

3-dimensional Bol loops as sections in non-solvable Lie groups

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1 Introduction

Differentiable diassociative loops can be treated following the ideas of Sophus Lie since for their tangential objects, binary Lie algebras, Lie's first theorem and Lie's third theorem are satisfied: To every binary Lie algebra there exists (up to local isomorphisms) a unique differentiable diassociative local loop. Kuzmin, Kerdman, and Nagy have proved that any local differentiable Moufang loop (these loops form a subclass of the class of diassociative loops) can uniquely be embedded into a connected simply connected global one (cf. [8], [12], [17]). This problem is more difficult for diassociative loops. Namely, there are local differentiable diassociative loops such that the corresponding global analytic loops do not exist (cf. [5]). In [5] the author gives in special cases the exact condition when a local diassociative loop can be embedded in a connected simply connected global one.

The present research on loops turns to such classes \mathcal{C} of differentiable loops which have local forms determined in a unique way by algebras the properties of which are deduced by the properties of the tangential objects of loops in the class \mathcal{C} . The most important class \mathcal{C} of such loops are the Bol loops. Their tangential objects, the Bol algebras, can be seen as Lie triple systems with an additional binary operation (cf. [18] pp. 84-86, Def. 6.10.). As known the Lie triple systems are in one-to-one correspondence to (global) simply connected symmetric spaces (cf. [14], [18] Sec. 6). Hence there is a strong connection between the theory of differentiable Bol loops and the theory of symmetric spaces. In particular the theory of connected differentiable Bruck loops (which form a subclass of the class of Bol loops) is essentially the theory of affine symmetric spaces (cf. [18] Sec. 11).

The 1-dimensional connected differentiable Bol loops are isomorphic either to the group \mathbb{R} or to $SO_2(\mathbb{R})$. The smallest connected differentiable proper Bol loops are realized on 2-dimensional manifolds. There exist precisely two isotopism classes of proper 2-dimensional connected differentiable global Bol

loops. As a representative of the one class we may choose the hyperbolic plane loop (cf. [18], Section 22); this loop is the unique Bruck loop in the isotopism class of loops, having the connected component of the group of hyperbolic motions as the group topologically generated by their left translations. As a representative of the other isotopism class may be chosen the 2-dimensional Bruck loop which is realized on the pseudo-euclidean affine plane such that its left translations group is the connected component of the group of pseudo-euclidean motions and the elements of L are the lines of positive slope in the pseudo-euclidean affine plane (cf. [18] Section 25).

Our aim in this paper is to classify the 3-dimensional connected differentiable global Bol loops, which have a non-solvable group as the group topologically generated by their left translations and to describe their relations to metric space geometries. The classification of global differentiable Bol loops significantly differs from the classification of local differentiable Bol loops. There exist much more local differentiable Bol loops than global differentiable Bol loops, as we will also show in this paper.

We treat the differentiable Bol loops as images of global differentiable sections $\sigma : G/H \rightarrow G$ such that for all $r, s \in \sigma(G/H)$ the element rsr lies in $\sigma(G/H)$, where H is the stabilizer of the identity e of L in G . Using the present theory of differentiable Bol loops it is not difficult to prove that G is four, five or sixdimensional. The images of sections for local Bol loops can be realized as the exponential images of Lie triple systems. But then we have to discuss which of these local Bol loops can be extended to global Bol loops. The results of our paper can summarized in the following

Theorem *There are precisely two isotopism classes \mathcal{C}_i , ($i = 1, 2$) of the 3-dimensional connected differentiable simple proper Bol loops L such that the group G generated by the left translations $\{\lambda_x; x \in L\}$ is a non-solvable Lie group.*

The class \mathcal{C}_1 consists of Bol loops having the simple Lie group $G = PSL_2(\mathbb{C})$ as the group topologically generated by their left translations and as the stabilizer H of $e \in L$ in G the group $SO_3(\mathbb{R})$.

Any loop in the class \mathcal{C}_1 can be characterized by a real parameter a , where $-1 < a < 1$. The loops L_a are isotopic to the hyperbolic space loop L_0 , which is realized on the hyperbolic space by the multiplication $x \cdot y = \tau_{e,x}(y)$, where $\tau_{e,x}$ is the hyperbolic translation moving e onto x . The loop L_0 is the only Bruck loop in \mathcal{C}_1 . The loops L_a and L_b in \mathcal{C}_1 are isomorphic if and only if the angles between the tangent 3-space \mathbf{m}_a , respectively \mathbf{m}_b and the tangent 3-space \mathbf{m}_0 are the same with respect to the Cartan-Killing form. This is the case if and only if $b = \pm a$.

The other class \mathcal{C}_2 of simple Bol loops consists of 3-dimensional connected

differentiable Bol loops such that the group G topologically generated by their left translations is the semidirect product $PSL_2(\mathbb{R}) \ltimes \mathbb{R}^3$, where the action of $PSL_2(\mathbb{R})$ on \mathbb{R}^3 is the adjoint action of $PSL_2(\mathbb{R})$ on its Lie algebra, and for the stabilizer H of $e \in L$ we have

$$\left\{ \left(\pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \begin{pmatrix} -x & y \\ y & x \end{pmatrix} \right); t \in [0, 2\pi], x, y \in \mathbb{R} \right\}.$$

Any loop in this class can be characterized by three real parameters a, b, c , with $a^2 + b^2 < 1$. In \mathcal{C}_2 there is only one isomorphism class of Bruck loops, which consists of the loops $\hat{L}_c = L_{0,0,c}$. As a representative of this isomorphism class may be chosen the pseudo-euclidean space loop $\hat{L} = L_{0,0,0}$, which is realized on the pseudo-euclidean affine space. The connected component of the group of pseudo-euclidean motions is the group topologically generated by the left translations of \hat{L} . The elements of \hat{L} are the planes on which the euclidean metric is induced. Two loops $L_{a,b,c}$ and $L_{a',b',c'}$ in the class \mathcal{C}_2 are isomorphic if and only if the angles between the tangent 3-space $\mathbf{m}_{a,b,c}$, respectively $\mathbf{m}_{a',b',c'}$ and the tangent 3-space $\hat{\mathbf{m}} = \mathbf{m}_{0,0,0}$ are the same with respect to the Cartan-Killing form. This is the case if and only if $a = \pm a'$. The non-simple 3-dimensional differentiable proper Bol loops are either the direct products of a 1-dimensional Lie group with a 2-dimensional Bol loop isotopic to the hyperbolic plane loop or the unique Scheerer extension of the Lie group $SO_2(\mathbb{R})$ by the 2-dimensional hyperbolic plane loop.

Another important class of differentiable loops which generalize the class of abelian groups is the class of left A-loops. These loops correspond strongly to reductive homogeneous spaces. We show that any 3-dimensional differentiable Bol loop which is also a left A-loop is either a Bruck loop or the unique Scheerer extension of the orthogonal group $SO_2(\mathbb{R})$ by the 2-dimensional hyperbolic plane loop (which is not a Bruck loop).

At the end of this paper we realize within the 3-dimensional pseudo-euclidean geometry on the set of euclidean planes differentiable loops which are not Bol loops but the corresponding section of which are given in a pure geometric manner.

2 Bol loops and their Bol algebras

A set L with a binary operation $(x, y) \mapsto x \cdot y$ is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution which we denote by $y = a \setminus b$ and $x = b / a$. The left translation $\lambda_a : y \mapsto a \cdot y : L \rightarrow L$ is a bijection of L for any $a \in L$. Two loops (L_1, \circ) and $(L_2, *)$ are called isotopic if there are three

bijections $\alpha, \beta, \gamma : L_1 \rightarrow L_2$ such that $\alpha(x) * \beta(y) = \gamma(x \circ y)$ holds for any $x, y \in L_1$. An isotopism is an equivalence relation. If $\alpha = \beta = \gamma$ then the isotopic loops (L_1, \circ) and $(L_2, *)$ are called isomorphic. Let (L_1, \cdot) and $(L_2, *)$ be two loops. The direct product $L = L_1 \times L_2 = \{(a, b) \mid a \in L_1, b \in L_2\}$ with the multiplication $(a_1, b_1) \circ (a_2, b_2) = (a_1 \cdot a_2, b_1 * b_2)$ is again a loop, which is called the direct product of L_1 and L_2 , and the loops (L_1, \cdot) , $(L_2, *)$ are subloops of (L, \circ) .

A loop L is called a Bol loop if for any two left translations λ_a, λ_b the product $\lambda_a \lambda_b \lambda_a$ is again a left translation of L .

Let G be the group generated by the left translations of L and let H be the stabilizer of $e \in L$ in the group G . The left translations of L form a subset of G acting on the cosets $\{xH; x \in G\}$ such that for any given cosets aH, bH there exists precisely one left translation λ_z with $\lambda_z aH = bH$.

Conversely let G be a group, H be a subgroup containing no normal non-trivial subgroup of G and $\sigma : G/H \rightarrow G$ be a section such that σ satisfies the following conditions:

1. The image $\sigma(G/H)$ forms a subset of G with $\sigma(H) = 1 \in G$.
2. $\sigma(G/H)$ generates G .
3. $\sigma(G/H)$ acts sharply transitively on the space G/H of the left cosets $\{xH, x \in G\}$ (cf. [18], p. 18).

Then the multiplication on the factor space G/H defined by $xH * yH = \sigma(xH)yH$ yields a loop $L(\sigma)$. This loop is a Bol loop if and only if for all $r, s \in \sigma(G/H)$ the element rsr is contained in $\sigma(G/H)$.

If L_1 and L_2 are Bol loops, then the direct product $L_1 \times L_2$ is again a Bol loop.

Proposition 1. *Let L be a loop and let G be the group generated by the left translations of L , and denote by H the stabilizer of $e \in L$ in G . If G and H are direct products $G = G_1 \times G_2$ and $H = H_1 \times H_2$ with $H_i \subset G_i$ ($i = 1, 2$) then the loop L is the direct product of two loops L_1 and L_2 , and L_i is isomorphic to a loop L_i^* having G_i as the group generated by its left translations and H_i as the corresponding stabilizer subgroup ($i = 1, 2$).*

In particular there is no 3-dimensional Bol loop L such that L is the direct product of a 1-dimensional and a 2-dimensional Bol loop and G is a 5- or 6-dimensional Lie group.

Proof. The first assertion is the Proposition 1.18. in [18]. The second assertion follows from the fact that a 1-dimensional Bol loop is a Lie group. Hence the dimension of the group G is at most 4. \square

If the elements of L are points of a differentiable manifold and the operations $(x, y) \mapsto x \cdot y$, $(x, y) \mapsto x/y$, $(x, y) \mapsto x \setminus y : L \times L \rightarrow L$ are differentiable

mappings then L is called a differentiable loop. Moreover the manifold L is parallelizable since the set of the left translations is sharply transitive. For a differentiable manifold L the group of all autohomeomorphisms of L becomes with respect to the compact-open topology a topological group. If L is a Bol loop then the group G topologically generated by the left translations within the group of autohomeomorphisms is a Lie group (cf. [18], p. 33; [16], pp. 414-416).

Every connected differentiable Bol loop is isomorphic to a Bol loop L realized on the factor space G/H , where G is a connected Lie group, H is a connected closed subgroup containing no normal subgroup $\neq \{1\}$ of G and the sharply transitive section σ of L is a differentiable map (cf. [18], p. 32).

Proposition 2. *If L a connected differentiable 3-dimensional Bol loop, then the pairs (G, H) for the factor space G/H must be different from the following cases:*

a) $G = SL_2(\mathbb{C})$ or $PSL_2(\mathbb{C})$, $H \in \{U_0, U_1\}$ respectively $H \in \{U_0/\mathbb{Z}_2, U_1/\mathbb{Z}_2\}$ with $U_r = \left\{ \begin{pmatrix} z & (r-1)w \\ -(r+1)\bar{w} & \bar{z} \end{pmatrix}; |z|^2 + (r^2-1)|w|^2 = 1 \right\}$.

b) G is locally isomorphic to $SO_3(\mathbb{R}) \times \mathbb{R}$ and H is a 1-dimensional subgroup of G .

c) The group $G = SO_3(\mathbb{R}) \times \mathbb{R}^3$ is the connected component of the euclidean motion group and H is a semidirect product of a 2-dimensional translation group \mathbb{R}^2 by a 1-dimensional rotation group $SO_2(\mathbb{R})$.

d) $G = SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$, and H is any 3-dimensional subgroup of G .

e) $G = SL_2(\mathbb{C})$ or $PSL_2(\mathbb{C})$ and $H = W_r$ respectively $W_r\mathbb{Z}_2/\mathbb{Z}_2$, where $W_r = \left\{ \begin{pmatrix} \exp((ri-1)x) & 0 \\ z & \exp(-(ri-1)x) \end{pmatrix}; x \in \mathbb{R}, z \in \mathbb{C} \right\}$ for $r \in \mathbb{R}$.

Proof. In the case b) every 1-dimensional subgroup H of G containing no non-trivial normal subgroup of G has one of the following shapes:

$$H_1 = \{K \times \{0\}\}, \quad \text{or} \quad H_2 = \{K \times \varphi(K)\},$$

where K is isomorphic to $SO_2(\mathbb{R})$ and φ is a non-trivial homomorphism.

In the case d) every 3-dimensional subgroup H of G , which does not contain any normal subgroup $\neq \{1\}$ of G is conjugate to $\{(a, a) | a \in SO_3(\mathbb{R})\}$.

In the cases a), c) and for H_1 in the case b) the factor space G/H is a topological product of spaces having as a factor the 2-sphere or the projective plane, which are non-parallelizable. In the cases d), e) and for H_2 in the case b) the factor space G/H is compact ([18], Section 16). \square

A real vector space V with a trilinear multiplication $(., ., .)$ is called a Lie triple system \mathcal{V} , if the following identities are satisfied:

$$\begin{aligned}
& (X, X, Y) = 0 \\
& (X, Y, Z) + (Y, Z, X) + (Z, X, Y) = 0 \\
(X, Y, (U, V, W)) = & ((X, Y, U), V, W) + (U, (X, Y, V), W) + (U, V, (X, Y, W)).
\end{aligned}$$

A Bol algebra A is a Lie triple system $(V, (., ., .))$ with a bilinear skew-symmetric operation $[[., .]], (X, Y) \mapsto [[X, Y]] : V \times V \rightarrow V$ such that the following identity is satisfied:

$$\begin{aligned}
& [[(X, Y, Z), W]] - [[(X, Y, W), Z]] + (Z, W, [[X, Y]]) \\
& - (X, Y, [[Z, W]]) + [[[[X, Y]], [[Z, W]]]] = 0.
\end{aligned}$$

With any connected differentiable Bol loop L we can associate a Bol algebra in the following way: Let G be the Lie group topologically generated by the left translations of L , and let $(\mathfrak{g}, [., .])$ be the Lie algebra of G . Denote by \mathfrak{h} the Lie algebra of the stabilizer H of $e \in L$ in G and by $\mathfrak{m} = T_1\sigma(G/H)$ the tangent space at $1 \in G$ of the image of the section $\sigma : G/H \rightarrow G$ corresponding to the Bol loop L . Then $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, $[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}] \subseteq \mathfrak{m}$ and \mathfrak{m} generates the Lie algebra \mathfrak{g} . The subspace \mathfrak{m} is the tangent Bol algebra of the Bol loop L the operations of which are defined by $(X, Y, Z) \mapsto [[X, Y], Z]$, $(X, Y) \mapsto [X, Y]_{\mathfrak{m}}$, where X, Y, Z are elements of the subspace \mathfrak{m} and $Z \mapsto Z_{\mathfrak{m}} : \mathfrak{g} \rightarrow \mathfrak{m}$ is the projection of \mathfrak{g} onto \mathfrak{m} along the subalgebra \mathfrak{h} . Miheev and Sabinin define for any Bol algebra A the standard enveloping Lie algebra of A (cf. [16], pp. 428-429). If L is a connected differentiable Bol loop having a Bol algebra A as its tangent Bol algebra, then the Lie algebra \mathfrak{g} of the Lie group G topologically generated by the left translations of L is isomorphic to the standard enveloping Lie algebra of A . If A is a n -dimensional Bol algebra, then the dimension of its standard enveloping Lie algebra is at most $n + \frac{n(n-1)}{2}$ (cf. [16], pp. 428-430).

From every triple $((\mathfrak{g}, [., .]), \mathfrak{h}, \mathfrak{m})$ we can construct in a canonical way a triple $((\mathfrak{g}^*, [., .]^*), \mathfrak{h}^*, \mathfrak{m})$ (cf. [16], pp. 424-425, and [18], Section 6). The Lie algebra $(\mathfrak{g}^*, [., .]^*)$ is the Lie algebra of the isometry group of a symmetric space and the Lie algebra \mathfrak{g} is an epimorphic image of \mathfrak{g}^* . Moreover the subspace $\mathfrak{m} \subset (\mathfrak{g}, [., .])$ determines a Lie triple system $A := (\mathfrak{m}, (., ., .))$, with $(., ., .) = [[., .], .]$. If the intersection of the centre of \mathfrak{g} with $[[\mathfrak{m}, \mathfrak{m}]]$ is trivial then the Lie algebra \mathfrak{g} is an epimorphic image of the Lie algebra of the isometry group for the symmetric space belonging to the Lie triple system A . Hence if $\mathfrak{g} = \mathfrak{g}^*$ then $\mathfrak{g} = \mathfrak{m} \oplus [[\mathfrak{m}, \mathfrak{m}]]$.

Remark. Let G be a connected Lie group and let \mathfrak{g} be its Lie algebra. For the classification of differentiable local Bol loops L having G as the group

topologically generated by the left translations such that $\mathfrak{g} = \mathfrak{m} \oplus [\mathfrak{m}, \mathfrak{m}]$ for the subspace \mathfrak{m} corresponding to the tangential space of L holds we may proceed in the following way: First we determine all symmetric space \mathfrak{m} having G as the isometry group. After this we have to find for any symmetric space \mathfrak{m} all subalgebras \mathfrak{h} with $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ such that \mathfrak{h} does not contain any non-trivial ideal of \mathfrak{g} . For the classification up to isotopisms it is enough to take for \mathfrak{h} a suitable representative of the set $\{g^{-1}\mathfrak{h}g; g \in G\}$.

Since the Bol loops are strongly left alternative (Definition 5.3. in [18]) every global Bol loop contains an exponential image of a complement \mathfrak{m} of the Lie algebra \mathfrak{h} of H in the Lie algebra \mathfrak{g} of G , such that \mathfrak{m} generates \mathfrak{g} and satisfies the relation $[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}] \subseteq \mathfrak{m}$.

Lemma 3. *Let L be a differentiable loop and denote by \mathfrak{m} the tangent space of L at $1 \in G$. Then \mathfrak{m} does not contain any element of $Ad_g\mathfrak{h}$ for some $g \in G$.*

Proof. For $g \in G$ the group $H^g = g^{-1}Hg$ is the stabilizer of the element $g^{-1}(e) \in L$ in G . □

The conjugation $\sigma(G/H)^g = g^{-1}\sigma(G/H)g$ of the section σ of L with an element $g \in G$ defines again a sharply transitive section $xH \mapsto \sigma(xH)^g$ and the loop L^* belonging to the triple $(G, H, \sigma(G/H)^g)$ is isotopic to L (cf. [18], Theorem 1.11 (iii) p. 22). Moreover this loop L^* is isomorphic to $L = (G, H, \sigma)$ if and only if there is an automorphism of G leaving H invariant and mapping the section σ corresponding to L onto the section $\sigma(G/H)^g$ corresponding to L^* (cf. [18], Theorem 1.11 (i) p. 21).

A loop L is called a left A-loop if each $\lambda_{x,y} = \lambda_{xy}^{-1}\lambda_x\lambda_y : L \rightarrow L$ is an automorphism of L . The group G topologically generated by the left translations of L is a Lie group (cf. [18], Proposition 5.20, p. 75). Let \mathfrak{g} be the Lie algebra of G and let \mathfrak{h} be the Lie subalgebra of the stabilizer H of $e \in L$ in G . Denote by $\sigma : x \mapsto \lambda_x : L \rightarrow G$ the section corresponding to L . Then the tangent space $\mathfrak{m} = T_1\sigma(L)$ of the image of the section σ at $1 \in G$ is complementary to \mathfrak{h} and satisfies the properties $\mathfrak{m} \oplus \mathfrak{h} = \mathfrak{g}$ and $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$ (cf. [18], Definition 5.18. and Proposition 5.20. pp. 74-75), i.e. \mathfrak{g} is reductive with respect to $(\mathfrak{h}, \mathfrak{m})$ in the sense of Kobayashi and Nomizu (cf. [11] Vol II, p. 190).

A differentiable loop L is called a Bruck loop if there is an involutory automorphism σ of the Lie algebra \mathfrak{g} of the connected Lie group G generated by the left translations of L such that the tangent space $T_e(L) = \mathfrak{m}$ is the

-1 -eigenspace and the Lie algebra \mathfrak{h} of the stabilizer H of $e \in L$ in G is the $+1$ -eigenspace of σ .

3 3-dimensional Lie triple systems

Let $(\mathfrak{m}, (\cdot, \cdot, \cdot))$ be a Lie triple system. Denote by $(\mathfrak{g}^*, [\cdot, \cdot])$ the Lie algebra of the group of displacements of the symmetric space belonging to a Lie triple system. We classify the 3-dimensional non-solvable Lie triple systems $(\mathfrak{m}, (\cdot, \cdot, \cdot))$ within the group \mathfrak{g}^* as subspaces with $(\cdot, \cdot, \cdot) = [[\cdot, \cdot], \cdot]$.

a) The semisimple Lie triple systems

If \mathfrak{m} is a 3-dimensional semisimple Lie triple system then the corresponding Lie algebra is semisimple with $4 \leq \dim \mathfrak{g}^* \leq 6$ ([13], p. 219, and Theorem 2.7. p. 222). Every semisimple Lie algebra \mathfrak{g}^* has dimension 6. To classify the 3-dimensional semisimple Lie triple systems we have to determine all involutory automorphisms of the Lie algebra \mathfrak{g}^* , which leave a 3-dimensional subalgebra of \mathfrak{g}^* elementwise fixed. This problem is equivalent to give all involutory automorphisms of the Lie group G^* , which fix elementwise a 3-dimensional subgroup of G^* .

Any 6-dimensional semisimple Lie group is locally isomorphic to one of the following Lie groups:

1. $PSL_2(\mathbb{R}) \times SO_3(\mathbb{R})$
2. $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$
3. $SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$
4. $PSL_2(\mathbb{C})$.

1. case: The automorphism group Γ of the Lie group $G^* = PSL_2(\mathbb{R}) \times SO_3(\mathbb{R})$ is the direct product of the automorphism group of $PSL_2(\mathbb{R})$ and the automorphism group of $SO_3(\mathbb{R})$. Since there is no involutory automorphism of Γ leaving a 3-dimensional subgroup of G^* elementwise fixed therefore there is no 3-dimensional Lie triple system corresponding to the Lie group $G^* = PSL_2(\mathbb{R}) \times SO_3(\mathbb{R})$.

2. case: The automorphism group Γ of the Lie group $G^* = PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ is the semidirect product of the normal automorphism group $\Gamma_1 \times \Gamma_1$, where Γ_1 is the automorphism group of $PSL_2(\mathbb{R})$, by the group generated by the automorphism $\sigma : G^* \rightarrow G^*$; $(u, v) \mapsto (v, u)$. The only involutory automorphisms centralizing a 3-dimensional subgroup of Γ are the conjugate elements to σ . Therefore there is up to conjugation precisely one 3-dimensional Lie triple system \mathfrak{m} having the shape $\{(X, -X) \mid X \in sl_2(\mathbb{R})\}$.

3. case: In the case $G^* = SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$ there is up to conjugation only one Lie triple system corresponding to G^* . This Lie triple system has the

form $\{(X, -X) \mid X \in so_3(\mathbb{R})\}$.

4. case: B1 A real basis of the Lie algebra $sl_2(\mathbb{C})$: Let us consider in \mathfrak{g}^* the following real basis $\{\mathcal{H}, \mathcal{T}, \mathcal{U}, i\mathcal{H}, i\mathcal{T}, i\mathcal{U}\}$, where $\mathcal{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

$\mathcal{T} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\mathcal{U} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The Lie algebra multiplication is given by the following rules: $[\mathcal{H}, \mathcal{T}] = 2\mathcal{U}$, $[\mathcal{H}, \mathcal{U}] = 2\mathcal{T}$, $[\mathcal{U}, \mathcal{T}] = 2\mathcal{H}$.

K1 The normalized complex Cartan-Killing form $k_{\mathbb{C}} : sl_2(\mathbb{C}) \times sl_2(\mathbb{C}) \rightarrow \mathbb{C}$ of $sl_2(\mathbb{C})$ is the bilinear form defined by $k_{\mathbb{C}}(X, Y) = \frac{1}{8}\text{trace}(\text{ad}X \text{ ad}Y)$. If $X \in sl_2(\mathbb{C})$ has the decomposition

$$X = \lambda_1 \mathcal{H} + \lambda_2 \mathcal{T} + \lambda_3 \mathcal{U} + \lambda_4 i\mathcal{H} + \lambda_5 i\mathcal{T} + \lambda_6 i\mathcal{U}$$

then the complex Cartan-Killing form $k_{\mathbb{C}}$ satisfies

$$k_{\mathbb{C}}(X) = \lambda_1^2 + \lambda_2^2 + \lambda_6^2 - \lambda_3^2 - \lambda_4^2 - \lambda_5^2 + i(2\lambda_1\lambda_4 + 2\lambda_2\lambda_5 - 2\lambda_3\lambda_6)$$

(cf. [20], Section 6.1, pp. 215-228). The normalized real Cartan-Killing form $k_{\mathbb{R}} : sl_2(\mathbb{C}) \times sl_2(\mathbb{C}) \rightarrow \mathbb{R}$ is the restriction of $k_{\mathbb{C}}$ to \mathbb{R} such that $k_{\mathbb{R}}(X) = \lambda_1^2 + \lambda_2^2 + \lambda_6^2 - \lambda_3^2 - \lambda_4^2 - \lambda_5^2$ and the basis $\{\mathcal{H}, \mathcal{T}, \mathcal{U}, i\mathcal{H}, i\mathcal{T}, i\mathcal{U}\}$ is orthonormal with respect to $k_{\mathbb{R}}$.

The automorphism group Γ of $PSL_2(\mathbb{C})$ is the semidirect product of the group of inner automorphisms by the group of order 2 generated by the outer automorphism $\tau : z \mapsto \bar{z}$. According to ([15], pp. 153-154) there are in Γ precisely two conjugacy classes of involutory automorphisms centralizing 3-dimensional subgroups. In the one class there is involutory automorphism fixing the group $\left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}; a, b \in \mathbb{C}, a\bar{a} + b\bar{b} = 1 \right\} / \mathbb{Z}_2 \cong SO_3(\mathbb{R})$ elementwise. In the other conjugacy class there is an involutory automorphism centralizing the subgroup $\left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}; a, b \in \mathbb{C}, a\bar{a} - b\bar{b} = 1 \right\} / \mathbb{Z}_2 \cong PSL_2(\mathbb{R})$.

4.1 The subalgebra \mathfrak{h}_1 isomorphic to $so_3(\mathbb{R})$ is generated by the basis elements $i\mathcal{H}, i\mathcal{T}, \mathcal{U}$. The Lie triple system \mathfrak{m} with the property $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}_1$ has as generators $\mathcal{H}, \mathcal{T}, i\mathcal{U}$.

4.2 The subalgebra \mathfrak{h}_2 isomorphic to $sl_2(\mathbb{R})$ has as basis elements $i\mathcal{H}, \mathcal{T}, i\mathcal{U}$. The Lie triple system \mathfrak{m} with the property $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}_2$ is generated by the elements $\mathcal{H}, i\mathcal{T}, \mathcal{U}$.

b) Since there is no non-solvable 3-dimensional Lie triple system with a 2-dimensional radical $\mathfrak{r}(\mathfrak{m})$ (cf. [13], Theorem 2.20. p. 227) we have to consider Lie triple systems having a 1-dimensional radical.

Let the subspace $\mathbb{R}e_1$ be the 1-dimensional radical $\mathfrak{r}(\mathfrak{m})$ of the Lie triple system \mathfrak{m} . One has $\mathfrak{m} = \mathfrak{r}(\mathfrak{m}) \oplus \mathfrak{u}_0$, where \mathfrak{u}_0 is a 2-dimensional semisimple subsystem generated by e_2 and e_3 . Since the isometry group of the sym-

metric space \mathcal{P} corresponding to u_0 is 3-dimensional we may assume that $[e_2, e_3] = e_4$ is the generator of the Lie algebra of the stabilizer of a point in \mathcal{P} .

According to Furness ([4], pp. 44-45) we have 3 types of 2-dimensional semisimple Lie triple systems. The elements e_2, e_3 generate the Lie algebra of $SO_3(\mathbb{R})$ if and only if

$$[e_4, e_2] = e_3, \quad [e_4, e_3] = -e_2.$$

The elements e_2, e_3 generate the Lie algebra of $PSL_2(\mathbb{R})$ if and only if either

$$[e_4, e_2] = -e_3, \quad [e_4, e_3] = e_2$$

and e_4 is elliptic with respect to the Cartan-Killing form, or

$$[e_4, e_2] = -e_3, \quad [e_4, e_3] = -e_2$$

and the element e_4 is hyperbolic.

Since \mathfrak{m} is a Lie triple system with the properties $[[\mathfrak{m}, e_1], e_1] = 0$ and $[[e_1, \mathfrak{m}], \mathfrak{m}] \subset \langle e_1 \rangle$ we obtain that

$$\begin{aligned} 1. \quad & [[e_1, e_2], e_2] = ae_1, \quad [[e_1, e_3], e_3] = be_1, \quad [[e_1, e_2], e_3] = ce_1 \\ & [[e_1, e_3], e_2] = de_1, \quad [[e_2, e_3], e_1] = (d - c)e_1, \end{aligned}$$

where a, b, c, d are real parameters.

5. First let $\mathbb{R}e_1$ be the centre of \mathfrak{m} . Then we have $[[e_1, \mathfrak{m}], \mathfrak{m}] = 0$ and $a = b = c = d = 0$. We obtain 3 types of Lie triple systems with 1-dimensional centre belonging to the different types of the 2-dimensional semisimple Lie triple systems. Now we characterize the corresponding Lie algebras \mathfrak{g}^* .

5.1 The Lie algebra \mathfrak{g}^* is isomorphic to the Lie algebra $so_3(\mathbb{R}) \oplus \mathbb{R}$. The multiplication table in \mathfrak{g}^* is defined by the following relations:

$$[e_4, e_2] = e_3, \quad [e_4, e_3] = -e_2, \quad [e_4, e_i] = 0 \quad (i = 2, 3, 4).$$

There are precisely two Lie triple systems $\mathfrak{m}_1, \mathfrak{m}_2$, such that the corresponding Lie algebra \mathfrak{g}^* is isomorphic to the Lie algebra $sl_2(\mathbb{R}) \oplus \mathbb{R}$.

B2 A real basis of the Lie algebra $sl_2(\mathbb{R}) \oplus \mathbb{R}$: Denote by e_1 the generator of the Lie algebra \mathbb{R} . Let $e_2 = (\mathcal{H}, 0)$, $e_3 = (\mathcal{T}, 0)$, $e_4 = (\mathcal{U}, 0)$, with $\mathcal{H}, \mathcal{T}, \mathcal{U}$ defined in **B1**, be a real basis of $sl_2(\mathbb{R}) \oplus \{0\}$. Then the multiplication in \mathfrak{g}^* is given by the following rules:

$$[e_2, e_3] = e_4, \quad [e_4, e_2] = -e_3, \quad [e_4, e_3] = e_2, \quad [e_1, e_i] = 0 \quad (i = 2, 3, 4).$$

5.2 The Lie triple system \mathbf{m}_1 is generated by the basis elements e_1, e_2, e_4 .

5.3 The Lie triple system \mathbf{m}_2 has as generators e_1, e_2, e_3 .

If the centre of \mathbf{m} is trivial then not all parameters a, b, c, d in the system 1) of b) are zero. Using the third property in the definition of a Lie triple system and an automorphism φ of the form: $\varphi(e_1) = \alpha e_1$, $\varphi(e_2) = \beta e_2 + \gamma e_3$, $\varphi(e_3) = \delta e_2 + \varepsilon e_3$, where $\beta\varepsilon - \gamma\delta \neq 0$, we have the following four cases:

6.1 The Lie triple system \mathbf{m}_1 is given as follows:

$$[e_4, e_2] = e_3, [e_4, e_3] = -e_2, [[e_1, e_2], e_2] = -e_1, [[e_1, e_3], e_3] = -e_1.$$

The corresponding Lie algebra \mathbf{g}^* is isomorphic to the Lie algebra $so_3(\mathbb{R}) \ltimes \mathbb{R}^3$. Since $\mathbb{R}e_2, \mathbb{R}e_3, \mathbb{R}e_4$ correspond to 1-dimensional rotations and $[[e_1, e_3], e_2] = 0$, $[[e_1, e_2], e_3] = 0$, $[e_4, e_1] = 0$ the vectors $[e_1, e_3]$, $[e_1, e_2]$, e_1 are the axes of the rotation groups corresponding to e_2, e_3 respectively e_4 . Hence $e_1, e_5 := [e_1, e_3]$, $e_6 := [e_1, e_2]$ form a basis of the radical of $\mathbf{g}^* \cong \mathbb{R}^3$. We have $[e_4, e_6] = e_5$, $[e_4, e_5] = -e_6$ because the rotation group belonging to e_4 with the axis e_1 leaves the 2-dimensional subspace $\langle e_5, e_6 \rangle$ orthogonal to $\langle e_1 \rangle$ invariant. Similarly the rotation groups corresponding to e_2 and to e_3 with axes e_5 respectively e_6 leaves the subspaces $\langle e_1, e_6 \rangle$, respectively $\langle e_1, e_5 \rangle$ invariant. Hence we have $[e_2, e_6] = e_1$, $[e_3, e_5] = e_1$. Therefore the multiplication in the Lie algebra \mathbf{g}^* is determined.

B3 Representing the Lie algebra \mathbf{g}^* in the Lie algebra of real (4×4) -matrices we may choose as basis the following matrices:

$$e_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, e_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, e_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

6.2

$$[e_4, e_2] = -e_3, [e_4, e_3] = e_2, [[e_1, e_2], e_2] = e_1, [[e_1, e_3], e_3] = e_1.$$

6.3 The Lie triple \mathbf{m}_3 is given by the following relations:

$$[e_4, e_2] = -e_3, [e_4, e_3] = -e_2, [[e_1, e_2], e_2] = e_1, [[e_1, e_3], e_3] = -e_1.$$

In both cases the Lie algebra \mathfrak{g}^* is isomorphic to $sl_2(\mathbb{R}) \ltimes \mathbb{R}^3$. From the relation in **6.3** we can deduce the multiplication table of the Lie algebra \mathfrak{g}^* :

$$\begin{aligned} [e_1, e_2] &=: e_6, [e_1, e_3] =: e_5, [e_2, e_3] =: e_4, [e_5, e_4] = -e_6, \\ [e_1, e_4] &= [e_1, e_5] = [e_1, e_6] = [e_2, e_5] = [e_3, e_6] = [e_6, e_5] = 0, \\ [e_2, e_6] &= [e_3, e_5] = -e_1, [e_2, e_4] = e_3, [e_3, e_4] = -e_2, [e_6, e_4] = e_5. \end{aligned}$$

B4 Representing \mathfrak{g}^* as pairs of real (2×2) -matrices we may choose as basis $e_1 = (0, -\mathcal{U})$, $e_2 = (\mathcal{H}, 0)$, $e_3 = (\mathcal{T}, 0)$, $e_4 = (\mathcal{U}, 0)$, $e_5 = (0, -\mathcal{H})$, $e_6 = (0, \mathcal{T})$ with $\mathcal{H}, \mathcal{T}, \mathcal{U}$ defined in **B1**. The Lie triple system \mathfrak{m}_2 defined by the relations **6.2** in the basis **B4** is generated by the basis elements e_2, e_4, e_6 . The Lie triple system \mathfrak{m}_3 defined by the relations **6.3** belonging to the basis **B4** has as generators e_1, e_2, e_3 .

If an element X of $\mathfrak{g}^* = sl_2(\mathbb{R}) \ltimes \mathbb{R}^3$ has the decomposition

$$X = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + \lambda_5 e_5 + \lambda_6 e_6$$

then the Cartan-Killing form k (cf. [20], p.193) on \mathfrak{g}^* satisfies the relation

$$\mathbf{K2} : \quad k(X, X) = \lambda_2^2 + \lambda_3^2 - \lambda_4^2.$$

7 The last Lie triple system and the corresponding Lie algebra \mathfrak{g}^* are defined by the following rules:

$$\begin{aligned} [e_4, e_2] &= -e_3, [e_4, e_3] = -e_2, [[e_1, e_2], e_2] = \frac{1}{4} e_1, [[e_1, e_3], e_3] = -\frac{1}{4} e_1, \\ [[e_1, e_2], e_3] &= \frac{1}{4} e_1, [[e_1, e_3], e_2] = -\frac{1}{4} e_1, [[e_2, e_3], e_1] = -\frac{1}{2} e_1. \end{aligned}$$

In this case the Lie algebra \mathfrak{g}^* is isomorphic to $sl_2(\mathbb{R}) \ltimes \mathbb{R}^2$. Moreover the elements e_2, e_3 and $e_4 = [e_2, e_3]$ form a basis of $sl_2(\mathbb{R})$, such that e_2 is a hyperbolic, e_3 an elliptic and e_4 again a hyperbolic element with respect to the Cartan-Killing form.

4 Bol loops having semisimple Lie groups as the groups generated by their left translations

In this section we classify all 3-dimensional connected differentiable Bol loops having semisimple Lie groups as the groups topologically generated by their

left translations and describe the symmetric spaces as well as the natural geometries associated with these loops.

Let L be a 3-dimensional proper Bol loop, let G be the group generated by the left translations of L and H be the stabilizer of $e \in L$ in G . Since G is semisimple and $4 \leq \dim G \leq 6$ we have $\dim G = 6$.

Proposition 4. *Let $G = G_1 \times G_2$ be topologically generated by the left translations of a 3-dimensional connected differentiable Bol loop L such that G_i ($i = 1, 2$) are 3-dimensional quasi-simple Lie groups. Then G_i ($i = 1, 2$) is isomorphic to $PSL_2(\mathbb{R})$ and the stabilizer H of $e \in L$ in G may be chosen either as*

$$(i) \quad H_1 = \{(x, x) \mid x \in PSL_2(\mathbb{R})\}$$

or

$$(ii) \quad H_2 = \left\{ \left(\begin{pmatrix} a & b_1 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} a & b_2 \\ 0 & a^{-1} \end{pmatrix} \right); a > 0, b_1, b_2 \in \mathbb{R} \right\}.$$

Proof. Denote by $\pi_i : G \rightarrow G_i$ the natural projection of G to G_i for $i = 1, 2$. Let $\dim \pi_1(H) \leq 1$. Since $H \leq \pi_1(H) \times \pi_2(H)$ one has $\dim \pi_2(H) \geq 2$. If $\dim \pi_2(H) = 2$ then H is the direct product of $\pi_1(H)$ with $\pi_2(H)$ and we obtain a contradiction to Proposition 1. Let $\pi_2(H) = G_2$. If $\dim \pi_1(H) = 0$ then $H = \{1\} \times G_2$ which is impossible. If $\pi_1(H)$ is a 1-dimensional subgroup then H would be 4-dimensional since there is no non-trivial homomorphism from G_2 into $\pi_1(H)$.

Let now $\dim \pi_1(H) = 2$. We may assume that $\dim \pi_2(H) \geq 2$ since interchanging the indices we would obtain the previous case. Therefore each of the factors of G is locally isomorphic to the group $PSL_2(\mathbb{R})$. Since the Lie algebra of G_2 is simple there is no non-trivial homomorphism from $\pi_2(H) = G_2$ into $\pi_1(H)$, so $\pi_2(H)$ cannot be G_2 . If $\dim \pi_2(H) = \dim \pi_1(H) = 2$ then $\pi_i(H) \cong \mathcal{L}_2 = \{ax + b \mid a > 0, b \in \mathbb{R}\}$ ($i = 1, 2$) and there exist homomorphisms φ_1, φ_2 with 1-dimensional kernels, such that $\varphi_1 : \pi_1(H) \rightarrow \pi_2(H)$ and $\varphi_2 : \pi_2(H) \rightarrow \pi_1(H)$. Since $H \cap G_i = \ker \varphi_i = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbb{R} \right\}$ and (up

to conjugation) $\text{im } \varphi_i = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a > 0 \right\}$ ($i = 1, 2$) we obtain that

$$\pi_1(H) = \ker \varphi_1 \text{ im } \varphi_2, \pi_2(H) = \ker \varphi_2 \text{ im } \varphi_1$$

and H has the shape (ii).

Finally let $\dim \pi_1(H) = 3$. From the previous arguments it follows that $\dim \pi_2(H) = 3$. If there is no homomorphism $\varphi : G_1 \rightarrow G_2$ then there is no 3-dimensional subgroup H of $G = G_1 \times G_2$. If there exists a homomorphism $\varphi : G_1 \rightarrow G_2$ then the stabilizer H has the shape $\{(x, \varphi(x)) \mid x \in G_1\}$. Moreover φ must be an isomorphism since otherwise G would contain a discrete central subgroup of G . Hence G can be identified with $G_1 \times G_1$, where G_1

is isomorphic either to $PSL_2(\mathbb{R})$ or $SO_3(\mathbb{R})$, and H may be chosen as the diagonal subgroup $\{(x, x) \mid x \in G_1\}$. According to Proposition 2 d) we have $G = PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$. \square

Now let G be locally isomorphic to the group $PSL_2(\mathbb{C})$. According to ([1], pp. 273-278) there are 4 conjugacy classes of the 3-dimensional subgroups of $G = SL_2(\mathbb{C})$, which we denote by W_r , U_0 , U_1 and $SU_2(\mathbb{C})$. Since $SU_2(\mathbb{C})$ contains central elements $\neq 1$ of $SL_2(\mathbb{C})$ it follows from Proposition 2 that no of these groups can be the stabilizer of $e \in L$ in G . Hence it is sufficient to consider the following cases:

1. $G \cong PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$, and H_i ($i = 1, 2$) is one of the subgroups given in Proposition 4.
2. G is isomorphic to $PSL_2(\mathbb{C})$ and H is isomorphic to $SO_3(\mathbb{R})$.

Now we deal with the case 1). There is up to isomorphism precisely one Lie triple system $\mathfrak{m} = \{(X, -X) \mid X \in sl_2(\mathbb{R})\}$ corresponding to the Lie group $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ (**2. case**, Section 3).

The Lie algebra of H_1 is $\mathfrak{h}_1 = \{(X, X) \mid X \in sl_2(\mathbb{R})\}$. The Lie algebra \mathfrak{h}_2 of H_2 is generated by the basis elements: $(\mathcal{H}, \mathcal{H})$, $(\mathcal{U} + \mathcal{T}, 0)$, $(0, \mathcal{U} + \mathcal{T})$, where \mathcal{H} , \mathcal{U} , \mathcal{T} are (2×2) -matrices defined in B1. We see that the intersection of \mathfrak{m} and \mathfrak{h}_1 is trivial. But the subalgebra \mathfrak{h}_1 contains the element $(\mathcal{H}, \mathcal{H})$, which is conjugate to the element $(\mathcal{H}, -\mathcal{H}) \in \mathfrak{m}$ under the element corresponding to the pair of matrices $\left(\pm 1, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$. This contradicts Lemma 3.

Since the intersection of \mathfrak{m} and \mathfrak{h}_2 is $\langle(\mathcal{U} + \mathcal{T}, -(\mathcal{U} + \mathcal{T}))\rangle$ the subspace \mathfrak{m} cannot be a complement to \mathfrak{h}_2 in \mathfrak{g} .

From this discussion it follows that there is no 3-dimensional connected differentiable Bol loop such that the group G generated by its left translations is isomorphic to the group $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$.

Now we consider the case 2). The Lie algebra \mathfrak{h} of H with respect to the basis **B1** (**4.case**, Section 3) has as generators $\mathcal{U}, i\mathcal{T}, i\mathcal{H}$. The Lie triple system $\mathfrak{m} = \langle i\mathcal{T}, \mathcal{U}, \mathcal{H} \rangle$ given in **4.2** cannot be a complement to \mathfrak{h} in \mathfrak{g} since $\mathfrak{h} \cap \mathfrak{m} = \langle i\mathcal{T}, \mathcal{U} \rangle$.

The Lie triple system \mathfrak{m} of **4.1** satisfies $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$, and $\mathfrak{g} = \mathfrak{m} \oplus [\mathfrak{m}, \mathfrak{m}]$. Hence it determines a 3-dimensional Riemann symmetric space (cf. [11], Chapter VI, Theorem 2.2 (iii)). According to Theorem 11.8 in ([18]) there exists a global differentiable Bruck loop L_0 homeomorphic to \mathbb{R}^3 and having $G \cong PSL_2(\mathbb{C})$ as the group topologically generated by its left translations such that $H \cong SO_3(\mathbb{R})$ is the stabilizer of $e \in L$ in G . Since the symmetric spaces with $PSL_2(\mathbb{C})$ as the group of displacements which correspond to

global differentiable Bol loops are isomorphic, there is precisely one isotopism class \mathcal{C} of differentiable Bol loops having $PSL_2(\mathbb{R})$ as the group topologically generated by their left translations (cf. Remark in section 3) and L_0 is a representative of \mathcal{C} . Since the subspace \mathfrak{m} corresponding to L_0 is the hyperbolic space we call L_0 the hyperbolic space loop.

In order to determine within the isotopism class \mathcal{C} the isomorphism classes we look for a suitable parametrization of the local Bol loops isotopic to the hyperbolic space loop. For this reason we consider all 3-dimensional complements to $\mathfrak{h} = so_3(\mathbb{R})$ in $\mathfrak{g} = sl_2(\mathbb{C})$ with the properties $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, $[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}] \subseteq \mathfrak{m}$ and \mathfrak{m} generates \mathfrak{g} . We can write \mathfrak{m} in the general form:

$$\mathfrak{m} = \langle \mathcal{T} + a\mathcal{U} + b i\mathcal{T} + c i\mathcal{H}, i\mathcal{U} + d\mathcal{U} + e i\mathcal{T} + f i\mathcal{H}, \mathcal{H} + g\mathcal{U} + h i\mathcal{T} + k i\mathcal{H} \rangle,$$

$a, b, c, d, e, f, g, h, k \in \mathbb{R}$. According to [21] (pp. 217-219) there are precisely two classes of the Bol complements \mathfrak{m} to \mathfrak{h} in \mathfrak{g} having the following shapes:

$$\mathfrak{m}_a = \langle \mathcal{T} + a\mathcal{U}, i\mathcal{U} + a i\mathcal{T}, \mathcal{H} \rangle \text{ for } a \in \mathbb{R} \setminus \{1, -1\},$$

$$\mathfrak{m}_d = \langle \mathcal{U} + \mathcal{T}, d i\mathcal{H} + i(\mathcal{U} + \mathcal{T}), \mathcal{H} + d \mathcal{U} \rangle \text{ for } d \in \mathbb{R} \setminus \{0\}.$$

A Bol complement belongs to a local Bol loop in the class \mathcal{C} if and only if $[\mathfrak{m}, \mathfrak{m}]$ is a compact subalgebra of \mathfrak{g} . Since the value of the complex Cartan-Killing form $\mathbf{K1}$ on the element $[\mathcal{U} + \mathcal{T}, d i\mathcal{H} + i(\mathcal{U} + \mathcal{T})] \in [\mathfrak{m}_d, \mathfrak{m}_d]$ is zero $[\mathfrak{m}_d, \mathfrak{m}_d]$ is not compact. The subalgebra $[\mathfrak{m}_a, \mathfrak{m}_a]$ is compact if and only if $a \in (-1, 1)$ since $[\mathfrak{m}_a, \mathfrak{m}_a] = \langle i\mathcal{H}, \mathcal{U} + a\mathcal{T}, i\mathcal{T} + a i\mathcal{U} \rangle$.

In the isotopism class $\mathcal{C} = \{L_a, a \in (-1, 1)\}$ only the Bol complement \mathfrak{m}_0 satisfies the relation $[\mathfrak{h}, \mathfrak{m}_0] \subseteq \mathfrak{m}_0$. Therefore only the hyperbolic space loop L_0 corresponding to the triple $(G, H, \exp \mathfrak{m}_0)$ is a left A-loop. It is the unique Bruck loop in the class \mathcal{C} .

The complement $\mathfrak{m}_0 = \langle \mathcal{T}, i\mathcal{U}, \mathcal{H} \rangle$ is orthogonal to \mathfrak{h} with respect to the Cartan-Killing form $k_{\mathbb{R}}$ on \mathfrak{g} (see $\mathbf{K1}$ in Section 3). Let $\mathfrak{m}_a \subset \mathfrak{g} = sl_2(\mathbb{C})$ be the tangent space $T_1\sigma_a(G/H)$ of the image $\sigma_a(G/H)$ of the differentiable section $\sigma_a : G/H \rightarrow G$ belonging to the loop L_a for $a \in (-1, 1)$. Two loops corresponding to $(G, H, \exp \mathfrak{m}_a)$ and $(G, H, \exp \mathfrak{m}_b)$ are isomorphic if and only if there exists an automorphism α of \mathfrak{g} such that $\alpha(\mathfrak{m}_a) = \mathfrak{m}_b$ and $\alpha(\mathfrak{h}) = \mathfrak{h}$. The automorphism group of \mathfrak{g} leaving \mathfrak{m}_0 and \mathfrak{h} invariant is the semidirect product Θ of Ad_H by the group generated by the involutory map $\varphi : z \mapsto \bar{z}$. The Cartan-Killing form $k_{\mathbb{R}}$ is invariant under Θ . Hence the 3 angles between \mathfrak{m}_a and \mathfrak{m}_0 respectively between \mathfrak{m}_b and \mathfrak{m}_0 are equal with respect to $k_{\mathbb{R}}$. The loops L_a and L_b are isomorphic if there exists a matrix $B = \begin{pmatrix} c_1 + c_2 i & d_1 + d_2 i \\ -d_1 + d_2 i & c_1 - c_2 i \end{pmatrix} \in H$ such that $\varphi(B^{-1} \mathfrak{m}_a B) = \mathfrak{m}_b$. This condition is equivalent with the following equations:

$$c_1^2 + c_2^2 + d_1^2 + d_2^2 = 1, \quad d_1 c_2 - c_1 d_2 = 0, \quad d_1 c_1 + c_2 d_2 = 0,$$

$$\begin{aligned}
c_1 d_2 + c_2 d_1 &= 0, & c_2 d_2 - c_1 d_1 &= 0, \\
(d_2^2 - d_1^2 - a(c_2^2 - c_1^2)) b &= c_1^2 - c_2^2 - a(d_2^2 - d_1^2), \\
(d_1 d_2 + c_1 c_2 a) b &= c_1 c_2 - a d_1 d_2, \\
b(-a(c_1^2 - c_2^2) + (d_2^2 - d_1^2)) &= -a(d_2^2 - d_1^2) - (c_1^2 - c_2^2), \\
b(c_1 c_2 a - d_1 d_2) &= d_1 d_2 a + c_1 c_2.
\end{aligned}$$

The solution of this system of equations is $b = \frac{1}{a}$ and $b = -a$. Since $b, a \in (-1, 1)$ it follows that $b = -a$. Therefore the loops L_a , $a \in [0, 1)$ can be chosen as the representatives of the isomorphism classes of the Bol loops L_a , $a \in (-1, 1)$.

The multiplication of the hyperbolic space loop L_0 may be given in the 3-dimensional hyperbolic geometry \mathbb{H}_3 explicitly. We choose a point e in \mathbb{H}_3 and define $x \circ y = \tau_{e,x}(y)$ for all points $x, y \in \mathbb{H}_3$, where $\tau_{e,x}$ is the hyperbolic translation along the line joining e with x and mapping e onto x .

The hyperbolic space geometry has an elementary model in the upper half space $\mathbb{R}^{3+} = \{(x, y, z) \in \mathbb{R}^3; z > 0\}$. We can identify the elements of \mathbb{R}^3 with the elements of the \mathbf{J} -quaternion space. The \mathbf{J} -quaternion space is the 3-dimensional subspace of the quaternion space which is orthogonal to the canonical basis quaternion k (cf. [3], p. 3). We can give the action of the group $SL_2(\mathbb{C})$ on \mathbb{R}^3 by the linear rational functions: $\gamma(w) = (aw + b)(cw + d)^{-1}$, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{C}$, $ad - bc = 1$, $w = x + jy \in \mathbf{J}$, $x \in \mathbb{C}$, $y \in \mathbb{R}$ (cf. [3], p. 26). The restriction of this action onto the subspace \mathbb{R}^{3+} defines the action of $SL_2(\mathbb{C})$ on the upper half space, and $(w, \pm\gamma) \mapsto \gamma(w)$ is the transitive action of $PSL_2(\mathbb{C})$ on the upper half space. Since the stabilizer subgroup $H = SO_3(\mathbb{R})$ leaves the point j fixed, we may choose this point as the identity element of the hyperbolic space loop. Summarizing our discussion we obtain

Theorem 5. *There is only one isotopism class \mathcal{C} of the 3-dimensional connected differentiable Bol loops L such that the group G generated by the left translations $\{\lambda_x; x \in L\}$ is a semisimple Lie group. The group G is isomorphic to $PSL_2(\mathbb{C})$ and the stabilizer H of $e \in L$ in G is isomorphic to $SO_3(\mathbb{R})$. Any loop L_a ($-1 < a < 1$) in this class \mathcal{C} is isotopic to the hyperbolic space loop L_0 . The tangent space $T_1\Lambda$ of the set Λ of the left translations of the hyperbolic space loop L_0 at the identity $1 \in G$ is the plane \mathfrak{m}_0 through 0 in the Lie algebra \mathfrak{g} of G such that \mathfrak{m}_0 is orthogonal to the 3-dimensional Lie algebra \mathfrak{h} of H with respect to the Cartan-Killing form $k_{\mathbb{R}}$ of \mathfrak{g} . In the class \mathcal{C} only the hyperbolic space loop is a Bruck loop.*

The tangential spaces $T_1\sigma_a(G/H)$ at $1 \in G$ for all $a \in (-1, 1)$ are the Bol complements \mathbf{m}_a . The loops L_a and L_b in \mathcal{C} are isomorphic if and only if the angles between the 3-spaces \mathbf{m}_a and \mathbf{m}_0 , respectively \mathbf{m}_b and \mathbf{m}_0 are the same with respect to $k_{\mathbb{R}}$. As representatives of the isomorphism classes of the loops belonging to \mathcal{C} we may choose the loops L_a , where $0 \leq a < 1$.

5 3-dimensional Bol loops corresponding to 4-dimensional non-solvable Lie groups

In this section we determine all 3-dimensional connected differentiable global Bol loops L having a 4-dimensional non-solvable Lie group G as the group topologically generated by their left translations.

Since according to Proposition 2 b) a group locally isomorphic to $SO_3(\mathbb{R}) \rtimes \mathbb{R}$ cannot be the group topologically generated by the left translations of a 3-dimensional Bol loop we have to consider only the case that G is locally isomorphic to $PSL_2(\mathbb{R}) \times \mathbb{R}$. Moreover, since the group G is isomorphic to the Lie group G^* described in **5.2** and **5.3** of section 3 we have to discuss the following situation.

Using the \mathbb{R} -basis **B2** (**5.2**, Section 3) the 1-dimensional subalgebras \mathbf{h} of $\mathfrak{g} = sl_2(\mathbb{R}) \times \mathbb{R}$ containing no non-zero ideal of \mathfrak{g} are the following:

$$\mathbf{h}_1 = \langle e_2 + ke_1 \rangle$$

$$\mathbf{h}_2 = \langle (e_3 + e_4) + ke_1 \rangle$$

$$\mathbf{h}_3 = \langle e_4 + ke_1 \rangle, \text{ where } k \in \mathbb{R}.$$

Using the automorphism $A : \mathfrak{g} \rightarrow \mathfrak{g}$ given by $A(e_1) = ke_1$, $A(e_i) = e_i$ for $2 \leq i \leq 4$ we may assume that $k = 0, 1$.

Since the Lie triple system \mathbf{m}_1 in the case **5.2** is generated by the basis elements e_1, e_2, e_4 , the Lie algebras \mathbf{h}_1 and \mathbf{h}_3 cannot be complements to \mathbf{m}_1 in \mathfrak{g} . Since the elements $e_2 + e_4 \in \mathbf{m}$ and $e_3 + e_4 \in \mathbf{h}$ are both parabolic in $sl_2(\mathbb{R})$, we have a contradiction to Lemma 3.

Now we consider the Lie triple system $\mathbf{m}_2 = \langle e_1, e_2, e_3 \rangle$ in the case **5.3**. The intersection of \mathbf{h}_1 and \mathbf{m}_2 is not trivial, therefore \mathbf{m}_2 is not a complement to \mathbf{h}_1 in \mathfrak{g} . Since $\mathbf{h}_2 \cap \mathbf{m}_2 = \mathbf{h}_3 \cap \mathbf{m}_2 = \{0\}$ and there is no element of \mathbf{m}_2 which is conjugate to an element of \mathbf{h}_i for $i = 2, 3$ we have to deal with both cases.

For $k = 0$ we have $H_2 = \exp \mathbf{h}_2 = \left\{ \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, 1 \right), b \in \mathbb{R} \right\}$ and

$H_3 = \exp \mathbf{h}_3 = \{(x, 1), x \in SO_2(\mathbb{R})\}$. Now G and H are direct products $G = G_1 \times G_2$, $H = H_1 \times H_2$ with $H_i \subset G_i$ ($i = 1, 2$). The Bol loop L corresponding to a such pair is the direct product of a 2-dimensional Bol loop L_1 and a 1-dimensional Lie group L_2 such that the group topologically generated by the left translations of the 2-dimensional Bol loop L_1 is isomorphic

to $PSL_2(\mathbb{R})$. Then $G = PSL_2(\mathbb{R}) \times G_2$, where G_2 is either \mathbb{R} or $SO_2(\mathbb{R})$, and $H = H_3$. Moreover L_1 is isotopic to the hyperbolic plane loop.

For $k = 1$ one has $G = PSL_2(\mathbb{R}) \times \mathbb{R}$, $H_2 = \exp \mathbf{h}_2 = \left\{ \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \right) \right\}$,

where $b \in \mathbb{R}$ and $\exp \mathbf{m} = \left\{ \left(\begin{pmatrix} x+y & z \\ z & x-y \end{pmatrix}, k \right) \right\}$ with $x, y, z, k \in \mathbb{R}$,

and $x^2 - y^2 - z^2 = 1$ ([6], p. 24). If for a section $\sigma : G/H \rightarrow G$ the image $\sigma(G/H)$ contains $\exp \mathbf{m}$ then any coset $\left(\begin{pmatrix} 1+c & 1 \\ c & 1 \end{pmatrix}, 0 \right) H$, $c \in \mathbb{R}$

should contain precisely one element s of $\sigma(G/H)$. If $c \neq -1$ then $s =$

$\left(\begin{pmatrix} 1+c & c \\ c & \frac{c^2+1}{1+c} \end{pmatrix}, 0 \right)$. Since $\sigma : G/H \rightarrow G$ is continuous we have

$$\sigma \left(\left(\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, 0 \right) H \right) = \lim_{c \rightarrow -1} \sigma \left(\left(\begin{pmatrix} 1+c & 1 \\ c & 1 \end{pmatrix}, 0 \right) H \right) =$$

$$\lim_{c \rightarrow -1} \left(\begin{pmatrix} 1+c & c \\ c & \frac{c^2+1}{1+c} \end{pmatrix}, 0 \right), \text{ which is a contradiction. This means}$$

that there is no global Bol loop corresponding to the triple $(G, H_2, \exp \mathbf{m})$.

The other case is $G = PSL_2(\mathbb{R}) \times SO_2(\mathbb{R})$, $H_3 = \{(x, x) \mid x \in SO_2(\mathbb{R})\}$ and $\exp \mathbf{m} = \exp\{(\lambda_2 e_2 + \lambda_3 e_3), \lambda_2, \lambda_3 \in \mathbb{R}\} \times \exp\{(\lambda_1 e_1), \lambda_1 \in \mathbb{R}\} = M \times SO_2(\mathbb{R})$, such that M is the image of the section σ_1 given by

$$\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \begin{pmatrix} \frac{a^{-1}+a}{\pm\sqrt{b^2+(a^{-1}+a)^2}} & \frac{b}{\pm\sqrt{b^2+(a^{-1}+a)^2}} \\ -\frac{b}{\pm\sqrt{b^2+(a^{-1}+a)^2}} & \frac{a^{-1}+a}{\pm\sqrt{b^2+(a^{-1}+a)^2}} \end{pmatrix},$$

choosing $\text{sign}(\pm\sqrt{b^2+(a^{-1}+a)^2}) = \text{sign } b$ if $b \neq 0$ and $+1$ for $b = 0$. The section σ_1 corresponds to the hyperbolic plane loop (cf. [18], pp. 283-284).

Since $[[\mathbf{m}, \mathbf{m}], \mathbf{m}] \subseteq \mathbf{m}$ and each element $g \in G$ can uniquely be represented as a product $g = mh$ with $m \in \exp \mathbf{m}$ and $h \in H_3$ we have a global Bol loop L defined on the factor space G/H_3 (cf. [9], Corollary 3.11, p. 51 and [18], Lemma 1.3, p. 17). In L there is a normal subgroup \tilde{G} isomorphic to $SO_2(\mathbb{R})$ and the factor loop L/\tilde{G} is isomorphic to the hyperbolic plane loop. Therefore is L the unique Scheerer extension of the Lie group $SO_2(\mathbb{R})$ by the hyperbolic plane loop (cf. [18], Section 2). This loop is a left A-loop, because of $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$. But it is not a Bruck loop since there is no involutory automorphism $\sigma : \mathbf{g} \rightarrow \mathbf{g}$ such that $\sigma(\mathbf{m}) = -\mathbf{m}$ and $\sigma(\mathbf{h}) = \mathbf{h}$.

Theorem 6. *There are precisely three isotopism classes $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ of connected differentiable Bol loops L with dimension 3 such that the group G generated by the left translations $\{\lambda_x; x \in L\}$ is a 4-dimensional non-solvable*

Lie group. Every loop in the class \mathcal{C}_1 , respectively \mathcal{C}_2 is the direct product of a 2-dimensional loop isotopic to the hyperbolic plane loop with the Lie group \mathbb{R} , respectively $SO_2(\mathbb{R})$. In the class \mathcal{C}_3 is contained up to isomorphisms only the Scheerer extension of the Lie group $SO_2(\mathbb{R})$ by the hyperbolic plane loop.

6 3-dimensional Bol loops belonging to 5-dimensional non-solvable Lie groups

In this section we seek for 3-dimensional connected differentiable global Bol loops having a 5-dimensional non-solvable Lie group G as the group topologically generated by their left translations.

Any 5-dimensional non-solvable Lie group is locally isomorphic to one of the following groups:

- 1) $PSL_2(\mathbb{R}) \times \mathbb{R}^2$
- 2) $PSL_2(\mathbb{R}) \times \mathcal{L}_2$, where $\mathcal{L}_2 \cong \{x \mapsto ax + b; a > 0, b \in \mathbb{R}\}$.
- 3) $SO_3(\mathbb{R}) \times \mathbb{R}^2$
- 4) $SO_3(\mathbb{R}) \times \mathcal{L}_2$
- 5) The semidirect product $G = SL_2(\mathbb{R}) \ltimes \mathbb{R}^2$ such that $SL_2(\mathbb{R})$ acts in the natural way on \mathbb{R}^2 . The group G may be seen as the connected component of the group of area preserving affinities of the real plane.

Since there is no epimorphism from a 6-dimensional Lie algebra \mathfrak{g}^* given in section 3 onto one of the 5-dimensional Lie algebras \mathfrak{g} listed above only the Lie algebra $\mathfrak{g} = sl_2(\mathbb{R}) \ltimes \mathbb{R}^2$, which is the Lie algebra of the isometry group of a symmetric space (see case 7 in section 3) can occur as the group topologically generated by the left translations of a 3-dimensional connected differentiable Bol loop. Every 2-dimensional connected subgroup H of G , which does not have any non-trivial normal subgroup of G , contains 1-parameter subgroups S fixing a point or contains a 1-dimensional translation group. The Lie algebra of S has as generator an elliptic, parabolic, or hyperbolic element of \mathfrak{g} . Using the relations of 7 we see that \mathfrak{m} contains elements corresponding to translations and $\mathfrak{m} \cap sl_2(\mathbb{R})$ has with e_2 a hyperbolic, with e_3 an elliptic and with $e_2 + e_3$ a parabolic element. Therefore $\exp \mathfrak{m}$ contains elements conjugate to elements of H . According to Lemma 3 there is no 3-dimensional Bol loop L as section in the group $PSL_2(\mathbb{R}) \ltimes \mathbb{R}^2$. This consideration yields the following

Theorem 7. *There is no 3-dimensional connected differentiable global Bol loop L having a 5-dimensional non-solvable Lie group as the group topologically generated by its left translations.*

7 3-dimensional Bol loops with 6-dimensional non-solvable Lie groups as their left translations groups

A 6-dimensional non-solvable Lie group G can occur as the group topologically generated by the left translations of a 3-dimensional connected differentiable Bol loop, if G is isomorphic to a 6-dimensional Lie group G^* listed in the section 3. Then we have to deal with the cases, where G is locally isomorphic to one of the following Lie groups:

- 1) The connected component of the euclidean motion group on \mathbb{R}^3 .
- 2) The semidirect product of $PSL_2(\mathbb{R}) \times \mathbb{R}^3$, where the action of $PSL_2(\mathbb{R})$ on \mathbb{R}^3 is the adjoint action of $PSL_2(\mathbb{R})$ on its Lie algebra.

Let us consider the first case. The 3-dimensional subgroups H of $G = SO_3(\mathbb{R}) \times \mathbb{R}^3$, which does not contain any non-trivial normal subgroup of G , are locally isomorphic to the following subgroups: Either H has the shape as in case c) of Proposition 2 and there is no 3-dimensional differentiable Bol loop corresponding to the pair (G, H) or $H = \{(a, 0), a \in SO_3(\mathbb{R})\}$. We consider the \mathbb{R} -basis **B3** of the Lie algebra $\mathfrak{g} = so_3(\mathbb{R}) \times \mathbb{R}^3$ given in the case **6.1** (Section 3). Then the Lie algebra \mathfrak{h} of H is generated by the basis elements e_2, e_3, e_4 and the Lie triple system \mathfrak{m} is generated by the basis elements e_1, e_2, e_3 . Then $\mathfrak{h} \cap \mathfrak{m} = \langle e_2, e_3 \rangle$ which is a contradiction.

Now we discuss the second case. Then the group multiplication is given by

$$(A_1, X_1) \circ (A_2, X_2) = (A_1 A_2, A_2^{-1} X_1 A_2 + X_2),$$

where (A_i, X_i) , $i = 1, 2$ are two elements of G such that X_i $i = 1, 2$ are represented by 2×2 real matrices with trace 0. We use the \mathbb{R} -basis **B4** of the Lie algebra $\mathfrak{g} = sl_2(\mathbb{R}) \times \mathbb{R}^3$ (**6.3**, Section 3). The 3-dimensional subalgebras \mathfrak{h} of \mathfrak{g} containing no non-zero ideal of \mathfrak{g} are the following:

- a) $\langle e_2, e_5, e_1 + e_6 \rangle$,
- b) $\langle e_2 + k e_5, e_1, e_6 \rangle$, where $k \in \mathbb{R}$,
- c) $\langle e_3 + e_4, e_5, e_1 - e_6 \rangle$,
- d) $\langle e_2, e_3 + e_4, e_1 - e_6 \rangle$,
- e) $\langle e_2, e_3, e_4 \rangle$,
- f) $\langle e_4, e_5, e_6 \rangle$.

First we deal with the Lie triple system \mathfrak{m} of (**6.2** in section 3) generated by the basis elements e_4, e_2, e_6 . If the Lie algebra \mathfrak{h} has the shape a), b), d), e) and f) then the intersection of \mathfrak{h} with \mathfrak{m} are the subspaces $\langle e_2 \rangle$, $\langle e_6 \rangle$, $\langle e_2 \rangle$, $\langle e_2, e_4 \rangle$, or $\langle e_4, e_6 \rangle$ respectively. In the case c) the elements $e_2 + e_4 \in \mathfrak{m}$ and

$e_3 + e_4 \in \mathfrak{h}$ are parabolic and the parabolic elements of $sl_2(\mathbb{R})$ are conjugate to each other. This is a contradiction to Lemma 3.

Let us treat the Lie triple system \mathfrak{m} of (6.3 in section 3) which is generated by the basis elements e_1, e_2, e_3 and the image of \mathfrak{m} under the exponential map.

The exponential map $\exp : \mathfrak{g} \rightarrow G$ is defined in the following way: For $X \in \mathfrak{g}$ we have $\exp X = \gamma_X(1)$, where $\gamma_X(t)$ is the 1-parameter subgroup of G with the property $\frac{d}{dt}\big|_{t=0} \gamma_X(t) = X$. In the 1-parameter subgroup $\alpha(t) = (\beta(t), \gamma(t))$ of G with the conditions $\alpha(t=0) = (1, 0)$ and $\frac{d}{dt}\big|_{t=0} \alpha(t) = (X_1, X_2) = X \in \mathfrak{g}$ the first component $\beta(t)$ is the 1-parameter subgroup of $PSL_2(\mathbb{R})$, and the second component satisfies

$$\begin{aligned} \frac{d}{dt} \gamma(t) &= \frac{d}{ds}\bigg|_{s=0} \gamma(t+s) = -\frac{d}{ds}\bigg|_{s=0} \beta(s) \gamma(t) + \gamma(t) \frac{d}{ds}\bigg|_{s=0} \beta(s) + \frac{d}{ds}\bigg|_{s=0} \gamma(s) = \\ &= -X_1 \gamma(t) + \gamma(t) X_1 + X_2. \end{aligned}$$

For $X_1 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, and $X_2 = \begin{pmatrix} k & u \\ y & -k \end{pmatrix}$, with $a, b, c, k, u, y \in \mathbb{R}$ and $\gamma(t) = \begin{pmatrix} r(t) & s(t) \\ v(t) & -r(t) \end{pmatrix}$, one has

$$\begin{aligned} \frac{d}{dt} \gamma(t) &= \begin{pmatrix} \frac{d}{dt} r(t) & \frac{d}{dt} s(t) \\ \frac{d}{dt} v(t) & -\frac{d}{dt} r(t) \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & a \end{pmatrix} \begin{pmatrix} r(t) & s(t) \\ v(t) & -r(t) \end{pmatrix} + \\ &+ \begin{pmatrix} r(t) & s(t) \\ v(t) & -r(t) \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} + \begin{pmatrix} k & u \\ y & -k \end{pmatrix} \end{aligned}$$

with the following properties:

$$r(0) = s(0) = v(0) = 0, \quad \frac{d}{dt}\bigg|_{t=0} r(t) = k, \quad \frac{d}{dt}\bigg|_{t=0} s(t) = u, \quad \frac{d}{dt}\bigg|_{t=0} v(t) = y,$$

$$\frac{d}{dt} r(t) = -bv(t) + cs(t) + k, \quad \frac{d}{dt} s(t) = 2(br(t) + as(t)) + u,$$

$$\frac{d}{dt} v(t) = 2(av(t) - cr(t)) + y.$$

The solution of this inhomogeneous system of linear differential equations is:

$$\begin{aligned} r(t) &= \frac{1}{8(a^2 + bc)^{\frac{3}{2}}} (e^{2\sqrt{a^2+bc} t} - e^{-2\sqrt{a^2+bc} t}) (-acu - bay + 2kcb) + \\ &+ \frac{1}{8(a^2 + bc)} [(e^{\sqrt{a^2+bc} t} - e^{-\sqrt{a^2+bc} t})^2 (-cu + by) + t(8ka^2 + 4acu + 4aby)], \end{aligned}$$

$$\begin{aligned}
s(t) &= \frac{1}{8(a^2 + bc)^{\frac{3}{2}}} (e^{2\sqrt{a^2+bc}t} - e^{-2\sqrt{a^2+bc}t}) (-b^2y + ubc - 2bak + 2a^2u) + \\
&\frac{1}{8(a^2 + bc)} [(e^{\sqrt{a^2+bc}t} - e^{-\sqrt{a^2+bc}t})^2 (-2bk + 2au) + t(4b^2y + 4ubc + 8kab)], \\
v(t) &= \frac{1}{8(a^2 + bc)^{\frac{3}{2}}} (e^{2\sqrt{a^2+bc}t} - e^{-2\sqrt{a^2+bc}t}) (2ya^2 - 2cka + bcy - c^2u) + \\
&\frac{1}{8(a^2 + bc)} [(e^{\sqrt{a^2+bc}t} - e^{-\sqrt{a^2+bc}t})^2 (-2ay + 2ck) + t(8cka + 4c^2u + 4bcy)].
\end{aligned}$$

It follows $\exp \mathbf{m} = \{\exp(X_1, X_2)\}$, where $X_1 \in \left\{ \begin{pmatrix} \lambda_2 & \lambda_3 \\ \lambda_3 & -\lambda_2 \end{pmatrix}, \lambda_2, \lambda_3 \in \mathbb{R} \right\}$ and $X_2 \in \left\{ \begin{pmatrix} 0 & -\lambda_1 \\ \lambda_1 & 0 \end{pmatrix}, \lambda_1 \in \mathbb{R} \right\}$.

The first component of $\exp \mathbf{m}$ is

$$(\exp(X_1, X_2))_1 = \begin{pmatrix} \cosh \sqrt{A} + \frac{\sinh \sqrt{A}}{\sqrt{A}} \lambda_2 & \frac{\sinh \sqrt{A}}{\sqrt{A}} \lambda_3 \\ \frac{\sinh \sqrt{A}}{\sqrt{A}} \lambda_3 & \cosh \sqrt{A} - \frac{\sinh \sqrt{A}}{\sqrt{A}} \lambda_2 \end{pmatrix},$$

where $A = \lambda_2^2 + \lambda_3^2$ (cf. [6], p. 24); the second component of $\exp \mathbf{m}$ is $(\exp(X_1, X_2))_2 = \begin{pmatrix} r(1) & s(1) \\ v(1) & -r(1) \end{pmatrix}$, where

$$r(1) = \frac{\lambda_3 \lambda_1}{4(\lambda_2^2 + \lambda_3^2)} (e^{\sqrt{\lambda_2^2 + \lambda_3^2}} - e^{-\sqrt{\lambda_2^2 + \lambda_3^2}})^2,$$

$$s(1) = \frac{-\lambda_1}{4\sqrt{\lambda_2^2 + \lambda_3^2}} (e^{2\sqrt{\lambda_2^2 + \lambda_3^2}} - e^{-2\sqrt{\lambda_2^2 + \lambda_3^2}}) - \frac{\lambda_2 \lambda_1}{4(\lambda_2^2 + \lambda_3^2)} (e^{\sqrt{\lambda_2^2 + \lambda_3^2}} - e^{-\sqrt{\lambda_2^2 + \lambda_3^2}})^2,$$

$$v(1) = \frac{\lambda_1}{4\sqrt{\lambda_2^2 + \lambda_3^2}} (e^{2\sqrt{\lambda_2^2 + \lambda_3^2}} - e^{-2\sqrt{\lambda_2^2 + \lambda_3^2}}) - \frac{\lambda_2 \lambda_1}{4(\lambda_2^2 + \lambda_3^2)} (e^{\sqrt{\lambda_2^2 + \lambda_3^2}} - e^{-\sqrt{\lambda_2^2 + \lambda_3^2}})^2.$$

The Lie algebra \mathfrak{h} of the stabilizer cannot have the shape a), b), d) and e) since in these cases the intersection of \mathfrak{h} with \mathfrak{m} is the subspace $\langle e_2 \rangle$, $\langle e_1 \rangle$, $\langle e_2 \rangle$ and $\langle e_2, e_3 \rangle$ respectively.

In the case c) we have $\mathfrak{h} \cap \mathfrak{m} = 0$, there is no element of \mathfrak{m} which is conjugate to an element of \mathfrak{h} and

$$H = \left\{ \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} e & f \\ 0 & -e \end{pmatrix} \right); b, e, f \in \mathbb{R} \right\}.$$

To prove that in this case there is no global Bol loop we may proceed as we did it for the group $G = PSL_2(\mathbb{R}) \times \mathbb{R}$ and for the stabilizer

$\exp \mathbf{h}_2 = \left\{ \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \right); b \in \mathbb{R} \right\}$ since the first component of H is equal to the first component of $\exp \mathbf{h}_2$ and the first components of $\exp \mathbf{m}$ are the same in both cases.

Now we assume that the stabilizer H has the shape f). Since the group $SL_2(\mathbb{R})$ has no 3-dimensional linear representation the group G is isomorphic to the semidirect product of $PSL_2(\mathbb{R}) \times \mathbb{R}^3$ and H may be chosen as

$$H = \left\{ \left(\pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \begin{pmatrix} -x & y \\ y & x \end{pmatrix} \right), t \in [0, 2\pi), x, y \in \mathbb{R} \right\}.$$

We have $[\mathbf{m}, \mathbf{m}] = \mathbf{h}$, $\mathbf{g} = \mathbf{m} \oplus [\mathbf{m}, \mathbf{m}]$ and therefore the corresponding loop L realized on G/H is a Bruck loop. We prove that this loop is global.

For the connected simple Lie group $PSL_2(\mathbb{R})$ there exists a unique decomposition $g = \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \left(\pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \right)$, where $a > 0, b \in \mathbb{R}, t \in [0, 2\pi)$ (cf. [7], p. 525). Since

$$\begin{aligned} \left(\pm \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \right) \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix} \left(\pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \right) + \begin{pmatrix} k & l \\ l & -k \end{pmatrix} = \\ \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix} + \begin{pmatrix} k & l \\ l & -k \end{pmatrix} = \begin{pmatrix} k & u+l \\ l-u & -k \end{pmatrix}, \end{aligned}$$

each element of G can uniquely be written as $\left(\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \right) = \left(\begin{pmatrix} a_1 & 0 \\ b_1 & a_1^{-1} \end{pmatrix}, \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix} \right) \circ \left(\pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \begin{pmatrix} k & l \\ l & -k \end{pmatrix} \right)$ with $a, b, c, d \in \mathbb{R}, ad - bc = 1, x, y, z \in \mathbb{R}, a_1 > 0, b_1, u, k, l \in \mathbb{R}, t \in [0, 2\pi)$, such that $k = x, l = \frac{y+z}{2}, u = \frac{y-z}{2}$. The submanifold $\exp \mathbf{m}$ is the image of a section $\tau : G/H \rightarrow G$ if and only if each element $g \in G$ can uniquely be represented as a product $g = m h$ with $m \in \exp \mathbf{m}$ and $h \in H$ or equivalently $m = g h^{-1}$. It is sufficient to prove this for each element $g \in G$ with the shape

$$\left\{ \left(\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix} \right), a > 0, b, u \in \mathbb{R} \right\}.$$

The first component of $\exp \mathbf{m}$ is precisely the section σ_1 of the hyperbolic plane loop given before Theorem 6. Therefore for given $a > 0, b \in \mathbb{R}$ we have unique $\lambda_2, \lambda_3 \in \mathbb{R}, t \in [0, 2\pi)$ such that

$$\begin{pmatrix} \cosh \sqrt{A} + \frac{\sinh \sqrt{A}}{\sqrt{A}} \lambda_2 & \frac{\sinh \sqrt{A}}{\sqrt{A}} \lambda_3 \\ \frac{\sinh \sqrt{A}}{\sqrt{A}} \lambda_3 & \cosh \sqrt{A} - \frac{\sinh \sqrt{A}}{\sqrt{A}} \lambda_2 \end{pmatrix} =$$

$$\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},$$

where $A = \lambda_2^2 + \lambda_3^2$. Hence we have to consider the second component of $\exp \mathbf{m}$. For given u, λ_2, λ_3 we have to find unique $\lambda_1, k, l \in \mathbb{R}$ such that

$$\begin{pmatrix} r(1) & s(1) \\ v(1) & -r(1) \end{pmatrix} = \begin{pmatrix} k & l + u \\ l - u & -k \end{pmatrix},$$

where $r(1), s(1), v(1)$ are values of functions, which depend on the variables $\lambda_1, \lambda_2, \lambda_3$. Since for λ_1 we obtain the equation

$$2u = \frac{-\lambda_1}{2\sqrt{\lambda_2^2 + \lambda_3^2}} (e^{2\sqrt{\lambda_2^2 + \lambda_3^2}} - e^{-2\sqrt{\lambda_2^2 + \lambda_3^2}})$$

we have the unique solutions $\lambda_1 = \frac{-4u\sqrt{\lambda_2^2 + \lambda_3^2}}{e^{2\sqrt{\lambda_2^2 + \lambda_3^2}} - e^{-2\sqrt{\lambda_2^2 + \lambda_3^2}}}$, $k = r(1)$ and $l = \frac{s(1) + v(1)}{2}$.

Summarizing the above discussion we see that $\exp \mathbf{m}$ is an image of a global section $\sigma : G/H \rightarrow G$. The section σ determines a global Bol loop $(\hat{L}, *)$ since $[[\mathbf{m}, \mathbf{m}], \mathbf{m}] \subseteq \mathbf{m}$ and therefore the equation $x * a = b$ has the solution $x = a^{-1} * [(a * b) * a^{-1}]$ for all $a, b \in \hat{L}$ (cf. [9], Corollary 3.11, p. 51 and [18], Lemma 1.3, p. 17). We call the loop $(\hat{L}, *)$ the pseudo-euclidean space loop.

In order to determine within the isotopism class \mathcal{C} of the pseudo-euclidean space loop $(\hat{L}, *)$ the isomorphism classes we consider arbitrary complements \mathbf{m} to the Lie algebra \mathbf{h} of the stabilizer H of $e \in L$ in \mathfrak{g} . These complements have the form:

$$\langle e_1 + a_1 e_4 + a_2 e_5 + a_3 e_6, e_2 + b_1 e_4 + b_2 e_5 + b_3 e_6, e_3 + c_1 e_4 + c_2 e_5 + c_3 e_6 \rangle$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$. Such complements are Bol complements if and only if

$$\mathbf{m} = \mathbf{m}_{b_3, c_3, c_2} = \langle e_1 - c_3 e_6 + b_3 e_5, e_2 + c_2 e_6 + b_3 e_4, e_3 + c_2 e_5 + c_3 e_4 \rangle,$$

where $c_2, c_3, b_3 \in \mathbb{R}$, and $b_3^2 + c_3^2 \neq 1$. According to remark in section 2 the subspace $\mathbf{m}_{b_3, c_3, c_2}$ belongs to a loop within the isotopism class \mathcal{C} if and only if the subalgebra $\mathbf{h}_{b_3, c_3, c_2} = [\mathbf{m}_{b_3, c_3, c_2}, \mathbf{m}_{b_3, c_3, c_2}]$ is the Lie algebra of the semidirect product of a 2-dimensional normal translation group T by a 1-dimensional rotation group S . It follows that then $\mathbf{h}_{b_3, c_3, c_2} = \mathfrak{t} \oplus \mathfrak{s}$, where \mathfrak{t} respectively \mathfrak{s} is the Lie algebra of T respectively of S and the Cartan-Killing form k given by **K2** in section 3 is zero on \mathfrak{t} and negative on \mathfrak{s} . Since

$[\mathfrak{m}_{b_3, c_3, c_2}, \mathfrak{m}_{b_3, c_3, c_2}] = \langle d_1, d_2, d_3 \rangle$ with

$$d_1 = (1 - b_3^2)e_6 - c_3e_1 - c_3b_3e_5, \quad d_2 = (1 - c_3^2)e_5 + b_3e_1 - c_3b_3e_6, \\ d_3 = e_4 + c_3e_3 + b_3e_2 + c_2c_3e_5 + b_3c_2e_6,$$

the subalgebra $\langle d_1, d_2 \rangle$ is an ideal of $\mathfrak{h}_{b_3, c_3, c_2}$ and $k(d_3) < 0$ if and only if $b_3^2 + c_3^2 < 1$.

The Bol complements $\mathfrak{m}_{b_3, c_3, c_2}$ with $b_3^2 + c_3^2 > 1$ are according to Remark in section 2 symmetric spaces belonging to the Lie triple system given in **6.2** (section 3). But the local Bol loop corresponding to $\mathfrak{g} = \mathfrak{m}_{b_3, c_3, c_2} \oplus \mathfrak{h}$ cannot be extended to a global Bol loop.

The loop L belonging to $(G, H, \exp \mathfrak{m}_{b_3, c_3, c_2})$, $b_3^2 + c_3^2 < 1$ and the loop L' corresponding to $(G, H, \exp \mathfrak{m}_{b'_3, c'_3, c'_2})$, $b'_3{}^2 + c'_3{}^2 < 1$ are isomorphic if and only if there exists an automorphism α of \mathfrak{g} such that $\alpha(\mathfrak{m}_{b'_3, c'_3, c'_2}) = \mathfrak{m}_{b_3, c_3, c_2}$ and $\alpha(\mathfrak{h}) = \mathfrak{h}$. The automorphism group A of the Lie algebra \mathfrak{g} leaving \mathfrak{h} invariant consists of linear mappings determined by the basis transformations:

$$\alpha(e_1) = a e_1,$$

$$\alpha(e_6) = b_2 e_6 + b_4 e_5,$$

$$\alpha(e_5) = -\varepsilon b_4 e_6 + \varepsilon b_2 e_5,$$

$$\alpha(e_4) = \varepsilon e_4 + d_5 e_6 + d_6 e_5,$$

$$\alpha(e_2) = \frac{b_2}{a} e_2 + \frac{b_4}{a} e_3 + \left(\frac{-\varepsilon d_5 b_4 + \varepsilon d_6 b_2}{a} \right) e_1 + f_5 e_6 + f_6 e_5,$$

$$\alpha(e_3) = \varepsilon \frac{b_2}{a} e_3 - \varepsilon \frac{b_4}{a} e_2 + \left(\frac{-d_5 b_2 - d_6 b_4}{a} \right) e_1 - \varepsilon f_6 e_6 + \varepsilon f_5 e_5,$$

where $\varepsilon \in \{1, -1\}$ and for the real parameters $a, b_2, b_4, d_5, d_6, f_5, f_6$ the properties $a^2 = b_2^2 + b_4^2 \neq 0$ and $b_2 f_6 = b_4 f_5$ are satisfied.

According to **K2** (Section 3) the Cartan-Killing form k is invariant under α . Hence the 3 angles between $\mathfrak{m}_{b_3, c_3, c_2}$ and $\hat{\mathfrak{m}} := \mathfrak{m}_{0,0,0}$ respectively between $\mathfrak{m}_{b'_3, c'_3, c'_2}$ and $\hat{\mathfrak{m}}$ are equal with respect to k .

Now we compute the isomorphism classes of the loops L_{b_3, c_3, c_2} . Since the automorphism γ of \mathfrak{g} given by $\gamma(e_1) = -e_1$, $\gamma(e_2) = -e_2 - 2c_2 e_4$, $\gamma(e_3) = -e_3 - 2c_2 e_5$, $\gamma(e_i) = e_i$ for $i = 4, 5, 6$ maps $\mathfrak{m}_{b_3, c_3, c_2}$ onto $\mathfrak{m}_{b_3, c_3, 0}$ we seek for which real triples $(b'_3, c'_3, 0)$ and $(b_3, c_3, 0)$ there is an automorphism α in A , such that $\alpha(\mathfrak{m}_{b'_3, c'_3, 0}) = \mathfrak{m}_{b_3, c_3, 0}$. This condition is equivalent to the following system of equations:

$$\begin{aligned} -a c_3 &= -c'_3 b_2 - \varepsilon b'_3 b_4, \\ a b_3 &= -c'_3 b_4 + \varepsilon b'_3 b_2, \\ a (f_5 + b'_3 d_5) &= (\varepsilon d_5 b_4 - \varepsilon d_6 b_2) c_3, \\ a (f_6 + b'_3 d_6) &= (-\varepsilon d_5 b_4 + \varepsilon d_6 b_2) b_3, \\ a (-\varepsilon f_6 + c'_3 d_5) &= (d_5 b_2 + d_6 b_4) c_3, \\ a (\varepsilon f_5 + c'_3 d_6) &= (-d_5 b_2 - d_6 b_4) b_3. \end{aligned}$$

Solving this system of equations we see that the loop $\hat{L}_d = L_{d,0,0}$ for $0 \leq d < 1$

is a representative of the isomorphism class consisting of loops L_{b_3, c_3, c_2} with $\sqrt{b_3^2 + c_3^2} = d$. The pseudo-euclidean space loop $(\hat{L}, *) = L_{0,0,0}$ is a representative of the isomorphism class consisting of the loops $L_{0,0,c_2}$, $c_2 \in \mathbb{R}$.

A loop isomorphic to \hat{L} is a Bruck loop and hence $\mathfrak{m}_{0,0,c_2}$ is reductive to \mathfrak{h} in \mathfrak{g} . If $\mathfrak{m}_{d,0,0}$ with $d \neq 0$ were reductive to \mathfrak{h} in the Lie algebra \mathfrak{g} then for $e_4 \in \mathfrak{h}$ and $e_1 + d e_5 \in \mathfrak{m}_{d,0,0}$ the element $[e_4, e_1 + d e_5] = d e_6 \in \mathfrak{h}$ would be contained in $\mathfrak{m}_{d,0,0}$, which is a contradiction.

By $E(2,1)$ we denote the pseudo-euclidean space the points of which are represented by the matrices

$$(I, Y) = \left(I, \begin{pmatrix} x & k+l \\ k-l & -x \end{pmatrix} \right),$$

where $x, k, l \in \mathbb{R}$, and which has as norm $\|(I, Y)\| = x^2 + k^2 - l^2$. The group G acts on the space $E(2,1)$ in the following way: For given $(A, X) \in G$ and $(I, Y) \in E(2,1)$

$$(*) \quad (A, X) * (I, Y) = (I, A^{-1}YA + X).$$

The norm is invariant under the action of G , therefore G is the connected component of the motion group of $E(2,1)$.

The 3-dimensional pseudo-euclidean geometry $E(2,1)$ has also a representation \mathcal{R} on the affine space \mathbb{R}^3 such that the motion group consists of the affine mappings

$$(B, b) : (x, y, z) \mapsto (x, y, z)B^T + b,$$

where $B = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ with $\det B = 1$, $a_1^2 + a_2^2 = a_3^2 + 1$, $b_1^2 + b_2^2 = b_3^2 + 1$, $c_1^2 + c_2^2 = c_3^2 - 1$, and $b = (b_1, b_2, b_3)$ ([2], Kapitel 6). The mappings

$$\omega : \left(I, \begin{pmatrix} k & l+n \\ l-n & -k \end{pmatrix} \right) \mapsto (k, l, n)$$

and

$$\Omega : \left(\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix} \right) \mapsto$$

$$\left(\begin{pmatrix} da+bc & cd-ba & cd+ba \\ bd-ca & \frac{a^2+d^2-b^2-c^2}{2} & \frac{d^2+b^2-a^2-c^2}{2} \\ bd+ca & \frac{d^2-b^2-a^2+c^2}{2} & \frac{d^2+b^2+a^2+c^2}{2} \end{pmatrix}, (x, y, z) \right)$$

establish an isometry from $E(2,1)$ onto \mathcal{R} ([10], pp. 97-103).

The stabilizer H which is the image of the Lie algebra of the shape f) under the exponential map leaves in $E(2,1)$ the plane P consisting of the points

$\left\{ \left(I, \begin{pmatrix} x & y \\ y & -x \end{pmatrix} \right), x, y \in \mathbb{R} \right\}$ invariant. The points of P satisfy $x^2 + y^2 > 0$.

The planes of \mathcal{R} the points (x, y, z) of which satisfy $x^2 + y^2 > z^2$ are called euclidean planes. The connected component $\Omega(G)$ of the motion group of \mathcal{R} acts transitively on the set Ψ of the euclidean planes and the set $\Omega(\exp \mathbf{m}_{0,0,0})$ is sharply transitive on Ψ . The planes of Ψ can be taken as the points of the pseudo-euclidean space loop $(\hat{L}, *)$ such that the multiplication is given by

$$(**) \quad Q_1 * Q_2 = \tau_{P, Q_1}(Q_2), \quad \text{for all } Q_1, Q_2 \in \Psi,$$

where τ_{P, Q_1} is the unique element of $\Omega(\exp \mathbf{m}_{0,0,0})$ mapping the plane P , which is the identity of \hat{L} onto Q_1 .

From the above discussion we obtain the

Theorem 8. *There is only one isotopism class \mathcal{C} of the 3-dimensional connected differentiable Bol loops L such that the group G generated by the left translations $\{\lambda_x; x \in L\}$ is a 6-dimensional non-semisimple and non-solvable Lie group. The group G is isomorphic to $PSL_2(\mathbb{R}) \times \mathbb{R}^3$, where the action of $PSL_2(\mathbb{R})$ on \mathbb{R}^3 is the adjoint action of $PSL_2(\mathbb{R})$ on its Lie algebra. The stabilizer H of $e \in L$ in G is isomorphic to*

$$H = \left\{ \left(\pm \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \begin{pmatrix} -x & y \\ y & x \end{pmatrix} \right); t \in [0, 2\pi], x, y \in \mathbb{R} \right\}.$$

The class \mathcal{C} consists of Bol loops L_{b_3, c_3, c_2} , where b_3, c_3, c_2 are real parameters and $b_3^2 + c_3^2 < 1$. As a representative of \mathcal{C} one may choose the pseudo-euclidean space loop $\hat{L} = L_{0,0,0}$. This loop is a Bruck loop.

The tangential spaces $T_1\sigma(G/H)$ of the loops L_{b_3, c_3, c_2} at $1 \in G$ are the Bol complements $\mathbf{m}_{b_3, c_3, c_2}$. The loops L and L' in \mathcal{C} are isomorphic if and only if the angles between the 3-spaces $\mathbf{m}_{b_3, c_3, c_2}$, $\mathbf{m}_{b'_3, c'_3, c'_2}$ and the 3-space $\hat{\mathbf{m}} = \mathbf{m}_{0,0,0}$ are the same with respect to the Cartan-Killing form. As representatives of the isomorphism classes of the Bol loops L_{b_3, c_3, c_2} may be chosen the Bol loops $\hat{L}_d = L_{d,0,0}$ for $d \in [0, 1)$.

The pseudo-euclidean space loop $(\hat{L}, *)$ has a representation on the manifold Ψ of euclidean planes in the 3-dimensional pseudo-euclidean geometry \mathcal{R} with the multiplication is given by the formula (**).

8 Construction of 3-dimensional differentiable loops within the pseudo-euclidean space

Let \mathcal{R} be the 3-dimensional pseudo-euclidean affine space. We embed the affine space \mathbb{R}^3 into the projective space $P_3(\mathbb{R})$ such that $(x, y, z) \mapsto$

$k(1, x, y, z)$, $k \in \mathbb{R} \setminus \{0\}$. Then the projective plane E at infinity consists of the points $\{k(0, x, y, z) \mid x, y, z \in \mathbb{R}, k \in \mathbb{R} \setminus \{0\}\}$. The cone $x^2 + y^2 - z^2 = 0$ of the affine space \mathbb{R}^3 intersects the plane E in the conic given by $C : \{k(0, x, y, z) \mid x^2 + y^2 = z^2, k(0, x, y, z) \neq (0, 0, 0, 0)\}$. The conic C divides the points of $E \setminus C$ in two regions R_1, R_2 . The points $(0, x, y, z)$ of R_1 respectively of R_2 are characterized by the property $x^2 + y^2 < z^2$, respectively $x^2 + y^2 > z^2$.

Let Ψ be the manifold of planes which intersect the plane E in lines contained in the region R_2 , i.e. in lines which do not meet C . There is a polarity π of E such that the absolute points of π are the points of the conic C . The polarity π interchanges the points of R_1 with the lines of R_2 . We may assume that the connected component of the motion group of $G = PSL_2(\mathbb{R}) \times \mathbb{R}^3$ on \mathcal{R} is given by the Lorentz transformations. Since any translation of \mathbb{R}^3 leaves E pointwise fixed the group induced by G on E is isomorphic to $PSL_2(\mathbb{R})$ and it is already induced by the stabilizer Γ of $(0, 0, 0)$ in G .

Let Σ be the submanifold of $\Gamma \cong PSL_2(\mathbb{R})$ which is the image of the section σ_1 defining the hyperbolic plane loop (cf. σ_1 before Theorem 6) and let Λ be a 1-dimensional translation group leaving a line S invariant which intersects any plane of Ψ in precisely one point and contains the point $(0, 0, 0)$. We show that the product $\Theta = \Lambda\Sigma$ acts sharply transitively on the planes of Ψ and may be taken as the image of a section for a differentiable 3-dimensional loop L_Λ having G as the group topologically generated by its left translations. Let D_1, D_2 be planes belonging to Ψ . We show that there is precisely an element $\rho = \omega\delta$ with $\delta \in \Sigma$ and $\omega \in \Lambda$ such that $\rho(D_1) = (\omega\delta)(D_1) = D_2$. Let $W_1 = D_1 \cap E$ and $W_2 = D_2 \cap E$. The lines W_1 and W_2 are contained in the region R_2 . Hence $\pi(W_1)$ and $\pi(W_2)$ are points in R_1 . In Σ there is precisely one element δ such that $\delta(\pi(W_1)) = \pi(W_2)$. But then $\delta(W_1) = W_2$ and $\delta(D_1)$ is parallel to D_2 . Let $s_1 = D_1 \cap S$ and $s_2 = D_2 \cap S$. Then there exists precisely one translation ω in Λ with $\omega(s_1) = s_2$ and $\omega\delta(D_1) = D_2$.

Let P be a plane of Ψ containing the point $(0, 0, 0)$ and H be the stabilizer of P in G . Then for the manifold Θ , which topologically generates G , one has $\Theta \cap H = \{1\}$. Let $J = P \cap E$ and $L(P)$ be the line joining the points $(0, 0, 0)$ and $\pi(J)$. This line intersects any plane of Ψ in precisely one point. Let $\Lambda_{L(P)}$ be the 1-dimensional translation group leaving the line $L(P)$ invariant. The multiplication for the loop $L_{\Lambda_{L(P)}}$ with the identity P may be defined as follows:

$$(X, Y) \mapsto \omega_{(\tau_{J,Q}(P), X)}(\tau_{J,Q}(Y)) : \Psi \times \Psi \rightarrow \Psi,$$

where $\tau_{J,Q}$ is the unique element in Σ mapping J onto the line $Q = X \cap E$ and $\omega_{(\tau_{J,Q}(P), X)}$ is the unique translation in $\Lambda_{L(P)}$ mapping the plane $\tau_{J,Q}(P)$ onto X .

Let K be the 1-dimensional compact subgroup of Γ which leaves the line J invariant. Then K fixes $\pi(J)$ and leaves the line $L(P)$ pointwise invariant. Since $\Sigma \subset \Gamma$ is the image of a section corresponding to the hyperbolic plane loop with the identity $\pi(J)$ we have $g\Sigma g^{-1} = \Sigma$ and $g\Lambda_{L(P)}g^{-1} = \Lambda_{L(P)}$ for all $g \in K$. Hence the loop $L_{\Lambda_{L(P)}}$ has a 1-dimensional compact group of automorphisms.

Let Λ^* be a 1-dimensional translation group such that the lines S which are invariant under Λ^* intersect any plane of Ψ , but $S \cap E \neq \pi(J)$. Then the loop L_{Λ^*} with identity P which corresponds to the section $\Lambda^*\Sigma$ and to the stabilizer H is not isomorphic to the loop $L_{\Lambda_{L(P)}}$. This can be seen in the following way: The group $K \subseteq H$ does not leave S invariant and hence does not normalize Λ^* . It follows that $g\Theta g^{-1} = g\Lambda^*\Sigma g^{-1} = (g\Lambda^*g^{-1})g\Sigma g^{-1} = (g\Lambda^*g^{-1})\Sigma \neq \Theta$ and K does not consist of automorphisms of the loop L_{Λ^*} . Every loop $L_{g^{-1}\Lambda^*g}$ with $g \in K$ is isomorphic to L_{Λ^*} .

No loop L_Λ is a Bol loop since otherwise for elements $1 \neq \lambda \in \Lambda$ and $1 \neq \rho \in \Sigma$ the element $(\lambda\rho) \cdot 1 \cdot \lambda\rho$ should be in $\Lambda\Sigma$. But one has $\lambda\rho\lambda\rho = \lambda\rho\lambda\rho^{-1}\rho^2 \notin \Lambda\Sigma$ since $\lambda\rho\lambda\rho^{-1} \notin \Lambda$. The submanifold Θ is the image of a differentiable section. Hence the multiplication of L_Λ as well as the mappings $(a, b) \mapsto a \setminus b : L_\Lambda \times L_\Lambda \rightarrow L_\Lambda$ are differentiable. Let D_1 and D_2 be elements of Λ_L and $D_i \cap E = W_i$ for $i = 1, 2$. Since the mapping $(W_1, W_2) \mapsto \tau_{J,Q} \in \Sigma$ with $\tau_{J,Q}(W_1) = W_2$ as well as for parallel planes D_1 and D_2 the map $(D_1, D_2) \rightarrow \omega_{D_1, D_2} \in \Lambda$ with $\omega_{D_1, D_2}(D_1) = D_2$ are differentiable any loop L_Λ is differentiable.

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