Multiplicative loops of 2-dimensional topological quasifields

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Abstract

We determine the algebraic structure of the multiplicative loops for locally compact 2-dimensional topological connected quasifields. In particular, our attention turns to multiplicative loops which have either a normal subloop of positive dimension or which contain a 1-dimensional compact subgroup. In the last section we determine explicitly the quasifields which coordinatize locally compact translation planes of dimension 4 admitting an at least 7-dimensional Lie group as collineation group.

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1. Introduction

Locally compact connected topological non-desarguesian translation planes have been a popular subject of geometrical research since the seventies of the last century ([18], [2]-[9], [13], [15]). These planes are coordinatized by locally compact quasifields Q such that the kernel of Q is either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers (cf. [11], IX.5.5 Theorem, p. 323). If the quasifield Q is 2-dimensional, then its kernel is $\mathbb{R}$.

The classification of topological translation planes $\mathcal{A}$ was accomplished by reconstructing the spreads corresponding to $\mathcal{A}$ from the translation complement which is the stabilizer of a point in the collineation group of $\mathcal{A}$. In this way all planes $\mathcal{A}$ having an at least 7-dimensional collineation group have been determined ([3]-[8], [15]).

Although any spread gives the lines through the origin and hence the multiplication in a 2-dimensional quasifield Q coordinatizing the plane $\mathcal{A}$, to the algebraic structure of the multiplicative loop Q^* of a proper quasifield Q

is not given special attention apart from the facts that the group topologically generated by the left translations of Q^* is the connected component of $\mathrm{GL}_2(\mathbb{R})$, the group topologically generated by the right translations of Q^* is an infinite-dimensional Lie group (cf. [14], Section 29, p. 345) and any locally compact 2-dimensional semifield is the field of complex numbers ([17]).

Since in the meantime some progress in the classification of compact differentiable loops on the 1-sphere has been achieved (cf. [10]), we believe that loops could have more space in the research concerning 4-dimensional translation planes. Using the images of differentiable sections $\sigma : G/H \to G$, where $H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}$, we classify the C^1-differentiable multiplicative loops Q^* of 2-dimensional locally compact quasifields Q by functions, the Fourier series of which are described in [10].

The multiplicative loops Q^* of 2-dimensional locally compact left quasifields Q for which the set of the left translations of Q^* is the product $\mathcal{T}\mathcal{K}$ with $|\mathcal{T} \cap \mathcal{K}| \leq 2$, where $\mathcal{T}$ is the set of the left translations of a 1-dimensional compact loop and $\mathcal{K}$ is the set of the left translations of Q^* corresponding to the kernel K_r of Q, form an important subclass of loops, that we call decomposable loops. Namely, if Q^* has a normal subloop of positive dimension or if it contains the group $\mathrm{SO}_2(\mathbb{R})$, then Q^* is decomposable. Moreover, we show that any 1-dimensional C^1-differentiable compact loop is a factor of a decomposable multiplicative loop of a locally compact connected quasifield coordinatizing a 4-dimensional translation plane. A 2-dimensional locally compact quasifield Q is the field of complex numbers if and only if the multiplicative loop Q^* contains a 1-dimensional normal compact subloop.

Till now mainly those simple loops have been studied for which the group generated by their left translations is a simple group. If the group generated by the left translations of a loop L is simple, then L is also simple (cf. Lemma 1.7 in [14]). The multiplicative loops Q^* of 2-dimensional locally compact quasifields show that there are many interesting 2-dimensional locally compact quasi-simple loops for which the group generated by their left translations has a one-dimensional centre.

In the last section we use Betten's classification to determine in our framework the multiplicative loops Q^* of the quasifields which coordinatize the 4-dimensional non-desarguesian translation planes $\mathcal{A}$ admitting an at least seven-dimensional collineation group and to study their properties. The results obtained there yield the following

Theorem *Let $\mathcal{A}$ be a 4-dimensional locally compact non-desarguesian translation plane which admits an at least 7-dimensional collineation group Γ. If the quasifield Q coordinatizing $\mathcal{A}$ is constructed with respect to two lines such that their intersection points with the line at infinity are contained in the*

1-dimensional orbit of Γ *or contain the set of the fixed points of* Γ*, then the multiplicative loop* Q^* *of* Q *is decomposable if and only if one of the following cases occurs:*
(a) Γ *is 8-dimensional, the translation complement* C *is the group* $\mathrm{GL}_2(\mathbb{R})$ *and acts reducibly on the translation group* $\mathbb{R}^4$*;*
(b) Γ *is 7-dimensional, the translation complement* C *fixes two distinct lines of* $\mathcal{A}$ *and leaves on one of them, one or two 1-dimensional subspaces invariant;*
(c) Γ *is 7-dimensional, the translation complement* C *fixes two distinct lines* $\{S, W\}$ *through the origin and acts transitively on the spaces* P_S *and* P_W *but does not act transitively on the product space* $P_S \times P_W$*, where* P_S *and* P_W *are the sets of all 1-dimensional subspaces of* S*, respectively of* W*.*

2. Preliminaries

A binary system $(L, \cdot)$ is called a quasigroup if for any given $a, b \in L$ the equations $a \cdot y = b$ and $x \cdot a = b$ have unique solutions which we denote by $y = a \backslash b$ and $x = b/a$. If a quasigroup L has an element 1 such that $x = 1 \cdot x = x \cdot 1$ holds for all $x \in L$, then it is called a loop and 1 is the identity element of L. The left translations $\lambda_a : L \to L, x \mapsto a \cdot x$ and the right translations $\rho_a : L \to L, x \mapsto x \cdot a$, $a \in L$, are bijections of L. Two loops $(L_1, \circ)$ and $(L_2, *)$ are called isotopic if there exist three bijections $\alpha, \beta, \gamma : L_1 \to L_2$ such that $\alpha(x) * \beta(y) = \gamma(x \circ y)$ holds for all $x, y \in L_1$. A binary system $(K, \cdot)$ is called a subloop of $(L, \cdot)$ if $K \subset L$, for any given $a, b \in K$ the equations $a \cdot y = b$ and $x \cdot a = b$ have unique solutions in K and $1 \in K$. The kernel of a homomorphism $\alpha : (L, \cdot) \to (L', *)$ of a loop L into a loop L' is a normal subloop N of L, i.e. a subloop of L such that

$$x \cdot N = N \cdot x, \ (x \cdot N) \cdot y = x \cdot (N \cdot y), \ (N \cdot x) \cdot y = N \cdot (x \cdot y) \qquad (1)$$

hold for all $x, y \in L$. A loop L is called simple if $\{1\}$ and L are its only normal subloops.
A loop L is called topological, if it is a topological space and the binary operations $(a, b) \mapsto a \cdot b$, $(a, b) \mapsto a \backslash b$, $(a, b) \mapsto b/a : L \times L \to L$ are continuous. Then the left and right translations of L are homeomorphisms of L. If L is a connected differentiable manifold such that the loop multiplication and the left division are continuously differentiable mappings, then we call L an almost $\mathcal{C}^1$-differentiable loop. If also the right division of L is continuously differentiable, then L is a $\mathcal{C}^1$-differentiable loop. A connected topological loop is quasi-simple if it contains no normal subloop of positive dimension.
Every topological, respectively almost $\mathcal{C}^1$-differentiable, connected loop L having a Lie group G as the group topologically generated by the left trans-

lations of L corresponds to a sharply transitive continuous, respectively $\mathcal{C}^1$-differentiable section $\sigma : G/H \to G$, where $G/H = \{xH | x \in G\}$ consists of the left cosets of the stabilizer H of $1 \in L$ such that $\sigma(H) = 1_G$ and $\sigma(G/H)$ generates G. The section σ is sharply transitive if the image $\sigma(G/H)$ acts sharply transitively on the factor space G/H, i.e. for given left cosets xH, yH there exists precisely one $z \in \sigma(G/H)$ which satisfies the equation $zxH = yH$.

A (left) quasifield is an algebraic structure $(Q, +, \cdot)$ such that $(Q, +)$ is an abelian group with neutral element 0, $(Q \setminus \{0\}, \cdot)$ is a loop with identity element 1 and between these operations the (left) distributive law $x\cdot(y+z) = x\cdot y+x\cdot z$ holds. A locally compact connected topological quasifield is a locally compact connected topological space Q such that $(Q, +)$ is a topological group, $(Q \setminus \{0\}, \cdot)$ is a topological loop, the multiplication $\cdot : Q \times Q \to Q$ is continuous and the mappings $\lambda_a : x \mapsto a \cdot x$ and $\rho_a : x \mapsto x \cdot a$ with $0 \neq a \in Q$ are homeomorphisms of Q. If for any given $a, b, c \in Q$ the equation $x \cdot a + x \cdot b = c$ with $a + b \neq 0$ has precisely one solution, then Q is called planar. A translation plane is an affine plane with transitive group of translations; this is coordinatized by a planar quasifield (cf. [16], Kap. 8).

The kernel K_r of a (left) quasifield Q is a skewfield defined by

$$K_r = \{k \in Q;\ (x+y)\cdot k = x\cdot k+y\cdot k \text{ and } (x\cdot y)\cdot k = x\cdot(y\cdot k) \text{ for all } x, y \in Q\}.$$

In this paper we consider left quasifields Q. Then Q is a right vector space over K_r. Moreover, for all $a \in Q$ the map $\lambda_a : Q \to Q, x \mapsto a \cdot x$ is K_r-linear. According to [12], Theorem 7.3, p. 160, every quasifield that has finite dimension over its kernel is planar.

Let F be a skewfield and let V be a vector space over F. A collection $\mathcal{B}$ of subspaces of V with $|\mathcal{B}| \geq 3$ is called a spread of V if for any two different elements U_1, $U_2 \in \mathcal{B}$ we have $V = U_1 \oplus U_2$ and every vector of V is contained in an element of $\mathcal{B}$.

If S and W are different subspaces of the spread $\mathcal{B}$, then V can be coordinatized in such a way that $S = \{0\} \times X$ and $W = X \times \{0\}$. Any spread of $V = X \times X$ can be described by a collection $\mathcal{M}$ of linear mappings $X \to X$ satisfying the following conditions:

(M_1) For any $\omega_1 \neq \omega_2 \in \mathcal{M}$ the mapping $\omega_1 - \omega_2$ is bijective.

(M_2) For all $x \in X \setminus \{0\}$ the mapping $\phi_x : \mathcal{M} \to X : \omega \mapsto \omega(x)$ is surjective.

Namely, if $\mathcal{M}$ is a collection of linear mappings satisfying (M_1) and (M_2), then the sets $U_\omega = \{(x, \omega(x)), x \in X\}$ and $\{0\} \times X$ yield a spread of $V = X \times X$. Conversely, every component $U \in \mathcal{B} \setminus \{S\}$ of V is the graph of a linear mapping $\omega_U : W \to S$ and the set of ω_U gives a collection $\mathcal{M}$ of linear mappings of X satisfying (M_1) and (M_2) (cf. [13], Proposition 1.11.).

The mapping ω_W is the zero mapping. For this reason any collection $\mathcal{M}$ of linear mappings of X which satisfy (M_1) and M_2 is called a spread set of X. Every translation plane can be obtained from a spread set of a suitable vector space $V = X \times X$ (cf. [13], Theorem 1.5, p. 7, and [1]). As every translation plane can be coordinatized by a quasifield and a quasifield contains 0 and 1, the associated spread set contains the zero endomorphism and the identity map. This is not true for arbitrary spread sets $\mathcal{M}$, but if $\omega_0, \omega_1 \in \mathcal{M}$ are distinct, then $\mathcal{M}' = \{(\omega - \omega_0)(\omega_1 - \omega_0)^{-1},\ \omega \in \mathcal{M}\}$ is a normalized spread set of X, i.e. a spread set which contains the zero and the identity map. The translation planes obtained from $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic (cf. [13], Lemma 1.15, p. 13). Let $\mathcal{M}$ be a normalized spread set of X, $e \in X \setminus \{0\}$ and let $\phi_e : \mathcal{M} \to X$ be defined by $\phi_e(\omega) = \omega(e)$. Then the multiplication $\circ : X \times X \to X$ defined by $m \circ x = (\phi_e^{-1}(m))(x)$ yields a multiplicative loop of a left quasifield Q coordinatizing the translation plane $\mathcal{A}$ belonging to the spread $\mathcal{M}$ of X.
If we fix a basis of Q over its kernel K_r and identify X with the vector space of pairs $\{(x, y)^t,\ x, y \in K_r\}$, then the set $\mathcal{M}$ consists of matrices $C(\alpha, \beta, \gamma, \delta) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \alpha, \beta, \gamma, \delta \in K_r$. If $e = (1, 0)^t$, then we get $\phi_e(C(\alpha, \beta, \gamma, \delta)) = C(\alpha, \beta, \gamma, \delta)(e) = (\alpha, \gamma)^t$. Since $\mathcal{M}$ is a spread of X the set of vectors $(\alpha, \gamma)^t$ consists of all vectors of X. Hence if $(\alpha, \gamma)^t$ is an element of X, then there exists a unique matrix of $\mathcal{M}$ having $(\alpha, \gamma)^t$ as the first column.

We consider multiplicative loops of locally compact connected topological quasifields Q of dimension 2 coordinatizing 4-dimensional non-desarguesian topological translation planes. Then the kernel K_r of Q is isomorphic to the field of the real numbers, $(Q, +)$ is the vector group $\mathbb{R}^2$ and the multiplicative loop $(Q \backslash \{0\}, \cdot)$ is homeomorphic to $\mathbb{R} \times S^1$, where S^1 is the circle.

4. Multiplicative loops of 2-dimensional quasifields

Let $(Q, +, *)$ be a real topological (left) quasifield of dimension 2. Let e_1 be the identity element of the multiplicative loop $Q^* = (Q \setminus \{0\}, *)$ of Q, which generates the kernel $K_r = \mathbb{R}$ of Q as a vector space and let $B = \{e_1, e_2\}$ be a basis of the right vector space Q over K_r. Once we fix B, we identify Q with the vector space of pairs $(x, y)^t \in \mathbb{R}^2$ and K_r with the subspace of pairs $(x, 0)^t$. The element $(1, 0)^t$ is the identity element of Q^*. According to [14], Theorem 29.1, p. 345, the group G topologically generated by the left translations of Q^* is the connected component of the group $\mathrm{GL}_2(\mathbb{R})$. As $\dim Q^* = 2$ and the stabilizer of the identity element of Q^* in G does not contain any non-trivial normal subgroup of G we may replace the stabilizer of the identity by the subgroup $H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}$. The elements g of G have a unique decomposition as the product

$$g = \begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} \begin{pmatrix} k & l \\ 0 & k^{-1} \end{pmatrix}$$

with suitable elements $u > 0$, $k > 0$, $l \in \mathbb{R}$, $t \in [0, 2\pi)$. Hence the loop Q^* corresponds to a continuous section $\sigma : G/H \to G$;

$$\begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} H \mapsto \begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} \begin{pmatrix} a(u,t) & b(u,t) \\ 0 & a^{-1}(u,t) \end{pmatrix} \quad (2)$$

where the pair of continuous functions $a(u,t), b(u,t) : \mathbb{R}_{>0} \times [0, 2\pi) \to \mathbb{R}$, where $\mathbb{R}_{>0}$ is the set of positive numbers, satisfies the following conditions:

$$a(u,t) > 0, \quad a(1,0) = 1, \quad b(1,0) = 0.$$

As Q is a left quasifield, any $(x,y)^t \in Q^*$ induces a linear transformation $M_{(x,y)} \in \sigma(G/H)$. More precisely one has

$$\begin{pmatrix} x \\ y \end{pmatrix} * \begin{pmatrix} u \\ v \end{pmatrix} = M_{(x,y)} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} r\cos\varphi & r\sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{pmatrix} \begin{pmatrix} a(r,\varphi) & b(r,\varphi) \\ 0 & a^{-1}(r,\varphi) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad (3)$$

where $x = r\cos(\varphi)a(r,\varphi)$, $y = -r\sin(\varphi)a(r,\varphi)$. The kernel K_r of Q consists of $(0,0)^t$ and $(r\cos(k\pi)a(r,k\pi),0)^t$, $r > 0$, $k \in \{0,1\}$, such that the matrices corresponding to the elements $(r\cos(k\pi)a(r,k\pi),0)^t$ have the form

$$M(r\cos(k\pi)a(r,k\pi),0) = \begin{pmatrix} r\cos(k\pi)a(r,k\pi) & r\cos(k\pi)b(r,k\pi) \\ 0 & r\cos(k\pi)a^{-1}(r,k\pi) \end{pmatrix}.$$

The identity matrix I corresponds to the identity $(1,0)^t$ of Q^*. As to each real number $r\cos(k\pi)a(r,k\pi)$ belongs precisely one matrix

$$M(r\cos(k\pi)a(r,k\pi),0)$$

the functions $f_1(r) = ra(r,0), r > 0$ and $f_2(r) = -ra(r,\pi), r > 0$ are strictly monotone. If the functions $a(r,0)$, $a(r,\pi)$ are differentiable, then for every $r > 0$ the derivatives $a(r,0) + ra'(r,0)$ and $-a(r,\pi) - ra'(r,\pi)$ are either always positive or negative. This is equivalent to the fact that the derivatives $[\ln(a(r,0))]'$, $[\ln(a(r,\pi))]'$ are always greater or smaller than $-r^{-1}$. Since the matrix $M(r\cos(k\pi)a(r,k\pi),0)$ is isomorphic to the group $(\mathbb{R},+)$ the function $a(r,0)$ is a homomorphism and for the function $b(r,0)$ the identity $b(r_1r_2,0) = a(r_1,0)b(r_2,0) + b(r_1,0)a^{-1}(r_2,0)$ is satisfied for all $r_1, r_2 > 0$.

Remark 1. *The set $\mathcal{K} = \{M(r\cos(k\pi)a(r,k\pi),0); r > 0, k \in \{0,1\}\}$ of the left translations of Q^* corresponding to the kernel K_r of Q is*

$$\left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, r \in \mathbb{R} \setminus \{0\} \right\}$$

if and only if one has $a(r,k\pi) = 1$, $b(r,k\pi) = 0$ for all $r > 0$, $k \in \{0,1\}$.

The section σ given by (2) is sharply transitive precisely if for all pairs (u_1,t_1), (u_2,t_2) in $\mathbb{R}_{>0}\times[0,2\pi)$ there exists precisely one $(u,t)\in\mathbb{R}_{>0}\times[0,2\pi)$ and $k>0$, $l\in\mathbb{R}$ such that

$$\begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix}\begin{pmatrix} a(u,t) & b(u,t) \\ 0 & a^{-1}(u,t) \end{pmatrix}\begin{pmatrix} u_1\cos t_1 & u_1\sin t_1 \\ -u_1\sin t_1 & u_1\cos t_1 \end{pmatrix} =$$
$$\begin{pmatrix} u_2\cos t_2 & u_2\sin t_2 \\ -u_2\sin t_2 & u_2\cos t_2 \end{pmatrix}\begin{pmatrix} k & l \\ 0 & k^{-1} \end{pmatrix}. \tag{4}$$

As the determinant of the matrices on both sides of (4) are equal we get that $u=u_1^{-1}u_2$. Therefore the system (4) of equations is uniquely solvable if and only if for any fixed $u>0$ the mapping

$$\sigma_u : \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} H \mapsto \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} a(u,t) & b(u,t) \\ 0 & a^{-1}(u,t) \end{pmatrix}$$

determines a quasigroup F_u homeomorphic to S^1. One may take as the points of F_u the vectors $(ua(u,t)a^{-1}(u,0)\cos t, -ua(u,t)a^{-1}(u,0)\sin t)^t$ and as the section the mapping

$$\sigma_u : \begin{pmatrix} ua(u,t)a^{-1}(u,0)\cos t \\ -ua(u,t)a^{-1}(u,0)\sin t \end{pmatrix} \mapsto \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} a(u,t)a^{-1}(u,0) & b(u,t) \\ 0 & a^{-1}(u,t)a(u,0) \end{pmatrix} =$$
$$\begin{pmatrix} a(u,t)a^{-1}(u,0)\cos t & b(u,t)\cos t + a^{-1}(u,t)a(u,0)\sin t \\ -a(u,t)a^{-1}(u,0)\sin t & -b(u,t)\sin t + a^{-1}(u,t)a(u,0)\cos t \end{pmatrix}. \tag{5}$$

In this way we see that the quasigroup F_u has the right identity $(u,0)^t$ since

$$\sigma_u\begin{pmatrix} ua(u,t)a^{-1}(u,0)\cos t \\ -ua(u,t)a^{-1}(u,0)\sin t \end{pmatrix}\cdot\begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} ua(u,t)a^{-1}(u,0)\cos t \\ -ua(u,t)a^{-1}(u,0)\sin t \end{pmatrix}.$$

The quasigroup F_u is a loop, i.e. $(u,0)^t$ is the left identity of F_u, if and only if

$$\sigma_u\begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} a(u,0)a^{-1}(u,0)\cos 0 & b(u,0)\cos 0 \\ 0 & a^{-1}(u,0)a(u,0)\cos 0 \end{pmatrix} = \begin{pmatrix} 1 & b(u,0) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which means $b(u,0)=0$ for all $u>0$. The almost $\mathcal{C}^1$-differentiable loop Q^* belonging to the sharply transitive $\mathcal{C}^1$-differentiable section σ given by (2) is $\mathcal{C}^1$-differentiable precisely if the mapping $(xH,yH)\mapsto z : G/H\times G/H\to\sigma(G/H)$ determined by $zxH=yH$ is $\mathcal{C}^1$-differentiable (cf. [14], p. 32), i.e. the solutions $u>0$, $t\in[0,2\pi)$ of the matrix equation (4) are continuously differentiable functions of $u_1,u_2\in\mathbb{R}_{>0}$, $t_1,t_2\in[0,2\pi)$. The function $u=u_1^{-1}u_2$ is continuously differentiable. If for each fixed $u>0$ the section σ_u given by (5) yields a 1-dimensional $\mathcal{C}^1$-differentiable compact loop, then the function $t(u_1,u_2,t_1,t_2)=t_{(u_1,u_2)}(t_1,t_2)$ is continuously differentiable (cf. [14], Examples 20.3, p. 258). Indeed, the function $t_{(u_1,u_2)}(t_1,t_2)$ is determined implicitly by equations which depend continuously differentiably also on the parameters u_1 and u_2. Applying the above discussion we can prove the following:

Theorem 2. *Let Q^* be the $\mathcal{C}^1$-differentiable multiplicative loop of a locally compact 2-dimensional connected topological quasifield Q. Then Q^* is diffeomorphic to $S^1 \times \mathbb{R}$ and belongs to a $\mathcal{C}^1$-differentiable sharply transitive section σ of the form*

$$\begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} H \mapsto \begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} \cdot \begin{pmatrix} a(u,t) & b(u,t) \\ 0 & a^{-1}(u,t) \end{pmatrix},$$

with $b(u,0) = 0$ for all $u > 0$ if and only if for each fixed $u > 0$ the function $a_u^{-1}(t) := a(u,0)a^{-1}(u,t)$ has the shape

$$a_u^{-1}(t) = e^t\Big(1 - \int_0^t R(s)e^{-s}\, ds\Big)$$

where $R(s)$ is a continuous function, the Fourier series of which is contained in the set $\mathcal{F}$ of Definition 1 in [10] and converges uniformly to R. Moreover, $b_u(t) := b(u,t)$ is a periodic $\mathcal{C}^1$-differentiable function with $b_u(0) = b_u(2\pi) = 0$ such that

$$b_u(t) > -a_u(t) \int_0^t \frac{(a_u^2(s) - a_u'^2(s))}{a_u^4(s)}\, ds \quad for \quad all \quad t \in (0, 2\pi).$$

Proof. The section σ_u given by (5) yields a 1-dimensional $\mathcal{C}^1$-differentiable compact loop having the group $\mathrm{SL}_2(\mathbb{R})$ as the group topologically generated by its left translations if and only if for each fixed $u > 0$ the continuously differentiable functions $a(u,0)a^{-1}(u,t) := \bar{a}_u(t)$, $-b(u,t) := \bar{b}_u(t)$ satisfy the conditions

$$\bar{a}_u'^2(t) + \bar{b}_u(t)\bar{a}_u'(t) + \bar{b}_u'(t)\bar{a}_u(t) - \bar{a}_u^2(t) < 0,\ \bar{b}_u'(0) < 1 - \bar{a}_u'^2(0) \qquad (6)$$

(cf. [14], Section 18, (C), p. 238, [10], pp. 132-139). The solution of the differential inequalities (6) is given by Theorem 6 in [10], pp. 138-139. This proves the assertion. □

Proposition 3. *Let Q^* be the $\mathcal{C}^1$-differentiable multiplicative loop of a locally compact 2-dimensional connected topological quasifield Q. Assume that for each fixed $u > 0$ the function $a_u(t) := a^{-1}(u,0)a(u,t)$ is the constant function 1 and that $b(u,0) = 0$ is satisfied for all $u > 0$. Then Q^* belongs to a $\mathcal{C}^1$-differentiable sharply transitive section σ of the form (2) if and only if for each fixed $u > 0$ one has $b_u(t) := b(u,t) > -t$ for all $0 < t < 2\pi$.*

Proof. If for each fixed $u > 0$ the function $a(u,0)a^{-1}(u,t) = a_u^{-1}(t) = \bar{a}_u(t)$ is constant with value 1, then the section σ_u given by (5) yields a $\mathcal{C}^1$-differentiable compact loop L if and only if for each fixed $u > 0$ the continuously differentiable function $\bar{b}_u(t) := -b_u(t)$ satisfies the differential inequality $\bar{b}'_u(t) < 1$ with the initial condition $\bar{b}'_u(0) < 1$ (cf. (6)). This is the case precisely if one has $b_u(t) > -t$ for all $0 < t < 2\pi$. □

Proposition 4. *Let Q^* be the $\mathcal{C}^1$-differentiable multiplicative loop of a locally compact 2-dimensional connected topological quasifield Q. Assume that for each fixed $u > 0$ the function $b(u,t)$ is the constant function 0. Then Q^* belongs to a $\mathcal{C}^1$-differentiable sharply transitive section σ of the form (2) precisely if for each fixed $u > 0$ one has $e^{-t} < a(u,t)a^{-1}(u,0) < e^t$ for all $0 < t < 2\pi$.*

Proof. If for each fixed $u > 0$ the function $b(u,t) = -\bar{b}_u(t)$ is constant with value 0, then the section σ_u given by (5) determines a $\mathcal{C}^1$-differentiable compact loop L if and only if for each fixed $u > 0$ the following inequalities are satisfied:

$$(\bar{a}'_u(t) - \bar{a}_u(t))(\bar{a}'_u(t) + \bar{a}_u(t)) < 0, \quad 0 < 1 - \bar{a}'^2_u(0),$$

where $\bar{a}_u(t) = a(u,0)a^{-1}(u,t)$. This is the case precisely if either one has $\bar{a}'_u(t) - \bar{a}_u(t) < 0$ and $\bar{a}'_u(t) + \bar{a}_u(t) > 0$ or one has $\bar{a}'_u(t) - \bar{a}_u(t) > 0$ and $\bar{a}'_u(t) + \bar{a}_u(t) < 0$. Now we consider the first case. Then the function $\bar{a}_u(t)$ determines a loop if and only if for each fixed $u > 0$ it is a subfunction of a differentiable function $h_u(t) := h(u,t)$ with $h_u(0) = 1$, $h'^2_u(0) = 1$, $h'_u(t) = h_u(t)$ and an upper function of a differentiable function $l_u(t) := l(u,t)$ with $l_u(0) = 1$, $l'^2_u(0) = 1$, $l'_u(t) = -l_u(t)$ (cf. [19], p. 66). Hence for each fixed $u > 0$ the function $\bar{a}_u(t)$ is a subfunction of the function e^t and an upper function of the function e^{-t} for all $t \in (0, 2\pi)$. Therefore, any continuously differentiable function $\bar{a}_u(t)$ such that for each fixed $u > 0$ and for all $t \in (0, 2\pi)$ one has $e^{-t} < \bar{a}_u(t)^{-1} < e^t$ determines a $\mathcal{C}^1$-differentiable compact loop L.

In the second case an analogous consideration as in the first case gives that for all fixed $u > 0$ the function $a(u,t)a^{-1}(u,0)$ must be a subfunction of the function e^{-t} and an upper function of the function e^t for all $t \in (0, 2\pi)$. Hence in this case the function $a(u,t)a^{-1}(u,0)$ does not exist. □

Proposition 5. *Let*

$$\begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} H \mapsto \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(1,t) & b(u,t) \\ 0 & a^{-1}(1,t) \end{pmatrix}, u > 0, t \in [0, 2\pi) \qquad (7)$$

with $b(u,0) = 0$ for all $u > 0$ be a section belonging to a multiplicative loop Q^* of a locally compact 2-dimensional connected topological quasifield Q. Then Q^* contains for any $u > 0$ a 1-dimensional compact subloop.

Proof. The image of the section (7) acts sharply transitively on the point set $\mathbb{R}^2 \setminus \{(0,0)^t\}$. Since the subgroup $\left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, u > 0 \right\}$ leaves any line through $(0,0)^t$ fixed, the subset

$$\mathcal{T} = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(1,t) & b(u,t) \\ 0 & a^{-1}(1,t) \end{pmatrix}, t \in [0, 2\pi) \right\} \tag{8}$$

acts sharply transitively on the oriented lines through $(0,0)^t$ for any $u > 0$. Therefore $\mathcal{T}$ corresponds to a 1-dimensional compact loop since $b(u,0) = 0$ for all $u > 0$. $\square$

As $\mathcal{T}$ given by (8) is the image of a section corresponding to a 1-dimensional compact subloop of Q^*, every element of $\mathcal{T}$ is elliptic.

Proposition 6. *Every element of the set $\mathcal{T}$ given by (8) is elliptic if and only if the following holds: $a(1, k\pi) = 1$ for $k \in \{0, 1\}$:*
1) if for all $t \in [0, 2\pi)$ and $u > 0$ one has $b(u,t) = 0$, then the function $a(1,t)$ satisfies the inequalities:

$$\frac{1 - |\sin(t)|}{|\cos(t)|} \leq a(1,t) \leq \frac{1 + |\sin(t)|}{|\cos(t)|}, \tag{9}$$

2) if the function $b(u,t)$ is different from the constant function 0*, then for* $\sin(t) > 0$ *one has*

$$\frac{(a(1,t) + a(1,t)^{-1})\cos(t) - 2}{\sin(t)} < b(u,t) < \frac{(a(1,t) + a(1,t)^{-1})\cos(t) + 2}{\sin(t)}, \tag{10}$$

for $\sin(t) < 0$ *we have*

$$\frac{(a(1,t) + a(1,t)^{-1})\cos(t) + 2}{\sin(t)} < b(u,t) < \frac{(a(1,t) + a(1,t)^{-1})\cos(t) - 2}{\sin(t)}. \tag{11}$$

Proof. Any element of (8) is elliptic if and only if the inequality

$$|\cos(t)(a(1,t) + a(1,t)^{-1}) - \sin(t)b(u,t)| \leq 2 \tag{12}$$

holds, where the equality sign occurs only for $t = k\pi$, $k \in \{0, 1\}$. Hence $a(1, k\pi) = 1$. If $b(u,t) = 0$, then inequality (12) reduces to $a^2(1,t)|\cos(t)| - 2a(1,t) + |\cos(t)| \leq 0$ which is equivalent to inequalities (9). If $b(u,t) \neq 0$, then inequality (12) is equivalent for all $t \neq k\pi$, $k \in \{0, 1\}$, to

$$(a(1,t) + a(1,t)^{-1})^2 \cos^2(t) - 2(a(1,t) + a(1,t)^{-1}) \sin(t) \cos(t) b(u,t) + \sin^2(t) b^2(u,t) < 4. \tag{13}$$

Solving the quadratic equation

$$(a(1,t)+a(1,t)^{-1})^2\cos^2(t)-2(a(1,t)+a(1,t)^{-1})\sin(t)\cos(t)x+\sin^2(t)x^2=4 \tag{14}$$

we get

$$x=\frac{2(a(1,t)+a(1,t)^{-1})\cos(t)\sin(t)\pm 4\sin(t)}{2\sin^2(t)}=\frac{(a(1,t)+a(1,t)^{-1})\cos(t)\pm 2}{\sin(t)}.$$

Comparing (13) and (14) one obtains

$$\left(b(u,t)-\frac{(a(1,t)+a(1,t)^{-1})\cos(t)-2}{\sin(t)}\right)\left(b(u,t)-\frac{(a(1,t)+a(1,t)^{-1})\cos(t)+2}{\sin(t)}\right)<0$$

which yields inequalities (10) and (11). □

Proposition 7. *Let Q^* be the multiplicative loop of a locally compact quasifield Q of dimension 2 containing a 1-dimensional compact normal subloop. The quasifield Q is the field $\mathbb{C}$ of complex numbers if and only if $\mathcal{T}$ is a normal subset in the set of all left translations of Q^*.*

Proof. If Q is the field of complex numbers, then Q^* is the group $\mathrm{SO}_2(\mathbb{R})\times\mathbb{R}$ and the assertion is true. If the set $\mathcal{T}$ is a normal subset in the set of the left translations of a proper loop Q^*, then it is normal in the connected component $\mathrm{GL}_2^+(\mathbb{R})$ of the group $\mathrm{GL}_2(\mathbb{R})$ because $\mathrm{GL}_2^+(\mathbb{R})$ is the group topologically generated by the left translations of Q^* (cf. [14], Section 29, p. 345). If $\mathcal{T}$ is normal in $\mathrm{GL}_2^+(\mathbb{R})$, then for $D=\begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}$ one has that

$$D^{-1}\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}DD^{-1}\begin{pmatrix}a(1,t) & b(1,t)\\ 0 & a^{-1}(1,t)\end{pmatrix}D=$$

$$\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}D^{-1}\begin{pmatrix}a(1,t) & b(1,t)\\ 0 & a^{-1}(1,t)\end{pmatrix}D=\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}\begin{pmatrix}a(1,t) & b(1,t)\\ 0 & a^{-1}(1,t)\end{pmatrix}$$

is satisfied for all $\varphi\in[0,2\pi)$ if and only if $a(1,t)=1$ and $b(1,t)=0$ or equivalently $\mathcal{T}=\mathrm{SO}_2(\mathbb{R})$. But the compact group $\mathrm{SO}_2(\mathbb{R})$ is not normal in $\mathrm{GL}_2^+(\mathbb{R})$. Hence Q^* is not proper and the assertion follows. □

Lemma 8. *If the multiplicative loop Q^* of a 2-dimensional locally compact connected topological quasifield Q is not quasi-simple, then the set $\mathcal{K}=\{M(r\cos(k\pi)a(r,k\pi),0); r>0, k\in\{0,1\}\}$ of the left translations of Q^* belonging to the kernel K_r of Q has the form $\mathcal{K}=\left\{\begin{pmatrix}r & 0\\ 0 & r\end{pmatrix}, 0\neq r\in\mathbb{R}\right\}$, which is a normal subgroup of the set Λ_{Q^*} of all left translations of Q^*.*

Proof. If Q is the field of complex numbers, then the assertion is true. If the loop Q^* is proper and not quasi-simple, then by Lemma 1.7, p. 19, in [14], the left translations of a normal subloop of Q^* generate a normal subgroup N of $\mathrm{GL}_2^+(\mathbb{R})$ which is the group topologically generated by all left translations of Q^*. Hence the set Λ_{Q^*} of the left translations of Q^* must contain the group $C = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix},\ 0 < u \in \mathbb{R} \right\}$ as a normal subgroup. It follows that the set $\mathcal{K}$ of the left translations corresponding to the elements of the kernel K_r of Q contained in Λ_{Q^*} has the form given in the assertion and $\mathcal{K} \cap \Lambda_{Q^*}$ is a normal subgroup of Λ_{Q^*}. $\square$

Assume that the set $\mathcal{K}$ of the left translations of the loop Q^* having $(1,0)^t$ as identity corresponding to the elements of the kernel K_r of Q has the form given in Lemma 8. According to (3) the element

$$\begin{pmatrix} ra(r,\varphi)\cos\varphi & rb(r,\varphi)\cos\varphi + ra^{-1}(r,\varphi)\sin\varphi \\ -ra(r,\varphi)\sin\varphi & -rb(r,\varphi)\sin\varphi + ra^{-1}(r,\varphi)\cos\varphi \end{pmatrix}$$

corresponds to the left translation of $(ra(r,\varphi)\cos\varphi, -ra(r,\varphi)\sin\varphi)^t$, $r > 0$, $\varphi \in [0, 2\pi)$. Let N^* be the subgroup of Q^* corresponding to the normal subgroup $\mathcal{K}$ of Λ_{Q^*}. We show that $N^* := \{(\hat{s}, 0)^t, \hat{s} \in \mathbb{R} \setminus \{0\}\}$ is normal in Q^*. For all elements $x := (\cos\varphi, -\sin\varphi)^t$, $y := (u, v)^t$ of Q^* the condition $(N^* * x) * y = N^* * (x * y)$ of (1) is satisfied if and only if we have

$$\left[\begin{pmatrix} \hat{s} \\ 0 \end{pmatrix} * \begin{pmatrix} cos\varphi \\ -sin\varphi \end{pmatrix}\right] * \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \hat{s}' \\ 0 \end{pmatrix} * \left[\begin{pmatrix} cos\varphi \\ -sin\varphi \end{pmatrix} * \begin{pmatrix} u \\ v \end{pmatrix}\right]$$

for all $\varphi \in [0, 2\pi)$, $(u, v) \in \mathbb{R}^2 \setminus \{(0,0)\}$ with suitable $\hat{s}, \hat{s}' \in \mathbb{R} \setminus \{0\}$ such that $\hat{s} = s\cos(k\pi)$, $\hat{s}' = s'\cos(k\pi)$, where $s, s' > 0$, $k \in \{0, 1\}$. This is the case precisely if one has

$$\begin{pmatrix} usa(s,\varphi+k\pi)\cos(\varphi+k\pi) + vsb(s,\varphi+k\pi)\cos(\varphi+k\pi) + vsa^{-1}(s,\varphi+k\pi)\sin(\varphi+k\pi) \\ -usa(s,\varphi+k\pi)\sin(\varphi+k\pi) - vsb(s,\varphi+k\pi)\sin(\varphi+k\pi) + vsa^{-1}(s,\varphi+k\pi)\cos(\varphi+k\pi) \end{pmatrix} =$$

$$\begin{pmatrix} s'\cos(k\pi)(ua(1,\varphi)\cos\varphi + vb(1,\varphi)\cos\varphi + va^{-1}(1,\varphi)\sin\varphi) \\ s'\cos(k\pi)(-ua(1,\varphi)\sin\varphi - vb(1,\varphi)\sin\varphi + va^{-1}(1,\varphi)\cos\varphi) \end{pmatrix}$$

or equivalently for all $\varphi \in [0, 2\pi)$, $(u, v) \in \mathbb{R}^2 \setminus \{(0,0)\}$ we have

$$[(ua(s,\varphi+k\pi)+vb(s,\varphi+k\pi))\cos(\varphi+k\pi)+va^{-1}(s,\varphi+k\pi)\sin(\varphi+k\pi)]\cdot[(-ua(1,\varphi)-vb(1,\varphi))\sin\varphi+va^{-1}(1,\varphi)\cos\varphi] =$$

$$[(-ua(s,\varphi+k\pi)-vb(s,\varphi+k\pi))\sin(\varphi+k\pi)+va^{-1}(s,\varphi+k\pi)\cos(\varphi+k\pi)]\cdot[(ua(1,\varphi)+vb(1,\varphi))\cos\varphi+va^{-1}(1,\varphi)\sin\varphi].$$

The last equation holds if and only if one has

$$(a(s,\varphi+k\pi)a^{-1}(1,\varphi) - a^{-1}(s,\varphi+k\pi)a(1,\varphi))uv + (b(s,\varphi+k\pi)a^{-1}(1,\varphi) - a^{-1}(s,\varphi+k\pi)b(1,\varphi))v^2 = 0.$$

Therefore we obtain $a(s,\varphi+k\pi)a^{-1}(1,\varphi) - a^{-1}(s,\varphi+k\pi)a(1,\varphi) = 0$ and $b(s,\varphi+k\pi)a^{-1}(1,\varphi) - a^{-1}(s,\varphi+k\pi)b(1,\varphi) = 0$. As $a(s,\varphi)$ is positive we have $a(s,\varphi+k\pi) = a(1,\varphi)$, $b(s,\varphi+k\pi) = b(1,\varphi)$ for all $s > 0$, $\varphi \in [0, 2\pi)$, $k \in \{0, 1\}$. By (1) the group N^* is a normal subgroup of Q^* if and only if for all $\varphi \in [0, 2\pi)$, $(u, v) \in \mathbb{R}^2 \setminus \{(0,0)\}$ one has

$$\left[\begin{pmatrix} \cos\varphi \\ -\sin\varphi \end{pmatrix} * \begin{pmatrix} \hat{s} \\ 0 \end{pmatrix}\right] * \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ -\sin\varphi \end{pmatrix} * \left[\begin{pmatrix} \hat{s}' \\ 0 \end{pmatrix} * \begin{pmatrix} u \\ v \end{pmatrix}\right] \quad \text{or}$$

$$\begin{pmatrix} sa(1,\varphi)a(1,\varphi)\cos(\varphi+k\pi) & sa(1,\varphi)b(1,\varphi)\cos(\varphi+k\pi)+ssin(\varphi+k\pi) \\ -sa(1,\varphi)a(1,\varphi)\sin(\varphi+k\pi) & -sa(1,\varphi)b(1,\varphi)\sin(\varphi+k\pi)+scos(\varphi+k\pi) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} =$$

$$\begin{pmatrix} a(1,\varphi)\cos\varphi & b(1,\varphi)\cos\varphi+a^{-1}(1,\varphi)sin\varphi \\ -a(1,\varphi)\sin\varphi & -b(1,\varphi)\sin\varphi+a^{-1}(1,\varphi)cos\varphi \end{pmatrix}\begin{pmatrix} s'\cos(k\pi)u \\ s'\cos(k\pi)v \end{pmatrix}$$

for suitable $\hat{s}, \hat{s}' \in \mathbb{R} \setminus \{0\}$ such that $\hat{s} = s\cos(k\pi)$, $\hat{s}' = s'\cos(k\pi)$, where $s, s' > 0$, $k \in \{0,1\}$. This is equivalent to

$$\begin{pmatrix} sua(1,\varphi)^2\cos(\varphi+k\pi)+sv[a(1,\varphi)b(1,\varphi)\cos(\varphi+k\pi)+\sin(\varphi+k\pi)] \\ -sua(1,\varphi)^2\sin(\varphi+k\pi)+sv[-a(1,\varphi)b(1,\varphi)\sin(\varphi+k\pi)+\cos(\varphi+k\pi)] \end{pmatrix} =$$

$$\begin{pmatrix} us'a(1,\varphi)\cos(\varphi+k\pi)+s'v[b(1,\varphi)\cos(\varphi+k\pi)+a^{-1}(1,\varphi)\sin(\varphi+k\pi)] \\ -us'a(1,\varphi)\sin(\varphi+k\pi)+s'v[-b(1,\varphi)\sin(\varphi+k\pi)+a^{-1}(1,\varphi)\cos(\varphi+k\pi)] \end{pmatrix}.$$

A direct computation yields that the above equality is true for all $(u,v) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $\varphi \in [0,2\pi)$, $k \in \{0,1\}$.

Using Proposition 7, Lemma 8 and the discussion above we have the following

Theorem 9. *The multiplicative loop Q^* of a locally compact 2-dimensional quasifield Q with $(1,0)^t$ as identity of Q^* is not quasi-simple if and only if for all $r > 0$, $\varphi \in [0,2\pi)$, $k \in \{0,1\}$ one has $a(r,k\pi) = 1$, $b(r,k\pi) = 0$, $a(r,\varphi+k\pi) = a(1,\varphi)$ and $b(r,\varphi+k\pi) = b(1,\varphi)$. Then Q^* is a split extension of a 1-dimensional normal subgroup N^* by a subloop homeomorphic to the 1-sphere. Moreover, one has*
a) N^* is isomorphic to $\mathbb{R} \times Z_2$, where Z_2 is the group of order 2.
b) This extension is the direct product precisely if Q is the field $\mathbb{C}$.

Proof. We have only to prove a) and b). According to Lemma 8 and the above discussion the only possibility for a normal subloop of positive dimension is the group N^*. The intersection of a compact subloop of Q^* with N^* has cardinality 2 (cf. Proposition 5 and Lemma 8). Hence the claim a) is proved. The claim of b) follows from Proposition 7. $\square$

From Proposition 7 and Theorem 9 it follows:

Corollary 10. *The multiplicative loop Q^* of a locally compact 2-dimensional quasifield Q with $(1,0)^t$ as identity of Q^* is the direct product of the group $\mathbb{R}$ and a subloop homeomorphic to the 1-sphere if and only if Q is the field of complex numbers.*

The set Λ_{Q^*} of the left translations of Q^* with a normal subloop of positive dimension and with $(1,0)^t$ as identity can be written into the form

$$\left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ua(1,t) & ub(1,t) \\ 0 & ua^{-1}(1,t) \end{pmatrix}, u > 0, t \in [0,2\pi) \right\} \qquad (15)$$

with $a(1,k\pi) = 1$, $b(1,k\pi) = 0$, $k \in \{0,1\}$.

5. Decomposable multiplicative loops of 2-dimensional quasifields

Definition 1. *We call the multiplicative loop Q^* of a locally compact connected topological 2-dimensional quasifield Q* decomposable, *if the set of all left translations of Q^* is a product $\mathcal{T}\mathcal{K}$ with $|\mathcal{T} \cap \mathcal{K}| \le 2$, where $\mathcal{T}$ is the set of all left translations of a 1-dimensional compact loop of the form (8) and $\mathcal{K}$ is the set of all left translations of Q^* belonging to the kernel K_r of Q.*

If the loop Q^* is decomposable, then it contains compact subloops for any $u > 0$ corresponding to the section (7). From now on we choose $u = 1$. Then the compact loop F_u of Q^* has identity $(1,0)^t$ and one has

$$\begin{pmatrix} \cos t a(1,t) & \cos t b(1,t) + \sin t a^{-1}(1,t) \\ -\sin t a(1,t) & -\sin t b(1,t) + \cos t a^{-1}(1,t) \end{pmatrix} \left[\begin{pmatrix} r\cos(k\pi) a(r,k\pi) & r\cos(k\pi) b(r,k\pi) \\ 0 & r\cos(k\pi) a^{-1}(r,k\pi) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] =$$

$$= \begin{pmatrix} r\cos(t+k\pi) a(r,t+k\pi) & r\cos(t+k\pi) b(r,t+k\pi) + r\sin(t+k\pi) a^{-1}(r,t+k\pi) \\ -r\sin(t+k\pi) a(r,t+k\pi) & -r\sin(t+k\pi) b(r,t+k\pi) + r\cos(t+k\pi) a^{-1}(r,t+k\pi) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (16)$$

Equation (16) yields that $a(r, t + k\pi) = a(1, t)a(r, k\pi)$.
Now we give sufficient and necessary conditions for the loop Q^* to be decomposable.

Proposition 11. *The multiplicative loop Q^* of a locally compact connected topological 2-dimensional quasifield Q with $(1,0)^t$ as identity of Q^* is decomposable if and only if for all $r > 0$, $t \in [0, 2\pi)$, $k \in \{0,1\}$ one has*

$$a(r, t+k\pi) = a(1,t)a(r,k\pi), \ \ b(r,t+k\pi) = a(1,t)b(r,k\pi) + a^{-1}(r,k\pi)b(1,t).$$

Proof. The point $(x,y)^t$ is the image of the point $(1,0)^t$ under the linear mapping $M_{(x,y)}$ and the set $\{M_{(x,y)}; (x,y)^t \in Q^*\}$ acts sharply transitively on Q^*. The matrix equation

$$\begin{pmatrix} \cos t a(1,t) & \cos t b(1,t) + \sin t a^{-1}(1,t) \\ -\sin t a(1,t) & -\sin t b(1,t) + \cos t a^{-1}(1,t) \end{pmatrix} \left[\begin{pmatrix} r\cos(k\pi) a(r,k\pi) & r\cos(k\pi) b(r,k\pi) \\ 0 & r\cos(k\pi) a^{-1}(r,k\pi) \end{pmatrix} \begin{pmatrix} u\cos\varphi a(u,\varphi) \\ -u\sin\varphi a(u,\varphi) \end{pmatrix} \right] =$$

$$= \begin{pmatrix} r\cos(t+k\pi) a(r,t+k\pi) & r\cos(t+k\pi) b(r,t+k\pi) + r\sin(t+k\pi) a^{-1}(r,t+k\pi) \\ -r\sin(t+k\pi) a(r,t+k\pi) & -r\sin(t+k\pi) b(r,t+k\pi) + r\cos(t+k\pi) a^{-1}(r,t+k\pi) \end{pmatrix} \begin{pmatrix} u\cos\varphi a(u,\varphi) \\ -u\sin\varphi a(u,\varphi) \end{pmatrix} \quad (17)$$

holds precisely if the identities of the assertion are satisfied. □

If Q^* is decomposable such that its compact subloop has identity $(1,0)^t$, then $|\mathcal{T} \cap \mathcal{K}| = 2$ because one has $\mathcal{T} \cap \mathcal{K} = \left\{ I, \begin{pmatrix} -1 & -b(1,\pi) \\ 0 & -1 \end{pmatrix} \right\}$. In this case the set of all left translations of Q^* is a product $\mathcal{T}\mathcal{W}$ with $\mathcal{T} \cap \mathcal{W} = I$, where $\mathcal{W}$ is the set of all left translations corresponding to the connected component of the kernel K_r of Q.

Theorem 12. *If the multiplicative loop Q^* of a locally compact connected topological 2-dimensional quasifield Q with $(1,0)^t$ as identity of Q^* is not quasi-simple, then Q^* is decomposable.*

Proof. By Theorem 9 the loop Q^* is not quasi-simple if and only if for all $r > 0$, $t \in [0, 2\pi)$, $k \in \{0, 1\}$ one has $a(r, k\pi) = 1$, $b(r, k\pi) = 0$, $a(r, t + k\pi) = a(1, t)$ and $b(r, t+k\pi) = b(1, t)$. Therefore the identities given in the assertion of Proposition 11 are satisfied. □

Proposition 13. *The $\mathcal{C}^1$-differentiable multiplicative loop Q^* of a locally compact connected topological 2-dimensional quasifield Q with $(1, 0)^t$ as identity of Q^* is decomposable precisely if for the inverse function $\bar{a}(1, t) = a^{-1}(1, t)$ and for $\bar{b}(1, t) = -b(1, t)$ the differential inequalities*

$$\bar{a}'^2(1, t) + \bar{b}(1, t)\bar{a}'(1, t) + \bar{b}'(1, t)\bar{a}(1, t) - \bar{a}^2(1, t) < 0, \ \bar{b}'(1, 0) < 1 - \bar{a}'^2(1, 0) \tag{18}$$

are satisfied.

Proof. If Q^* is a $\mathcal{C}^1$-differentiable multiplicative loop of a quasifield Q with $(1, 0)^t$ as identity, then for $u = 1$ the continuously differentiable functions $\bar{a}_u(t) := a(u, 0)a^{-1}(u, t)$, $\bar{b}_u(t) := -b(u, t)$ satisfy the conditions in (6). The set of all left translations of Q^* is a product $\mathcal{TK}$ if and only if one has $a(u, t+k\pi) = a(u, k\pi)a(1, t)$ and $b(u, t + k\pi) = a(1, t)b(u, k\pi) + a^{-1}(u, k\pi)b(1, t)$ for all $u > 0$, $t \in [0, 2\pi)$, $k \in \{0, 1\}$ (cf. Proposition 11). Putting this into (6) we get for $u = 1$

$$a'^2(1, t) + b(1, t)a'(1, t)a^2(1, t) - b'(1, t)a^3(1, t) - a^2(1, t) < 0 \text{ and}$$
$$b'(1, 0) > a'^2(1, 0) - 1 \tag{19}$$

since $a(1, k\pi) = 1$, $k \in \{0, 1\}$. Inequalities (19) are equivalent to the inequalities (18) with $\bar{a}(1, t) = a^{-1}(1, t)$ and $\bar{b}(1, t) = -b(1, t)$. □

Corollary 14. *Let T be any 1-dimensional $\mathcal{C}^1$-differentiable connected compact loop such that the set $\mathcal{T}$ of its left translations has the form (8) and let $\mathcal{K}$ be any set of matrices of the form*

$$\mathcal{K} = \left\{ \begin{pmatrix} u\cos(k\pi)a(u, k\pi) & u\cos(k\pi)b(u, k\pi) \\ 0 & u\cos(k\pi)a^{-1}(u, k\pi) \end{pmatrix}, u > 0, k \in \{0, 1\} \right\},$$

where $a(u, k\pi) > 0$ and $b(u, k\pi)$ are continuously differentiable functions such that $ua(u, 0)$, $-ua(u, \pi)$ are strictly monotone. Then the product $\mathcal{TK}$ is the set of all left translations of a $\mathcal{C}^1$-differentiable decomposable multiplicative loop Q^* of a 2-dimensional locally compact connected quasifield Q with $(1, 0)^t$ as identity of Q^*.

Proof. Since one has

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(1, t) & b(1, t) \\ 0 & a(1, t)^{-1} \end{pmatrix} \begin{pmatrix} u\cos(k\pi)a(u, k\pi) & u\cos(k\pi)b(u, k\pi) \\ 0 & u\cos(k\pi)a^{-1}(u, k\pi) \end{pmatrix} =$$

$$\begin{pmatrix} \cos(t+k\pi) & \sin(t+k\pi) \\ -\sin(t+k\pi) & \cos(t+k\pi) \end{pmatrix} \begin{pmatrix} ua(u,k\pi)a(1,t) & ub(u,k\pi)a(1,t)+ub(1,t)a^{-1}(u,k\pi) \\ 0 & ua^{-1}(u,k\pi)a(1,t)^{-1} \end{pmatrix} =$$

$$\begin{pmatrix} \cos(t+k\pi) & \sin(t+k\pi) \\ -\sin(t+k\pi) & \cos(t+k\pi) \end{pmatrix} \begin{pmatrix} ua(u,t+k\pi) & ub(u,t+k\pi) \\ 0 & ua^{-1}(u,t+k\pi) \end{pmatrix},$$

for the continuously differentiable functions $a(1,t)$, $b(1,t)$ the inequalities (19) hold and $a(1,k\pi)=1$, $k \in \{0,1\}$, for $u=1$ the continuously differentiable functions $\bar{a}_u(t+k\pi) = a(u,0)a^{-1}(u,t+k\pi) = a(u,0)a^{-1}(u,k\pi)a^{-1}(1,t)$, $\bar{b}_u(t+k\pi) = -b(u,t+k\pi) = -b(u,k\pi)a(1,t) - b(1,t)a^{-1}(u,k\pi)$ satisfy inequalities (6). Hence the product $\mathcal{TK}$ given in the assertion is the image of a $\mathcal{C}^1$-differentiable section of a multiplicative loop Q^* of a quasifield Q. □

Proposition 15. *The set Λ_{Q^*} of all left translations of the multiplicative loop Q^* for a locally compact connected topological 2-dimensional quasifield Q with $(1,0)^t$ as identity of Q^* contains the group $\mathrm{SO}_2(\mathbb{R})$ if and only if Λ_{Q^*} has the form*

$$\Lambda_{Q^*} = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ua(u,0) & ub(u,0) \\ 0 & ua^{-1}(u,0) \end{pmatrix}, u>0, t \in [0,2\pi) \right\} \tag{20}$$

where $a(u,0)$, $b(u,0)$ are arbitrary continuous functions with $a(u,0)>0$ such that $ua(u,0)$ is strictly monotone. In this case Q^* is decomposable.

Proof. If the set Λ_{Q^*} contains the group $\mathrm{SO}_2(\mathbb{R})$, then for each fixed $u>0$ the function $a(u,t)$ is constant with value 1 and the function $b(u,t)$ is constant with value 0. So the functions $a(u,t)=a(u,0)$, $b(u,t)=b(u,0)$ do not depend on the variable t. Hence the identities in Proposition 11 are satisfied and the set Λ_{Q^*} has the form $\mathcal{TW}$ as in the assertion.

Conversely, if $ua(u,0)$ is a strictly monotone continuous function, then for arbitrary continuous functions $a(u,0), b(u,0)$ with $a(u,0)>0$ the set given by (20) is the set Λ_{Q^*} of all left translations of the multiplicative loop Q^* of a locally compact quasifield such that Λ_{Q^*} contains the group $\mathrm{SO}_2(\mathbb{R})$ and Q^* has identity $(1,0)^t$. Hence Q^* is decomposable and the assertion is proved. □

6. Betten's classification of 4-dimensional translation planes

Using 2-dimensional spreads, Betten in [3], [4], [5], [6], [7], [8], see also [13] and [15], has classified all locally compact 4-dimensional translation planes which admit an at least 7-dimensional collineation group. His normalized 2-dimensional spreads are images of sharply transitive sections $\sigma' : G/H' \to G$, where G is the connected component of the group $\mathrm{GL}_2(\mathbb{R})$, H' is the subgroup $\left\{ \begin{pmatrix} 1 & c \\ 0 & d \end{pmatrix}, d>0, c \in \mathbb{R} \right\}$ (cf. [2], [3]) and $\sigma'(G/H')$ consists of the elements

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ra(r,t) & 0 \\ 0 & r^{-1}a^{-1}(r,t) \end{pmatrix} \begin{pmatrix} 1 & b(r,t)a^{-1}(r,t) \\ 0 & r^2 \end{pmatrix}.$$

With respect to the stabilizer $H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}$ the sharply transitive section σ' transforms into a sharply transitive section $\sigma : G/H \to G$ given by (2), because the elements of $\sigma'(G/H')$ coincide with

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} a(r,t) & b(r,t) \\ 0 & a^{-1}(r,t) \end{pmatrix}.$$

Proposition 16. *Let $\mathcal{A}$ be a 4-dimensional non-desarguesian translation plane admitting an 8-dimensional collineation group such that $\mathcal{A}$ is coordinatized by the locally compact topological quasifield Q. Then the multiplicative loop Q^* can be given by one of the following sets Λ_{Q^*} of the left translations of Q^*:*

a) Λ_{Q^*} has the form (15) with $a(1,t) = 1$ and $b(1,t) = 0$ for $0 \le t \le \pi$, $a(1,t) = 1/\sqrt{\cos^2 t + \frac{\sin^2 t}{w}}$ and $b(1,t) = a(1,t)\frac{1-w}{w} \sin t \cos t$ for $\pi < t < 2\pi$. The quasifields Q_w, $w > 1$, correspond to a one-parameter family of planes $\mathcal{A}_w$.

b) Λ_{Q^*} is the range of the section given by (2) such that for $\alpha \ge \frac{-3\beta^2}{4}$ one has $a(r,t) = \frac{\sqrt{\alpha^2+\beta^2}}{\alpha+\beta^2}$ and $b(r,t) = \frac{\beta(-\alpha+1)}{\sqrt{\alpha^2+\beta^2}}$ with $r\cos(t) = \alpha\frac{\alpha+\beta^2}{\sqrt{\alpha^2+\beta^2}}$, $r\sin(t) = -\beta\frac{\alpha+\beta^2}{\sqrt{\alpha^2+\beta^2}}$.

For $\alpha < \frac{-3\beta^2}{4}$ we have $a(r,t) = 3\sqrt{\frac{\alpha^2+\beta^2}{\alpha^2}}$ and $b(r,t) = \beta\sqrt{\frac{\alpha^2+\beta^2}{\alpha^2}} + \frac{\beta\alpha}{3\sqrt{\alpha^2(\alpha^2+\beta^2)}}$ with $r\cos(t) = \frac{\alpha}{3}\sqrt{\frac{\alpha^2}{\alpha^2+\beta^2}}$, $r\sin(t) = -\frac{\beta}{3}\sqrt{\frac{\alpha^2}{\alpha^2+\beta^2}}$. The quasifield Q coordinatizes a single plane.

c) Λ_{Q^*} is the range of the section given by (2) such that $a(r,t) = \sqrt{\frac{v^2+s^2}{\frac{s^4}{3}+s^2v+v^2}}$, $b(r,t) = \frac{-\frac{s^3v}{3}+s^3+sv}{\sqrt{(\frac{s^4}{3}+s^2v+v^2)(s^2+v^2)}}$ with

$$r\cos(t) = v\sqrt{\frac{\frac{s^4}{3}+s^2v+v^2}{s^2+v^2}}, \quad r\sin(t) = -s\sqrt{\frac{\frac{s^4}{3}+s^2v+v^2}{s^2+v^2}}.$$

The quasifield Q coordinatizes a single plane.

In case a) the multiplicative loop Q_w^* is decomposable and a split extension of the normal subgroup $\widetilde{N^*} \cong \mathbb{R}$ corresponding to the connected component of $\widetilde{\mathcal{K}} = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R} \right\}$ with a subloop homeomorphic to the 1-sphere. In case c) the set $\mathcal{K}$ of the left translations of Q^* corresponding to the kernel K_r of the quasifield Q has the form $\widetilde{\mathcal{K}}$. In case b) the set $\mathcal{K}$ has the form

$$\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \alpha > 0 \right\} \cup \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \frac{\alpha}{9} \end{pmatrix}, \alpha < 0 \right\}.$$

In cases b) and c) the multiplicative loops Q^* are not decomposable and quasi-simple.

Proof. If the translation complement of $\mathcal{A}$ is the group $\mathrm{GL}_2(\mathbb{R})$ and acts reducibly on $\mathbb{R}^4$, then one obtains the one-parameter family $\mathcal{A}_w$, $w > 1$, of the non-desarguesian translation planes corresponding to the following spreads:

$$\{S\} \cup \left\{ \begin{pmatrix} s & -v \\ v & s \end{pmatrix}, s, v \in \mathbb{R}, v \geq 0 \right\} \cup \left\{ \begin{pmatrix} s & \frac{-v}{w} \\ v & s \end{pmatrix}, s, v \in \mathbb{R}, v < 0 \right\},$$

$w > 1$ (cf. [3], Satz 5, p. 144). Any such spread coincides with the set Λ_{Q^*} in (15) with $a(1,t)$ and $b(1,t)$ as in assertion a). By Theorem 9 the multiplicative loop Q_w^* is a split extension of a normal subgroup N^* with a 1-dimensional compact loop. By Theorem 12 the loop Q_w^* is decomposable. Hence $\widetilde{N^*}$ has the form as in the assertion.
If the translation complement $\mathrm{GL}_2(\mathbb{R})$ acts irreducibly on $\mathbb{R}^4$, then one obtains a single plane $\mathcal{A}$ generated by the spread

$$\{S\} \cup \left\{ \begin{pmatrix} \alpha & -\alpha\beta - \beta^3 \\ \beta & \alpha + \beta^2 \end{pmatrix}, \alpha, \beta \in \mathbb{R}, \alpha \geq \frac{-3\beta^2}{4} \right\} \cup \left\{ \begin{pmatrix} \alpha & \frac{1}{3}\alpha\beta \\ \beta & \frac{\alpha}{9} + \frac{\beta^2}{3} \end{pmatrix}, \alpha, \beta \in \mathbb{R}, \alpha < \frac{-3\beta^2}{4} \right\}, \tag{21}$$

(cf. [5], Satz, p. 553).
If the translation complement is solvable, then one gets a single plane $\mathcal{A}$ generated by the spread

$$\{S\} \cup \left\{ \begin{pmatrix} v & -\frac{s^3}{3} \\ s & s^2 + v \end{pmatrix}, s, v \in \mathbb{R} \right\}, \tag{22}$$

(cf. [4], Satz 2 (b), p. 331).
The spread (21), respectively (22) coincides with the image of the section σ in (2) with the well defined functions $a(r,t)$ and $b(r,t)$ given in assertion b), respectively c). In case c) one has $a(r,k\pi) = 1$, $b(r,k\pi) = 0$, for all $r > 0$, $k \in \{0,1\}$ hence Remark 1 gives the form $\mathcal{K}$ of the assertion. In case b) we have $a(r,0) = 1$, $a(r,\pi) = 3$, $b(r,l\pi) = 0$, for all $r > 0$, $l \in \{0,1\}$. These give the form $\mathcal{K}$ of the assertion.
For decomposable Q^*, the identity $a(r, t + k\pi) = a(1,t)a(r,k\pi)$ holds for all $r > 0$, $t \in [0, 2\pi)$, $k \in \{0,1\}$ (cf. Proposition 11). In case b) for $-3 \leq \alpha \leq 1$ one has $a(1,t) = \sqrt{\alpha^2 - \alpha + 1}$ which yields a contradiction. In case c) we have $a(r, \frac{\pi}{4} + k\pi) = \sqrt{\frac{2}{1 - s + \frac{s^2}{3}}}$, $s \in \mathbb{R} \setminus \{0\}$ and the condition $a(r, \frac{\pi}{4} + k\pi) = a(1, \frac{\pi}{4})$ gives a contradiction. Hence in both cases Q^* is not decomposable and therefore quasi-simple (cf. Theorem 12). $\square$

If the translation complement of a 4-dimensional topological plane $\mathcal{A}$ has dimension 3, then the point ∞ of the line $S = \{(0,0,u,v), u, v \in \mathbb{R}\}$ is fixed under the seven-dimensional collineation group Γ of $\mathcal{A}$.

Proposition 17. *Let Q be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane $\mathcal{A}$ such that the 7-dimensional collineation group Γ of $\mathcal{A}$ acts transitively on the points of $W \setminus \{\infty\}$, where W is the translation axis of $\mathcal{A}$ and the kernel of the action of the translation complement on the line S has dimension 1. Then the multiplicative loop Q^* can be given by one of the following sets Λ_{Q^*} of the left translations of Q^*:*
a) Λ_{Q^*} is the range of the section (2) such that

$$a(r,t) = \sqrt{\frac{s^2+v^2}{s^2v+v^2+\frac{s^4}{3}+s^2}} \text{ and } b(r,t) = \frac{s^3-\frac{s^3v}{3}}{\sqrt{(s^2v+v^2+\frac{s^4}{3}+s^2)(s^2+v^2)}}$$

with $r\cos(t) = v\sqrt{\frac{s^2v+v^2+\frac{s^4}{3}+s^2}{s^2+v^2}}$, $r\sin(t) = -s\sqrt{\frac{s^2v+v^2+\frac{s^4}{3}+s^2}{s^2+v^2}}$. The quasifield Q corresponds to a single plane.
b) Λ_{Q^*} is the range of the section given by (2) such that

$$a(r,t) = \sqrt{\frac{v^2+u^2+2\gamma^2(1-\cos(u))-2v\gamma\sin(u)-2\gamma u\cos(u)+2\gamma u}{v^2+u^2-2\gamma^2+2\gamma^2\cos(u)}} \text{ and}$$

$$b(r,t) = \frac{-2u\gamma\sin u+2v\gamma\cos u-2v\gamma}{\sqrt{v^2+u^2+2\gamma^2(1-\cos u)-2v\gamma\sin u-2\gamma u\cos u+2\gamma u}\sqrt{v^2+u^2-2\gamma^2(1-\cos u)}}$$

with

$$r\cos(t) = (v-\gamma\sin(u))\sqrt{\frac{v^2+u^2-2\gamma^2+2\gamma^2\cos u}{v^2+u^2+2\gamma^2(1-\cos(u))-2v\gamma\sin(u)-2\gamma u\cos(u)+2\gamma u}},$$

$$r\sin(t) = (u-\gamma(\cos(u)-1))\sqrt{\frac{v^2+u^2-2\gamma^2+2\gamma^2\cos u}{v^2+u^2+2\gamma^2(1-\cos(u))-2v\gamma\sin(u)-2\gamma u\cos(u)+2\gamma u}}.$$

The quasifields Q_γ coordinatize a one-parameter family of planes $\mathcal{A}_\gamma, 0 < |\gamma| \leq 1$.
In all cases the multiplicative loop Q^* is not decomposable and quasi-simple. The set $\mathcal{K}$ of the left translations of Q^* corresponding to the kernel of the quasifield Q has the form $\left\{\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R}\right\}$.

Proof. If the translation complement C leaves a 1-dimensional subspace of S invariant, then one obtains a single plane $\mathcal{A}$ which corresponds to the following spread:

$$\{S\} \cup \left\{\begin{pmatrix} v & -\frac{s^3}{3}-s \\ s & s^2+v \end{pmatrix}, s, v \in \mathbb{R}\right\} \tag{23}$$

(cf. [18], 73.10., [4], pp. 330-331).
If the translation complement acts transitively on the 1-dimensional subspaces of S, then one gets a one-parameter family $E_\gamma, 0 < |\gamma| \le 1$, of planes which are generated by the normalized spread

$$\{S\} \cup \left\{ \begin{pmatrix} v - \gamma \sin u & u + \gamma(\cos u - 1) \\ \gamma(\cos u - 1) - u & v + \gamma \sin u \end{pmatrix}, u, v \in \mathbb{R} \right\}, \tag{24}$$

([8], Satz, p. 128, [13], Proposition 5.8). The spread (23), respectively (24) coincides with the image of the section σ in (2) such that the well defined functions $a(r,t)$ and $b(r,t)$ are given in assertion a), respectively b). Since in both cases one has $a(r, k\pi) = 1, b(r, k\pi) = 0$, for all $r > 0$, $k \in \{0, 1\}$, Remark 1 gives the form of $\mathcal{K}$. Moreover, in case a) one has $a(r, \frac{\pi}{4} + k\pi) = \sqrt{\frac{2}{2+v+\frac{v^2}{3}}}$, $v \in \mathbb{R} \setminus \{0\}$. In case b) for $v = 1$ we get

$$a(r_j, t_j) = \sqrt{\frac{1 + u^2 + 2\gamma^2(1 - \cos u) - 2\gamma \sin u - 2\gamma u \cos u + 2\gamma u}{1 + u^2 - 2\gamma^2 + 2\gamma^2 \cos u}}, \quad a(1, t_j) = 1.$$

For decomposable Q^* one has $a(r, t+k\pi) = a(1,t)a(r,k\pi)$ for all $r \in \mathbb{R}\setminus\{0\}$, $t \in [0, 2\pi)$, $k \in \{0, 1\}$ (cf. Proposition 11) which yields a contradiction. Thus in both cases Q^* is not decomposable and hence quasi-simple (cf. Theorem 12). □

Proposition 18. *Let Q be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane $\mathcal{A}$ such that the translation complement C of the 7-dimensional collineation group Γ of $\mathcal{A}$ has an orbit of dimension 1 on $W \setminus \{0\}$, C leaves in the set of lines through the origin only S fixed and the kernel of its action on S has positive dimension. Then the multiplicative loop Q^* can be given by one of the following sets Λ_{Q^*} of the left translations of Q^*:*
a) Λ_{Q^*} is the range of the section (2) such that for $\beta \ge 0$ one has

$$a(r,t) = \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2 + z\alpha\beta^{\frac{1}{1+s}} - w\beta^{\frac{2}{1+s}}}} \text{ and } b(r,t) = \frac{w\alpha\beta^{\frac{1-s}{1+s}} + \alpha\beta + z\beta^{\frac{2+s}{1+s}}}{\sqrt{\alpha^2+\beta^2}\sqrt{\alpha^2 + z\alpha\beta^{\frac{1}{1+s}} - w\beta^{\frac{2}{1+s}}}}$$

$$\text{with } r\cos(t) = \alpha\sqrt{\frac{\alpha^2 + z\alpha\beta^{\frac{1}{1+s}} - w\beta^{\frac{2}{1+s}}}{\alpha^2+\beta^2}}, \ r\sin(t) = -\beta\sqrt{\frac{\alpha^2 + z\alpha\beta^{\frac{1}{1+s}} - w\beta^{\frac{2}{1+s}}}{\alpha^2+\beta^2}}.$$

For $\beta < 0$ one gets

$$a(r',t) = \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2 + q\alpha(-\beta)^{\frac{1}{1+s}} + p(-\beta)^{\frac{2}{1+s}}}} \text{ and } b(r',t) = \frac{p\alpha(-\beta)^{\frac{1-s}{1+s}} + \alpha\beta - q(-\beta)^{\frac{2+s}{1+s}}}{\sqrt{\alpha^2+\beta^2}\sqrt{\alpha^2 + q\alpha(-\beta)^{\frac{1}{1+s}} + p(-\beta)^{\frac{2}{1+s}}}}$$

with

$$r'\cos(t) = \alpha\sqrt{\frac{\alpha^2 + q\alpha(-\beta)^{\frac{1}{1+s}} + p(-\beta)^{\frac{2}{1+s}}}{\alpha^2+\beta^2}} \text{ and } r'\sin(t) = -\beta\sqrt{\frac{\alpha^2 + q\alpha(-\beta)^{\frac{1}{1+s}} + p(-\beta)^{\frac{2}{1+s}}}{\alpha^2+\beta^2}}.$$

The quasifields $Q_{s,w,z,p,q}$ coordinatize a family of planes $\mathcal{A}_{s,w,z,p,q}$ such that the parameters s, w, z, p, q satisfy the conditions $0 < s < 1$, $z^2 + 4w(1 - s^2) \leq 0$, $q^2 - 4p(1 - s^2) \leq 0$.
b) Λ_{Q^*} is the range of the section (2) such that for $\beta \geq 0$ we have

$$a(r,t) = \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2 + z\alpha\beta - w\beta^2 + 2\alpha\beta\ln\beta + z\beta^2\ln\beta + \beta^2(\ln\beta)^2}} \ and$$

$$b(r,t) = \frac{(w+1)\alpha\beta + z\beta^2 - z\alpha\beta\ln\beta - \alpha\beta(ln\beta)^2 + 2\beta^2\ln\beta}{\sqrt{\alpha^2+\beta^2}\sqrt{\alpha^2 + z\alpha\beta + 2\alpha\beta\ln\beta - w\beta^2 + z\beta^2\ln\beta + \beta^2(\ln\beta)^2}}$$

with

$$r\cos(t) = \alpha\sqrt{\frac{\alpha^2 + z\alpha\beta - w\beta^2 + 2\alpha\beta\ln\beta + z\beta^2\ln\beta + \beta^2(\ln\beta)^2}{\alpha^2+\beta^2}},$$

$$r\sin(t) = -\beta\sqrt{\frac{\alpha^2 + z\alpha\beta - w\beta^2 + 2\alpha\beta\ln\beta + z\beta^2\ln\beta + \beta^2(\ln\beta)^2}{\alpha^2+\beta^2}}.$$

For $\beta < 0$ we obtain

$$a(r',t) = \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2 - q\alpha\beta + p\beta^2 + (2\alpha\beta - q\beta^2)\ln(-\beta) + \beta^2(\ln(-\beta))^2}} \ and$$

$$b(r',t) = \frac{(1-p)\alpha\beta - q\beta^2 + (2\beta^2 + q\alpha\beta)\ln(-\beta) - \alpha\beta(\ln(-\beta))^2}{\sqrt{\alpha^2+\beta^2}\sqrt{\alpha^2 - q\alpha\beta + p\beta^2 + (2\alpha\beta - q\beta^2)\ln(-\beta) + \beta^2(\ln(-\beta))^2}}$$

with

$$r'\cos(t) = \alpha\sqrt{\frac{\alpha^2 - q\alpha\beta + p\beta^2 + (2\alpha\beta - q\beta^2)\ln(-\beta) + \beta^2(\ln(-\beta))^2}{\alpha^2+\beta^2}},$$

$$r'\sin(t) = -\beta\sqrt{\frac{\alpha^2 - q\alpha\beta + p\beta^2 + (2\alpha\beta - q\beta^2)\ln(-\beta) + \beta^2(\ln(-\beta))^2}{\alpha^2+\beta^2}}.$$

The quasifields $Q_{w,z,p,q}$ coordinatize a family of planes $\mathcal{A}_{w,z,p,q}$ such that for the parameters w, z, p, q the relations $\left(\frac{z}{2}\right)^2 \leq -w - 1$, $\left(\frac{q}{2}\right)^2 \leq p - 1$ hold.
c) Λ_{Q^*} is the range of the section given by (2) such that $a(r, k\pi) = 1$ and $b(r, k\pi) = 0$, $k \in \{0, 1\}$ with $r = |\beta|$.
For $u \in \mathbb{R}$, $\beta > 0$, we get

$$a(r,t) = \sqrt{\frac{u^2 + \sin^2(l)(w^2 + 2zu + z^2) + \cos^2(l) - (2uw + 2u + 2z)\sin(l)\cos(l)}{u^2 + uz - w}},$$

$$b(r,t) = \frac{\cos^2(l)(2uw + 2u + 2z) + \sin(l)\cos(l)(1 - w^2 - z^2 - 2uz) - (u + z + uw)}{\sqrt{(u^2 + \sin^2(l)(w^2 + 2zu + z^2) + \cos^2(l) - (2uw + 2u + 2z)\sin(l)\cos(l))(u^2 + uz - w)}}$$

with

$$r\cos(t) = \beta\left(u - (w+1)\sin(l)\cos(l) + z\sin^2(l)\right)\sqrt{\frac{u^2 + uz - w}{u^2 + \sin^2(l)(w^2 + 2zu + z^2) + \cos^2(l) - (2uw + 2u + 2z)\sin(l)\cos(l)}},$$

$$r\sin(t) = \beta\left(w\sin^2(l) + z\sin(l)\cos(l) - \cos^2(l)\right)\sqrt{\frac{u^2 + uz - w}{u^2 + \sin^2(l)(w^2 + 2zu + z^2) + \cos^2(l) - (2uw + 2u + 2z)\sin(l)\cos(l)}},$$

where $l = \frac{1}{k}\ln\beta$. For $u \in \mathbb{R}$, $\beta < 0$ one obtains

$$a(r',t') = \sqrt{\frac{u^2 + \sin^2(l_1)(q^2 + 2qu + p^2) + \cos^2(l_1) + (2u + 2q - 2up)\sin(l_1)\cos(l_1)}{u^2 + uq + p}}$$

$$b(r',t') = \frac{\sin(l_1)\cos(l_1)(1-2uq-p^2-q^2)+\sin^2(l_1)(2q+2u-2up)+(up-q-u)}{\sqrt{(u^2+\sin^2(l_1)(q^2+2qu+p^2)+\cos^2(l_1)+(2q+2u-2up)\sin(l_1)\cos(l_1))(u^2+uq+p)}}$$

with

$$r'\cos(t') = \beta\left((p-1)\sin(l_1)\cos(l_1) - q\sin^2(l_1) - u\right)\cdot$$

$$\sqrt{\frac{u^2+uq+p}{u^2+\sin^2(l_1)(q^2+2qu+p^2)+\cos^2(l_1)+(2u+2q-2up)\sin(l_1)\cos(l_1)}},$$

$$r'\sin(t') = -\beta\left(\cos^2(l_1) + q\sin(l_1)\cos(l_1) + p\sin^2(l_1)\right)\cdot$$

$$\sqrt{\frac{u^2+uq+p}{u^2+\sin^2(l_1)(q^2+2qu+p^2)+\cos^2(l_1)+(2u+2q-2up)\sin(l_1)\cos(l_1)}},$$

where $l_1 = \frac{1}{k}\ln(-\beta)$.
The quasifields $Q_{k,w,z,p,q}$ *coordinatize a family of planes* $\mathcal{A}_{k,w,z,p,q}$ *such that for the parameters* k,w,z,p,q *one has* $k \neq 0$, $(4+k^2)(z^2+(w+1)^2) \leq k^2(1-w)^2$, $(4+k^2)(q^2+(p-1)^2) \leq k^2(p+1)^2$, $(w,z,p,q) \neq (-1,0,1,0)$.
In all cases Q^* *is not decomposable and quasi-simple. The set of the left translations of* Q^* *belonging to the kernel of* Q *is* $\mathcal{K} = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R} \right\}$.

Proof. If the translation complement C fixes two 1-dimensional subspaces of S, then we have a family of translation planes $\mathcal{A}_{s,w,z,p,q}$ such that the normalized spreads belonging to these planes are given as follows:

$$\{S\} \cup \left\{ \begin{pmatrix} \alpha & w\beta^{\frac{1-s}{1+s}} \\ \beta & z\beta^{\frac{1}{1+s}} + \alpha \end{pmatrix}, \alpha \in \mathbb{R}, \beta \geq 0 \right\} \cup \left\{ \begin{pmatrix} \alpha & p(-\beta)^{\frac{1-s}{1+s}} \\ \beta & q(-\beta)^{\frac{1}{1+s}} + \alpha \end{pmatrix}, \alpha \in \mathbb{R}, \beta < 0 \right\}, \tag{25}$$

(cf. [6], Satz 1, pp. 411-412).
If the translation complement C fixes only one 1-dimensional subspace of S, then there is a family of translation planes $\mathcal{A}_{w,z,p,q}$ such that the corresponding normalized spreads have the form:

$$\{S\} \cup \left\{ \begin{pmatrix} \alpha & w\beta - z\beta\ln\beta - \beta(\ln\beta)^2 \\ \beta & \alpha + z\beta + 2\beta\ln\beta \end{pmatrix}, \alpha \in \mathbb{R}, \beta \geq 0 \right\} \cup \left\{ \begin{pmatrix} \alpha & -p\beta - \beta(\ln(-\beta))^2 + q\beta\ln(-\beta) \\ \beta & q(-\beta) + \alpha + 2\beta\ln(-\beta) \end{pmatrix}, \alpha \in \mathbb{R}, \beta < 0 \right\} \tag{26}$$

(cf. Satz 2, [6], pp. 418-419).
If the translation complement C acts transitively on the 1-dimensional subspaces of S, then we have a family of translation planes $\mathcal{A}_{k,w,z,p,q}$ such that the normalized spreads belonging to these planes have the form

$$\{S\} \cup \left\{ \beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \beta \in \mathbb{R} \right\} \cup$$

$$\left\{ \beta \begin{pmatrix} u-(w+1)\sin(l)\cos(l)+z\sin^2(l) & w\cos^2(l)-z\sin(l)\cos(l)-\sin^2(l) \\ \cos^2(l)-z\sin(l)\cos(l)-w\sin^2(l) & z\cos^2(l)+(w+1)\sin(l)\cos(l)+u \end{pmatrix}, u \in \mathbb{R}, \beta > 0 \right\} \cup$$

$$\left\{ \beta \begin{pmatrix} (p-1)\sin(l_1)\cos(l_1)-q\sin^2(l_1)-u & q\sin(l_1)\cos(l_1)-p\cos^2(l_1)-\sin^2(l_1) \\ \cos^2(l_1)+q\sin(l_1)\cos(l_1)+p\sin^2(l_1) & (1-p)\sin(l_1)\cos(l_1)-q\cos^2(l_1)-u \end{pmatrix}, u \in \mathbb{R}, \beta < 0 \right\}, \tag{27}$$

where $l = \frac{1}{k}\ln\beta$, $l_1 = \frac{1}{k}\ln(-\beta)$ (cf. [15], Proposition 4.1, p. 6, and [6], Satz 3, pp. 422-423). The spreads (25), respectively (26), respectively (27) coincide with the image of the section σ in (2) such that the well defined functions $a(r,t)$ and $b(r,t)$ are given in assertion a), respectively b), respectively c).

Since in all three cases we have $a(r,k\pi) = 1$, $b(r,k\pi) = 0$, $r > 0$, $k \in \{0,1\}$, Remark 1 shows that $\mathcal{K}$ has the form as in the assertion. In case a), respectively b) for $\beta > 0$ one gets $a(r, \frac{\pi}{4}+\pi) = \frac{\sqrt{2}\beta}{\sqrt{\beta^2 - z\beta^{\frac{2+s}{1+s}} - w\beta^{\frac{2}{1+s}}}}$, respectively $a(r, \frac{\pi}{4}+\pi) = \sqrt{\frac{2}{1-z-w-2\ln\beta+z\ln\beta+(\ln\beta)^2}}$. In case c) for $u = 0$, $\beta > 0$ we get that $a(1,t_j)$ is constant. These relations give a contradiction to the condition $a(r, t+k\pi) = a(1,t)$, $r > 0$, $t \in [0, 2\pi)$, $k \in \{0,1\}$ of Proposition 11. Hence in all cases Q^* is not decomposable and quasi-simple (cf. Theorem 12). $\square$

Proposition 19. *Let Q be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane $\mathcal{A}$ such that the translation complement C of the 7-dimensional collineation group Γ of $\mathcal{A}$ has an orbit of dimension 1 on $W \setminus \{0\}$, C leaves only S in the set of lines through the origin fixed and the kernel of its action on S is zero-dimensional. Then the set Λ_{Q^*} of all left translations of the multiplicative loop Q^* is given by the range of the section (2) defined as follows: For $\alpha \geq -\frac{\beta^2}{2}$ one has*

$$a(r,t) = \sqrt{\frac{\alpha^2+\beta^2}{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} + \left(\alpha + \frac{\beta^2}{2}\right)\left(\alpha + \frac{q-1}{q}\beta^2\right) - \frac{p\beta}{q}\left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}}}},$$

$$b(r,t) = \frac{\frac{p}{q}\alpha\left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}} - \frac{p}{q}\left(\alpha^2+\beta^2\right) + \frac{1-q}{q}\beta\alpha^2 + \frac{\alpha\beta^3}{6q} - \frac{\beta^3\alpha}{2} + \frac{\beta^3}{2q} + \frac{\beta^3}{2} + \alpha\beta}{\sqrt{\alpha^2+\beta^2}\sqrt{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} + \left(\alpha + \frac{\beta^2}{2}\right)\left(\alpha + \frac{(q-1)}{q}\beta^2\right) - \frac{p\beta}{q}\left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}}}},$$

with

$$r\cos(t) = \alpha\sqrt{\frac{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} + \left(\alpha + \frac{\beta^2}{2}\right)\left(\alpha + \frac{q-1}{q}\beta^2\right) - \frac{p\beta}{q}\left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}}}{\alpha^2+\beta^2}},$$

$$r\sin(t) = -\beta\sqrt{\frac{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} + \left(\alpha + \frac{\beta^2}{2}\right)\left(\alpha + \frac{q-1}{q}\beta^2\right) - \frac{p\beta}{q}\left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}}}{\alpha^2+\beta^2}}.$$

For $\alpha < -\frac{\beta^2}{2}$ we get

$$a(r,t) = \sqrt{\frac{\alpha^2+\beta^2}{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} - \left(\alpha + \frac{\beta^2}{2}\right)\left(\frac{\alpha z}{q} + \frac{(z+1)\beta^2}{q}\right) - \frac{w\beta}{q}\left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}}}},$$

$$b(r,t) = \frac{\frac{w}{q}\alpha\left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}} + \frac{p}{q}\left(-\alpha^2 - \beta^2\right) + \left(\frac{z+1}{q}\alpha\beta - \frac{z\beta}{q}\right)\left(\alpha + \frac{\beta^2}{2}\right) - \frac{\alpha\beta^3}{3q} + \frac{\beta^3}{2q}}{\sqrt{\alpha^2+\beta^2}\sqrt{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} - \left(\alpha + \frac{\beta^2}{2}\right)\left(\frac{\alpha z}{q} + \frac{(z+1)\beta^2}{q}\right) - \frac{w\beta}{q}\left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}}}},$$

with

$$r\cos(t) = \alpha\sqrt{\frac{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} - \left(\alpha + \frac{\beta^2}{2}\right)\left(\frac{\alpha z}{q} + \frac{(z+1)\beta^2}{q}\right) - \frac{w\beta}{q}\left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}}}{\alpha^2+\beta^2}},$$

$$r\sin(t) = -\beta\sqrt{\frac{\frac{\alpha\beta^2}{2q} + \frac{\beta^4}{3q} - \left(\alpha + \frac{\beta^2}{2}\right)\left(\frac{\alpha z}{q} + \frac{(z+1)\beta^2}{q}\right) - \frac{w\beta}{q}\left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}}}{\alpha^2+\beta^2}}.$$

The quasifields $Q_{w,z,p,q}$ coordinatize a family of planes $\mathcal{A}_{w,z,p,q}$ such that the parameters w, z, p, q satisfy $(3w)^2 \leq -16z(z+1)$, $(3p)^2 \leq 16q(q-1)$, $q > 0$, $z < 0$ and $(w, z, p, q) \neq (0, -\frac{1}{3}, 0, 3)$.
The multiplicative loops $Q^*_{w,z,p,q}$ of the quasifields $Q_{w,z,p,q}$ are not decomposable and quasi-simple. The left translations of $Q^*_{w,z,p,q}$ corresponding to the kernel of $Q_{w,z,p,q}$ have the form $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$, $0 \neq r \in \mathbb{R}$, if and only if $w = p = 0$, $q = -z = 1$.

Proof. By Satz 5 in [6], the planes $\mathcal{A}_{w,z,p,q}$ are determined by the normalized spreads which have the form

$$\{S\} \cup \left\{ \begin{pmatrix} \alpha & \frac{-p}{q}\alpha + \frac{p}{q}\left(\alpha + \frac{\beta^2}{2}\right)^{\frac{3}{2}} + \frac{(1-q)}{q}\beta\left(\alpha + \frac{\beta^2}{2}\right) - \frac{\beta^3}{3q} \\ \beta & \frac{-p}{q}\beta + \frac{\beta^2}{2q} + \left(\alpha + \frac{\beta^2}{2}\right) \end{pmatrix}, \beta \in \mathbb{R}, \alpha \geq -\frac{\beta^2}{2} \right\} \cup$$

$$\left\{ \begin{pmatrix} \alpha & \frac{-p}{q}\alpha + \frac{w}{q}\left(-\alpha - \frac{\beta^2}{2}\right)^{\frac{3}{2}} + \frac{(z+1)}{q}\beta\left(\alpha + \frac{\beta^2}{2}\right) - \frac{\beta^3}{3q} \\ \beta & \frac{-p}{q}\beta + \frac{\beta^2}{2q} - \frac{z}{q}\left(\alpha + \frac{\beta^2}{2}\right) \end{pmatrix}, \beta \in \mathbb{R}, \alpha < -\frac{\beta^2}{2} \right\}.$$

These spreads coincide with the image of the section σ in (2) such that the well defined functions $a(r, t)$ and $b(r, t)$ are given in the assertion. One gets that $a(r, 0) = 1$ and $a(r, \pi) = \sqrt{\frac{-q}{z}}$ for all $r > 0$.

For $\beta > 2$ we obtain

$$a\left(r, \frac{\pi}{4} + \pi\right) = \frac{\sqrt{2}\beta}{\sqrt{\frac{\beta^4}{3q} - \frac{\beta^3}{2q} - \frac{w}{q}\beta\left(\beta - \frac{\beta^2}{2}\right)^{\frac{3}{2}} + \left(\beta - \frac{\beta^2}{2}\right)\left(\frac{z+1}{q}\beta^2 - \frac{\beta z}{q}\right)}}.$$

The loop $Q^*_{w,z,p,q}$ is not decomposable since we have a contradiction to the condition $a(r, \frac{\pi}{4} + k\pi) = a(1, \frac{\pi}{4})a(r, k\pi)$, $r > 0$, $k \in \{0, 1\}$ (cf. Proposition 11). Hence $Q^*_{w,z,p,q}$ is quasi-simple (cf. Theorem 12). As $a(r, k\pi) = 1$ and $b(r, k\pi) = 0$, $r > 0$, $k \in \{0, 1\}$ holds precisely if $w = p = 0$, $q = -z = 1$ the last assertion follows. □

Proposition 20. *Let Q be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane $\mathcal{A}$ such that the translation complement C of the 7-dimensional collineation group $\mathcal{A}$ fixes two distinct lines $\{S, W\}$ through the origin and leaves on S one or two 1-dimensional subspaces invariant. Then the multiplicative loop Q^* can be given by one of the following sets Λ_{Q^*} of the left translations of Q^* having the form (20):*
a)

$$a(r, 0) = r^{\frac{1-w}{1+w}}, \quad b(r, 0) = c\left(r^{\frac{w-1}{w+1}} - r^{\frac{1-w}{1+w}}\right),$$

with $r = s^{\frac{w+1}{2}}$, $s > 0$, $t = -\varphi$, where s and φ are variables of the spreads (28). The quasifields $Q_{w,c}$ coordinatize a family of planes $\mathcal{A}_{w,c}$ such that for the parameters $w \neq 1, c$ one has $0 < w$ and $(w-1)^2c^2 \leq 4w$.

b)

$$a(r,0)=1, \quad b(r,0)=\frac{\ln r}{d},$$

with $r=e^s$, $t=-\varphi$, where s and φ are variables of the spreads (29). The quasifields Q_d coordinatize a one-parameter family of planes $\mathcal{A}_d$ such that $4d^2 \geq 1$.
In both cases Q^* is decomposable and contains the group $\mathrm{SO}_2(\mathbb{R})$.

Proof. If the group C fixes two 1-dimensional subspaces of S, respectively only one 1-dimensional subspace of S, then one obtains a family of translation planes corresponding to the normalized spreads

$$\{S,W\} \cup \left\{ \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} s & c(s^w-s) \\ 0 & s^w \end{pmatrix}, \ s,\varphi \in \mathbb{R}, s>0 \right\} \quad (28)$$

(cf. [7], Satz 1 and [9], p. 15), respectively

$$\{S,W\} \cup \left\{ \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} e^s & e^s \frac{s}{d} \\ 0 & e^s \end{pmatrix}, s,\varphi \in \mathbb{R} \right\}, \quad (29)$$

(cf. [7], Satz 2 and [9], p. 15). In both cases these spreads coincide with the set $\Lambda = \mathrm{SO}_2(\mathbb{R})\mathcal{K}$ given in (20) such that the set $\mathcal{K}$ corresponding to the kernel K_r of Q is determined by the functions $a(r,0)$, $b(r,0)$ as in assertion a), respectively b). $\square$

Remark 21. *In [2] D. Betten constructed 4-dimensional locally compact non-desarguesian planes $\mathcal{A}_f$ corresponding to continuous, non-linear, strictly monotone functions f defined for $0 \leq u \in \mathbb{R}$ with $f(0)=0$ and $\lim\limits_{u\to\infty} f(u) = \infty$. The planes $\mathcal{A}_f$ are determined by the normalized spreads*

$$\left\{ \begin{pmatrix} u\cos\varphi & -\frac{f(u)\sin\varphi}{f(1)} \\ u\sin\varphi & \frac{f(u)\cos\varphi}{f(1)} \end{pmatrix}, u>0, \ \varphi \in [0,2\pi) \right\}.$$

These spreads coincide with the set $\Lambda = \mathrm{SO}_2(\mathbb{R})\mathcal{K}$ given in (20) such that the set $\mathcal{K}$ corresponding to the kernel K_r of the quasifield Q_f coordinatizing $\mathcal{A}_f$ is determined by the functions $a(r,0)=\sqrt{\frac{uf(1)}{f(u)}}$, $b(r,0)=0$ with $r=\sqrt{\frac{uf(u)}{f(1)}}$, $t=-\varphi$, $u\neq 0$. For $f(u)=f(1)u^w$ these planes are planes in Proposition 20 a) with $c=0$ and $a(r,0)=r^{\frac{1-w}{1+w}}$. Otherwise the full collineation group of the planes $\mathcal{A}_f$ has dimension 6.

Proposition 22. *Let Q be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane $\mathcal{A}$ such that the translation*

complement C of the 7-dimensional collineation group of $\mathcal{A}$ fixes two distinct lines $\{S, W\}$ through the origin and acts transitively on the spaces P_S and P_W of all 1-dimensional subspaces of S, respectively W. Then the multiplicative loop Q^* of Q can be given by one of the following sets Λ_{Q^*} of the left translations of Q^*:
a) Λ_{Q^*} is the range of the section (2) with

$$a(r,u) = \sqrt{\frac{dD}{de^{2(qt-ps)} + de^{2q\pi} + e^{qt-ps+q\pi}\,(2d\cos s\cos t + (c^2+1+d^2)\sin s\sin t)}},$$

$$b(r,u) = \frac{e^{2(qt-ps)}\left[(-c^2-1+d^2)\cos t\sin t - c(c^2+1+d^2)\sin^2 t\right]}{\sqrt{dD\left[d(e^{2(qt-ps)}+e^{2q\pi}) + e^{qt-ps+q\pi}(2d\cos s\cos t + (d^2+c^2+1)\sin s\sin t)\right]}} +$$

$$+ \frac{e^{qt-ps+q\pi}\,(\cos s\cos t + d\sin s\sin t + c\cos s\sin t)}{\sqrt{dD\left[d(e^{2(qt-ps)}+e^{2q\pi}) + e^{qt-ps+q\pi}(2d\cos s\cos t + (d^2+c^2+1)\sin s\sin t)\right]}},$$

such that

$$r\cos u = \frac{e^{qt-ps}(\cos s\cos t + c\sin t\cos s + d\sin t\sin s) + e^{q\pi}}{1+e^{q\pi}}\,a^{-1}(r,u),$$

$$r\sin u = -\frac{e^{qt-ps}(d\cos s\sin t - \sin s\cos t - c\sin s\sin t)}{1+e^{q\pi}}\,a^{-1}(r,u),$$

$$D = e^{2(qt-ps)}\left((\cos t + c\sin t)^2 + d^2\sin^2 t\right) + e^{2q\pi} + 2e^{qt-ps+q\pi}(\cos s\cos t + c\cos s\sin t + d\sin s\sin t).$$

The quasifields $Q_{p,q,c,d}$ coordinatize a family of planes $\mathcal{A}_{p,q,c,d}$ such that the parameters p, q, c, d satisfy the conditions

$$\begin{array}{ll} p = q > 0 & \text{and} \quad -1 \le d < 0, \\ q > 0,\, p = \frac{k-1}{k+1}q,\, k = 1,2,3,\cdots & \text{and} \quad d > 0, \end{array}$$

$$-(q+p)^2 A + (q-p)^2 B - 4AB \ge 0, \ \text{where } A = \frac{(d-1)^2+c^2}{4d} \ \text{and } B = \frac{(d+1)^2+c^2}{4d}.$$

The multiplicative loops Q^* of the quasifields $Q_{p,q,c,d}$ are not decomposable and quasi-simple.
b) Λ_{Q^*} has the form (15) with

$$a(1,u) = \sqrt{(\cos nt + c\sin nt)^2 + d^2\sin^2 nt},\ \ b(1,u) = \frac{\sin nt\cos nt(d^2-1-c^2) - c\sin^2 nt(d^2+1+c^2)}{d\sqrt{(\cos nt + c\sin nt)^2 + d^2\sin^2 nt}}$$

such that

$$r\cos u = \frac{s(\cos nt\cos mt + c\sin nt\cos mt + d\sin nt\sin mt)}{\sqrt{(\cos nt + c\sin nt)^2 + d^2\sin^2 nt}},\ \ r\sin u = \frac{s(d\sin nt\cos mt - \cos nt\sin mt - c\sin nt\sin mt)}{\sqrt{(\cos nt + c\sin nt)^2 + d^2\sin^2 nt}}$$

and $s \ge 0$.
The quasifields $Q_{m,n,c,d}$ coordinatize a family of planes $\mathcal{A}_{m,n,c,d}$ such that the parameters $m, n \in \mathbb{Z}$, $(m,n) = 1$, $c, d \in \mathbb{R}$ satisfy the conditions

$$\begin{array}{lll} m = n = 1 & & \text{and} \ -1 \le d < 0 \\ m = 1,2,3,\cdots & n = m+1 & \text{and} \ d > 0 \\ m = 1,3,5,\cdots & n = m+2 & \text{and} \ d > 0 \end{array}$$

$$(n-m)^2 B \ge (n+m)^2 A, \ \text{where } A = \frac{(d-1)^2+c^2}{4d} \ \text{and } B = \frac{(d+1)^2+c^2}{4d}.$$

The loops $Q^*_{m,n,c,d}$ are split extensions of the normal subgroup $\widetilde{N^*} \cong \mathbb{R}$ corresponding to the connected component of $\left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, 0 \ne u \in \mathbb{R} \right\}$ with a subloop homeomorphic to the 1-sphere.

Proof. If the translation complement C acts transitively on the product space $P_S \times P_w$, then there is a family of translation planes corresponding to the normalized spreads

$$\{S, W\} \cup \left\{ \begin{pmatrix} \frac{\alpha(s,t)+e^{q\pi}}{1+e^{q\pi}} & \frac{\gamma(s,t)-c\alpha(s,t)}{d(1+e^{q\pi})} \\ \frac{\beta(s,t)}{1+e^{q\pi}} & \frac{\delta(s,t)-c\beta(s,t)+de^{q\pi}}{d(1+e^{q\pi})} \end{pmatrix}, s, t \in \mathbb{R} \right\}$$

such that $\alpha(s,t) = e^{qt-ps}(\cos s \cos t + c \sin t \cos s + d \sin t \sin s)$,
$\beta(s,t) = e^{qt-ps}(d \cos s \sin t - \sin s \cos t - c \sin s \sin t)$,
$\gamma(s,t) = e^{qt-ps}(d \cos t \sin s - \sin t \cos s + c \cos t \cos s)$,
$\delta(s,t) = e^{qt-ps}(d \cos t \cos s + \sin t \sin s - c \cos t \sin s)$ (cf. [7], Satz 3, pp. 135-136). These spreads coincide with the image of the section σ in (2) with the well defined functions $a(r,u)$ and $b(r,u)$ as in assertion a). For $s = 0$ we get a contradiction to the condition $a(r_j, u_j) = a(r_j, 0)a(1, u_j)$ which must hold for decomposable Q^*. It follows that Q^* is not decomposable and hence quasi-simple (cf. Theorem 12).
If the translation complement C does not act transitively on the product space $P_S \times P_W$, then there is a family of translation planes which correspond to the normalized spreads

$$\{S, W\} \cup \left\{ \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} a_{11}(t) & -\frac{c}{d}a_{11}(t) + \frac{1}{d}a_{21}(t) \\ a_{12}(t) & -\frac{c}{d}a_{12}(t) + \frac{1}{d}a_{22}(t) \end{pmatrix}, s \geq 0, t \in \mathbb{R} \right\}$$

with $a_{11}(t) = \cos nt \cos mt + c \sin nt \cos mt + d \sin nt \sin mt$,
$a_{12}(t) = d \sin nt \cos mt - \cos nt \sin mt - c \sin nt \sin mt$,
$a_{21}(t) = d \cos nt \sin mt - \sin nt \cos mt + c \cos nt \sin mt$,
$a_{22}(t) = d \cos nt \cos mt + \sin nt \sin mt - c \cos nt \sin mt$ (cf. [7], Satz 4, pp. 142-144). These spreads coincide with the set Λ_{Q^*} in (15) such that the periodic functions $a(1,t)$ and $b(1,t)$ are given in assertion b). As in the proof of Proposition 16 a) it follows that the loop $Q^*_{m,n,c,d}$ is a split extension as in the assertion. □

Corollary 23. *Let $\mathcal{A}$ be a 4-dimensional locally compact non-desarguesian topological plane which admits an at least 7-dimensional collineation group Γ. If the quasifield Q coordinatizing $\mathcal{A}$ is constructed with respect to two lines such that their intersection points with the line at infinity are contained in the 1-dimensional orbit of Γ or contain the set of the fixed points of Γ, then for the multiplicative loop Q^* of Q one of the following holds:*
a) Q^* is quasi-simple and not decomposable. Such quasifields Q are described by Propositions 16 b), 16 c), 17), 18), 19) and in Proposition 22 a).
b) Q^* is quasi-simple but decomposable and it is a product $SO_2(\mathbb{R})B$, where B is a 1-dimensional loop homeomorphic to $\mathbb{R}$. The quasifields Q of this type

are described in Proposition 20.
c) Q^* *is a split extension of the group* $\widetilde{N^*} \cong \mathbb{R}$ *with a loop homeomorphic to the* 1*-sphere. The quasifields of this type are described in Propositions 16 a) and 22 b).*

Proof. A locally compact topological quasifield coordinatizing the translation plane $\mathcal{A}$ and constructed with respect to two lines satifying the assumptions is isotopic to a quasifield given in Betten's classification (cf. [11], p. 321, [3] Satz 5). For isotopic loops Q_1^* and Q_2^* the following holds: The group generated by their left translations, every subgroup and all nuclei of them are isomorphic (cf. [14], Lemmata 1.9, 1.10, p. 20). From these facts we get: If Q_1 is quasisimple and not decomposable, then also Q_2 is quasisimple and not decomposable. If Q_1 contains the subgroup $SO_2(\mathbb{R})$, then also Q_2 contains the group $SO_2(\mathbb{R})$. If Q_1 is a split extension of $\widetilde{N^*}$ with a 1-dimensional compact loop, then the same holds for Q_2. □

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