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Loops on spheres having a compact-free inner mapping group

By

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Abstract. We prove that any topological loop homeomorphic to a sphere or to a real projective space and having a compact-free Lie group as the inner mapping group is homeomorphic to the circle. Moreover, we classify the differentiable 1-dimensional compact loops explicitly using the theory of Fourier series.

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Introduction

The only known proper topological compact connected loops such that the groups G topologically generated by their left translations are locally compact and the stabilizers H of their identities in G have no non-trivial compact subgroups are homeomorphic to the 1-sphere. In [8], [9], [7], [10] it is shown that the differentiable 1-dimensional loops can be classified by pairs of real functions which satisfy a differential inequality containing these functions and their first derivatives. A main goal of this paper is to determine the functions satisfying this inequality explicitly in terms of Fourier series.

If L is a topological loop homeomorphic to a sphere or to a real projective space and having a Lie group G as the group topologically generated by the left translations such that the stabilizer of the identity of L is a compact-free Lie subgroup of G, then L is the 1-sphere and G is isomorphic to a finite covering of the group $PSL_2(\mathbb{R})$ (cf. Theorem 4).

To decide which sections $\sigma : G/H \to G$, where G is a Lie group and H is a (closed) subgroup of G containing no normal subgroup $\neq 1$ of G correspond to loops we use systematically a theorem of Baer (cf. [3] and [8], Proposition 1.6, p. 18). This statement says that σ corresponds to a loop if and only if the image $\sigma(G/H)$ is also the image for any section $G/H^a \to G$, where $H^a = a^{-1}Ha$ and $a \in G$. As one of the applications of this we derive in a different way the differential inequality in [8], p. 238, in which the necessary and sufficient conditions for the existence of 1-dimensional differentiable loops are hidden.

Basic facts in loop theory

A set L with a binary operation $(x,y) \mapsto x*y : L \times L \to L$ and an element $e \in L$ such that $e*x = x*e = x$ for all $x \in L$ is called a loop if for any given $a,b \in L$ the equations $a*y = b$ and $x*a = b$ have unique solutions which we denote by $y = a\backslash b$ and $x = b/a$. Every left translation $\lambda_a : y \mapsto a*y : L \to L$, $a \in L$, is a bijection of L and the set $\Lambda = \{\lambda_a,\ a \in L\}$ generates a group G such that Λ forms a system of representatives for the left cosets $\{xH,\ x \in G\}$, where H is the stabilizer of $e \in L$ in G. Moreover, the elements of Λ act on $G/H = \{xH,\ x \in G\}$ such that for any given cosets aH and bH there exists precisely one left translation λ_z with $\lambda_z aH = bH$.

Conversely, let G be a group, H be a subgroup containing no normal subgroup $\neq 1$ of G and let $\sigma : G/H \to G$ be a section with $\sigma(H) = 1 \in G$ such that the set $\sigma(G/H)$ of representatives for the left cosets of H in G generates G and acts sharply transitively on the space G/H (cf. [8], p. 18). Such a section we call a sharply transitive section. Then the multiplication defined by $xH * yH = \sigma(xH)yH$ on the factor space G/H or by $x*y = \sigma(xyH)$ on $\sigma(G/H)$ yields a loop $L(\sigma)$. The group G is isomorphic to the group generated by the left translations of $L(\sigma)$.

We call the group generated by the mappings $\lambda_{x,y} = \lambda_{xy}^{-1}\lambda_x\lambda_y : L \to L$, for all $x,y \in L$, the inner mapping group of the loop L (cf. [8], Definition 1.30, p. 33). According to Lemma 1.31 in [8], p. 33, this group coincides with the stabilizer H of the identity of L in the group generated by the left translations of L.

A locally compact loop L is almost topological if it is a locally compact space and the multiplication $* : L \times L \to L$ is continuous. Moreover, if the maps $(a,b) \mapsto b/a$ and $(a,b) \mapsto a\backslash b$ are continuous, then L is a topological loop. An (almost) topological loop L is connected if and only if the group topologically generated by the left translations is connected. We call the loop L strongly almost topological if the group topologically generated by its left translations is locally compact and the corresponding sharply transitive section $\sigma : G/H \to G$, where H is the stabilizer of $e \in L$ in G, is continuous.

If a loop L is a connected differentiable manifold such that the multiplication $* : L \times L \to L$ is continuously differentiable, then L is an almost $\mathscr{C}^1$-differentiable loop (cf. Definition 1.24 in [8], p. 31). Moreover, if the mappings $(a,b) \mapsto b/a$ and $(a,b) \mapsto a\backslash b$ are also continuously differentiable, then the loop L is a $\mathscr{C}^1$-differentiable loop. If an almost $\mathscr{C}^1$-differentiable loop has a Lie group G as the group topologically generated by its left translations, then the sharply transitive section $\sigma : G/H \to G$ is $\mathscr{C}^1$-differentiable. Conversely, any continuous, respectively $\mathscr{C}^1$-differentiable sharply transitive section $\sigma : G/H \to G$ yields an almost topological, respectively an almost $\mathscr{C}^1$-differentiable loop.

It is known that for any (almost) topological loop L homeomorphic to a connected topological manifold there exists a universal covering loop $\widetilde{L}$ such that the covering mapping $p : \widetilde{L} \to L$ is an epimorphism. The inverse image $p^{-1}(e) = \mathrm{Ker}(p)$ of the identity element e of L is a central discrete subgroup Z of $\widetilde{L}$ and it is naturally isomorphic to the fundamental group of L. If Z' is a subgroup of Z, then the factor loop $\widetilde{L}/Z'$ is a covering loop of L and any covering loop of L is isomorphic to a factor loop $\widetilde{L}/Z'$ with a suitable subgroup Z' (see [5]).

If L' is a covering loop of L, then Lemma 1.34 in [8], p. 34, clarifies the relation between the group topologically generated by the left translations of L' and the group topologically generated by the left translations of L:

Let L be a topological loop homeomorphic to a connected topological manifold. Let the group G topologically generated by the left translations λ_a, $a \in L$, of L be a Lie group. Let $\widetilde{L}$ be the universal covering of L and $Z \subseteq \widetilde{L}$ be the fundamental group of L. Then the group $\widetilde{G}$ topologically generated by the left translations $\widetilde{\lambda}_u, u \in \widetilde{L}$, of $\widetilde{L}$ is the covering group of G such that the kernel of the covering mapping $\varphi : \widetilde{G} \to G$ is $Z^* = \{\widetilde{\lambda}_z, z \in Z\}$ and Z^* is isomorphic to Z. If we identify $\widetilde{L}$ and L with the homogeneous spaces $\widetilde{G}/\widetilde{H}$ and G/H, where H or $\widetilde{H}$ is the stabilizer of the identity of L in G or of $\widetilde{L}$ in $\widetilde{G}$, respectively, then $\varphi(\widetilde{H}) = H$, $\widetilde{H} \cap Z^* = \{1\}$, and $\widetilde{H}$ is isomorphic to H.

Compact topological loops on the 3-dimensional sphere

Proposition 1. *There is no almost topological proper loop L homeomorphic to the 3-sphere $\mathscr{S}_3$ or to the 3-dimensional real projective space $\mathscr{P}_3$ such that the group G topologically generated by the left translations of L is isomorphic to the group $SL_2(\mathbb{C})$ or to the group $PSL_2(\mathbb{C})$, respectively.*

Proof. We assume that there is an almost topological loop L homeomorphic to $\mathscr{S}_3$ such that the group topologically generated by its left translations is isomorphic to $G = SL_2(\mathbb{C})$. Then there exists a continuous sharply transitive section $\sigma : SL_2(\mathbb{C})/H \to SL_2(\mathbb{C})$, where H is a connected compact-free 3-dimensional subgroup of $SL_2(\mathbb{C})$. According to [2], pp. 273–278, there is a one-parameter family of connected compact-free 3-dimensional subgroups H_r, $r \in \mathbb{R}$, of $SL_2(\mathbb{C})$ such that H_{r_1} is conjugate to H_{r_2} precisely if $r_1 = r_2$. Hence we may assume that the stabilizer H has one of the folowing shapes

$$H_r = \left\{ \begin{pmatrix} \exp[(ri-1)a] & b \\ 0 & \exp[(1-ri)a] \end{pmatrix}; a \in \mathbb{R}, b \in \mathbb{C} \right\}, \quad r \in \mathbb{R},$$

(cf. Theorem 1.11 in [8], p. 21). For each $r \in \mathbb{R}$ the section $\sigma_r : G/H_r \to G$ corresponding to a loop L_r is given by

$$\begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix} H_r \mapsto \begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix} \begin{pmatrix} \exp[(ri-1)f(x,y)] & g(x,y) \\ 0 & \exp[(1-ri)f(x,y)] \end{pmatrix},$$

where $x, y \in \mathbb{C}$, $x\bar{x} + y\bar{y} = 1$ such that $f(x,y) : S^3 \to \mathbb{R}$, $g(x,y) : S^3 \to \mathbb{C}$ are continuous functions with $f(1,0) = 0 = g(1,0)$. Since σ_r is a sharply transitive section for each $r \in \mathbb{R}$ the image $\sigma_r(G/H_r)$ forms a system of representatives for all cosets xH_r^{γ}, $\gamma \in G$. This means for all given $c, d \in \mathbb{C}^2$, $c\bar{c} + d\bar{d} = 1$ each coset

$$\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \begin{pmatrix} c & d \\ -\bar{d} & \bar{c} \end{pmatrix} H_r \begin{pmatrix} \bar{c} & -d \\ \bar{d} & c \end{pmatrix},$$

where $u, v \in \mathbb{C}$, $u\bar{u} + v\bar{v} = 1$, contains precisely one element of $\sigma_r(G/H_r)$. This is the case if and only if for all given $c, d, u, v \in \mathbb{C}$ with $u\bar{u} + v\bar{v} = 1 = c\bar{c} + d\bar{d}$ there exists a unique triple $(x, y, q) \in \mathbb{C}^3$ with $x\bar{x} + y\bar{y} = 1$ and a real number m such that the following matrix equation holds:

$$\begin{pmatrix} \bar{u}\bar{c} - \bar{v}d & -ud - v\bar{c} \\ \bar{v}c + \bar{u}\bar{d} & uc - v\bar{d} \end{pmatrix} \begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix} \begin{pmatrix} \exp[(ri-1)f(x,y)] & g(x,y) \\ 0 & \exp[(1-ri)f(x,y)] \end{pmatrix}$$
$$= \begin{pmatrix} \exp[(ri-1)m] & q \\ 0 & \exp[(1-ri)m] \end{pmatrix} \begin{pmatrix} \bar{c} & -d \\ \bar{d} & c \end{pmatrix}. \tag{1}$$

The (1, 1)- and (2, 1)-entry of the matrix equation (1) give the following system A of equations:

$$[(\bar{u}x + v\bar{y})\bar{c} + (u\bar{y} - \bar{v}x)d] \exp[(ri-1)f(x,y)] = \exp[(ri-1)m]\bar{c} + q\bar{d} \tag{2}$$

$$[(\bar{v}x - u\bar{y})c + (\bar{u}x + v\bar{y})\bar{d}] \exp[(ri-1)f(x,y)] = \exp[(1-ri)m]\bar{d}. \tag{3}$$

If we take c and d as independent variables, the system A yields the following system B of equations:

$$(\bar{u}x + v\bar{y}) \exp[irf(x,y)] \exp[-f(x,y)] = \exp(irm) \exp(-m) \tag{4}$$

$$(u\bar{y} - \bar{v}x) \exp[(ri-1)f(x,y)]d = \bar{d}q \tag{5}$$

$$(\bar{u}x + v\bar{y}) \exp[irf(x,y)] \exp[-f(x,y)] = \exp(m) \exp(-irm). \tag{6}$$

Since Eq. (5) must be satisfied for all $d \in \mathbb{C}$ we obtain $q = 0$. From Eq. (4) it follows

$$\bar{u}x + v\bar{y} = \exp(irm) \exp(-m) \exp[-irf(x,y)] \exp[f(x,y)]. \tag{7}$$

Putting (7) into (6) one obtains

$$\exp(irm) \exp(-m) = \exp(m) \exp(-irm) \tag{8}$$

which is equivalent to

$$\exp[2(ir-1)m] = 1. \tag{9}$$

Equation (9) is satisfied if and only if $m = 0$. Hence the matrix equation (1) reduces to the matrix equation

$$\begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix} \begin{pmatrix} \exp[(ri-1)f(x,y)] & g(x,y) \\ 0 & \exp[(1-ri)f(x,y)] \end{pmatrix} = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$$

and therefore the matrix

$$M = \begin{pmatrix} \exp[(ri-1)f(x,y)] & g(x,y) \\ 0 & \exp[(1-ri)f(x,y)] \end{pmatrix}$$

is an element of $SU_2(\mathbb{C})$. This is the case if and only if $f(x,y) = 0 = g(x,y)$ for all $(x,y) \in \mathbb{C}^2$ with $x\bar{x} + y\bar{y} = 1$. Since for each $r \in \mathbb{R}$ the loop L_r is isomorphic to the loop $L_r(\sigma_r)$, hence to the group $SU_2(\mathbb{C})$, there is no connected almost topological proper loop L homeomorphic to $\mathscr{S}_3$ such that the group topologically generated by its left translations is isomorphic to the group $SL_2(\mathbb{C})$.

The universal covering of an almost topological proper loop L homeomorphic to the real projective space $\mathscr{P}_3$ is an almost topological proper loop

$\widetilde{L}$ homeomorphic to $\mathscr{S}_3$. If the group topologically generated by the left translations of L is isomorphic to $PSL_2(\mathbb{C})$, then the group topologically generated by the left translations of $\widetilde{L}$ is isomorphic to $SL_2(\mathbb{C})$. Since no proper loop $\widetilde{L}$ exists the Proposition is proved. □

Proposition 2. *There is no almost topological proper loop L homeomorphic to the 3-dimensional real projective space $\mathscr{P}_3$ or to* the *3-sphere $\mathscr{S}_3$ such that the group G topologically generated by the left translations of L is isomorphic to the group $SL_3(\mathbb{R})$ or to the universal covering group $\widetilde{SL_3(\mathbb{R})}$, respectively.*

Proof. First we assume that there exists an almost topological loop L homeomorphic to $\mathscr{P}_3$ such that the group topologically generated by its left translations is isomorphic to $G = SL_3(\mathbb{R})$. Then there is a continuous sharply transitive section $\sigma : SL_3(\mathbb{R})/H \to SL_3(\mathbb{R})$, where H is a connected compact-free 5-dimensional subgroup of $SL_3(\mathbb{R})$. According to Theorem 2.7, p. 187, in [4] and to Theorem 1.11, p. 21, in [8] we may assume that

$$H = \left\{ \begin{pmatrix} a & k & v \\ 0 & b & l \\ 0 & 0 & (ab)^{-1} \end{pmatrix} ; a > 0, b > 0, k, l, v \in \mathbb{R} \right\}. \tag{10}$$

Using Euler angles, every element of $SO_3(\mathbb{R})$ can be represented by the following matrix

$$g(t,u,z) := \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos z & \sin z \\ 0 & -\sin z & \cos z \end{pmatrix} \begin{pmatrix} \cos u & \sin u & 0 \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos t \cos u - \sin t \cos z \sin u & \cos t \sin u + \sin t \cos z \cos u & \sin t \sin z \\ -\sin t \cos u - \cos t \cos z \sin u & -\sin t \sin u + \cos t \cos z \cos u & \cos t \sin z \\ \sin z \sin u & -\sin z \cos u & \cos z \end{pmatrix},$$

where $t, u \in [0, 2\pi]$ and $z \in [0, \pi]$.

The section $\sigma : SL_3(\mathbb{R})/H \to SL_3(\mathbb{R})$ is given by

$$g(t,u,z)H \mapsto g(t,u,z) \begin{pmatrix} f_1(t,u,z) & f_2(t,u,z) & f_3(t,u,z) \\ 0 & f_4(t,u,z) & f_5(t,u,z) \\ 0 & 0 & f_1^{-1}(t,u,z) f_4^{-1}(t,u,z) \end{pmatrix}, \tag{11}$$

where $t, u \in [0, 2\pi]$, $z \in [0, \pi]$ and $f_i(t,u,z) : [0, 2\pi] \times [0, 2\pi] \times [0, \pi] \to \mathbb{R}$ are continuous functions such that for $i \in \{1, 4\}$ the functions f_i are positive with $f_i(0,0,0) = 1$ and for $j = \{2, 3, 5\}$ the functions $f_j(t,u,z)$ satisfy that $f_j(0,0,0) = 0$. As σ is sharply transitive the image $\sigma(SL_3(\mathbb{R})/H)$ forms a system of representatives for all cosets xH^δ, $\delta \in SL_3(\mathbb{R})$. Since the elements x and δ can be chosen in the group $SO_3(\mathbb{R})$ we may take x as the matrix

$$\begin{pmatrix} \cos q \cos r - \sin q \sin r \cos p & \cos q \sin r + \sin q \cos r \cos p & \sin q \sin p \\ -\sin q \cos r - \cos q \sin r \cos p & -\sin q \sin r + \cos q \cos r \cos p & \cos q \sin p \\ \sin p \sin r & -\sin p \cos r & \cos p \end{pmatrix}$$

and δ as the matrix

$$\begin{pmatrix} \cos\alpha\,\cos\beta-\sin\alpha\,\sin\beta\,\cos\gamma & \cos\alpha\,\sin\beta+\sin\alpha\,\cos\beta\,\cos\gamma & \sin\alpha\,\sin\gamma \\ -\sin\alpha\,\cos\beta-\cos\alpha\,\sin\beta\,\cos\gamma & -\sin\alpha\,\sin\beta+\cos\alpha\,\cos\beta\,\cos\gamma & \cos\alpha\,\sin\gamma \\ \sin\gamma\,\sin\beta & -\sin\gamma\,\cos\beta & \cos\gamma \end{pmatrix},$$

where $q, r, \alpha, \beta \in [0, 2\pi]$ and $p, \gamma \in [0, \pi]$. The image $\sigma(SL_3(\mathbb{R})/H)$ forms for all given $\delta \in SO_3(\mathbb{R})$ and $x \in SO_3(\mathbb{R})$ a system of representatives for the cosets xH^δ if and only if there exists unique angles $t, u \in [0, 2\pi]$ and $z \in [0, \pi]$ and unique positive real numbers a, b as well as unique real numbers k, l, v such that the following equation holds

$$\delta x^{-1} g(t,u,z) f = h\delta, \tag{12}$$

where the matrices $\delta, x, g(t,u,z)$ have the form as above,

$$f = \begin{pmatrix} f_1(t,u,z) & f_2(t,u,z) & f_3(t,u,z) \\ 0 & f_4(t,u,z) & f_5(t,u,z) \\ 0 & 0 & f_1^{-1}(t,u,z) f_4^{-1}(t,u,z) \end{pmatrix}$$

and

$$h = \begin{pmatrix} a & k & v \\ 0 & b & l \\ 0 & 0 & (ab)^{-1} \end{pmatrix}.$$

Comparing the first column of the left and the right side of the Eq. (12) we obtain the following three equations:

$$\begin{aligned} f_1(t,u,z)\{&[(\cos\alpha\,\cos\beta-\sin\alpha\,\sin\beta\,\cos\gamma)(\cos r\,\cos q-\sin r\,\sin q\,\cos p) \\ &+(\cos\alpha\,\sin\beta+\sin\alpha\,\cos\beta\,\cos\gamma)(\sin r\,\cos q+\cos r\,\sin q\,\cos p) \\ &+\sin\alpha\sin\gamma\sin p\sin q](\cos t\,\cos u-\sin t\,\sin u\,\cos z) \\ &-[-(\cos\alpha\,\cos\beta-\sin\alpha\,\sin\beta\,\cos\gamma)(\cos r\,\sin q+\sin r\,\cos q\,\cos p) \\ &+(\cos\alpha\,\sin\beta+\sin\alpha\,\cos\beta\,\cos\gamma)(-\sin r\,\sin q+\cos r\,\cos q\,\cos p) \\ &+\sin\alpha\sin\gamma\sin p\cos q](\sin t\,\cos u+\cos t\,\sin u\,\cos z) \\ &+[(\cos\alpha\,\cos\beta-\sin\alpha\,\sin\beta\,\cos\gamma)\sin r\,\sin p \\ &-(\cos\alpha\,\sin\beta+\sin\alpha\,\cos\beta\,\cos\gamma)\cos r\sin p+\sin\alpha\,\sin\gamma\,\cos p]\sin z\,\sin u\} \\ =a(&\cos\alpha\,\cos\beta-\sin\alpha\,\sin\beta\,\cos\gamma)-k(\sin\alpha\,\cos\beta+\cos\alpha\,\sin\beta\,\cos\gamma) \\ &+v\sin\gamma\,\sin\beta, \end{aligned}$$

$$\begin{aligned} f_1(t,u,z)\{&[-(\sin\alpha\,\cos\beta+\cos\alpha\,\sin\beta\,\cos\gamma)(\cos r\,\cos q-\sin r\,\sin q\,\cos p) \\ &-(-\sin\alpha\,\sin\beta+\cos\alpha\,\cos\beta\,\cos\gamma)(\sin r\,\cos q+\cos r\,\sin q\,\cos p) \\ &+\cos\alpha\sin\gamma\sin p\,\sin q](\cos t\,\cos u-\sin t\,\sin u\,\cos z) \\ &-[(\sin\alpha\,\cos\beta+\cos\alpha\,\sin\beta\,\cos\gamma)(\cos r\,\sin q+\sin r\,\cos q\,\cos p) \\ &+(-\sin\alpha\,\sin\beta+\cos\alpha\,\cos\beta\,\cos\gamma)(-\sin r\,\sin q+\cos r\,\cos q\,\cos p) \\ &+\cos\alpha\sin\gamma\sin p\cos q](\sin t\,\cos u+\cos t\,\sin u\,\cos z) \end{aligned}$$

$$+ [-(\sin\alpha \ \cos\beta + \cos\alpha \ \sin\beta \ \cos\gamma)\sin r \ \sin p - (\cos\alpha \ \cos\beta \ \cos\gamma$$
$$- \sin\alpha \ \sin\beta)\cos r \sin p + \cos\alpha \ \sin\gamma \ \cos p]\sin z \ \sin u\}$$
$$= -b(\sin\alpha \ \cos\beta + \cos\alpha \ \sin\beta \ \cos\gamma) + l\sin\gamma \ \sin\beta,$$

$$f_1(t,u,z)\{[(\cos r \ \cos q - \sin r \ \sin q \ \cos p)\sin\gamma \ \sin\beta$$
$$- (\sin r \ \cos q + \cos r \ \sin q \ \cos p)\sin\gamma \ \cos\beta + \cos\gamma \sin p \sin q]$$
$$\times (\cos t \ \cos u - \sin t \ \sin u \ \cos z) + [(\cos r \ \sin q + \sin r \ \cos q \ \cos p)\sin\gamma \ \sin\beta$$
$$+ (-\sin r \ \sin q + \cos r \ \cos q \ \cos p)\sin\gamma \ \cos\beta - \cos\gamma \sin p \cos q]$$
$$\times (\sin t \ \cos u + \cos t \ \sin u \ \cos z)$$
$$+ [(\sin\gamma \ \sin\beta \ \sin r \ \sin p + \sin\gamma \ \cos\beta \ \cos r \ \sin p) + \cos\gamma \cos p]\sin z \ \sin u\}$$
$$= (ab)^{-1}\sin\gamma \ \sin\beta.$$

If we take $\sin\gamma \ \sin\beta$ and $\cos\gamma$ as independent variables the third equation turns to the following equations

$$0 = f_1(t,u,z)[\sin p \ \sin q(\cos t \ \cos u - \sin t \ \sin u \ \cos z)$$
$$- \sin p \ \cos q(\sin t \ \cos u + \cos t \ \sin u \ \cos z) + \cos p \ \sin z \ \sin u] \quad (13)$$
$$(ab)^{-1} = \{[(\cos r \ \cos q - \sin r \ \sin q \ \cos p)(\cos t \ \cos u - \sin t \ \sin u \ \cos z)$$
$$+ (\cos r \ \sin q + \sin r \ \cos q \ \cos p)(\sin t \ \cos u + \cos t \ \sin u \ \cos z)$$
$$+ \sin r \ \sin p \ \sin z \ \sin u] - \frac{\cos\beta}{\sin\beta}[(\sin r \ \cos q + \cos r \ \sin q \ \cos p)$$
$$\times (\cos t \ \cos u - \sin t \ \sin u \ \cos z) - (-\sin r \ \sin q + \cos r \ \cos q \ \cos p)$$
$$\times (\sin t \ \cos u + \cos t \ \sin u \ \cos z) - \cos r \ \sin p \ \sin z \ \sin u]\}f_1(t,u,z). \quad (14)$$

If we take $\cos\alpha \ \sin\beta \ \cos\gamma$ and $\sin\beta \ \sin\gamma$ as independent variables it follows from the second equation that

$$l = \frac{\cos\alpha}{\sin\beta}f_1(t,u,z)[\sin p \ \sin q(\cos t \ \cos u - \sin t \ \sin u \ \cos z)$$
$$- \sin p \ \cos q(\sin t \ \cos u + \cos t \ \sin u \ \cos z) + \cos p \ \sin z \ \sin u] \quad (15)$$
$$-b = \{[-(\cos r \ \cos q - \sin r \ \sin q \ \cos p)(\cos t \ \cos u - \sin t \ \sin u \ \cos z)$$
$$- (\cos r \ \sin q + \sin r \ \cos q \ \cos p)(\sin t \ \cos u + \cos t \ \sin u \ \cos z)$$
$$- \sin r \ \sin p \ \sin z \ \sin u] - \frac{\cos\beta}{\sin\beta}[(\sin r \ \cos q + \cos r \ \sin q \ \cos p)$$
$$\times (\cos t \ \cos u - \sin t \ \sin u \ \cos z) - (-\sin r \ \sin q + \cos r \ \cos q \ \cos p)$$
$$\times (\sin t \ \cos u + \cos t \ \sin u \ \cos z) - \cos r \ \sin p \ \sin z \ \sin u]\}f_1(t,u,z). \quad (16)$$

If we choose $\sin\alpha \ \sin\beta \ \cos\gamma$, $\sin\beta \ \sin\gamma$ as independent variables the first equation yields

$$v = \frac{\sin\alpha}{\sin\beta}f_1(t,u,z)[\sin p \ \sin q(\cos t \ \cos u - \sin t \ \sin u \ \cos z)$$
$$- \sin p \ \cos q(\sin t \ \cos u + \cos t \ \sin u \ \cos z) + \cos p \ \sin z \ \sin u] \quad (17)$$

$$a + k\frac{\cos\alpha}{\sin\alpha} = \{[(\cos r\ \cos q - \sin r\ \sin q\ \cos p)(\cos t\ \cos u - \sin t\ \sin u\ \cos z) - (\cos r\ \sin q + \sin r\ \cos q\ \cos p)(\sin t\ \cos u + \cos t\ \sin u\ \cos z) + \sin r\ \sin p\ \sin z\ \sin u] - \frac{\cos\beta}{\sin\beta}[(\sin r\ \cos q + \cos r\ \sin q\ \cos p) \times (\cos t\ \cos u - \sin t\ \sin u\ \cos z) - (-\sin r\ \sin q + \cos r\ \cos q\ \cos p) \times (\sin t\ \cos u + \cos t\ \sin u\ \cos z) - \cos r\ \sin p\ \sin z\ \sin u]\} f_1(t,u,z). \tag{18}$$

Since $f_1(t,u,z) > 0$ it follows from Eq. (13) that

$$0 = \sin p\ \sin q(\cos t\ \cos u - \sin t\ \sin u\ \cos z) - \sin p\ \cos q(\sin t\ \cos u + \cos t\ \sin u\ \cos z) + \cos p\ \sin z\ \sin u. \tag{19}$$

Using this, it follows from (15) that $l = 0$ holds and from Eq. (17) that $v = 0$. Since the Eq. (14) must be satisfied for all $\beta \in [0, 2\pi]$, we have

$$(ab)^{-1} = [(\cos r\ \cos q - \sin r\ \sin q\ \cos p)(\cos t\ \cos u - \sin t\ \sin u\ \cos z) + (\cos r\ \sin q + \sin r\ \cos q\ \cos p)(\sin t\ \cos u + \cos t\ \sin u\ \cos z) + \sin r\ \sin p\ \sin z\ \sin u] f_1(t,u,z) \tag{20}$$

$$0 = [(\sin r\ \cos q + \cos r\ \sin q\ \cos p)(\cos t\ \cos u - \sin t\ \sin u\ \cos z) - (-\sin r\ \sin q + \cos r\ \cos q\ \cos p)(\sin t\ \cos u + \cos t\ \sin u\ \cos z) - \cos r\ \sin p\ \sin z\ \sin u]. \tag{21}$$

Using Eq. (21) and comparing the Eqs. (20) and (16), we obtain that $(ab)^{-1} = b$. With Eq. (21) the Eq. (18) turns to

$$a + k\frac{\cos\alpha}{\sin\alpha} = [(\cos r\ \cos q - \sin r\ \sin q\ \cos p)(\cos t\ \cos u - \sin t\ \sin u\ \cos z) - (\cos r\ \sin q + \sin r\ \cos q\ \cos p)(\sin t\ \cos u + \cos t\ \sin u\ \cos z) + \sin r\ \sin p\ \sin z\ \sin u] f_1(t,u,z). \tag{22}$$

Since the Eq. (22) must be satisfied for all $\alpha \in [0, 2\pi]$, we obtain $k = 0$. Using this, the Eqs. (22) and (20) yield $(ab)^{-1} = a$. Since $1 = ab(ab)^{-1} = a^3$ it follows that $a = 1$ and hence the matrix h is the identity. But then the matrix equation (12) turns to the matrix equation

$$g(t,u,z)f = x.$$

As x and $g(t,u,z)$ are elements of $SO_3(\mathbb{R})$ one has $f = g^{-1}(t,u,z)x \in SO_3(\mathbb{R})$. But then f is the identity, which means that

$$f_1(t,u,z) = 1 = f_4(t,u,z), \quad f_2(t,u,z) = f_3(t,u,z) = f_5(t,u,z) = 0,$$

for all $t, u \in [0, 2\pi]$ and $z \in [0, \pi]$. Since the loop L is isomorphic to the loop $L(\sigma)$ and $L(\sigma) \cong SO_3(\mathbb{R})$ there is no connected almost topological proper loop L homeomorphic to $\mathcal{P}_3$ such that the group topologically generated by its left translations is isomorphic to $SL_3(\mathbb{R})$.

Now we assume that there is an almost topological loop L homeomorphic to $\mathcal{S}_3$ such that the group G topologically generated by its left translations is isomorphic to the universal covering group $\widetilde{SL_3(\mathbb{R})}$. Then the stabilizer H of the identity of L may be chosen as the group (10). Then there exists a local section $\sigma : U/H \to G$, where U is a suitable neighbourhood of H in G/H which has the shape (11) with sufficiently small $t, u \in [0, 2\pi]$, $z \in [0, \pi]$ and continuous functions $f_i(t, u, z) : [0, 2\pi] \times [0, 2\pi] \times [0, \pi] \to \mathbb{R}$ satisfying the same conditions as there. The image $\sigma(U/H)$ is a local section for the space of the left cosets $\{xH^\delta; x \in G, \delta \in G\}$ precisely if for all suitable matrices $x := g(q, r, p)$ with sufficiently small $(q, r, p) \in [0, 2\pi] \times [0, 2\pi] \times [0, \pi]$ there exist a unique element $g(t, u, z) \in \mathrm{Spin}_3(\mathbb{R})$ with sufficiently small $(t, u, z) \in [0, 2\pi] \times [0, 2\pi] \times [0, \pi]$ and unique positive real numbers a, b as well as unique real numbers k, l, v such that the matrix Eq. (12) holds. Then we see as in the case of the group $SL_3(\mathbb{R})$ that for small x and $g(t, u, z)$ the matrix f is the identity. Therefore any subloop T of L which is homeomorphic to $\mathcal{S}_1$ is locally commutative. Then according to [8], Corollary 18.19, p. 248, each subloop T is isomorphic to a 1-dimensional torus group. It follows that the restriction of the matrix f to T is the identity. Since L is covered by such 1-dimensional tori the matrix f is the identity for all elements of $\mathcal{S}_3$. Hence there is no proper loop L homeomorphic to $\mathcal{S}_3$ such that the group G topologically generated by its left translations is isomorphic to the universal covering group $\widetilde{SL_3(\mathbb{R})}$. □

Compact loops with compact-free inner mapping groups

Proposition 3. *Let L be an almost topological loop homeomorphic to a compact connected Lie group K. Then the group G topologically generated by the left translations of L cannot be isomorphic to a split extension of a solvable group R homeomorphic to* $\mathbb{R}^n$ $(n \geqslant 1)$ *by the group K.*

Proof. Denote by H the stabilizer of the identity of L in G. If G has the structure as in the assertion, then the elements of G can be represented by the pairs (k, r) with $k \in K$ and $r \in R$. Since L is homeomorphic to K the loop L is isomorphic to the loop $L(\sigma)$ given by a sharply transitive section $\sigma : G/H \to G$ the image of which is the set $\mathfrak{S} = \{(k, f(k));\ k \in K\}$, where f is a continuous function from K into R with $f(1) = 1 \in R$. The multiplication of $(L(\sigma), *)$ on $\mathfrak{S}$ is given by $(x, f(x)) * (y, f(y)) = \sigma((xy, f(x)f(y))H)$.

Let T be a 1-dimensional torus of K. Then the set $\{(t, f(t));\ t \in T\}$ topologically generates a compact subloop $\widetilde{T}$ of $L(\sigma)$ such that the group topologically generated by its left translations has the shape TU with $T \cap U = 1$, where U is a normal solvable subgroup of TU homeomorphic to $\mathbb{R}^n$ for some $n \geqslant 1$. The multiplication $*$ in the subloop $\widetilde{T}$ is given by

$$(x, f(x)) * (y, f(y)) = \sigma((xy, f(x)f(y))H) = (xy, f(xy)),$$

where $x, y \in T$. Hence $\widetilde{T}$ is a subloop homeomorphic to a 1-sphere which has a solvable Lie group S as the group topologically generated by the left translations. It follows that $\widetilde{T}$ is a 1-dimensional torus group since otherwise the group S would be not solvable (cf. [8], Proposition 18.2, p. 235). As $f : \widetilde{T} \to U$ is a homomorphism and U is homeomorphic to $\mathbb{R}^n$ it follows that the restriction of f to $\widetilde{T}$ is the constant function $f(\widetilde{T}) = 1$. Since the exponential map of a compact group is surjective any element of K is contained in a one-parameter subgroup of K. It follows $f(K) = 1$ and L is the group K which is a contradiction. □

Theorem 4. *Let L be an almost topological proper loop homeomorphic to a sphere or to a real projective space. If the group G topologically generated by the left translations of L is a Lie group and the stabilizer H of the identity of L in G is a compact-free subgroup of G, then L is homeomorphic to the* 1*-sphere and G is a finite covering of the group* $PSL_2(\mathbb{R})$.

Proof. If dim $L = 1$ then according to Brouwer's theorem (cf. [11], 96.30, p. 639) the transitive group G on $\mathcal{S}_1$ is a finite covering of $PSL_2(\mathbb{R})$.

Now let dim $L > 1$. Since the universal covering of the n-dimensional real projective space is the n-sphere $\mathcal{S}_n$ we may assume that L is homeomorphic to $\mathcal{S}_n$, $n \geqslant 2$. Since L is a multiplication with identity e on $\mathcal{S}_n$ one has $n \in \{3, 7\}$ (cf. [1]).

Any maximal compact subgroup K of G acts transitively on L (cf. [11], 96.19, p. 636). As $H \cap K = \{1\}$ the group K operates sharply transitively on L. Since there is no compact group acting sharply transitively on the 7-sphere (cf. [11], 96.21, p. 637), the loop L is homeomorphic to the 3-sphere. The only compact group homeomorphic to the 3-sphere is the unitary group $SU_2(\mathbb{C})$. If the group G were not simple, then G would be a semidirect product of the at most 3-dimensional solvable radical R with the group $SU_2(\mathbb{C})$ (cf. [4], p. 187 and Theorem 2.1, p. 180). But according to Proposition 3 such a group cannot be the group topologically generated by the left translations of L. Hence G is a non-compact Lie group the Lie algebra of which is simple. But then G is isomorphic either to the group $SL_2(\mathbb{C})$ or to the universal covering of the group $SL_3(\mathbb{R})$. It follows from Proposition 1 and 2 that no of these groups can be the group topologically generated by the left translations of an almost topological proper loop L. □

The classification of 1-dimensional compact connected $\mathscr{C}^1$-loops

If L is a connected strongly almost topological 1-dimensional compact loop, then L is homeomorphic to the 1-sphere and the group topologically generated by its left translations is a finite covering of the group $PSL_2(\mathbb{R})$ (cf. Proposition 18.2 in [8], p. 235). We want to classify explicitly all 1-dimensional $\mathscr{C}^1$-differentiable compact connected loops which have either the group $PSL_2(\mathbb{R})$ or $SL_2(\mathbb{R})$ as the group topologically generated by the left translations.

First we classify the 1-dimensional compact connected loops having $G = SL_2(\mathbb{R})$ as the group topologically generated by their left translations. Since the stabilizer H is compact-free and may be chosen as the group of upper triangular matrices (see Theorem 1.11, in [8], p. 21) this is equivalent to the classification of all loops $L(\sigma)$ belonging to the sharply transitive $\mathscr{C}^1$-differentiable sections

$$\sigma : \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}; a>0, b\in\mathbb{R} \right\} \\ \to \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} f(t) & g(t) \\ 0 & f^{-1}(t) \end{pmatrix} \quad \text{with } t\in\mathbb{R}. \tag{23}$$

Definition 1. Let $\mathscr{F}$ be the set of series

$$a_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt), \quad t\in\mathbb{R},$$

such that

$$1 - a_0 = \sum_{k=1}^{\infty} \frac{a_k + kb_k}{1+k^2}, \tag{i}$$

$$a_0 > \sum_{k=1}^{\infty} \frac{ka_k - b_k}{1+k^2} \sin kt - \frac{a_k + kb_k}{1+k^2} \cos kt \quad \text{for all } t\in[0,2\pi], \tag{ii}$$

$$2a_0 \geqslant \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \frac{k^2-1}{k^2+1}. \tag{iii}$$

Remark. The conditions (i) and (iii) of Definition 1 are equivalent to the condition

$$\sum_{k=1}^{\infty} \frac{(a_k^2+b_k^2)(k^2-1) + 2(a_k+kb_k)}{1+k^2} \leqslant 2. \tag{iv}$$

With $a_0 = 1 - \sum_{k=1}^{\infty} \frac{(a_k+kb_k)}{1+k^2}$ if a_k, b_k are non-negative, $b_k \leqslant ka_k$ for all $k \geqslant 1$ and

$$\sum_{k=1}^{\infty} \frac{(k+2)a_k + (2k-1)b_k}{1+k^2} < 1, \tag{v}$$

the inequality (ii) is satisfied since from (v) it follows

$$\sum_{k=1}^{\infty} \frac{(ka_k - b_k)}{1+k^2} + \sum_{k=1}^{\infty} \frac{(a_k+kb_k)}{1+k^2} < a_0$$

and

$$\left| \sum_{k=1}^{\infty} \frac{ka_k - b_k}{1+k^2} \sin kt - \frac{a_k+kb_k}{1+k^2} \cos kt \right| \leqslant a_0 \quad \text{for all } t\in[0,2\pi].$$

In particular, taking $\sum_{k=1}^{\infty} a_k \leqslant \frac{2}{5}$, $b_k = \frac{a_k}{k}$, we see that the inequalities (iv) and (v) are satisfied. Hence the set $\mathscr{F}$ contains a multitude of trigonometric series.

Lemma 5. *The set $\mathscr{F}$ consists of Fourier series of continuous functions.*

Proof. Since $\sum_{k=2}^{\infty} a_k^2 + b_k^2 < \frac{10}{3} a_0$ it follows from [14], p. 4, that any series in $\mathcal{F}$ converges uniformly to a continuous function f and hence it is the Fourier series of f (cf. [14], Theorem 6.3, p. 12). $\square$

Let σ be a sharply transitive section of the shape (23). Then $f(t)$, $g(t)$ are periodic continuously differentiable functions $\mathbb{R} \to \mathbb{R}$, such that $f(t)$ is strictly positive with $f(2k\pi) = 1$ and $g(2k\pi) = 0$ for all $k \in \mathbb{Z}$.

As σ is sharply transitive the image $\sigma(G/H)$ forms a system of representatives for the cosets xH^ρ for all $\rho \in G$ (cf. [3]). All conjugate groups H^ρ can be already obtained if ρ is an element of

$$K = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, t \in \mathbb{R} \right\}.$$

Since $K^\kappa H^\kappa = KH^\kappa$ for any $\kappa \in K$ the group K forms a system of representatives for the left cosets xH^κ.

We want to determine the left coset $x(t)H^\kappa$ containing the element

$$\varphi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} f(t) & g(t) \\ 0 & f^{-1}(t) \end{pmatrix},$$

where

$$\kappa = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \quad \text{and} \quad x(t) = \begin{pmatrix} \cos\ \eta(t) & \sin\ \eta(t) \\ -\sin\ \eta(t) & \cos\ \eta(t) \end{pmatrix}.$$

The element $\varphi(t)$ lies in the left coset $x(t)H^\kappa$ if and only if $\varphi(t)^{\kappa^{-1}} \in x(t)^{\kappa^{-1}} H = x(t)H$. Hence we have to solve the following matrix equation

$$\begin{aligned} &\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \left[\kappa \begin{pmatrix} f(t) & g(t) \\ 0 & f^{-1}(t) \end{pmatrix} \kappa^{-1} \right] \\ &= \begin{pmatrix} \cos\ \eta(t) & \sin\ \eta(t) \\ -\sin\ \eta(t) & \cos\ \eta(t) \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \end{aligned} \tag{24}$$

for suitable $a > 0, b \in \mathbb{R}$. Comparing both sides of the matrix equation (24) we have

$$\begin{aligned} &f(t)\cos\beta(\sin t\cos\beta - \cos t\sin\beta) - g(t)\sin\beta(\sin t\cos\beta - \cos t\sin\beta) \\ &\quad + f(t)^{-1}\sin\beta(\sin t\sin\beta + \cos t\cos\beta) = \sin\eta(t)a \end{aligned}$$

and

$$\begin{aligned} &f(t)\cos\beta(\cos t\cos\beta + \sin t\sin\beta) - g(t)\sin\beta(\cos t\cos\beta + \sin t\sin\beta) \\ &\quad + f(t)^{-1}\sin\beta(\cos t\sin\beta - \sin t\cos\beta) = \cos\eta(t)a. \end{aligned}$$

From this it follows that

$$\tan\eta_\beta(t) = \frac{(f(t) - g(t)\tan\beta)(\tan t - \tan\beta) + f^{-1}(t)\tan\beta(1 + \tan t\tan\beta)}{(f(t) - g(t)\tan\beta)(1 + \tan t\tan\beta) + f^{-1}(t)\tan\beta(\tan\beta - \tan t)}.$$

Since β can be chosen in the intervall $0 \leqslant \beta < \frac{\pi}{2}$ and $\frac{\pi}{2} < \beta < \pi$, we may replace the parameter $\tan\beta$ by any $w \in \mathbb{R}$.

A $\mathscr{C}^1$-differentiable loop L corresponding to σ exists if and only if the function $t \mapsto \eta_w(t)$ is strictly increasing, i.e. if $\eta_w'(t) > 0$ (cf. Proposition 18.3, p. 238, in [8]). The function $a_w(t) : t \mapsto \tan \eta_w(t) : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is strictly increasing if and only if $\eta_w'(t) > 0$ since

$$\frac{d}{dt} \tan(\eta_w(t)) = \frac{1}{\cos^2(\eta_w(t))} \eta_w'(t).$$

A straightforward calculation shows that

$$\begin{aligned} \frac{d}{dt} \tan(\eta_w(t)) = \frac{w^2+1}{\cos^2(t)} [& w^2(g'(t)f(t) + g(t)f'(t) + g^2(t)f^2(t) + 1) \\ & + w(-2f(t)f'(t) - 2g(t)f^3(t)) + f^4(t)]. \end{aligned} \tag{25}$$

Hence the loop $L(\sigma)$ exists if and only if for all $w \in \mathbb{R}$ the inequality

$$\begin{aligned} 0 < & w^2(g'(t)f(t) + g(t)f'(t) + g^2(t)f^2(t) + 1) \\ & + w(-2f(t)f'(t) - 2g(t)f^3(t)) + f^4(t) \end{aligned} \tag{26}$$

holds. For $w = 0$ the expression (26) equals to $f^4(t) > 0$. Therefore the inequality (26) is satisfied for all $w \in \mathbb{R}$ if and only if one has

$$f'^2(t) + g(t)f^2(t)f'(t) - g'(t)f^3(t) - f^2(t) < 0 \quad \text{and} \quad g'(0) < f'^2(0) - 1 \tag{27}$$

for all $t \in \mathbb{R}$. Putting $f(t) = \hat{f}^{-1}(t)$ and $g(t) = -\hat{g}(t)$ these conditions are equivalent to the conditions

$$\hat{f}'^2(t) + \hat{g}(t)\hat{f}'(t) + \hat{g}'(t)\hat{f}(t) - \hat{f}^2(t) < 0 \quad \text{and} \quad \hat{g}'(0) < 1 - \hat{f}'^2(0) \tag{28}$$

(cf. [8], Section 18, (C), p. 238).

Now we treat the differential inequality (28). The solution $h(t)$ of the linear differential equation

$$h'(t) + h(t)\frac{\hat{f}'(t)}{\hat{f}(t)} + \frac{\hat{f}'^2(t)}{\hat{f}(t)} - \hat{f}(t) = 0 \tag{29}$$

with the initial conditions $h(0) = 0$ and $h'(0) = 1 - \hat{f}'^2(0)$ is given by

$$h(t) = \hat{f}(t)^{-1} \int_0^t (\hat{f}^2(u) - \hat{f}'^2(u)) du.$$

Since $\hat{g}(0) = h(0) = 0$ and $\hat{g}'(0) < h'(0)$ it follows from VI in [13] (p. 66) that $\hat{g}(t)$ is a subfunction of the differential equation (29), i.e. that $\hat{g}(t)$ satisfies the differential inequality (28). Moreover, according to Theorem V in [13] (p. 65) one has $\hat{g}(t) < h(t)$ for all $t \in (0, 2\pi)$. Since the functions $\hat{g}(t)$ and $h(t)$ are continuous $0 = \hat{g}(2\pi) \leqslant h(2\pi)$. This yields the following integral inequality

$$\int_0^{2\pi} (\hat{f}^2(t) - \hat{f}'^2(t)) dt \geqslant 0. \tag{30}$$

We consider the real function $R(t)$ defined by $R(t) = \hat{f}(t) - \hat{f}'(t)$. Since $\hat{f}(0) = \hat{f}(2\pi) = 1$ and $\hat{f}'(0) = \hat{f}'(2\pi)$ we have $R(0) = 1 - \hat{f}'(0) = 1 - \hat{f}'(2\pi) = R(2\pi)$.

The linear differential equation

$$y'(t) - y(t) + R(t) = 0 \quad \text{with } y(0) = 1 \tag{31}$$

has the solution

$$y(t) = e^t(1 - \int_0^t R(u)e^{-u}du). \tag{32}$$

This solution is unique (cf. [6], p. 2) and hence it is the function $\hat{f}(t)$. The condition $\hat{f}(2\pi) = 1$ is satisfied if and only if $\int_0^{2\pi} R(u)e^{-u}du = 1 - \frac{1}{e^{2\pi}}$. Since $R(t)$ has periode 2π its Fourier series is given by

$$a_0 + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt), \tag{33}$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} R(t)\, dt$, $a_k = \frac{1}{\pi}\int_0^{2\pi} R(t)\cos kt\, dt$, and $b_k = \frac{1}{\pi}\int_0^{2\pi} R(t)\sin kt\, dt$. Partial integration yields

$$\int_0^t \sin ku\ e^{-u}du = \frac{k - k\cos kt\ e^{-t} - \sin kt\ e^{-t}}{1+k^2} \tag{34}$$

$$\int_0^t \cos ku\ e^{-u}du = \frac{1 + k\sin kt\ e^{-t} - \cos kt\ e^{-t}}{1+k^2}. \tag{35}$$

Using (34) and (35), we obtain by partial integration

$$\begin{aligned}\int_0^t R(u)e^{-u}\, du &= a_0 - a_0e^{-t} + \sum_{k=1}^{\infty}\left[\int_0^t a_k \cos ku\ e^{-u}du + \int_0^t b_k \sin ku\ e^{-u}du\right]\\ &= a_0 - a_0e^{-t} + \sum_{k=1}^{\infty}\frac{a_k(1 + k\sin kt\ e^{-t} - \cos kt\ e^{-t})}{1+k^2}\\ &\quad + \frac{b_k(k - k\cos kt\ e^{-t} - \sin kt\ e^{-t})}{1+k^2}.\end{aligned} \tag{36}$$

Now, for the real coefficients a_0, a_k, b_k $(k \geqslant 1)$ it follows that

$$1 - \frac{1}{e^{2\pi}} = \int_0^{2\pi} R(u)e^{-u}du = \left(a_0 + \sum_{k=1}^{\infty}\frac{a_k + kb_k}{1+k^2}\right)\left(1 - \frac{1}{e^{2\pi}}\right).$$

Hence one has

$$a_0 + \sum_{k=1}^{\infty}\frac{a_k + kb_k}{1+k^2} = 1. \tag{37}$$

The function $\hat{f}(t)$ is positive if and only if

$$1 > \int_0^t R(u)e^{-u}du \quad \text{for all } t \in [0, 2\pi]. \tag{38}$$

Applying (34) and (35) again we see that the inequality (38) is equivalent to

$$a_0 > \sum_{k=1}^{\infty}\left[\frac{a_k k - b_k}{1+k^2}\sin kt - \frac{a_k + b_k k}{1+k^2}\cos kt\right]. \tag{39}$$

Since $\hat{f}'(t)+\hat{f}(t)=2e^t\big(1-\int_0^t R(u)e^{-u}du\big)-R(t)$ the function $\hat{f}(t)$ satisfies the integral inequality (30) if and only if

$$\int_0^{2\pi} R(t)\left[2e^t\left(1-\int_0^t R(u)e^{-u}du\right)-R(t)\right]dt \geqslant 0. \tag{40}$$

The left side of (40) can be written as

$$2\int_0^{2\pi} R(t)e^t dt - 2\int_0^{2\pi} R(t)e^t\left(\int_0^t R(u)e^{-u}du\right)dt - \int_0^{2\pi} R^2(t)dt. \tag{41}$$

Using partial integration and representing $R(u)$ by a Fourier series (33) we have

$$\int_0^{2\pi} R(t)e^t dt = \left(a_0+\sum_{k=1}^{\infty}\frac{a_k-b_k k}{1+k^2}\right)(e^{2\pi}-1). \tag{42}$$

From (36) it follows

$$\begin{aligned}
&\int_0^{2\pi} R(t)e^t\left(\int_0^t R(u)e^{-u}du\right)dt\\
&\quad = a_0\int_0^{2\pi} R(t)e^t dt - a_0\int_0^{2\pi} R(t)dt + \sum_{k=1}^{\infty}\int_0^{2\pi}\left(\frac{a_k+kb_k}{1+k^2}\right)R(t)e^t dt\\
&\qquad + \sum_{k=1}^{\infty}\int_0^{2\pi}\left(\frac{ka_k-b_k}{1+k^2}\right)R(t)\sin kt\ dt - \sum_{k=1}^{\infty}\int_0^{2\pi}\left(\frac{a_k+kb_k}{1+k^2}\right)R(t)\cos kt\ dt. \qquad (43)
\end{aligned}$$

Substituting for $R(t)$ its Fourier series and applying the relation (a) in [12] (p. 10) we have

$$\int_0^{2\pi} R(t)dt = 2\pi a_0.$$

Futhermore, one has

$$\begin{aligned}
&\sum_{k=1}^{\infty}\int_0^{2\pi}\left(\frac{ka_k-b_k}{1+k^2}\right)R(t)\sin kt\ dt\\
&\quad = \sum_{k=1}^{\infty}\int_0^{2\pi}\left(\frac{ka_k-b_k}{1+k^2}\right)\left[a_0+\sum_{l=1}^{\infty}(a_l\cos lt+b_l\sin lt)\right]\sin kt\ dt\\
&\quad = a_0\sum_{k=1}^{\infty}\int_0^{2\pi}\left(\frac{ka_k-b_k}{1+k^2}\right)\sin kt\ dt + \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\int_0^{2\pi}\left(\frac{ka_k-b_k}{1+k^2}\right)a_l\cos lt\ \ \sin kt\ dt\\
&\qquad + \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\int_0^{2\pi}\left(\frac{ka_k-b_k}{1+k^2}\right)b_l\sin lt\ \ \sin kt\ dt.
\end{aligned}$$

The relations (a), (b), (c), (d) in [12], p. 10, yield

$$\begin{aligned}
\sum_{k=1}^{\infty}\int_0^{2\pi}\left(\frac{ka_k-b_k}{1+k^2}\right)R(t)\sin kt\ dt &= \sum_{k=1}^{\infty}\int_0^{2\pi}\left(\frac{ka_k-b_k}{1+k^2}\right)b_k\sin^2 kt\ dt\\
&= \sum_{k=1}^{\infty}\left(\frac{ka_k-b_k}{1+k^2}\right)b_k\pi.
\end{aligned}$$

Analogously we obtain that

$$\sum_{k=1}^{\infty}\int_0^{2\pi}\left(\frac{a_k+kb_k}{1+k^2}\right)R(t)\cos kt\ dt=\sum_{k=1}^{\infty}\int_0^{2\pi}\left(\frac{ka_k+b_k}{1+k^2}\right)b_k\cos^2 kt\ dt$$
$$=\sum_{k=1}^{\infty}\left(\frac{a_k+kb_k}{1+k^2}\right)a_k\pi.$$

Using the equality (37) one has

$$\int_0^{2\pi}R(t)e^t\left(\int_0^t R(u)e^{-u}du\right)dt$$
$$=\left[a_0+\sum_{k=1}^{\infty}\frac{a_k-kb_k}{1+k^2}\right](e^{2\pi}-1)-\pi\sum_{k=1}^{\infty}\frac{b_k^2+a_k^2}{1+k^2}-2\pi a_0^2. \tag{44}$$

Substituting for $R(t)$ its Fourier series we have

$$\int_0^{2\pi}R^2(t)\ dt=\int_0^{2\pi}a_0^2\ dt+2a_0\sum_{k=1}^{\infty}\int_0^{2\pi}(a_k\cos kt\ +b_k\sin kt)\ dt$$
$$-\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\int_0^{2\pi}(a_ka_l\cos kt\cos lt+a_kb_l\cos kt\sin lt$$
$$+b_ka_l\sin kt\cos lt+b_kb_l\sin kt\sin lt)\ dt.$$

Applying the relations (a), (b), (c), (d) in [12] (p. 10) we obtain

$$\int_0^{2\pi}R^2(t)\ dt=2\pi a_0^2+\pi\sum_{k=1}^{\infty}(a_k^2+b_k^2).$$

Hence the integral inequality (30) holds if and only if

$$2a_0\geqslant\sum_{k=1}^{\infty}(a_k^2+b_k^2)\frac{k^2-1}{k^2+1}.$$

Since the Fourier series of $R(t)$ lies in the set $\mathscr{F}$ of series the Fourier series of R converges uniformly to R (Lemma 5).

Summarizing our discussion we obtain the main part of the following

Theorem 6. *Let L be a 1-dimensional connected $\mathscr{C}^1$-differentiable loop such that the group topologically generated by its left translations is isomorphic to the group $SL_2(\mathbb{R})$. Then L is compact and belongs to a $\mathscr{C}^1$-differentiable sharply transitive section σ of the form*

$$\sigma:\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\left\{\begin{pmatrix}a & b\\ 0 & a^{-1}\end{pmatrix};a>0,b\in\mathbb{R}\right\}$$
$$\to\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\begin{pmatrix}f(t) & g(t)\\ 0 & f^{-1}(t)\end{pmatrix}\quad \textit{with}\quad t\in\mathbb{R} \tag{45}$$

such that the inverse function f^{-1} *has the shape*

$$f^{-1}(t) = e^t\left(1 - \int_0^t R(u)e^{-u}\,du\right) = a_0 + \sum_{k=1}^{\infty} \frac{(ka_k - b_k)\sin kt + (a_k + kb_k)\cos kt}{1+k^2}, \tag{46}$$

where $R(u)$ *is a continuous function the Fourier series of which is contained in the set* $\mathscr{F}$ *and converges uniformly to* R, *and* g *is a periodic* $\mathscr{C}^1$*-differentiable function with* $g(0) = g(2\pi) = 0$ *such that*

$$g(t) > -f(t)\int_0^t \frac{(f^2(u) - f'^2(u))}{f^4(u)}\,du \quad \textit{for all } t \in (0, 2\pi). \tag{47}$$

Conversely, if $R(u)$ *is a continuous function the Fourier series of which is contained in* $\mathscr{F}$, *then the section* σ *of the form* (45) *belongs to a loop iff* f *is defined by* (46) *and* g *is a* $\mathscr{C}^1$*-differentiable periodic function with* $g(0) = g(2\pi) = 0$ *satisfying* (47).

The isomorphism classes of loops defined by σ *are in one-to-one correspondence to the* 2*-sets* $\{(f(t), g(t)), (f(-t), -g(-t))\}$.

Proof. The only part of the assertion which has to be discussed is the isomorphism question. It follows from [7], Theorem 3, p. 3, that any isomorphism class of the loops L contains precisely two pairs (f_1, g_1) and (f_2, g_2). If $(f_1, g_1) \neq (f_2, g_2)$ and if (f_1, g_1) satisfy the inequality (27), then we have

$$f_2'^2(-t) + g_2(-t)f_2^2(-t)f_2'(-t) - g_2'(-t)f_2^3(-t) - f_2^2(-t) < 0$$

since from $f_1(t) = f_2(-t)$ and $g_1(t) = -g_2(-t)$ we have $f_1'(t) = -f_2'(-t)$ and $g_1'(t) = g_2'(-t)$. □

Remark. A loop $\widetilde{L}$ belonging to a section σ of shape (45) is a 2-covering of a $\mathscr{C}^1$-differentiable loop L having the group $PSL_2(\mathbb{R})$ as the group topologically generated by the left translations if and only if for the functions f and g one has $f(\pi) = 1$ and $g(\pi) = 0$ (cf. [9], p. 5106). Moreover, L is the factor loop $\widetilde{L}/\left\{\left(\begin{smallmatrix}\epsilon & 0\\ 0 & \epsilon\end{smallmatrix}\right); \epsilon = \pm 1\right\}$. Any n-covering of L is a non-split central extension $\widehat{L}$ of the cyclic group of order n by L. The loop $\widehat{L}$ has the n-covering of $PSL_2(\mathbb{R})$ as the group topologically generated by its left translations.

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