3-dimensional Bol loops corresponding to solvable Lie triple systems

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Abstract

We classify the connected 3-dimensional differentiable Bol loops L having a solvable Lie group as the group topologically generated by the left translations of L using 3-dimensional solvable Lie triple systems. Together with [4] our results complete the classification of all 3-dimensional differentiable Bol loops.

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1 Introduction

The present research on differentiable loops is focused to such loops which have local forms determined in a unique way by their tangential objects. The most important and most studied class of differentiable loops are the Bol loops. Their tangential objects, the Bol algebras, may be seen as Lie triple systems with an additional binary operation (cf. [15] pp. 84–86, Def. 6.10). As known the Lie triple systems are in one-to-one correspondence to (global) simply connected symmetric spaces (cf. [10], [15] Section 6). Hence there is a strong connection between the theory of differentiable Bol loops and the theory of symmetric spaces. In particular the theory of connected differentiable Bruck loops (which form a subclass of the class of Bol loops) is essentially the theory of affine symmetric spaces (cf. [15] Section 11).

The 2-dimensional differentiable Bol loops are classified in [15] (Section 25). My goal is to classify differentiable multiplications satisfying the left Bol identity on 3-dimensional connected manifolds since these manifolds also play an exceptional role.

The 3-dimensional differentiable Bol loops having a non-solvable Lie group as the group topologically generated by the left translations have been determined in [4]. In this paper I classify all 3-dimensional connected differentiable (global) Bol loops in which the left translations generate a solvable Lie group. Since for differentiable Bol loops the group topologically generated by the left translations is always a Lie group with the results of this paper the classification of 3-dimensional differentiable Bol loops is complete.

We treat the differentiable Bol loops as images of global differentiable sections $\sigma : G/H \to G$, where G is a connected Lie group, H is a closed subgroup containing no non-trivial normal subgroup of G and for all $r, s \in \sigma(G/H)$ the element rsr lies in $\sigma(G/H)$. In this treatment the exponential images of Lie triple systems form local Bol loops. Hence for the classification of 3-dimensional differentiable Bol loops L having a solvable Lie group G as the group topologically generated by the left translations we proceed in the following way: First we determine all solvable 3-dimensional Lie triple systems $\mathbf{m}$ and all enveloping Lie algebras $\mathbf{g}$ of $\mathbf{m}$. We show that $\mathbf{g}$ and therefore the solvable Lie group G topologically generated by the left translations of a differentiable Bol loop has dimension four or five. Then we find for any pair $(\mathbf{g}, \mathbf{m})$ all subalgebras $\mathbf{h}$ containing no non-trivial ideal of $\mathbf{g}$ such that $\mathbf{g} = \mathbf{m} \oplus \mathbf{h}$ and we prove that global Bol loops L correspond precisely to those exponential images of $\mathbf{m}$, which form a system of representatives for the cosets of $\exp \mathbf{h}$ in G.

If the group G is nilpotent then G is the 4-dimensional non-decomposable nilpotent Lie group and the corresponding 3-dimensional Bol loops form only one isotopism class containing precisely two isomorphism classes (Theorem 4, Section 5.1).

If the solvable Lie group G is 4-dimensional and not nilpotent then it is a central extension of a 1-dimensional Lie group N either by the 3-dimensional solvable Lie group G_1 with precisely two 1-dimensional normal subgroups or by the direct product G_2 of $\mathbb{R}$ and the 2-dimensional non-abelian Lie group. All loops L corresponding to the extensions of N by G_1 are extensions of N by a loop isotopic to the pseudo-euclidean plane loop (Theorem 6 in Section 5.2 and Theorem 9 in Section 5.3). The 3-dimensional Bol loops having the central extension of $\mathbb{R}$ by G_2 as the group topologically generated by their left translations are all isomorphic (Theorem 6 in Section 5.2).

If the solvable Lie group G is 5-dimensional then it is either a semidirect product G of $\mathbb{R}^4$ by the group $S = \mathbb{R}$ such that either no element of S different from the identity has a real eigenvalue in $\mathbb{R}^4$ or such that G has a 1-dimensional centre and precisely two 1-dimensional non-central normal subgroups. We prove that for both groups G there exist infinitely many non-isotopic 3-dimensional differentiable Bol loops corresponding to G (Theorem 7 in Section 5.2 and Theorem 11 in Section 6).

The variety of the 3-dimensional differentiable Bol loops having a solvable Lie group as the group topologically generated by their left translations contains families of loops depending on up to four real parameters. The size of this variety is so enormous that a classification of 4-dimensional differentiable Bol loops having a solvable Lie group as the group generated by the left translations seems to be not attainable.

2 Some basic notions of the theory of Bol loops

A set L with a binary operation $(x, y) \mapsto x \cdot y$ is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution which we denote by $y = a \backslash b$ and $x = b/a$. The left translation $\lambda_a : y \mapsto a \cdot y : L \to L$ is a bijection of L for any $a \in L$. Two loops $(L_1, \circ)$ and $(L_2, *)$ are called isotopic if there are three bijections $\alpha, \beta, \gamma : L_1 \to L_2$ such that $\alpha(x) * \beta(y) = \gamma(x \circ y)$ holds for any $x, y \in L_1$. Isotopy is an equivalence relation. If $\alpha = \beta = \gamma$ then the isotopic loops $(L_1, \circ)$ and $(L_2, *)$ are called isomorphic. Let $(L_1, \cdot)$ and $(L_2, *)$ be two loops. The set $L = L_1 \times L_2 = \{(a, b) \mid a \in L_1,\ b \in L_2\}$ with the componentwise multiplication is again a loop, which is called the direct product of L_1 and L_2, and the loops $(L_1, \cdot)$, $(L_2, *)$ are subloops of L.

A loop L is called a Bol loop if for any two left translations λ_a, λ_b the product $\lambda_a \lambda_b \lambda_a$ is again a left translation of L. If L_1 and L_2 are Bol loops, then the direct product $L_1 \times L_2$ is again a Bol loop.

If the elements of L are points of a differentiable manifold and the operations $(x, y) \mapsto x \cdot y$, $(x, y) \mapsto x/y$, $(x, y) \mapsto x \backslash y : L \times L \to L$ are differentiable mappings then L is called a differentiable loop.

If L is a connected differentiable Bol loop then the group G topologically generated by the left translations is a connected Lie group (cf. [15], p. 33; [11], pp. 414–416).

Every connected differentiable Bol loop is isomorphic to a loop L realized on the factor space G/H, where G is a connected Lie group, H is a connected closed subgroup containing no normal subgroup $\neq \{1\}$ of G and $\sigma : G/H \to G$ is a differentiable section with $\sigma(H) = 1 \in G$ such that the subset $\sigma(G/H)$ generates G and for all $r, s \in \sigma(G/H)$ the element rsr is contained in $\sigma(G/H)$ (cf. [15], p. 18 and Lemma 1.3, p. 17, [8], Corollary 3.11, p. 51). The multiplication of L on the factor space G/H is defined by $xH * yH = \sigma(xH)yH$.

Let L_1 be a loop in the factor space G/H with respect to the section $\sigma : G/H \to G$. The loops L_2 isomorphic to L_1 and having the same set of left translations $\sigma(G/H)$ and the same group G as the group generated by $\sigma(G/H)$ correspond to automorphisms α of G, which leave $\sigma(G/H)$ invariant. The loop L_2 corresponding to α is realized on $G/\alpha(H)$ such that the multiplication of L_2 is given by $x\alpha(H) * y\alpha(H) = [\alpha \circ \sigma \circ \alpha_H^{-1}(x\alpha(H))]y\alpha(H)$, where the mapping $\alpha_H : G/H \to G/\alpha(H)$ is defined by $kH \to \alpha(k)\alpha(H)$. Moreover, let L and L' be loops having the same group G generated by their left translations. Then L and L' are isotopic if and only if there is a loop L'' isomorphic to L' having G again as the group generated by its left translations such that there exists an inner automorphism τ of G mapping the stabilizer H'' of $e'' \in L''$ onto the stabilizer H of $e \in L$ (cf. [15], Theorem 1.11, p. 21).

A real vector space V with a trilinear multiplication $(.\,,.\,,.\,)$ is called a Lie triple system $\mathcal{V}$, if the following identities are satisfied:

$$(X, X, Y) = 0 \tag{1}$$
$$(X, Y, Z) + (Y, Z, X) + (Z, X, Y) = 0 \tag{2}$$
$$\begin{aligned}(X, Y, (U, V, W)) = ((X, Y, U), V, W)& \\ + (U, (X, Y, V), W) + (U, V, (X, Y, W)).&\end{aligned} \tag{3}$$

A Bol algebra A is a Lie triple system $(V, (.\,,.\,,.\,))$ with a bilinear skew-symmetric operation $[[.\,,.\,]]$, $(X, Y) \mapsto [[X, Y]] : V \times V \to V$ such that the following identity is satisfied:

$$\begin{aligned}[[(X, Y, Z), W]] - [[(X, Y, W), Z]] + \big(Z, W, [[X, Y]]\big)& \\ - \big(X, Y, [[Z, W]]\big) + \big[[[X, Y]], [[Z, W]]\big] = 0.&\end{aligned}$$

With any connected differentiable Bol loop L we can associate a Bol algebra in the following way: Let G be the Lie group topologically generated by the left translations of L, and let $(\mathbf{g}, [.\,,.\,])$ be the Lie algebra of G. Denote by $\mathbf{h}$ the Lie algebra of the stabilizer H of the identity $e \in L$ in G and by $\mathbf{m} = T_1\sigma(G/H)$ the tangent space at $1 \in G$ of the image of the section $\sigma : G/H \to G$ corresponding to the Bol loop L. Then $\mathbf{g} = \mathbf{m} \oplus \mathbf{h}$, $\big[[\mathbf{m}, \mathbf{m}], \mathbf{m}\big] \subseteq \mathbf{m}$ and $\mathbf{m}$ generates the Lie algebra $\mathbf{g}$. The subspace $\mathbf{m}$ with the operations defined by $(X, Y, Z) \mapsto \big[[X, Y], Z\big]$, $(X, Y) \mapsto [X, Y]_{\mathbf{m}}$, where X, Y, Z are elements of $\mathbf{m}$ and $Z \mapsto Z_{\mathbf{m}} : \mathbf{g} \to \mathbf{m}$ is the projection of $\mathbf{g}$ onto $\mathbf{m}$ along $\mathbf{h}$, is the Bol algebra of L. The Lie algebra $\mathbf{g}$ of G is isomorphic to an enveloping Lie algebra of the Lie triple system $\mathbf{m}$ corresponding to L.

An imbedding T of a Lie triple system $\mathcal{V}$ into a Lie algebra $\mathcal{L}^T$ is a linear mapping $X \mapsto X^T$ of $\mathcal{V}$ into $\mathcal{L}^T$ such that

(i) $(X, Y, Z)^T = [[X^T, Y^T], Z^T]$ holds for all $X, Y, Z \in \mathcal{V}$ and

(ii) the image $\mathcal{V}^T$ generates $\mathcal{L}^T$.

The Lie algebra $\mathcal{L}^T$ is called enveloping Lie algebra of the imbedding T. An imbedding U of a Lie triple system $\mathcal{V}$ is called universal and $\mathcal{L}^U = \mathcal{V}^U \oplus [\mathcal{V}^U, \mathcal{V}^U]$ is a universal Lie algebra of $\mathcal{V}$ if and only if, for every imbedding T of $\mathcal{V}$ the mapping $X^U \mapsto X^T$ is single-valued and can be extended to a Lie algebra homomorphism of $\mathcal{L}^U$ onto $\mathcal{L}^T$ ([7], p. 519, and [9], p. 219).

In [7] (pp. 517–518) it is shown that for every Lie triple system $\mathcal{V}$ there exists a particular imbedding S such that $\sum_i [X_i^S, Y_i^S] = 0$ for $X_i, Y_i \in \mathcal{V}$ if and only if $\sum_i (X_i, Y_i, Z) = 0$ for every $Z \in \mathcal{V}$. Moreover $\mathcal{L}^S = \mathcal{V}^S \oplus [\mathcal{V}^S, \mathcal{V}^S]$. This imbedding is called the standard imbedding of $\mathcal{V}$ and the Lie algebra

$\mathcal{L}^S$ is the smallest enveloping algebra. Using the standard imbedding the existence and the uniqueness of a universal imbedding U of every Lie triple system $\mathcal{V}$ follows ([7], p. 519). Moreover if $\mathcal{V}$ is a n-dimensional Lie triple system then the universal Lie algebra $\mathcal{L}^U$ of $\mathcal{V}$ and therefore every enveloping Lie algebra $\mathcal{L}^T$ of $\mathcal{V}$ has dimension at least n and at most $n(n+1)/2$.

A loop L is called a left A-loop if each $\lambda_{x,y} = \lambda_{xy}^{-1}\lambda_x\lambda_y : L \to L$ is an automorphism of L. If L is a differentiable left A-loop then the group G topologically generated by its left translations is a Lie group (cf. [15], Proposition 5.20, p. 75). If $\mathbf{g}$ is the Lie algebra of G and $\mathbf{h}$ is the Lie algebra of the stabilizer H of the identity $e \in L$ in G then one has $\mathbf{m} \oplus \mathbf{h} = \mathbf{g}$ and $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$, where $\mathbf{m}$ is the tangent space $T_e L$ (cf. [15], Definition 5.18. and Proposition 5.20. pp. 74–75).

A differentiable loop L is called a Bruck loop if there is an involutory automorphism σ of the Lie algebra $\mathbf{g}$ of the connected Lie group G generated by the left translations of L such that the tangent space $T_e(L) = \mathbf{m}$ is the -1-eigenspace and the Lie algebra $\mathbf{h}$ of the stabilizer H of $e \in L$ in G is the $+1$-eigenspace of σ.

Let L_1 be a loop defined on the factor space G_1/H_1 with respect to a section $\sigma_1 : G_1/H_1 \to G_1$ the image of which is the set $M_1 \subset G_1$. Let G_2 be a group, let $\varphi : H_1 \to G_2$ be a homomorphism and $(H_1, \varphi(H_1)) = \{(x, \varphi(x)); x \in H_1\}$. A loop L is called a Scheerer extension of G_2 by L_1 if L is defined on the factor space $(G_1 \times G_2)/(H_1, \varphi(H_1))$ with respect to the section $\sigma : (G_1 \times G_2)/(H_1, \varphi(H_1)) \to G_1 \times G_2$ the image of which is the set $M_1 \times G_2$ ([15], Section 2).

From [4] we will use often the following fact:

Lemma 1. *Let L be a differentiable global loop and denote by $\mathbf{m}$ the tangent space of $T_1\sigma(G/H)$, where $\sigma : G/H \to G$ is the section corresponding to L. Then $\mathbf{m}$ does not contain any element of $Ad_{g^{-1}}\mathbf{h} = g\mathbf{h}g^{-1}$ for some $g \in G$. Moreover, every element of G can be written uniquely as a product of an element of $\sigma(G/H)$ with an element of H.*

3 3-dimensional solvable Lie triple systems

Let $(\mathbf{m}, [[.,.],.])$ be a Lie triple system and let $(\mathbf{g}^*, [.,.])$ be the standard enveloping Lie algebra of $(\mathbf{m}, [[.,.],.])$ ([9], p. 219). The isomorphism classes of the 3-dimensional solvable Lie triple systems and their standard enveloping Lie algebras may be classified as follows:

1. If the Lie triple system $\mathbf{m}$ is abelian then it is the 3-dimensional abelian Lie algebra, which is also the standard enveloping Lie algebra of $\mathbf{m}$ (see Theorem 4.1, Type I in [1]).

2. Since a 3-dimensional Lie triple system cannot have a 2-dimensional centre we consider now the case that $\mathbf{m}$ has a 1-dimensional centre $\langle e_1 \rangle$.

Then the factor Lie triple system $\mathbf{m}/\langle e_1\rangle$ is 2-dimensional and according to [5] (pp. 44–45) it is either abelian or satisfies one of the following relations:

$$(i) \quad [[e_2, e_3], e_3] = e_2, \qquad (ii) \quad [[e_2, e_3], e_3] = -e_2.$$

It follows that for $\mathbf{m}$ and for the corresponding Lie algebra $\mathbf{g}^*$ we have the following possibilities.

2 a. If $\mathbf{m}/\langle e_1\rangle$ is abelian then we have $[[e_2, e_3], e_2] = e_1$, since $\mathbf{m}$ is not abelian. This Lie triple system is isomorphic to the Lie triple system belonging to the relation $[[e_2, e_3], e_3] = e_1$ under the isomorphism α given by $\alpha(e_1) = e_1$, $\alpha(e_2) = e_3$, $\alpha(e_3) = -e_2$ (see Theorem 4.1, Type II in [1]). Then the Lie algebra $\mathbf{g}^*$ is defined by the following non-trivial relations

$$[e_2, e_3] = e_4, \quad [e_4, e_3] = e_1.$$

According to [12] (p. 162) this is the unique 4-dimensional nilpotent Lie algebra with 2-dimensional commutator algebra.

2 b. The Lie triple system is the direct product of $\langle e_1\rangle$ with the 2-dimensional Lie triple system satisfying in **2** either (i) or (ii) respectively. Using the isomorphism α given by $\alpha(e_1) = e_3$, $\alpha(e_2) = e_1$, $\alpha(e_3) = e_2$ the Lie triple system with the relation (i) changes into the Lie triple system $\mathbf{m}^+ \times \langle e_3\rangle$ satisfying $[[e_1, e_2], e_2] = e_1$ (Type III in [1]) and the Lie triple system with the relation (ii) becomes the Lie triple system $\mathbf{m}^- \times \langle e_3\rangle$ satisfying $[[e_1, e_2], e_2] = -e_1$ (Type III in [1]). The Lie algebra $\mathbf{g}^*_{(+)}$ corresponding to $\mathbf{m}^+ \times \langle e_3\rangle$ is given by

$$[e_1, e_2] = e_4, \quad [e_4, e_2] = e_1,$$

whereas the other products are zero. This shows that $\mathbf{g}^*_{(+)}$ is the direct product of the 3-dimensional solvable Lie algebra having precisely two 1-dimensional ideals ([6], pp. 12–14) and the 1-dimensional Lie algebra.
The Lie algebra $\mathbf{g}^*_{(-)}$ belonging to $\mathbf{m}^- \times \langle e_3\rangle$ is defined by

$$[e_1, e_2] = e_4, \quad [e_4, e_2] = -e_1,$$

which shows that $\mathbf{g}^*_{(-)}$ is the direct product of the 3-dimensional solvable Lie algebra having no 1-dimensional ideal ([6], pp. 12–14) and the 1-dimensional Lie algebra.

2 c. The Lie triple system is a non-split extension of $\langle e_1\rangle$ by the 2-dimensional Lie triple system belonging to the relation (i) or (ii) in **2** respectively. Hence it is characterized by

$$\mathbf{m}^+ : [[e_2, e_3], e_2] = e_1, \qquad [[e_2, e_3], e_3] = e_2 \qquad \text{or}$$

$$\mathbf{m}^- : [[e_2, e_3], e_2] = e_1, \qquad [[e_2, e_3], e_3] = -e_2$$

(Type V in [1]).

The Lie algebra $\mathbf{g}^*_{(+)}$ of $\mathbf{m}^+$ is given by

$$[e_2, e_3] = e_4, \quad [e_4, e_2] = e_1, \quad [e_4, e_3] = e_2$$

which shows that $\mathbf{g}^*_{(+)}$ contains the 3-dimensional nilpotent ideal $\langle e_1, e_2, e_4\rangle$ and the factor Lie algebra $\mathbf{g}^*_{(+)}/\langle e_1\rangle$ is the 3-dimensional Lie algebra having precisely two 1-dimensional ideals. This Lie algebra is isomorphic to $g_{4,8}$ with $h = -1$ in [13] (p. 121).

The Lie algebra $\mathbf{g}^*_{(-)}$ of $\mathbf{m}^-$ is defined by

$$[e_2, e_3] = e_4, \quad [e_4, e_2] = e_1, \quad [e_4, e_3] = -e_2,$$

which shows that it contains the 3-dimensional nilpotent ideal $\langle e_1, e_2, e_4\rangle$ and the basis element e_3 acts as a euclidean rotation in the 2-dimensional subspace $\langle e_2, e_4\rangle$. This Lie algebra is isomorphic to $g_{4,9}$ with $p = 0$ in [13] (p. 121).

3. It remains to discuss that $\mathbf{m}$ has only trivial centre. In this case $\mathbf{m}$ is determined by

$$[[e_2, e_3], e_3] = e_1, \quad [[e_3, e_1], e_3] = e_2$$

(Type VI in [1]).

The corresponding Lie algebra $\mathbf{g}^*$ is defined by:

$$[e_2, e_3] = e_4, \quad [e_4, e_3] = e_1, \quad [e_1, e_3] = e_5, \quad [e_5, e_3] = -e_2,$$

and the other products are zero. The Lie algebra $\mathbf{g}^*$ has two 2-dimensional ideals which are invariant under the action of e_3.

Remark 1. Our classification of the 3-dimensional Lie triple system is a slight modification of Bouetou's classification ([1]). He has two classes more, namely

$$\text{a)} \qquad [[e_2, e_3], e_1] = e_1, \qquad [[e_3, e_1], e_2] = -e_1$$

$$\text{b)} \qquad [[e_1, e_2], e_2] = \varepsilon e_1, \qquad [[e_1, e_2], e_3] = e_1$$

$$[[e_3, e_1], e_2] = -e_1, \qquad [[e_3, e_1], e_3] = -\varepsilon e_1,$$

where $\varepsilon = \pm 1$.

The case a) does not satisfy the property (3) in the definition of a Lie triple system and the case b) is isomorphic to the case **2 b** using the isomorphism

$$\alpha(e_1) = e_1, \quad \alpha(e_2) = \varepsilon e_2 - e_3, \quad \alpha(e_3) = -\varepsilon e_2 + (\varepsilon + 1)e_3.$$

4 3-dimensional Bol loops corresponding to the abelian Lie triple system are abelian groups

Lemma 2. *The universal Lie algebra* $\mathbf{g}^U$ *of the abelian Lie triple system* $\mathbf{m}$ *is given by the following multiplication table:*

$$[e_1, e_2] = e_4, \quad [e_1, e_3] = e_5, \quad [e_2, e_3] = e_6,$$

and the other products are zero.

Proof. According to the definition of $\mathbf{g}^U$ we have $\mathbf{m}^U \cap [\mathbf{m}^U, \mathbf{m}^U] = 0$. Thus we can choose the elements e_1, e_2, e_3 as a basis of $\mathbf{m}^U$ and the elements $e_4 := [e_1, e_2]$, $e_5 := [e_1, e_3]$ and $e_6 := [e_2, e_3]$ as the generators of $[\mathbf{m}^U, \mathbf{m}^U]$. Since $\mathbf{m}$ is abelian we obtain the assertion. $\square$

The centre Z of $\mathbf{g}^U$ is generated by the elements e_4, e_5, e_6 and is equal to $[\mathbf{m}^U, \mathbf{m}^U]$. Therefore the Lie group G^U of $\mathbf{g}^U$ is a 6-dimensional nilpotent Lie group of nilpotency class 2. Every enveloping Lie algebra $\mathbf{g}^T$ of $\mathbf{m}$ is an epimorphic image of $\mathbf{g}^U$. The 4- or 5-dimensional epimorphic images of $\mathbf{g}^U$ are also nilpotent and has nilpotency class 2. It follows from [15] (p. 311) that any global connected differentiable proper Bol loop L having a Lie group of nilpotency class 2 as the group topologically generated by its left translations contains an at least 3-dimensional nilpotent subgroup. Hence there does not exist any differentiable proper 3-dimensional Bol loop L corresponding to the abelian Lie triple system.

5 3-dimensional Bol loops belonging to a Lie triple system with 1-dimensional centre

5.1 Bol loops corresponding to the non-decomposable nilpotent standard enveloping Lie algebra with dimension 4

We consider the Lie triple system $\mathbf{m}$ of type **2 a** in Section 3.

Lemma 3. *The universal Lie algebra* $\mathbf{g}^U$ *of the Lie triple system* $\mathbf{m}$ *of type* **2 a** *is the* 5*-dimensional nilpotent Lie algebra defined by the following non-trivial products:*

$$[e_2, e_3] = e_4, \quad [e_4, e_3] = e_1, \quad [e_3, e_1] = e_5.$$

The unique 4*-dimensional epimorphic image of* $\mathbf{g}^U$ *(up to isomorphisms) is the standard enveloping Lie algebra* $\mathbf{g}^*$ *described in* **2 a**.

Proof. Since $\mathbf{g}^U = \mathbf{m}^U \oplus [\mathbf{m}^U, \mathbf{m}^U]$ we may assume that the set $\{e_1, e_2, e_3\}$ is the set of the generators of $\mathbf{m}^U$ and the elements $e_4 := [e_2, e_3]$, $e_5 := [e_3, e_1]$ and $e_6 := [e_1, e_2]$ are basis elements of $[\mathbf{m}^U, \mathbf{m}^U]$. The relations of the Lie triple system of type **2 a** yield the following multiplication table:

$$[e_2, e_3] = e_4, \quad [e_4, e_3] = e_1, \quad [e_3, e_1] = e_5, \quad [e_1, e_2] = e_6.$$

Since $[[e_4, e_3], e_2] + [[e_3, e_2], e_4] + [[e_2, e_4], e_3] = e_6$ this multiplication satisfies the Jacobi identity if and only if $[e_1, e_2] = 0$ and this is the first assertion. The Lie algebra $\mathbf{g}^U$ is nilpotent hence every epimorphic images of $\mathbf{g}^U$ is also nilpotent. If $\mathbf{g}$ is a 4-dimensional epimorphic image of $\mathbf{g}^U$ then the commutator subalgebra of $\mathbf{g}$ is image of the commutator subalgebra $(\mathbf{g}^U)'$. Since $\dim(\mathbf{g}^U)' = 3$ we have $\dim \mathbf{g}' = 2$ and $\mathbf{g}$ is the standard enveloping Lie algebra $\mathbf{g}^*$ (cf. **2 a**). $\square$

Denote by G the Lie group of the standard enveloping Lie algebra $\mathbf{g}^*$. Using the Campbell–Hausdorff series the multiplication of G is defined by:

$$(x_1, x_2, x_3, x_4) * (y_1, y_2, y_3, y_4)$$
$$= \begin{pmatrix} x_1 + y_1 + \frac{1}{2}(x_4 y_3 - x_3 y_4) + \frac{1}{12}(x_3^2 y_2 - x_3 x_2 y_3) + \frac{1}{12}(x_2 y_3^2 - x_3 y_3 y_2) \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 + \frac{1}{2}(x_2 y_3 - x_3 y_2) \end{pmatrix}$$

([2], p. 77). A 1-dimensional subalgebra $\mathbf{h}$ of $\mathbf{g}^*$ such that $\mathbf{h}$ does not contain any non-trivial ideal of $\mathbf{g}$ and $\mathbf{h} \cap \mathbf{m} = \{0\}$ holds has the form

$$\mathbf{h} = \langle e_4 + a_1 e_1 + a_2 e_2 + a_3 e_3 \rangle, \quad a_i \in \mathbb{R}.$$

The automorphism group of $\mathbf{g}$ consisting of the linear mappings

$$\alpha(e_1) = bf^2 e_1, \quad \alpha(e_2) = ae_1 + be_2, \quad \alpha(e_3) = de_1 + le_2 + fe_3, \quad \alpha(e_4) = bfe_4,$$

where $a, b, d, l, f \in \mathbb{R}$ and $bf \neq 0$, leaves the subspace $\mathbf{m} = \langle e_1, e_2, e_3 \rangle$ invariant and maps the subalgebra $\mathbf{h}$ onto one of the following subalgebras

$$\mathbf{h}_1 = \langle e_4 \rangle, \quad \mathbf{h}_2 = \langle e_4 + e_1 \rangle, \quad \mathbf{h}_3 = \langle e_4 + e_2 \rangle, \quad \mathbf{h}_4 = \langle e_4 + e_3 \rangle$$

(see [2]). Since the element $e_4 + e_2 \in \mathbf{h}_3$ is conjugate to the element $e_2 - \frac{1}{2} e_1 \in \mathbf{m}$ under $g = (0, 0, -1, 0) \in G$ and the element $e_4 + e_3 \in \mathbf{h}_4$ is conjugate to the element $e_3 \in \mathbf{m}$ under $g = (0, 1, 0, 0) \in G$ we have a contradiction to Lemma 1. Therefore we have to consider only the cases $(\mathbf{g}^*, \mathbf{h}_1)$ and $(\mathbf{g}^*, \mathbf{h}_2)$. In [2] it is proved that for these 2 cases global Bol loops exist. The loop L belonging to the triple

$$(G, H_1 = \exp \mathbf{h}_1 = \{(0, 0, 0, h) \mid h \in \mathbb{R}\},\ \exp \mathbf{m} = \{(a, b, c, 0) \mid a, b, c \in \mathbb{R}\})$$

is a Bruck loop. The loop L^* corresponding to

$$(G, H_2 = \exp \mathbf{h}_2 = \{(h, 0, 0, h) \mid h \in \mathbb{R}\},\ \exp \mathbf{m} = \{(a, b, c, 0) \mid a, b, c \in \mathbb{R}\})$$

is a left A-loop, because of $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$. But it is not a Bruck loop since there is no involutory automorphism $\sigma : \mathbf{g} \to \mathbf{g}$ such that $\sigma(\mathbf{m}) = -\mathbf{m}$ and $\sigma(\mathbf{h}_2) = \mathbf{h}_2$.

Since the conjugation by the element $g = (0, 0, -1, 0) \in G$ maps the subalgebra $\mathbf{h}_1$ of H_1 onto the subalgebra $\mathbf{h}_2$ of H_2 the loop L is isotopic to L^*.

Now we consider the universal Lie algebra $\mathbf{g}^U$ defined in Lemma 3, which is the Lie algebra L_5^2 in [12] (p. 162). Using the Campbell–Hausdorff series ([16]) the multiplication of the Lie group G^U of $\mathbf{g}^U$ is given as follows:

$$(x_1, x_2, x_3, x_4, x_5) * (y_1, y_2, y_3, y_4, y_5)$$

$$= \begin{pmatrix} x_1 + y_1 + \frac{1}{2}(x_4 y_3 - x_3 y_4) + \frac{1}{12}(x_3^2 y_2 - x_3 x_2 y_3) + \frac{1}{12}(x_2 y_3^2 - x_3 y_3 y_2) \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 + \frac{1}{2}(x_2 y_3 - x_3 y_2) \\ x_5 + y_5 + \frac{1}{2}(x_3 y_1 - x_1 y_3) + \frac{1}{12}(-x_3^2 y_4 + x_3 x_4 y_3) \\ + \frac{1}{12}(-x_4 y_3^2 + x_3 y_3 y_4) + \frac{1}{24}(x_2 x_3 y_3^2 - x_3^2 y_2 y_3) \end{pmatrix}.$$

The class of the 2-dimensional subalgebras $\mathbf{h}$ of $\mathbf{g}_1$, which does not contain any non-trivial ideal and $\mathbf{h} \cap \mathbf{m} = \{0\}$ has the following shape:

$$\mathbf{h}_{a,b,a',b'} = \langle e_4 + a e_1 + b e_2, e_5 + a' e_1 + b' e_2 \rangle, \quad a, b, a', b' \in \mathbb{R},\ (a', b') \neq (0, 0)$$

([2], p. 80). There is no Bol loop L such that the group topologically generated by the left translations of L is the group G^U and the stabilizer of the identity $e \in L$ in G^U is the group

$$H_{a,b,a',b'} = \{(\lambda_1 a + \lambda_2 a', \lambda_1 b + \lambda_2 b', 0, \lambda_1, \lambda_2), \lambda_1, \lambda_2 \in \mathbb{R}\}, \quad a, b, a', b' \in \mathbb{R},$$

where $(a', b') \neq (0, 0)$. Namely we show that for given $a, b, a', b' \in \mathbb{R}$ with $(a', b') \neq (0, 0)$ we can find $(0, 0) \neq (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and an element $x = (x_1, x_2, x_3, x_4, x_5) \in G^U$ such that

$$Ad_x(\lambda_1(e_4 + a e_1 + b e_2) + \lambda_2(e_5 + a' e_1 + b' e_2)) \in \mathbf{m} \backslash \{0\}$$

where $\mathbf{m} = \{y_1 e_1 + y_2 e_2 + y_3 e_3;\ y_1, y_2, y_3 \in \mathbb{R}\}$. This is a consequence of the fact that the following system of equations:

$$y_1 = \lambda_1 \left(a - \frac{1}{2} x_3 \right) + \lambda_2 a', \quad y_2 = \lambda_1 b + \lambda_2 b', \quad y_3 = 0$$

$$\lambda_1(1-x_3b)-\lambda_2 b' x_3=0, \quad \lambda_2(1+x_3a')+\lambda_1\left(x_3a-\frac{1}{3}x_3^2\right)=0$$

has a solution $x_3 \neq 0$, $(\lambda_1,\lambda_2)\neq(0,0)$ and $(y_1,y_2,y_3)\neq(0,0,0)$ which holds true since there exists $x_3 \neq 0$ such that

$$1+x_3(a'-b)+x_3^2(b'a-ba')-\frac{1}{3}b'x_3^3=0.$$

Summarizing our discussion we obtain

Theorem 4. *There is only one isotopism class $\mathcal{C}$ of 3-dimensional connected differentiable Bol loops such that the group G topologically generated by their left translations is a nilpotent Lie group. The group G is isomorphic to the 4-dimensional non-decomposable nilpotent Lie group. The class $\mathcal{C}$ consists of precisely two isomorphism classes $\mathcal{C}_1$ and $\mathcal{C}_2$ which may be represented by the Bruck loop L having the group $H=\{(0,0,0,h)\mid h\in\mathbb{R}\}$ as the stabilizer of $e\in L$ in G respectively by the left A-loop L^* having the group $H=\{(h,0,0,h)\mid h\in\mathbb{R}\}$ as the stabilizer of $e\in L^*$ in G.*

5.2 Bol loops corresponding to a Lie triple system which is a direct product of its centre and a non-abelian Lie triple system

We discuss here the Lie triple systems characterized in **2 b** in Section 3.

Lemma 5. *The universal Lie algebras $\mathbf{g}^U_{(+)}$ and $\mathbf{g}^U_{(-)}$ of the Lie triple systems $\mathbf{m}^+\times\langle e_3\rangle$ or $\mathbf{m}^-\times\langle e_3\rangle$ respectively, are defined by:*

$$[e_1,e_2]=e_4, \quad [e_4,e_2]=\varepsilon\ e_1, \quad [e_2,e_3]=e_5,$$

where $\varepsilon=1$ for $\mathbf{g}^U_{(+)}$ and -1 for $\mathbf{g}^U_{(-)}$, and the other products are zero.

The unique 4-dimensional epimorphic image of $\mathbf{g}^U_{(-)}$ is (up to isomorphisms) the standard enveloping Lie algebra $\mathbf{g}^*_{(-)}$ described in **2 b**.

The 4-dimensional epimorphic images of $\mathbf{g}^U_{(+)}$ are (up to isomorphisms) either the standard enveloping Lie algebra $\mathbf{g}^*_{(+)}$ given in **2 b** *or the Lie algebra* $\mathbf{g}$ *given by:*

$$[e_1,e_2]=e_1, \quad [e_2,e_3]=e_4,$$

whereas the other products are zero.

Proof. For a basis of the universal Lie algebras $\mathbf{g}^U=\mathbf{m}^U\oplus[\mathbf{m}^U,\mathbf{m}^U]$ one can choose the elements e_1, e_2, e_3, e_4, e_5, e_6, where e_1,e_2,e_3 are the generators of $\mathbf{m}^U$ and $e_4:=[e_1,e_2]$, $e_5:=[e_2,e_3]$, $e_6:=[e_1,e_3]$ are the generators of $[\mathbf{m}^U,\mathbf{m}^U]$. Using the relations of the Lie triple systems of type **2 b** we obtain the following multiplication table:

$$[e_1,e_2]=e_4, \quad [e_4,e_2]=\pm e_1, \quad [e_2,e_3]=e_5, \quad [e_1,e_3]=e_6$$

and the other products are zero. Since for the elements e_2, e_3, e_4 one has

$$[[e_2, e_3], e_4] + [[e_3, e_4], e_2] + [[e_4, e_2], e_3] = \pm e_6,$$

this multiplication satisfies the Jacobi identity precisely if $[e_1, e_3] = 0$, and we obtain the universal Lie algebras $\mathbf{g}^U_{(\pm)}$. The unique 1-dimensional ideal of $\mathbf{g}^U_{(-)}$ is the centre of $\mathbf{g}^U_{(-)}$, which is generated by e_5. Moreover, the epimorphic image $\alpha(\mathbf{g}^U_{(-)})$ under the mapping $\alpha(e_i) = e_i$, $i = 1, 2, 3, 4$, $\alpha(e_5) = 0$ is the Lie algebra $\mathbf{g}^*_{(-)}$.

The 1-dimensional ideals of $\mathbf{g}^U_{(+)}$ are $i_1 = \langle e_5 \rangle$, $i_2 = \langle e_1 + e_4 \rangle$, $i_3 = \langle e_4 - e_1 \rangle$. The image of $\mathbf{g}^U_{(+)}$ under the epimorphism $\beta(e_i) = e_i$, $i = 1, 2, 3, 4$ and $\beta(e_5) = 0$ is the Lie algebra $\mathbf{g}^*_{(+)}$. The Lie algebras $\mathbf{g}^U_{(+)}/\langle e_1 + e_4 \rangle$ and $\mathbf{g}^U_{(+)}/\langle e_4 - e_1 \rangle$ are determined by

$$[e_1, e_2] = -e_1, \qquad [e_2, e_3] = e_4; \qquad \text{and by}$$

$$[e_1, e_2] = e_1, \qquad [e_2, e_3] = e_4$$

respectively. This shows that $\mathbf{g}^U_{(+)}/\langle e_1 + e_4 \rangle$ is isomorphic to $\mathbf{g}^U_{(+)}/\langle e_4 - e_1 \rangle$ under the isomorphism $\gamma(e_i) = e_i$, $i = 1, 4$ and $\gamma(e_j) = -e_j$, $j = 2, 3$, and the assertion follows. □

First we seek for Bol loops having the standard enveloping Lie algebra $\mathbf{g}^*_{(+)}$ given in **2 b** as the Lie algebra of the group topologically generated by their left translations. The Lie group G of $\mathbf{g}^*_{(+)}$ is the direct product $G = G_1 \times G_2$, where G_1 is the 3-dimensional solvable Lie group having precisely two 1-dimensional normal subgroups and G_2 is a 1-dimensional Lie group. Since the Lie triple system is the direct product of its centre C and a 2-dimensional non-abelian Lie triple system A one has $\exp \mathbf{m} = \exp \mathbf{m}_1 \times \exp \mathbf{m}_2$, where $\exp \mathbf{m}_1$ respectively $\exp \mathbf{m}_2$ corresponds to A respectively to C. Moreover, $\exp \mathbf{m}_1 \subseteq G_1$ and $\exp \mathbf{m}_2 = G_2$.

First we assume that the 1-dimensional Lie group $H = \exp \mathbf{h}$ is contained in $G_1 \times \{1\}$. Then the loop L is the direct product of a 2-dimensional Bol loop L_1 and a 1-dimensional Lie group ([15], Proposition 1.19, p. 28). The loop L_1 has G_1 as the group generated by its left translations, and it is isomorphic to precisely one of the non-isomorphic loops L_α, $\alpha \in \mathbb{R}$ with $\alpha \leq -1$ given in Theorem 23.1 of [15]. All loops L_α and hence also L_1 are isotopic to the pseudo-euclidean plane loop ([15], Remark 25.4, p. 326).

If the 1-dimensional Lie group $H = \exp \mathbf{h}$ is not contained in $G_1 \times \{1\}$ then H is isomorphic to $\mathbb{R}$ since G_1 does not contain any discrete normal subgroup $\neq 1$. Therefore $G_2 \cong \mathbb{R}$, $\exp \mathbf{m} = \exp \mathbf{m}_1 \times \mathbb{R}$ and H has the shape $\{(h_1, \varphi(h_1) \mid h_1 \in H_1\}$, where $H_1 \cong \mathbb{R}$ is a subgroup of G_1 and $\varphi : H_1 \to G_2$ is a monomorphism. For a loop L corresponding to the pair (G, H) the group G_2 is a normal subgroup of L and the factor loop L/G_2 is isomorphic to a loop L_1 defined on the factor space G_1/H_1. According to Theorem 23.1 in [15] the loop L_1 is isomorphic to a loop L_α. Then the Proposition 2.4 in [15] yields that L is a Scheerer extension of the group $\mathbb{R}$ by a loop L_α.

Now we deal with the standard enveloping Lie algebra $\mathbf{g}^*_{(-)}$ given in **2 b**. The Lie group G of $\mathbf{g}^*_{(-)}$ is the direct product $G = G_1 \times G_2$ of the 3-dimensional solvable Lie group G_1 having no non-trivial normal subgroup and a 1-dimensional Lie group G_2. Since $\exp \mathbf{m}$ decomposes into the topological product $\exp \mathbf{m} = \exp \mathbf{m}_1 \times \exp \mathbf{m}_2$ with $\exp \mathbf{m}_1 \subset G_1$ and $\exp \mathbf{m}_2 = G_2$ the 1-dimensional Lie group H has the form $(H_1, \varphi(H_1))$, where $\varphi : H_1 \to G_2$ is a homomorphism. Hence the loop belonging to $(G, H, \exp \mathbf{m})$ is a Scheerer extension of a 1-dimensional Lie group and a 2-dimensional loop $\tilde{L}$ (cf. [15] Proposition 1.19, p. 28 and Proposition 2.4, p. 44). But the group G_1 cannot be the group topologically generated by the left translations of $\tilde{L}$ (cf. [15] Lemma 23.15, p. 312). Therefore there is no differentiable Bol loop corresponding to the group G.

Now we investigate the Lie algebra $\mathbf{g}$ in Lemma 5, which consists of the matrices

$$ve_1 + ue_2 + ze_3 + ke_4 \mapsto \begin{pmatrix} 0 & v & 0 & 0 & 0 \\ 0 & u & 0 & 0 & 0 \\ 0 & 0 & 0 & u & k \\ 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad u, v, k, z \in \mathbb{R}.$$

It is a central extension of $\mathbb{R}$ by the direct product of $\mathbb{R}$ and the non-abelian 2-dimensional Lie algebra (see [13], pp. 120–121). The multiplication of the Lie group G of $\mathbf{g}$ is defined by

$$(x_1, x_2, x_3, x_4) * (y_1, y_2, y_3, y_4) = (y_1 + x_1 e^{y_2}, x_2 + y_2, x_3 + y_3, x_4 + y_4 + x_2 y_3).$$

The 1-dimensional subalgebras $\mathbf{h}$ of $\mathbf{g}$ which complement $\mathbf{m} = \langle e_1, e_2, e_3 \rangle$ have the shapes:

$$\mathbf{h}_{a_1,a_2,a_3} = \langle e_4 + a_1 e_1 + a_2 e_2 + a_3 e_3 \rangle,$$

where $a_1, a_2, a_3 \in \mathbb{R}$. For $a_1 = a_2 = a_3 = 0$ the Lie algebra $\mathbf{h}_{0,0,0} = \langle e_4 \rangle$ is an ideal of $\mathbf{g}$. Therefore we have $(a_1, a_2, a_3) \neq (0, 0, 0)$. The automorphisms γ of $\mathbf{g}$ leaving $\mathbf{m}$ invariant are determined by the linear mappings

$$\gamma(e_1) = ae_1, \quad \gamma(e_2) = b_1 e_1 + e_2 + b_3 e_3, \quad \gamma(e_3) = de_3, \quad \gamma(e_4) = de_4,$$

such that $a, d \in \mathbb{R}\backslash\{0\}$ and $b_1, b_3 \in \mathbb{R}$. A suitable automorphism γ of $\mathbf{g}$ with $\gamma(\mathbf{m}) = \mathbf{m}$ maps the subalgebra $\mathbf{h}_{a_1,a_2,a_3}$ onto one of the following Lie algebras:

$$\mathbf{h}_1 = \langle e_4 + e_2 \rangle, \ \mathbf{h}_2 = \langle e_4 + a_3 e_3 \rangle, \ a_3 \in \mathbb{R}\backslash\{0\}, \quad \mathbf{h}_3 = \langle e_4 + e_1 + a_3 e_3 \rangle, \ a_3 \in \mathbb{R}.$$

Because of $e_2 = Ad_g(e_4 + e_2) \in \mathbf{m}$ with $g = (0, 0, -1, 0) \in G$ the Lie algebra $\mathbf{h}_1$ is excluded. Since for $a_3 \neq 0$ and $g = (0, a_3^{-1}, 0, 0) \in G$ one has $a_3 e_3 = Ad_g(e_4 + a_3 e_3) \in \mathbf{m}$ and $[\exp(a_3^{-1})]e_1 + a_3 e_3 = Ad_g(e_4 + e_1 + a_3 e_3) \in \mathbf{m}$ we have to investigate only the triple $(\mathbf{g}, \mathbf{h} = \langle e_4 + e_1 \rangle, \mathbf{m})$ (cf. Lemma 1). For the exponential image of $\mathbf{m} = \langle e_1, e_2, e_3 \rangle$ we obtain

$$\begin{aligned} \exp\ \mathbf{m} &= \exp\{k_1 e_1 + k_2 e_2 + k_3 e_3; k_1, k_2, k_3 \in \mathbb{R}\} \\ &= \left\{ \left(k_1 \frac{e^{k_2} - 1}{k_2}, k_2, k_3, \frac{1}{2} k_2 k_3 \right),\ k_i \in \mathbb{R},\ i = 1, 2, 3 \right\}, \end{aligned}$$

and the subgroup $H = \exp\{a(e_4 + e_1), a \in \mathbb{R}\}$ consists of the elements $(a, 0, 0, a)$, $a \in \mathbb{R}$.

Since any element of G decomposes uniquely as $(0, y_1, y_2, y_3)(a, 0, 0, a)$ we can conclude that $\exp \mathbf{m}$ determines a global Bol loop if and only if each element $g = (0, y_1, y_2, y_3) \in G$, $y_i \in \mathbb{R}$, $i = 1, 2, 3$ can be written uniquely as a product $g = mh$ or equivalently $m = gh^{-1}$ with $m \in \exp \mathbf{m}$ and $h \in H$. This is the case since for all given $y_1, y_2, y_3 \in \mathbb{R}$ the following system of equations

$$y_1 = k_2,\ y_2 = k_3,\ y_3 - a = \frac{1}{2} k_2 k_3,\ a = -k_1 \frac{e^{k_2} - 1}{k_2}$$

has a unique solution $(a, k_1, k_2, k_3) \in \mathbb{R}^4$ given by

$$k_2 := y_1,\ k_3 := y_2,\ a := y_3 - \frac{1}{2} y_1 y_2,\ k_1 := \frac{\frac{1}{2} y_1 y_2 - y_3}{\frac{e^{y_1} - 1}{y_1}}.$$

Hence the pair $(G, H = \{(a, 0, 0, a), a \in \mathbb{R}\})$ corresponds to a 3-dimensional Bol loop L. Because of $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$ the loop L is a left A-loop.

Now we summarize the discussion in

Theorem 6. *Let L be a 3-dimensional connected differentiable Bol loop corresponding to a Lie triple system which is a direct product of its centre and a non-abelian 2-dimensional Lie triple system. If the group G topologically generated by the left translations of L is 4-dimensional, then for L and for G precisely one of the following cases occur:*

1) *G is the direct product of the 3-dimensional solvable Lie group having precisely two 1-dimensional normal subgroups and a 1-dimensional Lie group and L is either the direct product of the 1-dimensional compact Lie group $SO_2(\mathbb{R})$ with a 2-dimensional Bol loop L_α defined in Theorem 23.1 of [15] or a Scheerer extension of the group $\mathbb{R}$ by a loop L_α.*

2) *G is the 4-dimensional solvable Lie group with the multiplication*

$$(x_1, x_2, x_3, x_4) * (y_1, y_2, y_3, y_4) = (y_1 + x_1 e^{y_2}, x_2 + y_2, x_3 + y_3, x_4 + y_4 + x_2 y_3)$$

and L is isomorphic to the left A-loop having $H = \{(a, 0, 0, a) \mid a \in \mathbb{R}\}$ as the stabilizer of the identity of L.

Finally we treat the universal Lie algebras $\mathbf{g}^U_{(\pm)}$ defined in Lemma 5. (The Lie algebra $\mathbf{g}^U_{(+)}$ is isomorphic to the Lie algebra $g_{5,8}$ with $\gamma = -1$ and $\mathbf{g}^U_{(-)}$ is isomorphic to the Lie algebra $g_{5,14}$ with $p = 0$ in [14], p. 105.) The multiplication of the Lie group $G^U_{(\pm)}$ corresponding to $\mathbf{g}^U_{(\pm)}$ is given by:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} * \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} y_1 + x_1 \mathbf{cos}\, y_2 + \varepsilon x_4 \mathbf{sin}\, y_2 \\ y_2 + x_2 \\ y_3 + x_3 \\ y_4 + x_1 \mathbf{sin}\, y_2 + x_4 \mathbf{cos}\, y_2 \\ y_5 + x_5 + x_2 y_3 \end{pmatrix}.$$

The triple $(\mathbf{cos}\, y_2, \mathbf{sin}\, y_2, \varepsilon)$ denotes $(\cosh y_2, \sinh y_2, 1)$ in case $G^U_{(+)}$ and $(\cos y_2, \sin y_2, -1)$ in case $G^U_{(-)}$.

The 2-dimensional subalgebras $\mathbf{h}$ of $\mathbf{g}^U_{(\pm)}$ which are complements to $\mathbf{m} = \langle e_1, e_2, e_3 \rangle$ have the shapes:

$$\mathbf{h}_{a_1,a_3,b_1,b_3} = \langle e_4 + a_1 e_1 + a_3 e_3, e_5 + b_1 e_1 + b_3 e_3 \rangle,$$

where $a_1, a_3, b_1, b_3 \in \mathbb{R}$. Since the ideal $\langle e_5 \rangle$ of $\mathbf{g}^U_{(\pm)}$ lies in $\mathbf{h}_{a_1,a_3,0,0}$ and the ideal $\langle e_4 \pm e_1 \rangle$ of $\mathbf{g}^U_{(+)}$ is contained in $\mathbf{h}_{\pm 1,0,b_1,b_3}$ we may suppose that $(b_1, b_3) \neq (0,0)$ in the case of $\mathbf{g}^U_{(+)}$ as well as of $\mathbf{g}^U_{(-)}$ and $(a_1, a_3) \neq (\pm 1, 0)$ in the case $\mathbf{g}^U_{(+)}$.

For $b_1 = 0$ the element $0 \neq b_3 e_3 \in \mathbf{m}$ is conjugate to $e_5 + b_3 e_3 \in \mathbf{h}$ under $g = (0, -b_3^{-1}, 0, 0, 0) \in G^U_{(\pm)}$ which contradicts Lemma 1.

If $b_1 \neq 0$ then the linear mapping α defined by

$$\alpha(e_1) = \frac{1}{b_1} e_1,\ \alpha(e_2) = e_2,\ \alpha(e_3) = e_3,\ \alpha(e_4) = \frac{1}{b_1} e_4,\ \alpha(e_5) = e_5$$

is an automorphism of $\mathbf{g}^U_{(\pm)}$. This automorphism leaves the subspace $\mathbf{m}$ invariant and reduces $\mathbf{h}_{a_1,a_3,b_1,b_3}$ to $\mathbf{h}_{a_1,a_3,1,b_3}$.

The Lie group $H_{a_1,a_3,1,b_3} = \exp \mathbf{h}_{a_1,a_3,1,b_3}$ consists of the elements

$$\{(la_1 + k, 0, la_3 + kb_3, l, k),\ l, k \in \mathbb{R}\}$$

and the exponential image of the subspace $\mathbf{m}$ has the form

$$\begin{aligned} \exp\ \mathbf{m} &= \exp\{k_1 e_1 + k_2 e_2 + k_3 e_3; k_1, k_2, k_3 \in \mathbb{R}\} \\ &= \left\{ \left(\frac{k_1 \mathbf{sin}\, k_2}{k_2}, k_2, k_3, \varepsilon \frac{k_1(\mathbf{cos}\, k_2 - 1)}{k_2}, \frac{1}{2} k_2 k_3 \right),\ k_1, k_2, k_3 \in \mathbb{R} \right\}. \end{aligned}$$

Every element of the Lie group $G^U_{(\pm)}$ can be written uniquely as a product

$$(x_1, x_2, x_3, x_4, x_5) = (0, f_2, f_3, 0, f_5)(la_1 + k, 0, la_3 + kb_3, l, k),$$

where $(la_1+k, 0, la_3+kb_3, l, k) \in H_{a_1,a_3,1,b_3}$. Each element $g = (0, f_2, f_3, 0, f_5)$, $f_i \in \mathbb{R}$ for $i = 2, 3, 5$, has in $G^U_{(\pm)}$ a unique decomposition as $g = m\ h$ or equivalently $m = g\ h^{-1}$ with $m \in \exp \mathbf{m}$, $h \in H_{a_1,a_3,1,b_3}$ if and only if for all given $f_2, f_3, f_5, a_1, a_3, b_3 \in \mathbb{R}$ the following system of equations

$$-la_1 - k = \frac{k_1 \sin f_2}{f_2}, \quad k_3 = f_3 - la_3 - kb_3, \quad l = -\varepsilon \frac{k_1(\cos f_2 - 1)}{f_2},$$
$$-k + f_5 + f_2(k_3 - f_3) = \tfrac{1}{2} f_2 k_3, k_2 = f_2 \qquad (*)$$

has a unique solution $(k_1, k_2, k_3, k, l) \in \mathbb{R}^5$.

In the group $G^U_{(-)}$ we find

$$k_2 = f_2, \quad k_1 = \tfrac{f_2(-2f_5+f_2f_3)}{\tilde{n}},$$
$$k_3 = \tfrac{2[(\cos f_2-1)(f_3a_1-f_3a_3f_2+f_3b_3f_2a_1+a_3f_5-b_3a_1f_5)+\sin f_2(f_3+f_3b_3f_2-b_3f_5)]}{\tilde{n}},$$
$$k = \tfrac{(2f_5-f_2f_3)[\sin f_2+a_1(\cos f_2-1)]}{\tilde{n}}, \quad l = \tfrac{(\cos f_2-1)(-2f_5+f_2f_3)}{\tilde{n}}, \qquad (1)$$

where $\tilde{n} = (\cos f_2 - 1)(2a_1 - a_3 f_2 + b_3 f_2 a_1) + (2 + b_3 f_2) \sin f_2$.

In $G^U_{(+)}$ the system $(*)$ has the following solution:

$$k_2 = f_2, \quad k_1 = \tfrac{2f_2e^{f_2}(-2f_5+f_2f_3)}{(e^{f_2}-1)n},$$
$$k_3 = \tfrac{2[(e^{f_2}-1)(-f_3a_1+f_3a_3f_2-f_3b_3f_2a_1-a_3f_5+b_3a_1f_5)+(e^{f_2}+1)(f_3+f_3b_3f_2-b_3f_5)]}{n},$$
$$k = \tfrac{(-2f_5+f_2f_3)(a_1e^{f_2}-e^{f_2}-a_1-1)}{n}, \quad l = \tfrac{(e^{f_2}-1)(2f_5-f_2f_3)}{n}, \qquad (2)$$

where $n = (1 - e^{f_2})(2a_1 - a_3 f_2 + b_3 f_2 a_1) + (e^{f_2} + 1)(2 + b_3 f_2)$.

The solution (1) respectively the solution (2) is unique if and only if $\tilde{n} \neq 0$ respectively $n \neq 0$. If for a value f_2 one has $n(f_2) = 0$ respectively $n'(f_2) = 0$ then the coset $(0, f_2, f_3, 0, f_5)H_{a_1,a_3,1,b_3}$ contains no element of $\exp \mathbf{m}$.

Considering f_2 as a variable x for the function $\tilde{n}(f_2) = \tilde{n}(x)$ one has $\tilde{n}(x) = 0$ if and only if $a_3(x) = \left(\frac{2}{x} + b_3\right)\left(a_1 + \frac{\sin x}{\cos x - 1}\right)$, where $a_1, b_3 \in \mathbb{R}$ and $x \in \mathbb{R}\backslash\{2\pi l\}$, $l \in \mathbb{Z}$. For all $a_1 \in \mathbb{R}$ the function $h(x) := a_1 + \frac{\sin x}{\cos x - 1}$ has period 2π. It is continuous and strictly increasing on the intervals $(2\pi l, 2\pi + 2\pi l)$, $l \in \mathbb{Z}$ such that $\lim_{x \searrow 2\pi l} h(x) = -\infty$ and $\lim_{x \nearrow 2\pi + 2\pi l} h(x) = \infty$. The function $\frac{2}{x} + b_3$ is for $b_3 \leq -\frac{2}{3\pi}$ continuous and negative in $(4\pi, 6\pi)$ and for $b_3 > -\frac{2}{3\pi}$ it is continuous and positive in $(0, 2\pi)$. Hence the restriction of the function $a_3(x)$ to $(4\pi, 6\pi)$ respectively to $(0, 2\pi)$ takes all real numbers as values. This means that for all given a_1, a_3, b_3 there is a value $p \in \mathbb{R}\backslash\{2\pi l\}$, $l \in \mathbb{Z}$ such that $\tilde{n}(p) = 0$.

Replacing f_2 by the variable x we investigate the function $n(f_2) = n(x)$. We have $n(0) = 4$. We seek for $p \in \mathbb{R}\backslash\{0\}$ with $n(p) = 0$. Since $n(x)$ is continuous it is enough to prove that there is $x \in \mathbb{R}\backslash\{0\}$ with $n(x) < 0$.

This happens for the following triples

a) $(b_3 = 0,\ a_3 = 0,\ a_1 \notin [-1,1])$ b) $(b_3 = 0, a_3 < 0,\ a_1 \in \mathbb{R})$

c) $(b_3 \in \mathbb{R}\backslash\{0\},\ a_3 \leq 0,\ a_1 \in \mathbb{R})$ d) $\Big(b_3 < 0,\ a_3 > 0,\ a_1 < \dfrac{a_3}{b_3} + 1\Big)$

e) $\Big(b_3 > 0,\ a_3 > 0,\ a_1 > \dfrac{a_3}{b_3} - 1\Big)$.

Namely, in the case a) $\lim_{x\to -\infty} \frac{n(x)}{e^x+1} < 0$ for $a_1 < -1$ and $\lim_{x\to\infty} \frac{n(x)}{e^x+1} < 0$ for $a_1 > 1$. In the cases b) and e) we have $\lim_{x\to -\infty} \frac{n(x)}{e^x+1} = -\infty$ and in the case d) one obtains $\lim_{x\to\infty} \frac{n(x)}{e^x+1} = -\infty$. Moreover, in the case c) one has $n(-\frac{2}{b_3}) \leq 0$. Thus for the above triples (a_1, a_3, b_3) there is $p \in \mathbb{R}\backslash\{0\}$ such that $n(p) = 0$.

Let $\sigma : G^U_{(\pm)}/H_{a_1,a_3,1,b_3} \to G^U_{(\pm)}$ be a section belonging to a differentiable Bol loop L with dimension 3. If $\sigma(G^U_{(\pm)}/H_{a_1,a_3,1,b_3})$ contains $\exp \mathbf{m}$ then any coset $(0, f_2, 0, 0, 1)H_{a_1,a_3,1,b_3}$, $(f_2 \in \mathbb{R})$ should contain precisely one element s of $\sigma(G^U_{(\pm)}/H_{a_1,a_3,1,b_3})$. For $f_2 \neq p$ we obtain in the case $G^U_{(-)}$

$$s = \left(-2\frac{\sin f_2}{\tilde{n}}, f_2, k_3, \frac{2(\cos f_2 - 1)}{\tilde{n}}, \frac{1}{2} f_2 k_3\right)$$

and in the case $G^U_{(+)}$

$$s = \left(\frac{-2(e^{f_2}+1)}{n}, f_2, k_3, \frac{-4e^{f_2}(\cosh f_2 - 1)}{(e^{f_2}-1)n}, \frac{1}{2} f_2 k_3\right).$$

Since σ is continuous one has

$$\sigma((0,p,0,0,1)H_{a_1,a_3,1,b_3}) = \lim_{f_2\to p} \sigma((0,f_2,0,0,1)H_{a_1,a_3,1,b_3}) = \lim_{f_2\to p} s.$$

But $\lim_{f_2\to p} \frac{2(\cos f_2 - 1)}{\tilde{n}} = \infty$ as well as $\lim_{f_2\to p} \frac{-2(e^{f_2}+1)}{n} = \infty$ which are contradictions. Therefore the group $G^U_{(-)}$ cannot be the group topologically generated by the left translations of a differentiable 3-dimensional Bol loop and for the group $G^U_{(+)}$ the parameters satisfying the conditions a) till e) are excluded.

Now for $G^U_{(+)}$ it remains to investigate the triples

(i) $(b_3 = 0,\ a_3 = 0,\ -1 < a_1 < 1)$ (ii) $(b_3 = 0,\ a_3 > 0,\ a_1 \in \mathbb{R})$

(iii) $\Big(b_3 < 0,\ a_3 > 0,\ a_1 > \dfrac{a_3}{b_3} + 1\Big)$ (iv) $\Big(b_3 > 0,\ a_3 > 0,\ a_1 < \dfrac{a_3}{b_3} - 1\Big)$

(v) $\Big(b_3 > 0,\ a_3 > 0,\ a_1 = \dfrac{a_3}{b_3} - 1\Big)$ (vi) $\Big(b_3 < 0,\ a_3 > 0, a_1 = \frac{a_3}{b_3} + 1\Big)$.

In the case (i) the function $n(x)$ is positive. Therefore there is a connected differentiable 3-dimensional Bol loop, which is realized on the factor space $G^U_{(+)}/H_{a_1,0,1,0}$ with $-1 < a_1 < 1$.

In the case (ii) we have

$$n(x) = e^x(xa_3 - 2a_1 + 2) - xa_3 + 2a_1 + 2$$

and for the derivations we obtain

$$\begin{aligned} n'(x) &= e^x(xa_3 - 2a_1 + 2 + a_3) - a_3 \\ n''(x) &= e^x(xa_3 - 2a_1 + 2 + 2a_3) \\ n'''(x) &= e^x(xa_3 - 2a_1 + 2 + 3a_3). \end{aligned}$$

Since $n''(x) = 0$ only for $u = \frac{2a_1-2-2a_3}{a_3}$ holds and $n'''(u) = a_3 > 0$ the function $n'(x)$ assumes in u its unique minimum. Moreover, we have

$$\begin{aligned} &\lim_{x\to\infty} n'(x) = \infty, \qquad &&\lim_{x\to-\infty} n'(x) < 0, \qquad \text{and} \\ &\lim_{x\to\infty} n(x) = \infty, &&\lim_{x\to-\infty} n(x) = \infty. \end{aligned}$$

Therefore there is only one value p for which $n'(p) = 0$ and in p the function $n(x)$ achieves its unique minimum. One obtains $n'(p) = 0$ if and only if $a_1 = \frac{1}{2}(pa_3 + 2 + a_3 - \frac{a_3}{e^p})$. Furthermore, we have $n(p) > 0$ if and only if $p = 0$ or $0 < a_3 < \frac{4e^p}{(e^p-1)^2}$ if $p \in \mathbb{R}\backslash\{0\}$. Thus for the parameters (a_3, a_1) satisfying the properties

$$0 < a_3 \quad \text{and} \quad a_1 = 1$$

or

$$0 < a_3 < \frac{4e^p}{(e^p-1)^2} \text{ and } a_1 = \frac{1}{2}\left(pa_3 + 2 + a_3 - \frac{a_3}{e^p}\right)$$

there is a connected differentiable 3-dimensional Bol loop corresponding to the pair $(G^U_{(+)}, H_{a_1,a_3,1,0})$.

In the cases (iii) and (iv) we have

$$n(x) = (e^x + 1)(b_3x + 2) + (1 - e^x)(xb_3a_1 - xa_3 + 2a_1)$$

and for the derivations one obtains

$$\begin{aligned} n'(x) &= e^x(x(b_3 + a_3 - b_3a_1) + b_3 + a_3 - b_3a_1 + 2 - 2a_1) + b_3 + b_3a_1 - a_3 \\ n''(x) &= e^x(x(b_3 + a_3 - b_3a_1) + 2b_3 + 2a_3 - 2b_3a_1 + 2 - 2a_1) \\ n'''(x) &= e^x(x(b_3 + a_3 - b_3a_1) + 3b_3 + 3a_3 - 3b_3a_1 + 2 - 2a_1). \end{aligned}$$

The same arguments as above show that the function $n'(x)$ has only one minimum in $\frac{2(b_3a_1-b_3-a_3+a_1-1)}{b_3+a_3-b_3a_1}$ and that there exists only one value p such that $n'(p) = 0$; for this value p the function $n(x)$ takes its unique minimum.

We have $n'(p) = 0$ if and only if

$$p = 0 \quad \text{and} \quad a_1 = b_3 + 1$$

or

$$a_3 = \frac{e^p(1+p)(b_3a_1 - b_3) + e^p(2a_1 - 2) - b_3 - b_3a_1}{e^p(1+p) - 1} \quad \text{if } p \in \mathbb{R}\backslash\{0\}.$$

Putting a_3 into the expression of $n(x)$ we obtain the following: For the value p one has $n(p) > 0$ if and only if one of the following cases is satisfied

(I) $\quad p = 0$ and $a_1 = b_3 + 1$

(II) $\quad e^p(1+p) - 1 < 0$ and $p^2e^pb_3 - a_1(e^{2p}+1) + e^{2p} + 2pe^p - 1 + 2a_1e^p < 0$

(III) $\quad e^p(1+p) - 1 > 0$ and
$p^2e^pb_3 - a_1(e^{2p}+1) + e^{2p} + 2pe^p - 1 + 2a_1e^p > 0.$

In the case (I) the conditions in (iii) reduce to
(iii) a) $\quad b_3 < 0,\ a_1 = b_3 + 1,\ b_3^2 < a_3$
and from the conditions in (iv) one gets
(iv) b) $\quad b_3 > 0,\ a_1 = b_3 + 1,\ b_3^2 + 2b_3 < a_3.$
In both cases there is a connected differentiable 3-dimensional Bol loop L realized on the factor space $G^U_{(+)}/H_{a_1,a_3,1,b_3}$.

Now we discuss the case (II). For the parameters satisfying (iii) it is equivalent to the following system of inequalities

$(\alpha) \quad p < 0,\ \ b_3 < 0,\ \ a_1b_3 < a_3 + b_3, \quad (\beta) \quad a_3 > 0,$

$$(\gamma) \quad b_3 < \frac{a_1(e^p-1)^2 - e^{2p} - 2pe^p + 1}{p^2e^p},$$

$$(\delta) \quad a_3 = \frac{e^p(1+p)(b_3a_1 - b_3) + e^p(2a_1 - 2) - b_3 - b_3a_1}{e^p(1+p) - 1}.$$

Using (δ) the condition (α) may replaced by

$(\alpha') \quad a_1 < 1, \quad e^p(a_1 - 1) < b_3 < 0, \quad p < 0.$

The condition (β) is satisfied if and only if

$$(\beta') \quad \varepsilon b_3 < \varepsilon \frac{e^p(2-2a_1)}{e^p(1+p)(a_1-1) - (1+a_1)} \quad \text{and} \quad \varepsilon a_1 < \varepsilon \frac{1+e^p(1+p)}{-1+e^p(1+p)}$$

with $\varepsilon \in \{1,-1\}$ holds. Since $p < 0$ the condition $a_1 < 1$ gives in (β') for $\varepsilon = 1$ that $a_1 < \frac{1+e^p(1+p)}{-1+e^p(1+p)}$ and for $\varepsilon = -1$ that $\frac{1+e^p(1+p)}{-1+e^p(1+p)} < a_1 < 1$. Therefore the expression $\frac{e^p(2-2a_1)}{e^p(1+p)(a_1-1)-(1+a_1)}$ is positive for $\varepsilon = 1$ and negative for $\varepsilon = -1$.

Let $f(p)$, $l(p,a_1)$ and $k(p,a_1)$ be the following functions

$$f(p) := \frac{1+e^p(1+p)}{-1+e^p(1+p)}, \quad l(p,a_1) := e^p(a_1-1),$$

$$k(p, a_1) := \frac{e^p(2-2a_1)}{e^p(1+p)(a_1-1)-(1+a_1)}.$$

Thus for $\varepsilon = 1$ the conditions (α') and (β') yield

a) $\quad l(p, a_1) < b_3 < 0 \quad \text{and} \quad a_1 < f(p)$

whereas for $\varepsilon = -1$ the conditions (α') and (β') give

b) $\quad f(p) < a_1 < 1 \quad \text{and} \quad k(p, a_1) < b_3 < 0$ which satisfy (γ).

The function $n(p, a_1) := \frac{a_1(e^p-1)^2-e^{2p}-2pe^p+1}{p^2e^p}$ in (γ) is non negative if and only if

$$\text{(A)} \qquad a_1 \geq \frac{e^{2p}+2pe^p-1}{(e^p-1)^2}.$$

Denote by $g(p)$ the function $g(p) = \frac{e^{2p}+2pe^p-1}{(e^p-1)^2}$. Using for all $p < 0$ the inequality

(B) $\quad g(p) < f(p)$

one sees that the condition a) holds if and only if one of the following systems of inequalities is satisfied:

c) $\quad g(p) \leq a_1 < f(p) \quad \text{and} \quad l(p, a_1) < b_3 < 0,$

d) $\quad a_1 < g(p) \quad \text{and} \quad l(p, a_1) < b_3 < n(p, a_1) \quad \text{if} \quad l(p, a_1) < n(p, a_1).$

Because of $p^2e^{2p}-(e^p-1)^2 < 0$ for all $p < 0$, the condition $l(p, a_1) < n(p, a_1)$ is satisfied if and only if

$$\frac{p^2e^{2p}-e^{2p}-2pe^p+1}{p^2e^{2p}-e^{2p}+2e^p-1} < a_1.$$

Let $h(p)$ be the function $h(p) = \frac{p^2e^{2p}-e^{2p}-2pe^p+1}{p^2e^{2p}-e^{2p}+2e^p-1}$. Since $h(p) < g(p)$ for all $p < 0$ the condition d) is satisfied if and only if

e) $\quad h(p) < a_1 < g(p) \quad \text{and} \quad l(p, a_1) < b_3 < n(p, a_1)$ holds.

Thus for $p < 0$ and $b_3 < 0$ there is a connected differentiable Bol loop L such that the group topologically generated by its left translations is the group $G^U_{(+)}$ and the stabilizer of $e \in L$ is the subgroup $H_{a_1,a_3,1,b_3}$ if and only if the parameters a_1, a_3, b_3 satisfy one of the systems of inequalities b), c) or e) and the condition (δ).

For the parameters (iv) the case (II) yields the following system of inequalities

$$(\alpha) \quad p < 0,\ b_3 > 0,\ a_1b_3 < a_3 - b_3, \qquad (\beta) \quad a_3 > 0,$$

$$(\gamma) \quad b_3 < n(p,a_1), \qquad (\delta) \quad a_3 = \frac{e^p(1+p)(b_3a_1 - b_3) + e^p(2a_1-2) - b_3 - b_3a_1}{e^p(1+p)-1}.$$

Using (δ) the condition (α) holds if and only if one of the following cases is satisfied

$$(\alpha') \quad p < -1, \quad a_1 < 1, \quad 0 < b_3 < \frac{a_1-1}{1+p},$$

$$(\alpha'') \quad p = -1, \quad a_1 < 1, \quad 0 < b_3,$$

$$(\alpha''') \quad -1 < p < 0, \quad \max\left\{0, \frac{a_1-1}{1+p}\right\} < b_3.$$

The condition (β) may be replaced by
(β') $\varepsilon b_3 < \varepsilon k(p,a_1)$ and $\varepsilon a_1 < \varepsilon f(p)$
with $\varepsilon \in \{1,-1\}$. Denote by $m(p,a_1)$ the function $\frac{a_1-1}{1+p}$. The conditions (α') and (β'), (α'') and (β'), (α''') and (β') yield for $\varepsilon = 1$ the corresponding conditions

a) $p < -1,\ a_1 < f(p),\ 0 < b_3 < \min\{k(p,a_1), m(p,a_1)\},$

b) $p = -1,\ a_1 < -1,\ 0 < b_3 < k(-1,a_1),$

c) $-1 < p < 0,\ a_1 < f(p),\ 0 < b_3 < k(p,a_1)$

and for $\varepsilon = -1$ the conditions

d) $p < -1,\ f(p) < a_1 < 1,\ \ 0 < b_3 < k(p,a_1),$

e) $p = -1,\ a_1 < 1,\ 0 < b_3,$

f) $-1 < p < 0,\ 1 < a_1,\ \max\{m(p,a_1), k(p,a_1)\} < b_3,$

g) $-1 < p < 0,\ f(p) < a_1 \le 1,\ \ 0 < b_3.$

Now we deal with the condition (γ). Using the inequalities (A) and (B) the conditions a) till g) hold if and only if the following conditions in the same order as a) till g) are satisfied:

a') $p < -1,\ g(p) \le a_1 < f(p),\ 0 < b_3 < \min\{k(p,a_1), m(p,a_1), n(p,a_1)\},$

b') $p = -1,\ g(-1) \le a_1 < -1,\ 0 < b_3 < \min\{k(-1,a_1), n(-1,a_1)\},$

c') $-1 < p < 0,\ g(p) \le a_1 < f(p),\ 0 < b_3 < \min\{k(p,a_1), n(p,a_1)\},$

d') $p < -1,\ f(p) < a_1 < 1,\ \ 0 < b_3 < \min\{k(p,a_1), n(p,a_1)\},$

e') $p = -1,\ g(-1) \le a_1 < 1,\ 0 < b_3 < n(-1,a_1),$

f') $-1 < p < 0,\ 1 < a_1,$ and $\max\{m(p,a_1), k(p,a_1)\} < b_3 < n(p,a_1),$

if $\max\{m(p,a_1),k(p,a_1)\} < b_3 < n(p,a_1)$,

g') $-1 < p < 0$, $f(p) < a_1 \le 1$, $0 < b_3 < n(p,a_1)$.

Since for $-1 < p < 0$ and $1 < a_1$ one has $k(p,a_1) < m(p,a_1)$ as well as $(1+p)(e^p-1)^2 - p^2e^p < 0$ the inequality $\max\{m(p,a_1),k(p,a_1)\} < b_3 < n(p,a_1)$ in f') is satisfied if and only if

$$a_1 < \frac{(1+p)(e^{2p}+2pe^p-1)-p^2e^p}{(1+p)(e^{2p}-2e^p+1)-p^2e^p}.$$

The function $v(p) = \frac{(1+p)(e^{2p}+2pe^p-1)-p^2e^p}{(1+p)(e^{2p}-2e^p+1)-p^2e^p}$ is greater than 1 for $-1 < p < 0$. Hence the condition f') is equivalent to

h') $-1 < p < 0$, $1 < a_1 < v(p)$, $m(p,a_1) < b_3 < n(p,a_1)$. It follows that for $p < 0$ and $b_3 > 0$ there is a differentiable Bol loop L defined on the factor space $G^U_{(+)}/H_{a_1,a_3,1,b_3}$ if and only if the parameters a_1, a_3, b_3 satisfy one of the systems of inequalities a') till h') and the condition (δ).

Now we discuss the case (III). For (iii) we obtain the following system of inequalities

(α) $p > 0$, $b_3 < 0$, $a_1b_3 < a_3 + b_3$,

(β) $a_3 > 0$, (γ) $b_3 > n(p,a_1)$,

(δ) $$a_3 = \frac{e^p(1+p)(b_3a_1-b_3)+e^p(2a_1-2)-b_3-b_3a_1}{e^p(1+p)-1}.$$

Using (δ) the condition (α) yields

(α') $b_3 < \min\{0, e^p(a_1-1)\}$ and $p > 0$.

Furthermore, (β) is satisfied if and only if

(β') $\varepsilon b_3 > \varepsilon k(p,a_1)$ and $\varepsilon a_1 > \varepsilon f(p)$

with $\varepsilon \in \{1,-1\}$ holds. Since $p > 0$ the conditions (α') and (β') give for $\varepsilon = 1$

a) $a_1 > f(p)$ and $k(p,a_1) < b_3 < 0$

whereas for $\varepsilon = -1$ we obtain one of the following conditions

b) $1 < a_1 < f(p)$ and $b_3 < 0$

c) $a_1 < 1$ and $b_3 < \min\{l(p,a_1),k(p,a_1)\}$.

Since for $a_1 < 1$ and $p > 0$ we have $l(p,a_1) < k(p,a_1)$ the condition c) yields

d) $a_1 < 1$ and $b_3 < l(p,a_1)$.

Now we investigate the condition (γ). The function $n(p,a_1)$ is non negative if and only if

(C) $a_1 \ge \frac{e^{2p}+2pe^p-1}{(e^p-1)^2}$.

Because of

(D) $\quad f(p) < g(p)$ for all $p > 0$

the condition a) may be replaced by

e) $\quad f(p) < a_1 < g(p)$ and $\max\{k(p, a_1), n(p, a_1)\} < b_3 < 0$.

Moreover the condition b) is equivalent to

f) $\quad 1 < a_1 < f(p)$ and $n(p, a_1) < b_3 < 0$

whereas the condition d) is equivalent to

g) $\quad a_1 < 1$ and $n(p, a_1) < b_3 < l(p, a_1)$ for $n(p, a_1) < l(p, a_1)$.

Since for $p > 0$ one has

$$p^2 e^{2p} - (e^p - 1)^2 > 0 \quad \text{and} \quad h(p) < 1$$

the relation $n(p, a_1) < l(p, a_1)$ holds if and only if $h(p) < a_1$. Using this inequality and $h(p) < 1$ the condition g) is equivalent to

h) $\quad h(p) < a_1 < 1$ and $n(p, a_1) < b_3 < l(p, a_1)$.

Thus for $p > 0$ and $b_3 < 0$ there exists a differentiable Bol loop, which is realized on the factor space $G^U_{(+)}/H_{a_1,a_3,1,b_3}$ if and only if a_1, a_3, b_3 satisfy one of the systems of inequalities e), f) or h) and the condition (δ).

For the parameters (iv) the case (III) is equivalent to the following system of inequalities

$(\alpha) \qquad p > 0, \quad b_3 > 0, \quad a_1 b_3 < a_3 - b_3,$

$(\beta) \qquad a_3 > 0, \qquad\qquad (\gamma) \qquad b_3 > n(p, a_1),$

$$(\delta) \qquad a_3 = \frac{e^p(1+p)(b_3 a_1 - b_3) + e^p(2a_1 - 2) - b_3 - b_3 a_1}{e^p(1+p) - 1}.$$

Using (δ) the condition (α) may be replaced by the condition

$(\alpha') \quad 1 < a_1, \; 0 < b_3 < m(p, a_1)$ and $p > 0$.

Furthermore, the condition (β) is satisfied if and only if

$(\beta') \quad \varepsilon b_3 > \varepsilon k(p, a_1)$ and $\varepsilon a_1 > \varepsilon f(p)$

with $\varepsilon \in \{1, -1\}$ holds. Since $p > 0$ the conditions (α') and (β') give for $\varepsilon = 1$

a) $\quad f(p) < a_1$ and $0 < b_3 < m(p, a_1)$

and for $\varepsilon = -1$

b) $\quad 1 < a_1 < f(p)$ and $0 < b_3 < k(p, a_1)$.

Now we deal with the property (γ). Using the inequalities (C) and (D) one sees that the inequalities in b) satisfy (γ) and that the condition a) holds if and only if one of the following cases is true:

c) $\quad f(p) < a_1 \leq g(p)$ and $0 < b_3 < m(p, a_1)$

d) $\quad g(p) < a_1$ and $n(p, a_1) < b_3 < m(p, a_1)$ if $n(p, a_1) < m(p, a_1)$.

Since $(1 + p)(e^p - 1)^2 - p^2 e^p > 0$ for $p > 0$ the condition $n(p, a_1) < m(p, a_1)$

is equivalent to $a_1 < v(p)$. Moreover, for $p > 0$ one has $g(p) < v(p)$ and the condition d) is satisfied if and only if

e) $\quad g(p) < a_1 < v(p)$ and $n(p, a_1) < b_3 < m(p, a_1)$.

Hence for $p > 0$ and $b_3 > 0$ there exists a differentiable Bol loop L having $G^U_{(+)}$ as the group topologically generated by the left translations and the subgroup $H_{a_1,a_3,1,b_3}$ as the stabilizer of $e \in L$ in $G^U_{(+)}$ if and only if the parameters a_1, a_3, b_3 satisfy one of the conditions b), c) or e) and (δ).

For the parameters (v) we have $n'(p) = 0$ if and only if

$$p = 0 \text{ and } \frac{a_3}{b_3} = b_3 + 2 \text{ or } a_3 = b_3(pb_3 + b_3 + 2) \quad \text{if } p \in \mathbb{R}\backslash\{0\}.$$

Hence $n(p) > 0$ if and only if one of the following cases holds true:

1) $\quad b_3 > 0$, $a_3 = b_3(b_3 + 2)$, $a_1 = \frac{a_3}{b_3} - 1$ if $p = 0$

and

2) $\quad b_3(p + 1 - e^p) + 2 > 0$ for $p \in \mathbb{R}\backslash\{0\}$.

For the parameters in 1) there is a differentiable Bol loop L having $G^U_{(+)}$ as the group topologically generated by its left translations and the group $H_{a_1,a_3,1,b_3}$ as the stabilizer in $G^U_{(+)}$.

The case 2) is equivalent to the following system of inequalities

(α) $\quad b_3 > 0$, $\quad b_3(p + 1 - e^p) + 2 > 0$, $\quad (\beta)$ $\quad a_3 > 0$, $\quad a_3 = b_3(pb_3 + b_3 + 2)$.

Because of $p+1-e^p < 0$ for all $p \in \mathbb{R}\backslash\{0\}$ the condition (α) may be replaced by

(α') $\quad 0 < b_3 < -\frac{2}{p+1-e^p}$.

The condition (β) is satisfied if and only if one of the following holds:

(β') $\quad p > -1$ and $b_3 > -\dfrac{2}{p+1}$, $\qquad (\beta'')$ $\quad p = -1$ and $b_3 > 0$,

(β''') $\quad p < -1$ and $b_3 < -\dfrac{2}{p+1}$.

Comparing the conditions (α') and (β') respectively (α') and (β'') we obtain that for $p \geq -1$ one has $0 < b_3 < -\frac{2}{p+1-e^p}$. Since $-\frac{2}{p+1} > -\frac{2}{p+1-e^p}$ for all $p < -1$ holds (α') and (β''') reduces to $0 < b_3 < -\frac{2}{p+1-e^p}$. Hence for $p \in \mathbb{R}\backslash\{0\}$ there exists a differentiable Bol loop realized on the factor space $G^U_{(+)}/H_{a_1,a_3,1,b_3}$ if and only if

$$0 < b_3 < -\frac{2}{p + 1 - e^p}, \quad a_3 = b_3(pb_3 + b_3 + 2), \quad a_1 = \frac{a_3}{b_3} - 1.$$

For the parameters (vi) we have $n''(x) = -2e^x a_3 b_3^{-1} > 0$ for all $x \in \mathbb{R}$. Hence the function $n'(x) = -2e^x a_3 b_3^{-1} + 2b_3$ is strongly monotone increasing.

Thus $n'(x) = 0$ is satisfied only for $p = \ln(b_3^2 a_3^{-1})$ and $n(p) > 0$ if and only if

$$b_3\left(\ln\frac{b_3^2}{a_3} - 1\right) + 2 + \frac{a_3}{b_3} > 0.$$

This condition is necessary and sufficient that a group $H_{a_1,a_3,1,b_3}$ with parameters in (vi) is the stabilizer of a differentiable Bol loop realized on the factor spaces $G^U_{(+)}/H_{a_1,a_3,1,b_3}$.

From the above discussion we obtain the main part of the following

Theorem 7. *Let L be a 3-dimensional connected differentiable Bol loop corresponding to a solvable Lie triple system which is the direct product of its centre and a non-abelian 2-dimensional Lie triple system. If the group G topologically generated by the left translations of L is at least 5-dimensional then G is the 5-dimensional solvable Lie group defined by:*

$$(x_1, x_2, x_3, x_4, x_5) * (y_1, y_2, y_3, y_4, y_5) = (y_1 + x_1 \cosh y_2 + x_4 \sinh y_2,$$
$$y_2 + x_2, y_3 + x_3, y_4 + x_1 \sinh y_2 + x_4 \cosh y_2, y_5 + x_5 + x_2 y_3).$$

Let

(a) $H_{a,0,0} = \{(la + k, 0, 0, l, k);\ l, k \in \mathbb{R}\}$, $-1 < a < 1$,

(b) $H_{a_1,a_3,0} = \{(la_1 + k, 0, la_3, l, k);\ l, k \in \mathbb{R}\}$, $a_3 > 0$, *such that either* $a_1 = 1$ *or* $a_3 < \frac{4e^p}{(e^p-1)^2}$ *and* $a_1 = \frac{1}{2}(pa_3 + 2 + a_3 - \frac{a_3}{e^p})$ *with* $p \in \mathbb{R}\backslash\{0\}$.

(c) $H_{a_1,a_3,b_3} = \{(la_1 + k, 0, la_3 + kb_3, l, k);\ l, k \in \mathbb{R}\}$ *such that for the real parameters* a_1, a_3, b_3 *one of the following conditions is satisfied:*

(α) $b_3 < 0$, $b_3^2 < a_3$, $a_1 = b_3 + 1$,

(β) $b_3 > 0$, $b_3^2 + 2b_3 \leq a_3$, $a_1 = b_3 + 1$,

(γ) $b_3 < 0$, $a_3 > 0$, $a_1 = a_3 b_3^{-1} + 1$, $b_3(\ln\frac{b_3^2}{a_3} - 1) + 2 + \frac{a_3}{b_3} > 0$.

Any subgroup in (a), (b) *and* (c) *is the stabilizer of the identity e of L in G. No loop having the stabilizer of e in* (a) *is isotopic to a loop having the stabilizer in* (b). *Moreover, the loops L_a and L_b corresponding to the stabilizers $H_{a,0,0}$ respectively $H_{b,0,0}$ are isomorphic if and only if $b = \pm a$. The loops $L_{1,a_3,0}$ and $L_{1,a'_3,0}$ corresponding to the stabilizers $H_{1,a_3,0}$ respectively $H_{1,a'_3,0}$ in* (b) *are isotopic precisely if $a_3 = a'_3$. No loop having the stabilizer of e in* (c) *is isotopic to a loop with the stabilizer of e in* (a) *as well as to a loop $L_{1,a_3,0}$. There are infinitely many non-isotopic loops having stabilizers in* (c).

Denote by $f(p)$, $g(p)$, $h(p)$, $k(p, a_1)$, $l(p, a_1)$, $n(p, a_1)$, $m(p, a_1)$ and $v(p)$ the following functions of the real variables p and a_1:

$$f(p) = \frac{1 + e^p(1 + p)}{-1 + e^p(1 + p)}, \quad g(p) = \frac{e^{2p} + 2pe^p - 1}{(e^p - 1)^2}, \quad h(p) = \frac{p^2e^{2p} - e^{2p} - 2pe^p + 1}{p^2e^{2p} - e^{2p} + 2e^p - 1},$$

$$k(p,a_1)=\frac{e^p(2-2a_1)}{e^p(1+p)(a_1-1)-(1+a_1)},\quad l(p,a_1)=e^p(a_1-1),\quad m(p,a_1)=\frac{a_1-1}{1+p}$$

$$n(p,a_1)=\frac{a_1(e^p-1)^2-e^{2p}-2pe^p+1}{p^2e^p},\quad v(p)=\frac{(1+p)(e^{2p}+2pe^p-1)-p^2e^p}{(1+p)(e^{2p}-2e^p+1)-p^2e^p}.$$

If a loop L has a stabilizer H of e not contained in (a), (b) *or* (c) *then* $H=H_{a_1,a_3,b_3}=\{(la_1+k,0,la_3+kb_3,l,k);\ l,k\in\mathbb{R}\}$ *and there exists either a real number $p<0$ such that one of the following conditions is satisfied:*

(i) $f(p)<a_1<1,\ k(p,a_1)<b_3<0,$

(ii) $g(p)\le a_1<f(p),\ l(p,a_1)<b_3<0,$

(iii) $h(p)<a_1<g(p),\ l(p,a_1)<b_3<n(p,a_1),$

(iv) $p<-1,\ g(p)\le a_1<f(p),\ 0<b_3<\min\{k(p,a_1),m(p,a_1),n(p,a_1)\}$

(v) $p=-1,\ g(-1)\le a_1<-1,\ 0<b_3<\min\{k(-1,a_1),n(-1,a_1)\},$

(vi) $-1<p<0,\ g(p)\le a_1<f(p),\ 0<b_3<\min\{k(p,a_1),n(p,a_1)\},$

(vii) $p<-1,\quad f(p)<a_1<1,\quad 0<b_3<\min\{k(p,a_1),n(p,a_1)\},$

(viii) $p=-1,\quad g(-1)\le a_1<1,\quad 0<b_3<n(-1,a_1),$

(ix) $-1<p<0,\quad f(p)<a_1\le 1,\quad 0<b_3<n(p,a_1),$

(x) $-1<p<0,\quad 1<a_1<v(p),\quad m(p,a_1)<b_3<n(p,a_1),$

(xi) $0<b_3<-\dfrac{2}{p+1-e^p},$

or there exists a real number $p>0$ such that one of the following conditions holds:

(xii) $f(p)<a_1<g(p),\ \max\{k(p,a_1),n(p,a_1)\}<b_3<0,$

(xiii) $1<a_1<f(p),\ n(p,a_1)<b_3<0,$

(xiv) $h(p)<a_1<1,\ n(p,a_1)<b_3<l(p,a_1),$

(xv) $1<a_1<f(p),\ 0<b_3<k(p,a_1),$

(xvi) $f(p)<a_1\le g(p),\ 0<b_3<m(p,a_1),$

(xvii) $g(p)<a_1<v(p),\ n(p,a_1)<b_3<m(p,a_1),$

(xviii) $0<b_3<-\dfrac{2}{p+1-e^p}.$

Moreover, one has $a_3=\frac{e^p(1+p)(b_3a_1-b_3)+e^p(2a_1-2)-b_3-b_3a_1}{e^p(1+p)-1}$ *in the cases* (i) *till* (x) *and* (xii) *till* (xvii), *whereas* $a_3=b_3(pb_3+b_3+2)$ *and* $a_1=\frac{a_3}{b_3}-1$ *holds true in the cases* (xi) *and* (xviii).

There are infinitely many non-isotopic loops L having stabilizers H_{a_1,a_3,b_3} such that the parameters a_1, a_3 and b_3 satisfy one of the conditions (i) *till* (xviii).

No loop for which the parameters a_1, a_3 *and* b_3 *satisfy one of* (i) till (xviii) *is isotopic to a loop corresponding to a stabilizer contained in* (a). *Moreover, no loop for which the parameters* a_1, a_3 *and* b_3 *satisfy one of the conditions* (i) till (iii), (x) *and* (xii) *till* (xviii) *is isotopic to a loop having as stabilizer* $H_{1,a_3,0}$ *of* (b).

Proof. It remains to prove the assertions concerning the isotopisms between loops having G as the group topologically generated by the left translations.

The loops L_{a_1,a_3,b_3} and $L_{a'_1,a'_3,b'_3}$ corresponding to the pairs (G, H_{a_1,a_3,b_3}) and $(G, H_{a'_1,a'_3,b'_3})$ are isotopic if there exists an element $g{\in}G$ such that $g^{-1}\mathbf{h}_{a_1,a_3,b_3}g = \mathbf{h}_{a'_1,a'_3,b'_3}$, where $\mathbf{h}_{a_1,a_3,b_3}$ is the Lie algebra of the stabilizer H_{a_1,a_3,b_3}. The group G is the semidirect product of the 4-dimensional normal abelian subgroups
$\{(x_1, 0, x_3, x_4, x_5);\ x_1, x_3, x_4, x_5 \in \mathbb{R}\}$ by the 1-dimensional subgroup
$\{(0, x_2, 0, 0, 0);\ x_2 \in \mathbb{R}\}$. Hence $\mathbf{h}_{a_1,a_3,b_3}$ and $\mathbf{h}_{a'_1,a'_3,b'_3}$ are conjugate if and only if they are conjugate under an element $(0, x_2, 0, 0, 0) \in G$. This is the case if and only if there exists $x_2 \in \mathbb{R}$ such that the following system (I) of equations

$$-a_3 + (a_1a'_3 + b'_3 - a_1b'_3a'_1)\sinh x_2 + (a'_3 + a_1b'_3 - a'_1b'_3)\cosh x_2 = 0 \quad (1)$$
$$(a'_3 - a'_1b'_3)\sinh x_2 - b_3 + b'_3\cosh x_2 = 0 \quad (2)$$
$$(a_1 - a'_1 + a'_3x_2 - a'_1b'_3x_2 + a_1b'_3x_2)\cosh x_2 + (1 - a_1a'_1 + a'_3a_1x_2 + b'_3x_2 - a_1a'_1b'_3x_2)\sinh x_2 = 0 \quad (3)$$
$$(1 + b'_3x_2)\cosh x_2 - 1 + (a'_3x_2 - a'_1 - a'_1b'_3x_2)\sinh x_2 = 0 \quad (4)$$

has a solution. From the equation (2) we obtain that for $\sinh x_2 \neq 0$

$$a'_3 = \frac{b_3 - b'_3\cosh x_2 + a'_1b'_3\sinh x_2}{\sinh x_2}.$$

Putting this expression into the equations (1), (3) and (4) one obtains

$$b'_3 = -a_3\sinh x_2 - a_1b_3\sinh x_2 - b_3\cosh x_2 \quad (1')$$
$$(a_1 - a'_1)\cosh x_2\sinh x_2 + a_2b_3x_2\sinh x_2 - 1 + a_1a'_1 + x_2b_3\cosh x_2 + (1 - a_1a'_1)(\cosh x_2)^2 - x_2b'_3 = 0 \quad (3')$$
$$-1 + \cosh x_2 - a'_1\sinh x_2 + x_2b_3 = 0. \quad (4')$$

The equation (4') yields for $\sinh x_2 \neq 0$ that

$$a'_1 = \frac{\cosh x_2 + x_2b_3 - 1}{\sinh x_2}.$$

Using this expression for a'_1 the equation (3') reduces to

$$-1 + \cosh x_2 - x_2b'_3 + a_1\sinh x_2 = 0. \quad (3'')$$

If we substitute for b'_3 from the equation (1') in (3") we see that the system (I) is solvable if and only if x_2 is the solution of the equation

$$(a_1b_3x - a_3x - a_1)(e^{2x} - 1) - (e^x - 1)^2 + b_3x(e^{2x} + 1) = 0, \qquad \text{(i)}$$

the parameters b'_3 respectively a'_1 satisfies (1') respectively (4') and $a'_3 = a_3$ holds.

The condition $a_3 = a'_3$ yields the following claims:
No loop with stabilizer in (a) can be isotopic to a loop having the stabilizer of e not in (a).
The loops $L_{1,a_3,0}$ and $L_{1,a'_3,0}$ are not isotopic if $a_3 \neq a'_3$.
The loops having the stabilizers $H_{b+1,b^2+1,b}$ and $H_{b'+1,b'^2+1,b'}$ with $b, b' < 0$ and $b \neq b'$ are not isotopic.
Among the loops having the stabilizers H_{a_1,a_3,b_3} such that the parameters a_1, a_3, b_3 satisfy one of the conditions (i) till (xviii) there are infinitely many corresponding to different values of a_3. Hence there are infinitely many isotopism classes of such loops.

For $b_3 = a_3 = 0$ and $0 \leq a_1 < 1$ the equation (i) reduces to

$$(e^x - 1)[(1 + e^x)a_1 + (e^x - 1)] = 0.$$

The solutions of this equation are $x_2 = 0$ and $x_2 = ln\ \frac{1-a_1}{1+a_1}$. Therefore the loop L_{a_1} with the stabilizer $H_{a_1,0,0}$ in (a) is isotopic to the loop L_{-a_1} having the stabilizer $H_{-a_1,0,0}$. Since the automorphism α of the Lie algebra $\mathbf{g}$ of G given by

$$\alpha(e_1) = -e_1, \alpha(e_5) = -e_5, \alpha(e_i) = e_i, i = 2, 3, 4$$

leaves the subspace $\mathbf{m}$ invariant and changes the Lie algebra $\mathbf{h}_{a_1,0,0}$ to $\mathbf{h}_{-a_1,0,0}$ the loops L_{a_1} and L_{-a_1} are already isomorphic.

For $b_3 = 0$, $a_1 = 1$ and $a_3 > 0$ the equation (i) reduces to

$$(e^x - 1)[(1 + e^x)(xa_3 + 1) + (e^x - 1)] = 0. \qquad \text{(ii)}$$

We consider the function

$$f(y) = (1 + e^y)(ya_3 + 1) + (e^y - 1), \text{ where } a_3 > 0.$$

For the derivations of $f(y)$ one has

$$\begin{aligned} f'(y) &= e^y(ya_3 + a_3 + 2) + a_3, \\ f''(y) &= e^y(ya_3 + 2a_3 + 2), \\ f'''(y) &= e^y(ya_3 + 3a_3 + 2). \end{aligned}$$

Since $f''(y) = 0$ only for $p = -2 - \frac{2}{a_3}$ holds and $f'''(p) > 0$, the function $f'(y)$ assumes in p its unique minimum. The function $f(y)$ is monotone increasing since $f'(p) = a_3(1 - e^p) > 0$. We have $\lim_{y\to\infty} f(y) = \infty$ and

$\lim_{y\to-\infty} f(y) = -\infty$. Hence there is only one value u for which $f(u) = 0$. Since $f(y) > 0$ for all $y \ge 0$ we obtain that $u < 0$ and thus the equation (ii) has precisely two solutions $x_2 = 0$ and $x_2 = u$. The unique loop isotopic to the loop $L_{1,a_3,0}$ corresponds to the stabilizer $H_{a'_1,a'_3,b'_3}$ the parameters a'_1, a'_3, b'_3 of which satisfy

$$a'_3 = a_3 > 0, \quad a'_1 = \frac{e^u - 1}{e^u + 1} < 0, \quad b'_3 = \frac{a_3(1 - e^{2u})}{2e^u} > 0.$$

But for such parameters none of the conditions (α), (β), (γ) in (c) and none of the conditions (i) till (iii), (x) and (xii) till (xviii) is satisfied. $\square$

5.3 Bol loops corresponding to a Lie triple system which is a non-split extension of its centre

Now we treat the Lie triple systems described in the case **2 c** in Section 3.

Lemma 8. *The universal Lie algebras $\mathbf{g}^U_\pm$ of the Lie triple systems $\mathbf{m}^\pm = \langle e_1, e_2, e_3\rangle$ of type* **2 c** *coincide with the standard enveloping Lie algebras $\mathbf{g}^*{}_{(\pm)}$ given in* **2 c**.

Proof. Since for $\mathbf{g}^U$ one has $\mathbf{m}^U \cap [\mathbf{m}^U, \mathbf{m}^U] = 0$ we may assume that $\mathbf{m}^U = \langle e_1, e_2, e_3\rangle$ and that a basis of $[\mathbf{m}^U, \mathbf{m}^U]$ consists of $e_4 := [e_2, e_3]$, $e_5 := [e_1, e_3]$ and $e_6 := [e_1, e_2]$. Using the Lie triple system relations given in **2 c** we have the following multiplication:

$$[e_2, e_3] = e_4, \quad [e_4, e_2] = e_1, \quad [e_4, e_3] = \pm e_2, \quad [e_1, e_3] = e_5, \quad [e_1, e_2] = e_6$$

and the other products are zero. Moreover, one has

$$\begin{aligned} &[[e_2, e_3], e_4] + [[e_3, e_4], e_2] + [[e_4, e_2], e_3] = e_5 \\ &[[e_4, e_3], e_1] + [[e_3, e_1], e_4] + [[e_1, e_4], e_3] = \mp e_6. \end{aligned}$$

Hence the Jacobi identity is satisfied if and only if $[e_1, e_3] = [e_1, e_2] = 0$. From this the assertion follows. $\square$

The Lie groups $G_{(+)}$ and $G_{(-)}$ corresponding to the Lie algebras $\mathbf{g}^*_{(+)}$ or $\mathbf{g}^*_{(-)}$ respectively, are the semidirect products of the 1-dimensional Lie group

$$C = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos t & \sin t & 0 \\ 0 & \epsilon \sin t & \cos t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad t \in \mathbb{R} \right\}$$

and the 3-dimensional nilpotent Lie group

$$B = \left\{ \begin{pmatrix} 1 & -x_2 & x_4 & x_1 \\ 0 & 1 & 0 & x_4 \\ 0 & 0 & 1 & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x_1, x_2, x_4 \in \mathbb{R} \right\},$$

where the triple ($\mathbf{cos}\, t,\ \mathbf{sin}\, t,\ \epsilon$) denotes $(\cosh t, \sinh t, 1)$ in case $G_{(+)}$ and $(\cos t,\ \sin t,\ -1)$ in case $G_{(-)}$.
Denoting the elements of $G_{(\pm)}$ by $g(t, x_1, x_2, x_4)$ we see that the multiplication in $G_{(\pm)}$ is given by

$$\begin{aligned} &g(t_1, x_1, x_2, x_4) \cdot g(t_2, y_1, y_2, y_4) \\ &\quad = g(t_1 + t_2, x_1 + y_1 + \epsilon y_4(x_2 \,\mathbf{cos}\, t_2 - \epsilon x_4 \,\mathbf{sin}\, t_2) - \epsilon y_2(x_4 \,\mathbf{cos}\, t_2 - x_2 \,\mathbf{sin}\, t_2), \\ &y_2 + x_2 \,\mathbf{cos}\, t_2 - \epsilon x_4 \,\mathbf{sin}\, t_2, y_4 - x_2 \,\mathbf{sin}\, t_2 + x_4 \,\mathbf{cos}\, t_2). \end{aligned}$$

A 1-dimensional subalgebra $\mathbf{h}$ of $\mathbf{g}^*_{(\pm)}$ which complements $\mathbf{m} = \langle e_1, e_2, e_3 \rangle$, can be written as:

$$\mathbf{h} = \langle e_4 + \alpha e_1 + \beta e_2 + \gamma e_3 \rangle \quad \text{with} \quad \alpha, \beta, \gamma \in \mathbb{R}.$$

Any automorphism α of $\mathbf{g}^*_{(\pm)}$ leaving $\mathbf{m} = \langle e_1, e_2, e_3 \rangle$ invariant is given by

$$\alpha(e_1) = \pm a^2 e_1, \quad \alpha(e_2) = \pm \epsilon a c e_1 + a e_2, \quad \alpha(e_3) = b e_1 + c e_2 \pm e_3, \quad \alpha(e_4) = \pm a e_4,$$

where $a \in \mathbb{R}\backslash\{0\}$, $b, c \in \mathbb{R}$, $\epsilon = 1$ in the case $\mathbf{g}^*_{(+)}$ and $\epsilon = -1$ for $\mathbf{g}^*_{(-)}$. Using suitable automorphisms of this form we can reduce $\mathbf{h}$ to one of the following:

$$\mathbf{h}_1 = \langle e_4 \rangle, \quad \mathbf{h}_2 = \langle e_4 + e_3 \rangle, \quad \mathbf{h}_{3,y} = \langle e_4 + y e_2 \rangle, y > 0, \quad \mathbf{h}_4 = \langle e_4 + e_1 \rangle.$$

The exponential image of the subspace $\mathbf{m}$ has the shape

$$\begin{aligned} \exp \mathbf{m} &= \exp\{n e_1 + k e_2 + t e_3, t, n, k \in \mathbb{R}\} \\ &= \left\{ g(t, n + \frac{k^2}{t} - \frac{k^2}{t^2} \,\mathbf{sin}\, t, \frac{k}{t} \,\mathbf{sin}\, t, \frac{k}{t}(1 - \mathbf{cos}\, t)),\ t, n, k \in \mathbb{R} \right\} \end{aligned}$$

(cf. [3] p. 11 and p. 12) if we identify $\mathbf{m}$ with the subspace generated by

$$\left\langle \left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & \epsilon t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -k & 0 & n \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) \right\rangle.$$

First we investigate the group $G_{(-)}$. The element $g(\frac{\pi}{2}, 0, 1, 0) \in G_{(-)}$ conjugates $\exp e_4 \in H_1$ to $\exp(-2e_1) \in \exp \mathbf{m}$ and $\exp(e_4 + e_1) \in H_4$ to $\exp(-e_2 - e_1) \in \exp\ \mathbf{m}$. Moreover, $\exp \pi(e_4 + e_3) \in H_2$ is conjugate to $\exp \pi(e_1 + e_3) \in \exp \mathbf{m}$ under $g(0, 0, -1, 0) \in G_{(-)}$ and $\exp(e_4 + y e_2) \in H_{3,y}$ is for all $y \in \mathbb{R}\backslash\{0\}$ conjugate to $\exp[(\sin \operatorname{arc\,ctg} y)^{-1} e_2] \in \exp \mathbf{m}$ under $g(-\operatorname{arc\,ctg} y, 0, 0, 0) \in G_{(-)}$. Hence there is no 3-dimensional differentiable Bol loop L such that the group topologically generated by its left translations is the Lie group $G_{(-)}$ (cf. Lemma 1).

Finally we deal with the group $G_{(+)}$. The element $\exp(e_4+e_3) \in H_2$ is conjugate to $\exp(e_3-e_1)$ of $\exp \mathbf{m}$ under $g(0,0,1,0) \in G_{(+)}$. The element $\exp l(e_4+ye_2) \in H_{3,y}$ with $l = -\sinh\left(\frac{1}{2}\ln\frac{y-1}{y+1}\right)$ is conjugate to $\exp e_2 \in \exp \mathbf{m}$ under $g\left(\frac{1}{2}\ln\frac{y-1}{y+1},0,0,0\right) \in G_{(+)}$ for all $y>1$. Therefore we may suppose that the stabilizer of the identity of a Bol loop L is either the Lie group H_1 or H_4 or $H_{3,y}$, where $0<y\le 1$.

Each element $g \in G_{(+)}$ can be represented uniquely as a product $g=mh$, where $m \in \exp \mathbf{m}$ and h is an element of H_1, H_4 or $H_{3,y}$ with $0<y\le 1$ respectively, if and only if for given $t_1,x_1,x_2,x_4 \in \mathbb{R}$ the equation

$$g(t_1,x_1,x_2,x_4) = g\Big(t, n+\frac{k^2}{t}-\frac{k^2}{t^2}\sinh t, \frac{k}{t}\sinh t, \frac{k}{t}(1-\cosh t)\Big)\cdot h$$

is uniquely solvable for

$$h = g(0,0,0,a) \in H_1,\ h=g(0,a,0,a) \in H_4,\ \text{and}\ h=g(0,0,ly,l) \in H_{3,y}.$$

In the case of H_1 the unique solution is given by:

$$t := t_1, \quad k := \frac{x_2}{\frac{\sinh t_1}{t_1}}, \quad a := x_4 - \frac{x_2(1-\cosh t_1)}{\sinh t_1},$$
$$n := x_1 - x_4x_2 + \frac{x_2^2(1-\cosh t_1)}{\sinh t_1} - \frac{x_2^2(t_1-\sinh t_1)}{\sinh^2 t_1}.$$

In the case of H_4 we obtain as unique solution

$$t := t_1, \quad k := \frac{x_2}{\frac{\sinh t_1}{t_1}}, \quad a := x_4 - \frac{x_2(1-\cosh t_1)}{\sinh t_1},$$
$$n := x_1 - (1+x_2)\left[x_4 - \frac{x_2(1-\cosh t_1)}{\sinh t_1}\right] - \frac{x_2^2(t_1-\sinh t_1)}{\sinh^2 t_1}.$$

Moreover, in the case of $H_{3,y}$ for $y \in (0,1]$ the unique solution is given as follows:
For $t_1=0$ we have $t=0$, $l=x_4$, $k=x_2-x_4y$, $n=x_1-x_4(x_2-yx_4)$, whereas for $t_1 \neq 0$ we obtain

$$t=t_1, l=\frac{x_4\sinh t_1 + x_2\cosh t_1 - x_2}{\sinh t_1 - y + y\cosh t_1}, \quad k = \frac{(x_2-yx_4)t_1}{\sinh t_1 - y + y\cosh t_1},$$

$$n = x_1 + \frac{(\sinh t_1 - t_1)(x_2-yx_4)^2}{(\sinh t_1 - y + y\cosh t_1)^2} - \frac{(x_4\sinh t_1 + x_2\cosh t_1 - x_2)(x_2-yx_4)}{(\sinh t_1 - y + y\cosh t_1)}.$$

It follows that the group $G_{(+)}$ is the group topologically generated by the left translations of infinitely many non-isomorphic differentiable 3-dimensional Bol loops L. Every such loop L has a normal subgroup $N = \exp\{\lambda e_1,\ \lambda \in \mathbb{R}\} = \{g(0,\lambda,0,0),\ \lambda \in \mathbb{R}\}$ isomorphic to $\mathbb{R}$ and the factor loop L/N is isomorphic to a loop L_α with $\alpha \leq -1$ defined in Theorem 23.1 of [15] and thus isotopic to the pseudo-euclidean plane loop. Hence L is an extension of the group $\mathbb{R}$ by a loop L_α.

The loop L_1 having H_1 as the stabilizer of $e \in L_1$ in $G_{(+)}$ is a Bruck loop. The loop L_2 which is realized on the factor space G/H_4 is a left A-loop. The stabilizer H_1 is conjugate to H_4 under $g(0,0,-\frac{1}{2},0) \in G_{(+)}$ and to $H_{3,y}$ under $g(\operatorname{artanh}(-y),0,y,1) \in G_{(+)}$ with $y \in (0,1)$. Hence the loops corresponding to these stabilizers are isotopic. In contrast to this the loop corresponding to $H_{3,1} = \{g(0,0,l,l);\, l \in \mathbb{R}\}$ does not belong to the isotopism class of L_1.

These considerations yield the following

Theorem 9. *If L is a 3-dimensional connected differentiable Bol loop corresponding to a Lie triple system, which is a non-split extension of its centre and a 2-dimensional non-abelian Lie triple system, then the group G topologically generated by the left translations of L is the semidirect product of the normal group $\mathbb{R}$ and the 3-dimensional non-abelian nilpotent Lie group such that the multiplication of G is given by*

$$\begin{aligned} &g(t_1,x_1,x_2,x_4)\cdot g(t_2,y_1,y_2,y_4)\\ &\quad = g(t_1+t_2, x_1+y_1+y_4(x_2\cosh t_2 - x_4\sinh t_2) - y_2(x_4\cosh t_2 - x_2\sinh t_2),\\ &y_2 + x_2\cosh t_2 - x_4\sinh t_2, y_4 - x_2\sinh t_2 + x_4\cosh t_2). \end{aligned}$$

All loops L are extensions of the Lie group $\mathbb{R}$ by a loop L_α described in Theorem 23.1 of [15] and form precisely two isotopism classes $\mathcal{C}_1$, $\mathcal{C}_2$.

All loops in $\mathcal{C}_1$ are isomorphic and may be represented by the loop L which has the group $H = \{g(0,0,l,l);\ l \in \mathbb{R}\}$ as the stabilizer of its identity in G.

The class $\mathcal{C}_2$ contains (up to isomorphisms) a Bruck loop L_1 corresponding to $H_1 = \{g(0,0,0,a),\ a \in \mathbb{R}\}$, a left A-loop L_2 corresponding to $H_2 = \{g(0,a,0,a),\ a \in \mathbb{R}\}$ and the loops L_y with $y \in (0,1)$ corresponding to the groups $H_y = \{g(0,0,ly,l),\ l \in \mathbb{R}\}$ as the stabilizers of the identity.

6 Bol loops corresponding to the Lie triple system having trivial centre

Now we deal with the case **3** in Section 3.

Lemma 10. *The universal Lie algebra $\mathbf{g}^U$ of the Lie triple system $\mathbf{m} = \langle e_1,e_2,e_3\rangle$ of type* **3** *is the standard enveloping Lie algebra $\mathbf{g}^*$ characterized in* **3** *of Section 3.*

Proof. Because of $\mathbf{m}^U \cap [\mathbf{m}^U,\mathbf{m}^U] = 0$ we may assume that $\mathbf{m}^U = \langle e_1,e_2,e_3\rangle$ and take for a basis of $[\mathbf{m}^U,\mathbf{m}^U]$ the vectors $e_4 := [e_2,e_3]$, $e_5 := [e_1,e_3]$ and

$e_6 := [e_1, e_2]$. The relations of the Lie triple system of type **3** yield the following multiplication:

$$[e_2, e_3] = e_4, \quad [e_4, e_3] = e_1, \quad [e_1, e_3] = e_5, \quad [e_1, e_2] = e_6,$$

whereas the other products are zero. For e_2, e_3, e_4 one has

$$[[e_2, e_3], e_4] + [[e_3, e_4], e_2] + [[e_4, e_2], e_3] = e_6$$

and the Jacobi identity is satisfied if and only if $[e_1, e_2] = 0$. This is the assertion. □

The mapping β

$$\begin{aligned}
\beta(e_1) &= \frac{1}{2}\sqrt{2}e_1 - \frac{1}{2}\sqrt{2}e_4 - \frac{1}{2}\sqrt{2}e_2 + \frac{1}{2}\sqrt{2}e_5, \\
\beta(e_2) &= \frac{1}{2}\sqrt{2}e_1 - \frac{1}{2}\sqrt{2}e_4 + \frac{1}{2}\sqrt{2}e_2 - \frac{1}{2}\sqrt{2}e_5, \\
\beta(e_3) &= \frac{1}{2}\sqrt{2}e_3, \quad \beta(e_4) = e_1 + e_4, \quad \beta(e_5) = -e_2 - e_5
\end{aligned}$$

yields an isomorphism of $\mathbf{g}^*$ onto the Lie algebra $\mathbf{g}$ defined by the following non-trivial products:

$$[e_1, e_3] = e_1 - e_2, \quad [e_2, e_3] = e_1 + e_2, \quad [e_4, e_3] = -e_4 + e_5, \quad [e_5, e_3] = -e_5 - e_4.$$

(We remark, that $\mathbf{g}$ is isomorphic to the Lie algebra $g_{5,17}$ for $s = -1$, $q = -1$, $p = 1$ in [14] (p. 105)). The elements $xe_1 + ye_2 + ze_3 + ue_4 + ve_5$ of $\mathbf{g}$ can be identify with the matrices

$$\begin{pmatrix} 0 & y & x & 0 & 0 & 0 \\ 0 & z & z & 0 & 0 & 0 \\ 0 & -z & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v & u \\ 0 & 0 & 0 & 0 & -z & -z \\ 0 & 0 & 0 & 0 & z & -z \end{pmatrix}; \quad x, y, z, u, v \in \mathbb{R}.$$

Then the multiplication in G is determined by

$$\begin{aligned}
&g(a_1, b_1, c_1, d_1, f_1)g(a_2, b_2, c_2, d_2, f_2) \\
&\qquad = g(a_2 + b_1 e^{c_2} \sin c_2 + a_1 e^{c_2} \cos c_2, b_2 + b_1 e^{c_2} \cos c_2 - a_1 e^{c_2} \sin c_2, \\
&c_1 + c_2, d_2 - f_1 e^{-c_2} \sin c_2 + d_1 e^{-c_2} \cos c_2, f_2 + f_1 e^{-c_2} \cos c_2 + d_1 e^{-c_2} \sin c_2).
\end{aligned}$$

The isomorphism β maps the Lie triple system $\langle e_1, e_2, e_3 \rangle$ onto the Lie triple system $\mathbf{m} = \langle e_1 - e_4, e_2 - e_5, e_3 \rangle$ and one has

$$\exp \mathbf{m} = \exp\{n(e_1 - e_4) + m(e_2 - e_5) + se_3; n, m, s \in \mathbb{R}\}$$

$$= \left\{ g\left(\frac{(n-m)(e^s\cos s-1)+(n+m)e^s\sin s}{2s}, \right.\right.$$
$$\frac{(n+m)(e^s\cos s-1)+(m-n)e^s\sin s}{2s}, s,$$
$$\frac{(n-m)(e^{-s}\cos s-1)-(m+n)e^{-s}\sin s}{2s},$$
$$\left.\left. \frac{(n+m)(e^{-s}\cos s-1)+(n-m)e^{-s}\sin s}{2s} \right),\ n,m,s\in\mathbb{R} \right\}.$$

The 2-dimensional subalgebras $\mathbf{h}$ of $\mathbf{g}$ with the property $\mathbf{h}\cap\langle e_3\rangle = \{0\}$ have the following forms:

$$\begin{aligned} \mathbf{h}_{a_2,a_4,b_2} &= \langle e_5+a_2e_2+a_4e_4, e_1+b_2e_2\rangle && \text{with } a_2,a_4,b_2\in\mathbb{R}, \\ \mathbf{h}_{a_1,a_4,b_1} &= \langle e_5+a_1e_1+a_4e_4, b_1e_1+e_2\rangle, && \text{where } a_1,a_4,b_1\in\mathbb{R}, \\ \mathbf{h}_{a_1,a_2,b_1,b_2} &= \langle e_5+a_1e_1+a_2e_2, e_4+b_1e_1+b_2e_2\rangle, && \text{where } a_1,a_2,b_1,b_2\in\mathbb{R}. \end{aligned}$$

The automorphism $\alpha:\mathbf{g}\to\mathbf{g}$ given by

$$\alpha(e_1)=b_2e_1+e_2,\quad \alpha(e_2)=-e_1+b_2e_2,\quad \alpha(e_3)=e_3,$$
$$\alpha(e_4)=b_2e_4-e_5,\quad \alpha(e_5)=e_4+b_2e_5,$$

where $b_2\in\mathbb{R}$, and the automorphism $\beta:\mathbf{g}\to\mathbf{g}$ determined by

$$\beta(e_1)=e_1+b_1e_2,\quad \beta(e_2)=-b_1e_1+e_2,\quad \beta(e_3)=e_3,$$
$$\beta(e_4)=e_4-b_1e_5,\quad \beta(e_5)=b_1e_4+e_5$$

where $b_1\in\mathbb{R}$, leave the subspace $\mathbf{m}$ invariant. If $b_2\neq a_4$ then α maps $\mathbf{h}_{a_2,a_4,b_2}$ onto

$$\mathbf{h}_{a,b}=\langle e_5+ae_1+be_4, e_2\rangle \quad \text{with} \quad a,b\in\mathbb{R}$$

and if $b_2=a_4$ then α reduces $\mathbf{h}_{a_2,a_4,b_2}$ to

$$\mathbf{h}_a=\langle e_4+ae_1, e_2\rangle \quad \text{with} \quad a\in\mathbb{R}.$$

For $b_1\neq\frac{1}{a_4}$ the automorphism β maps $\mathbf{h}_{a_1,a_4,b_1}$ to $\mathbf{h}_{a,b}$, whereas for $b_1=\frac{1}{a_4}$ the subalgebras $\mathbf{h}_{a_1,a_4,b_1}$ reduce to $\mathbf{h}_a$. Since $\mathbf{h}_{a,b}\cap\mathbf{m}$ is not trivial if $a=-b$ we may assume that for $\mathbf{h}_{a,b}$ one has $a\neq -b$.

For $a_1=a_2=b_1=b_2=0$ the subalgebra $\mathbf{h}_{0,0,0,0}=\langle e_5,e_4\rangle$ is an ideal of $\mathbf{g}$. Therefore we suppose that in $\mathbf{h}_{a_1,a_2,b_1,b_2}$ not all parameters a_1,a_2,b_1,b_2 are 0. Moreover, $(a_2+1)(1+b_1)-a_1b_2\neq 0$, since otherwise $\mathbf{h}_{a_1,a_2,b_1,b_2}\cap\mathbf{m}\neq 0$.

The Lie groups corresponding to the Lie algebras $\mathbf{h}_a$, $\mathbf{h}_{a,b}$, $\mathbf{h}_{a_1,a_2,b_1,b_2}$ have the forms

a) $H_a=\exp\mathbf{h}_a=\{g(ka,l,0,k,0);\ k,l\in\mathbb{R}\},\ a\in\mathbb{R}$

b) $H_{a,b} = \exp \mathbf{h}_{a,b} = \{g(ka, l, 0, kb, k);\ k, l \in \mathbb{R}\},\ \ a, b \in \mathbb{R},\ a \neq b$

c) $H_{a_1,a_2,b_1,b_2} = \exp \mathbf{h}_{a_1,a_2,b_1,b_2} = \{g(ka_1 + lb_1, ka_2 + lb_2, 0, l, k);\ k, l \in \mathbb{R}\}$,

where $(a_2 + 1)(1 + b_1) - a_1 b_2 \neq 0$ and not all a_1, a_2, b_1, b_2 are equal 0.

Each element of G has a unique decomposition as

$$
\begin{aligned}
g(x_1, x_2, x_3, x_4, x_5) &= g(y_1, 0, y_2, 0, y_3)g(ka, l, 0, k, 0) \text{ in the case a)} \\
g(x_1, x_2, x_3, x_4, x_5) &= g(y_1, 0, y_2, y_3, 0)g(ka, l, 0, kb, k) \text{ in the case b)} \\
g(x_1, x_2, x_3, x_4, x_5) &= g(y_1, y_3, y_2, 0, 0)g(ka_1 + lb_1, ka_2 + lb_2, 0, l, k); l, k \in \mathbb{R}\} \\
&\quad \text{in the case c).}
\end{aligned}
$$

A differentiable Bol loop L exists precisely if in the case a) every element $g(y_1, 0, y_2, 0, y_3)$, in the case b) every element $g(y_1, 0, y_2, y_3, 0)$ and in the case c) every element $g(y_1, y_3, y_2, 0, 0)$, $y_i \in \mathbb{R}$, $i = 1, 2, 3$, can be written uniquely as a product $g = mh$ or equivalently $m = gh^{-1}$, where $m \in \exp \mathbf{m}$ and h is a suitable element of the stabilizer H_a, $H_{a,b}$ or H_{a_1,a_2,b_1,b_2} respectively. This happens if and only if for given $y_1, y_2, y_3 \in \mathbb{R}$ the following system of equations

$$
s = y_2, \quad A = \frac{u(e^s \cos s - 1) + ve^s \sin s}{2s}, \quad B = \frac{v(e^s \cos s - 1) - ue^s \sin s}{2s},
$$
$$
C = \frac{u(e^{-s} \cos s - 1) - ve^{-s} \sin s}{2s}, \quad D = \frac{v(e^{-s} \cos s - 1) + ue^{-s} \sin s}{2s}, \quad \text{(I)}
$$

with $A = y_1 - ka$, $B = -l$, $C = -k$, $D = y_3$ in the case a),
with $A = y_1 - ka$, $B = -l$, $C = y_3 - kb$, $D = -k$ in the case b) and
$A = y_1 - ka_1 - lb_1$, $B = y_3 - ka_2 - lb_2$, $C = -l$, $D = -k$, in the case c)
has a unique solution $(u, v, s, k, l) \in \mathbb{R}^5$.

Assuming $y_2 \neq 0$ and putting

$$
\begin{aligned}
m_{11} &= e^{y_2} \cos y_2 - 1 - a(e^{-y_2} \cos y_2 - 1), & m_{21} &= e^{-y_2} \sin y_2, \\
m_{12} &= e^{y_2} \sin y_2 + ae^{-y_2} \sin y_2, & m_{22} &= e^{-y_2} \cos y_2 - 1
\end{aligned}
$$

in the case a),

$$
\begin{aligned}
m_{11} &= e^{y_2} \cos y_2 - 1 + ae^{-y_2} \sin y_2, & m_{12} &= e^{y_2} \sin y_2 - a(e^{-y_2} \cos y_2 - 1), \\
m_{21} &= e^{-y_2} \cos y_2 - 1 - be^{-y_2} \sin y_2, & m_{22} &= -e^{-y_2} \sin y_2 - b(e^{-y_2} \cos y_2 - 1)
\end{aligned}
$$

in the case b) and

$$
\begin{aligned}
m_{11} &= e^{y_2} \cos y_2 + 1 - a_1 e^{-y_2} \sin y_2 - b_1(e^{-y_2} \cos y_2 - 1), \\
m_{12} &= e^{y_2} \sin y_2 - a_1(e^{-y_2} \cos y_2 - 1) + b_1 e^{-y_2} \sin y_2, \\
m_{21} &= -e^{y_2} \sin y_2 - a_2 e^{-y_2} \sin y_2 - b_2(e^{-y_2} \cos y_2 - 1),
\end{aligned}
$$

$$m_{22} = e^{y_2} \cos y_2 + 1 - a_2(e^{-y_2} \cos y_2 - 1) + b_2 e^{-y_2} \sin y_2$$

in the case c), we see that the system (I) yields the following system of linear equations

$$\begin{aligned} m_{11}u + m_{12}v &= 2y_1y_2 \\ m_{21}u + m_{22}v &= 2y_2y_3. \end{aligned} \tag{II}$$

If $y_1 = y_3 = 0$ and $\det(m_{ij}) = 0$ $i,j \in \{1,2\}$ then the system (II) has infinitely many solutions.

The condition $\det(m_{ij}) = 0$ holds if and only if in the case a) the function

$$f(x) = -(e^x + e^{-x}) \cos x - a(e^{-2x} - 2e^{-x} \cos x + 1) + 2\cos^2 x,$$

in the case b) the function

$$\begin{aligned} g(x) = {} & (2ae^{-2x} - 2b) \cos^2 x - (2 + 2abe^{-2x}) \cos x \sin x + be^x \cos x \\ & + (b - 2a)e^{-x} \cos x + e^x \sin x + (2ab + 1)e^{-x} \sin x + a - ae^{-2x} \end{aligned}$$

and in the case c) the function

$$\begin{aligned} h(x) = {} & e^{2x} + e^{-2x}(b_1a_2 - a_1b_2) + (e^x + e^{-x}) \sin x(a_1 - b_2) \\ & + e^x \cos x(a_2 + b_1 - 2) + e^{-x} \cos x(2a_1b_2 - 2a_2b_1 + b_1 + a_2) \\ & + 1 + (2b_2 - 2a_1) \sin x \cos x - (2b_1 + 2a_2) \cos^2 x + b_1a_2 - b_2a_1 \end{aligned}$$

assumes the value 0.

If $k = \max\{100, 2|a|\}$ then for $x = 2\pi k$ and $y = \pi + 2\pi k$ we obtain that $f(x) < 0$ and $f(y) > 0$. Hence in the open interval $\big(2\pi k, \pi + 2\pi k\big)$ there is a value y_2 such that $f(y_2) = 0$.

For $p_1 = \frac{\pi}{2} + 2\pi k$ and $p_2 = \frac{3\pi}{2} + 2\pi k$ with $k = \max\{100, 2|a|, 4|ab|\}$ one has $g(p_1) > 0$ and $g(p_2) < 0$. Hence the open interval $(\frac{\pi}{2} + 2\pi k, \frac{3\pi}{2} + 2\pi k)$ contains a value y_2 such that $g(y_2) = 0$.

Therefore there is no 3-dimensional differentiable Bol loop L such that the group topologically generated by its left translations is the group G and the stabilizer of $e \in L$ in G is a subgroup H_a or $H_{a,b}$.

In the case c) one has

a) $\lim_{x \to +\infty} h(x) = +\infty$,

b) $\lim_{x \to -\infty} h(x) = -\infty$ if $b_1a_2 - a_1b_2 < 0$

c) $\lim_{x \to -\infty} h(x) = \infty$ if $b_1a_2 - a_1b_2 > 0$.

The first and second derivative of $h(x)$ are

$$h'(x) = 2e^{2x} + (a_1 - b_2)[(e^x - e^{-x}) \sin x + (e^x + e^{-x}) \cos x]$$

$$
\begin{aligned}
&+ (a_2 + b_1 - 2)(e^x \cos x - e^x \sin x) - 2e^{-2x}(b_1 a_2 - a_1 b_2)\\
&- (b_1 + a_2 - 2a_2 b_1 + 2a_1 b_2)(e^{-x} \cos x + e^{-x} \sin x)\\
&+ (2b_2 - 2a_1)(\cos^2 x - \sin^2 x) + 4 \cos x \sin x(b_1 + a_2)
\end{aligned}
$$

and

$$
\begin{aligned}
h''(x) = 4e^{2x} &+ 4e^{-2x}(b_1 a_2 - a_1 b_2) + 2(a_1 - b_2)(e^x - e^{-x}) \cos x\\
&+ 2(b_1 + a_2 - 2a_2 b_1 + 2a_1 b_2)e^{-x} \sin x - 2(a_2 + b_1 - 2)e^x \sin x\\
&+ 4(b_1 + a_2)(\cos^2 x - \sin^2 x) - 8(b_2 - a_1) \cos x \sin x.
\end{aligned}
$$

One obtains $h(0) = h'(0) = 0$ and $h''(0) = 4 + 4(b_1 + a_2) + 4(a_2 b_1 - a_1 b_2)$. Since $h''(0) \neq 0$ we have two possibilities: $h''(0) < 0$ or $h''(0) > 0$. The function $h(x)$ has in 0 a maximum or a minimum according as $h''(0) < 0$ or $h''(0) > 0$. Now from the properties a) and b) it follows that for $h''(0) < 0$ and for $h''(0) > 0$ with $b_1 a_2 - a_1 b_2 < 0$ there is a value $p \in \mathbb{R}\backslash\{0\}$ such that $h(p) = 0$.

For $b_1 a_2 - a_1 b_2 = 0$ one has

$$
\begin{aligned}
h(x) = e^{2x} &+ (e^x + e^{-x}) \sin x(a_1 - b_2) + e^x \cos x(a_2 + b_1 - 2)\\
&+ e^{-x} \cos x(b_1 + a_2) + 1 + (2b_2 - 2a_1) \sin x \cos x - (2b_1 + 2a_2) \cos^2 x.
\end{aligned}
$$

First we assume that $a_1 - b_2 \neq 0$. Then we have $\varepsilon h(p_1) > 0$ and $\varepsilon h(p_2) < 0$ if $p_1 = -(\frac{\pi}{2} + 2\pi k)$ and $p_2 = -(\frac{3\pi}{2} + 2\pi k)$, where $k = \max\{100, \frac{4}{|a_1 - b_2|}\}$ and $\varepsilon = 1$ if $a_1 - b_2 < 0$, whereas $\varepsilon = -1$ for $a_1 - b_2 > 0$. Hence in the open interval $\big(-\frac{3\pi}{2} - 2\pi k, -\frac{\pi}{2} - 2\pi k\big)$ the function h assumes 0.

For $a_1 = b_2$ we obtain

$$
h(x) = e^{2x} + (b_1 + a_2)(e^x + e^{-x}) \cos x - 2e^x \cos x + 1 - (2b_1 + 2a_2) \cos^2 x.
$$

If $p_1 = -2\pi k$ and $p_2 = -\pi - 2\pi k$ then we have $\varepsilon h(p_1) > 0$ and $\varepsilon h(p_2) < 0$, where $k = \max\{100, \frac{4|1-2b_1-2a_2|}{|b_1+a_2|}\}$ and $\varepsilon = 1$ or $\varepsilon = -1$ according as $b_1 + a_2 > 0$ or $b_1 + a_2 < 0$. Therefore the interval $(-\pi - 2\pi k, -2\pi k)$ contains a value $p \in \mathbb{R}\backslash\{0\}$ such that $h(p) = 0$.

It follows that a differentiable Bol loop L does not exist if the parameters a_1, a_2, b_1, b_2 satisfy either

$$
1 + b_1 + a_2 + a_2 b_1 - a_1 b_2 < 0
$$

or

$$
1 + b_1 + a_2 + a_2 b_1 - a_1 b_2 > 0 \quad \text{and} \quad a_2 b_1 - a_1 b_2 \leq 0.
$$

For $y_2 = 0 = s$ the system (I) reduces to

$$
y_1 - k a_1 - l b_1 = n,\ y_3 - k a_2 - l b_2 = m,\ l = n,\ k = m, \quad \text{with } n, m \in \mathbb{R}. \tag{III}
$$

Since for the parameters a_1, a_2, b_1, b_2 one has $(a_2+1)(1+b_1)-a_1b_2 \neq 0$ the system (III) has precisely one solution for all $y_1, y_3 \in \mathbb{R}$. Namely, if $b_1 \neq -1$ we obtain

$$l = n = \frac{y_1 - ma_1}{1+b_1}, \quad k = m = \frac{y_3(1+b_1) - f_1 b_2}{(a_2+1)(1+b_1) - a_1 b_2},$$

whereas for $b_1 = -1$ one has $a_1 b_2 \neq 0$ and

$$k = m = \frac{y_1}{a_1}, \quad l = n = \frac{y_3 a_1 - y_1 a_2 - y_1}{b_2 a_1}.$$

The above discussion yields the following

Theorem 11. *If L is a 3-dimensional connected differentiable Bol loop corresponding to a Lie triple system which has trivial centre, then the group topologically generated by its left translations is the 5-dimensional Lie group G the multiplication of which is given by*

$$\begin{aligned} &g(a_1, b_1, c_1, d_1, f_1) g(a_2, b_2, c_2, d_2, f_2) \\ &\qquad = g(a_2 + b_1 e^{c_2} \sin c_2 + a_1 e^{c_2} \cos c_2, b_2 + b_1 e^{c_2} \cos c_2 - a_1 e^{c_2} \sin c_2, \\ &c_1 + c_2, d_2 - f_1 e^{-c_2} \sin c_2 + d_1 e^{-c_2} \cos c_2, f_2 + f_1 e^{-c_2} \cos c_2 + d_1 e^{-c_2} \sin c_2). \end{aligned}$$

Moreover, the stabilizer of the identity of L in G is the subgroup

$$H_{a_1,a_2,b_1,b_2} = \{g(ka_1 + lb_1, ka_2 + lb_2, 0, l, k);\ k, l \in \mathbb{R}\}$$

such that the parameters a_1, a_2, b_1, b_2 satisfy

$$1 + b_1 + a_2 + a_2 b_1 - a_1 b_2 > 0 \quad \text{and} \quad a_2 b_1 - a_1 b_2 > 0$$

and the function

$$\begin{aligned} h(x) = {} & e^{2x} + e^{-2x}(b_1 a_2 - a_1 b_2) + (e^x + e^{-x}) \sin x (a_1 - b_2) \\ & + e^x \cos x (a_2 + b_1 - 2) + e^{-x} \cos x (2a_1 b_2 - 2a_2 b_1 + b_1 + a_2) \\ & + 1 + (2b_2 - 2a_1) \sin x \cos x - (2b_1 + 2a_2) \cos^2 x + b_1 a_2 - b_2 a_1 \end{aligned}$$

is positive for all $x \in \mathbb{R} \setminus \{0\}$.

There are many differentiable 3-dimensional Bol loops on the factor space $G/H_{a_1,a_2,b_1,b_2}$. For instance choosing $a_1, a_2, b_1, b_2 \in \mathbb{R}$ such that $a_1 = b_2$, $a_2 = 2 - b_1$ and $c = b_1 a_2 - a_1 b_2 = -(b_1 - 1)^2 - b_2^2 + 1$ with $\frac{3}{7} < c \leq 1$ the function

$$h(x) = e^{2x} + (2 - 2c)e^{-x} \cos x + ce^{-2x} + 1 + c - 4\cos^2 x$$

of Theorem 11 is positive for all $x \in \mathbb{R} \setminus \{0\}$. To prove this it is enough to show that the function

$$k(x) = e^{2x} + (2 - 2c)e^{-x} \cos x + ce^{-2x} + c - 3$$

is positive for all $x \in \mathbb{R}\backslash\{0\}$. The second derivative

$$k''(x) = 4e^{2x} + 4ce^{-2x} + 4(1-c)e^{-x}\sin x$$

is positive if and only if

$$4e^{2x} + 4ce^{-2x} - 4(1-c)e^{-x} > 0$$

or

$$l(x) = e^{4x} + (c-1)e^{x} + c > 0 \quad \text{for all } x \in \mathbb{R}.$$

For the derivations of $l(x)$ we obtain

$$l'(x) = 4e^{4x} + (c-1)e^{x}, \quad l''(x) = 16e^{4x} + (c-1)e^{x}.$$

One has $l'(p) = 0$ if and only if $p = \frac{1}{3}\ln\frac{1-c}{4}$. For this value p the function $l(x)$ takes its unique minimum since $l''(p) = \left(\frac{1-c}{4}\right)^{\frac{1}{3}}(3-3c) > 0$. Because of $\frac{3}{7} < c \leq 1$ we get $l(p) = c - 3\left(\frac{1-c}{4}\right)^{\frac{4}{3}} \geq c - \frac{3}{4}(1-c) > 0$. It follows $k''(x) > 0$ for all $x \in \mathbb{R}$ and therefore $k'(x)$ is a strictly monotone increasing function. Since $k'(0) = 0$ the value 0 is the unique minimum of $k(x)$. Furthermore one has $k(x) \geq 0$ because of $k(0) = 0$ and $\lim_{x\to-\infty} k(x) = \lim_{x\to+\infty} k(x) = +\infty$.

Let L_{a_1,a_2} be the Bol loop belonging to the triple $(G, H_{a_1,a_2,2-a_2,a_1}, \exp\mathbf{m})$, where $-\frac{4}{7} < -(a_2-1)^2 - a_1^2 \leq 0$. Among these loops only the loop $L_{0,1}$ is a left A-loop. Since there is no element $g \in G$ such that $g^{-1}\mathbf{h}_{a_1,a_2,2-a_2,a_1}g = \mathbf{h}_{a_1',a_2',2-a_2',a_1'}$ for two different pairs (a_1,a_2), (a_1',a_2') holds the loops L_{a_1,a_2} and $L_{a_1',a_2'}$ are not isotopic. Therefore there are infinitely many non-isotopic Bol loops L_{a_1,a_2}.

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