H. (eds.) Quasigroups and Loops: Theory and Applications. Sigma Series in Pure Math. 8, pp. 357–430. Heldermann-Verlag, Berlin (1990)
[11] Nagy, G.P.: A class of simple proper Bol loops. Manuscr. Math. **127**, 81–88 (2002)
[12] Nagy, P.T., Strambach, K.: Loops in Group Theory and Lie Theory. de Gruyter Expositions in Mathematics, 35. Berlin, New York (2002)
[13] Nazaryan, R.O.: Minimal factorization of simple real Lie groups (Russian). Izv. Akad. Nauk Arm. SSR, **10**(5), 455–477 (1975)
[14] Onishchik, A.L.: Decompositions of reductive Lie groups (Russian). Mat. Sb., Nov. Ser. **80**(4), 553–599. English transl.: Math. USSR, Sb. **9**, 515–554 (1969)
[15] Ozeki, H.: On a transitive transformation group of a compact group manifold. Osaka J. Math. **14**, 519–531 (1977)
[16] Tits, J.: Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen. Lecture Notes in Mathematics, 40. Springer, Berlin (1967)