Quasi-simple Lie groups as multiplication groups of topological loops

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Abstract

We prove that among the quasi-simple Lie groups only the group $SL_4(\mathbb{R})$ occurs as the multiplication group of 3-dimensional connected topological loops L. These loops L are homeomorphic to the sphere S^3. Moreover, there does not exist any connected topological loop having an at most 8-dimensional quasi-simple Lie groups as its multiplication group.

1. Introduction

The multiplication group $Mult(L)$ and the inner mapping group $Inn(L)$ of a loop L are important tools for research in loop theory (cf. [4]). If the group $Mult(L)$ is simple, then the loop L is also simple and the group $Inn(L)$ is a maximal subgroup of $Mult(L)$ (cf. [2], [17]). Since the knowledge of simple loops is fundamental for the construction of all other loops it is a legitimate problem to investigate which simple groups can be represented as multiplication groups of loops. If L is a finite loop of order n, then $Mult(L)$ is prevalently the symmetric or the alternating group of degree n. The question, which finite simple groups are multiplication groups of finite loops, is discussed in many papers (cf. [7], [17], [18], [22], [23]). The negative answer for most of the investigated finite groups shows that there are hard obstructions for simple groups to be the multiplication group of loops. This observation is supported also by investigations for low-dimensional topological loops having Lie groups as the groups G_l topologically generated by their left translations (cf. [19]). Their multiplication group $Mult(L)$ is mostly a differentiable transformation group of infinite dimension. The condition that the group $Mult(L)$ is a Lie group gives strong restrictions for the isomorphism types of $Mult(L)$ and of L. For topological proper loops L of dimension ≤ 2 only special nilpotent groups are the groups $Mult(L)$ of

[1]Keywords: multiplication group of topological loops, topological transformation groups, sections in Lie groups, quasi-simple Lie groups.

[2]2010 Mathematics Subject Classification: 57S20, 57M60, 20N05, 22A30, 53C30

L (cf. [9]). For 3-dimensional loops L the situation changes; besides nilpotent groups there are many solvable Lie groups as the groups $Mult(L)$. (cf. [11]). In contrast to this in the present paper we show that quasi-simple Lie groups up to one exception are not multiplication groups of 3-dimensional topological loops. Namely, we have

Theorem 1. *Let L be a 3-dimensional connected topological proper loop. The multiplication group $Mult(L)$ of L is a quasi-simple Lie group if and only if L is homeomorphic to S^3 and $Mult(L)$ is the Lie group $SL_4(\mathbb{R})$.*

The group $SL_4(\mathbb{R})$ occurs as the multiplication group of a topological proper loop homeomorphic to the 3-sphere since there exist 4-dimensional locally compact proper semifields Q such that the multiplicative loop of Q is the direct product of $\mathbb{R}$ and a compact loop (cf. [8], [14]).
For quasi-simple Lie groups of dimension ≤ 8 we show more:

Theorem 2. *There does not exist any connected topological proper loop L having an at most 8-dimensional quasi-simple Lie group as its multiplication group.*

2. Preliminaries

A binary system $(L, \cdot)$ is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution, which we denote by $y = a \backslash b$ and $x = b/a$. A loop L is proper if it is not a group.

The left and right translations $\lambda_a : y \mapsto a \cdot y : L \times L \to L$ and $\rho_a : y \mapsto y \cdot a : L \times L \to L$, $a \in L$, are bijections of L. The permutation group $Mult(L)$ generated by all left and right translations of the loop L is called the multiplication group of L and the stabilizer of $e \in L$ in the group $Mult(L)$ is called the inner mapping group $Inn(L)$ of L.

A loop L is called topological if L is a topological space and the operations $(x, y) \mapsto x \cdot y$, $(x, y) \mapsto x \backslash y$, $(x, y) \mapsto y/x : L \times L \to L$ are continuous. Let G_l be a connected Lie group, let H_l be a subgroup of G_l. We denote by $Co_{G_l}(H_l)$ the core of H_l in G_l i.e. the largest normal subgroup of G_l contained in H_l. A continuous section $\sigma : G_l/H_l \to G_l$ is called sharply transitive, if the set $\sigma(G_l/H_l)$ operates sharply transitively on G_l/H_l, which means that for any xH_l and yH_l there exists precisely one $z \in \sigma(G_l/H_l)$ with $zxH_l = yH_l$. Every connected topological loop L having a Lie group G_l as the group topologically generated by the left translations is obtained on a homogeneous space G_l/H_l, where H_l is a closed subgroup of G_l with $Co_{G_l}(H_l) = 1$ and $\sigma : G_l/H_l \to G_l$ is a continuous sharply transitive section with $\sigma(H_l) = 1 \in G_l$ such that the subset $\sigma(G_l/H_l)$ generates G_l. The multiplication of L on the manifold G_l/H_l is defined by $xH_l * yH_l = \sigma(xH_l)yH_l$

and the group G_l is the group topologically generated by the left translations of L. Moreover, the subgroup H_l is the stabilizer of the identity element $e \in L$ in the group G_l.

Let K be a group, let $S \leq K$, and let A and B be two left transversals to S in K. We say that A and B are S-connected if $a^{-1}b^{-1}ab \in S$ for every $a \in A$ and $b \in B$. The connection between multiplication groups of loops and transversals is given in [17], Theorem 4.1.

Lemma 3. *A group K is isomorphic to the multiplication group of a loop L if and only if there exists a subgroup S with $Co_K(S) = 1$ and S-connected transversals A and B satisfying $K = \langle A, B \rangle$.*

In this theorem the subgroup S is the inner mapping group of L and A and B are the sets of left and right translations of L, respectively. Hence Lemma 3 yields part a) of the following

Lemma 4. *Let L be a loop and $\Lambda(L)$ be the set of left translations of L. Let K be a group containing $\Lambda(L)$ and S be a subgroup of K with $Co_K(S) = 1$ such that $\Lambda(L)$ is a left transversal to S in K.*
a) The group K is isomorphic to the multiplication group $Mult(L)$ of L if and only if there is a left transversal T to S in K such that $\Lambda(L)$ and T are S-connected and $K = \langle \Lambda(L), T \rangle$.
b) Let L be a locally compact connected topological loop such that the multiplication group $Mult(L)$ of L is a Lie group. Then, $Mult(L)$ is homeomorphic to the topological product $L \times Inn(L)$ as well as to $\Lambda(L) \times Inn(L)$.

Proof. We prove assertion b). The group $Mult(L)$ acts transitively and effectively as topological transformation group on L and hence the map $m \mapsto m(1) : Mult(L) \to L$ is open and it induces a homeomorphism $Mult(L)/Inn(L) \approx L$ (cf. [21], 96.8, 96.9 (a)). As the map $x \mapsto \lambda_x : L \to \Lambda(L)$ is a homeomorphism the assertion follows. □

The following lemma is proved in [17], Lemma 2.6.

Lemma 5. *Let S be a proper subgroup of a simple group K and let A and B be S-connected transversals in K. Then S is maximal in K.*

Lemma 6. *There does not exist a connected topological loop L which is homeomorphic to a topological product of spaces having as a factor the 2-sphere or the projective plane.*

Proof. By [1] there does not exist a multiplication with identity on the sphere S^2. This yields the assertion. □

Lemma 7. *a) For any connected topological loop there exists a universal covering loop. This loop is simply connected.*
b) Let L be a 3-dimensional connected simply connected topological loop such that the group $Mult(L)$ is a Lie group. Then L is homeomorphic either to S^3 or to $\mathbb{R}^3$.

Proof. Assertion a) is proved in [12], IX.1. Assertion b) is shown in [10], p. 388. □

For our computation we often use the following assertion which was proved in [19], Proposition 18.16, p. 246.

Lemma 8. *Let L be a loop and let G_l and G_r the groups generated by the left translations and by the right translations of L, respectively. We have $G_l = G_r = Mult(L)$ if and only if for the stabilizers H_l, respectively H_r of $e \in L$ in G_l, respectively in G_r one has $H_l = H_r = Inn(L)$ and for all $x \in L$ the map $f(x) : y \mapsto \lambda_x^{-1}\lambda_y x : L \to L$ is an element of $Inn(L)$.*

A connected Lie group is simple if it has no non-trivial normal subgroup. A connected Lie group G is called quasi-simple if any normal subgroup of G is discrete and central in G. A connected loop L is quasi-simple if any normal subloop of L is discrete in L. According to [12], p. 216, all discrete normal subloops of a connected loop are central. Since for any normal subgroup N of the multiplication group $Mult(L)$ of L, the orbit $N(e)$ is a normal subloop of L (cf. [4], IV.1, p. 62) it follows that:

Lemma 9. *If L is a connected loop having a quasi-simple Lie group as its multiplication group $Mult(L)$, then L is quasi-simple.*

Proposition 10. *Let L be a connected topological loop homeomorphic to the sphere S^3 such that the group topologically generated by the left translations of L is a compact Lie group. Then L is either the group $Spin(\mathbb{R})$ or $SO_3(\mathbb{R})$.*

Proof. Using Proposition 2.4 in [13] and Ascoli's Theorem, in [12] IX.2.9 Theorem it is deduced that the loop L has a left invariant uniformity. Therefore by IX.3.14 Theorem in [12] the assertion follows. □

Theorem 11. *Let L be a connected topological loop such that the group G_l of L is the group $\left\{ \pm \begin{pmatrix} z & 0 \\ u & z^{-1} \end{pmatrix}, z, u \in \mathbb{C}, z \neq 0 \right\}$ and the stabilizer H_l of $e \in L$ in G_l is the subgroup $\left\{ \pm \begin{pmatrix} \exp(ib) & 0 \\ 0 & \exp(-ib) \end{pmatrix}, b \in \mathbb{R} \right\}$. Any such loop is isomorphic to a loop L_{t_k} which is determined by the following set*

$$\Lambda = \left\{ \pm \begin{pmatrix} k & 0 \\ l & k^{-1} \end{pmatrix} \begin{pmatrix} \exp(it(k)) & 0 \\ 0 & \exp(-it(k)) \end{pmatrix}, k > 0 \right\} \tag{1}$$

of the left translations of L_{t_k}, where $t(k) : \mathbb{R}_+ \to \mathbb{R}$ is a non-constant continuous function with $t(1) = 0$.

Proof. A continuous section $\sigma : G_l/H_l \to G_l$ may be given by

$$\begin{pmatrix} a & 0 \\ u & a^{-1} \end{pmatrix} H_l \mapsto \pm \begin{pmatrix} a & 0 \\ u & a^{-1} \end{pmatrix} \begin{pmatrix} \exp(it(a,u)) & 0 \\ 0 & \exp(-it(a,u)) \end{pmatrix},$$

where $a > 0$, $u \in \mathbb{C}$ and $t(a,u) : \mathbb{R}_+ \times \mathbb{C} \to \mathbb{R}$ is a continuous function with $t(1,0) = 0$. The set $\sigma(G_l/H_l)$ acts sharply transitively on G_l/H_l if and only if for given numbers $a_1, a_2 > 0$ and $z_1, z_2 \in \mathbb{C}$ there exist precisely one $k > 0$ and $l \in \mathbb{C}$ such that the matrix equation

$$\begin{pmatrix} k & 0 \\ l & k^{-1} \end{pmatrix} \begin{pmatrix} \exp(it(k,l)) & 0 \\ 0 & \exp(-it(k,l)) \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ z_1 & a_1^{-1} \end{pmatrix} =$$

$$\begin{pmatrix} a_2 & 0 \\ z_2 & a_2^{-1} \end{pmatrix} \begin{pmatrix} \exp(is) & 0 \\ 0 & \exp(-is) \end{pmatrix} \tag{2}$$

holds for a suitable real number s. Comparing the $(1,1)$-entries of the matrix on the left hand side and on the right hand side of (2) we get $0 < ka_1a_2^{-1} = \exp(i(s - t(k,l)))$. Hence we obtain $s = t(k,l)$ and $k = a_2a_1^{-1}$. Using this and comparing the $(2,1)$-entries of the matrix on the left hand side and on the right hand side of (2) we have $z_2a_1^{-1} = l + a_2^{-1}z_1e^{-2it(a_2a_1^{-1},l)}$. This equation has a unique solution for given $a_1, a_2 > 0$ and $z_1, z_2 \in \mathbb{C}$ precisely if for every $a_0 = a_2a_1^{-1} > 0$, $a_2 > 0$ and $z_1 \in \mathbb{C}$ the function $g : l \mapsto l + a_2^{-1}z_1e^{-2it(a_2a_1^{-1},l)}$ is bijective. This is the case if and only if the function $t(k,l) = t(k)$ does not depend on l. Hence every continuous function $t(k)$ with $t(1) = 0$ determines a loop $L_{t(k)}$ realized on G_l/H_l. This loop is proper if and only if the set $\sigma(G_l/H_l)$ generates the group G_l. This is the case precisely if the function t is not the constant function 0. Hence the set Λ given by (1) is the set of left translations of a loop $L_{t(k)}$. □

Using [5], Satz 1, p. 251 and [6], Section 5, p. 276, we obtain the following:

Lemma 12. *Any connected closed maximal subgroup of $PSU_3(\mathbb{C},1)$ is one of the following groups*
(1) H_1 is isomorpic to $Spin_3 \times SO_2(\mathbb{R})$,
(2) H_2 is isomorphic to the 5-dimensional solvable group NG, where

$$N = \left\{ \begin{pmatrix} 1 & zi & z \\ \bar{z}i & 1 + it - \frac{z\bar{z}}{2} & t + \frac{z\bar{z}}{2} \\ \bar{z} & t + \frac{z\bar{z}}{2} & 1 - it + \frac{z\bar{z}}{2} \end{pmatrix}; z \in \mathbb{C}, t \in \mathbb{R} \right\} \text{ and}$$

$$G = \left\{ \begin{pmatrix} e^{-ik} & 0 & 0 \\ 0 & \frac{1}{2}(e^{-u} + e^{u})e^{\frac{1}{2}ik} & \frac{1}{2}(e^{u} - e^{-u})ie^{\frac{1}{2}ik} \\ 0 & \frac{1}{2}(e^{-u} - e^{u})ie^{\frac{1}{2}ik} & \frac{1}{2}(e^{-u} + e^{u})e^{\frac{1}{2}ik} \end{pmatrix}; k, u \in \mathbb{R} \right\},$$

(3) H_3 is isomorphic to the group $SU_2(\mathbb{C},1) \times SO_2(\mathbb{R}) \cong SL_2(\mathbb{R}) \times SO_2(\mathbb{R})$,
(4) H_4 is isomorphic to the group $SO_0(2,1) \cong PSL_2(\mathbb{R})$.

Lemma 13. *There does not exist any connected topological proper loop L such that its multiplication group $Mult(L)$ is the group $PSU_3(\mathbb{C},1)$ and its inner mapping group $Inn(L)$ is either the subgroup H_3 or H_4 in Lemma 12.*

Proof. The factor spaces $SU_3(\mathbb{C},1)/H_3$ and $SU_3(\mathbb{C},1)/H_4$ are homeomorphic to a topological product of spaces having as a factor S^2. This is a contradiction to Lemma 6. □

Lemma 14. *There does not exist a connected topological loop L such that for the pair $(Mult(L), Inn(L))$ one has $(PSU_3(\mathbb{C},1), Spin_3(\mathbb{R}) \times SO_2(\mathbb{R}))$.*

Proof. The group $Inn(L)$ can be represented as the group of matrices

$$\left\{ g(a_1,a_2,b_1,b_2,e^{ic}) = \begin{pmatrix} a_1+ia_2 & (b_1+ib_2)e^{ic} & 0 \\ -b_1+ib_2 & (a_1-ia_2)e^{ic} & 0 \\ 0 & 0 & e^{-ic} \end{pmatrix}, \ a_1,a_2,b_1,b_2,c \in \mathbb{R}, a_1^2+a_2^2+b_1^2+b_2^2=1 \right\},$$

the elements of L can be parameterized by the products of the matrices

$$l(z,t,u) = \begin{pmatrix} 1 & zi & z \\ \bar{z}i & 1+it-\frac{z\bar{z}}{2} & t+\frac{z\bar{z}}{2} \\ \bar{z} & t+\frac{z\bar{z}}{2} & 1-it+\frac{z\bar{z}}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(e^{-u}+e^{u}) & \frac{1}{2}i(e^{u}-e^{-u}) \\ 0 & \frac{1}{2}i(e^{-u}-e^{u}) & \frac{1}{2}(e^{-u}+e^{u}) \end{pmatrix},$$

where $z \in \mathbb{C}$, $t,u \in \mathbb{R}$. Arbitrary left transversals to the group $Inn(L)$ in $Mult(L)$ are

$$A = \{l(z,t,u)g(a_1(z,t,u),a_2(z,t,u),b_1(z,t,u),b_2(z,t,u),e^{ic(z,t,u)}), z \in \mathbb{C}, t,u \in \mathbb{R}\}$$

and

$$B = \{l(n,v,k)g(r_1(n,v,k),r_2(n,v,k),q_1(n,v,k),q_2(n,v,k),e^{ip(n,v,k)}), n \in \mathbb{C}, v,k \in \mathbb{R}\}$$

such that the continuous functions $a_i(z,t,u)$, $b_i(z,t,u)$, $c(z,t,u)$, $r_i(n,v,k)$, $q_i(n,v,k)$, $p(n,v,k) : \mathbb{C} \times \mathbb{R}^2 \to \mathbb{R}$ satisfy the properties $a_1(0,0,0) = 1 = r_1(0,0,0)$, $a_2(0,0,0) = b_1(0,0,0) = b_2(0,0,0) = 0$, $r_2(0,0,0) = q_1(0,0,0) = q_2(0,0,0) = 0$, $a_1(z,t,u)^2 + a_2(z,t,u)^2 + b_1(z,t,u)^2 + b_2(z,t,u)^2 = 1 = r_1(n,v,k)^2 + r_2(n,v,k)^2 + q_1(n,v,k)^2 + q_2(n,v,k)^2$. By Lemma 3 the group $G = PSU_3(\mathbb{C},1)$ is isomorphic to the multiplication group $Mult(L)$ of L precisely if for all $a \in A$ and $b \in B$ one has $a^{-1}b^{-1}ab \in Inn(L)$ and the set $\{A, B\}$ generates the group G. The product $a^{-1}b^{-1}ab$ with

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+it & t \\ 0 & t & 1-it \end{pmatrix} \begin{pmatrix} a_1(t)+ia_2(t) & (b_1(t)+ib_2(t))e^{ic(t)} & 0 \\ -b_1(t)+ib_2(t) & (a_1(t)-ia_2(t))e^{ic(t)} & 0 \\ 0 & 0 & e^{-ic(t)} \end{pmatrix} \in A$$

and

$$b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(e^{-k}+e^{k}) & \frac{1}{2}i(e^{k}-e^{-k}) \\ 0 & \frac{1}{2}i(e^{-k}-e^{k}) & \frac{1}{2}(e^{-k}+e^{k}) \end{pmatrix} \begin{pmatrix} r_1(k)+ir_2(k) & (q_1(k)+iq_2(k))e^{ip(k)} & 0 \\ -q_1(k)+iq_2(k) & (r_1(k)-ir_2(k))e^{ip(k)} & 0 \\ 0 & 0 & e^{-ip(k)} \end{pmatrix} \in B$$

is contained in $Inn(L)$ precisely if there exist elements

$$m(t,k) = \begin{pmatrix} w_1(t,k)+iw_2(t,k) & (v_1(t,k)+iv_2(t,k))e^{id(t,k)} & 0 \\ -v_1(t,k)+iv_2(t,k) & (w_1(t,k)-iw_2(t,k))e^{id(t,k)} & 0 \\ 0 & 0 & e^{-id(t,k)} \end{pmatrix} \in Inn(L)$$

such that for all $t,k \in \mathbb{R}$ the matrix equation

$$ab = bam \tag{3}$$

is satisfied. Comparing the $(2,3)$-entries and the $(3,3)$-entries on both sides of (3) we get for all $t,k \in \mathbb{R}$ that

$$(1+it)(a_1(t)-ia_2(t))e^{ic(t)}i\left(\frac{e^k}{2}-\frac{e^{-k}}{2}\right)e^{-ip(k)} + te^{-ic(t)}\left(\frac{e^{-k}}{2}+\frac{e^k}{2}\right)e^{-ip(k)} =$$

$$\left[\left(\frac{e^{-k}}{2}+\frac{e^{k}}{2}\right)(r_1(k)-ir_2(k))e^{ip(k)}te^{-ic(t)}+i\left(\frac{e^{k}}{2}-\frac{e^{-k}}{2}\right)e^{-ip(k)}(1-it)e^{-ic(t)}\right]e^{-id(t,k)}, \quad (4)$$

$$t(a_1(t)-ia_2(t))ie^{ic(t)}\left(\frac{e^{k}}{2}-\frac{e^{-k}}{2}\right)e^{-ip(k)}+(1-it)e^{-ic(t)}\left(\frac{e^{-k}}{2}+\frac{e^{k}}{2}\right)e^{-ip(k)}=$$

$$\left[te^{-ic(t)}i\left(\frac{e^{-k}}{2}-\frac{e^{k}}{2}\right)(r_1(k)-ir_2(k))e^{ip(k)}+(1-it)e^{-ic(t)}\left(\frac{e^{-k}}{2}+\frac{e^{k}}{2}\right)e^{-ip(k)}\right]e^{-id(t,k)}. \quad (5)$$

Expressing $e^{-id(t,k)}$ from (4) and putting it into (5) we have

$$\left[(1+it)(a_1(t)-ia_2(t))e^{ic(t)}i\left(\frac{e^{k}}{2}-\frac{e^{-k}}{2}\right)e^{-ip(k)}+te^{-ic(t)}\left(\frac{e^{-k}}{2}+\frac{e^{k}}{2}\right)e^{-ip(k)}\right]\cdot$$

$$\left[te^{-ic(t)}i(r_1(k)-ir_2(k))\left(\frac{e^{-k}}{2}-\frac{e^{k}}{2}\right)e^{ip(k)}+(1-it)e^{-ic(t)}\left(\frac{e^{-k}}{2}+\frac{e^{k}}{2}\right)e^{-ip(k)}\right]=$$

$$t(a_1(t)-ia_2(t))ie^{ic(t)}\left(\frac{e^{k}}{2}-\frac{e^{-k}}{2}\right)e^{-ip(k)}+(1-it)e^{-ic(t)}\left(\frac{e^{-k}}{2}+\frac{e^{k}}{2}\right)e^{-ip(k)}.$$

$$\left[te^{-ic(t)}(r_1(k)-ir_2(k))\left(\frac{e^{-k}}{2}+\frac{e^{k}}{2}\right)e^{ip(k)}+(1-it)e^{-ic(t)}i\left(\frac{e^{k}}{2}-\frac{e^{-k}}{2}\right)e^{-ip(k)}\right] \quad (6)$$

for all $t,k\in\mathbb{R}$. As $1,t,t^2$ are independent variables of (6) one has

$$i\left(\frac{e^{2k}}{4}-\frac{e^{-2k}}{4}\right)\left[e^{-2ip(k)}\left((a_1(t)-ia_2(t))-e^{-2ic(t)}\right)\right]=0 \quad (7)$$

$$t(a_1(t)-ia_2(t))(-r_1(k)+ir_2(k))\left(\frac{e^{2k}-e^{-2k}}{4}\right)+te^{-2ic(t)}(r_1(k)-ir_2(k))\left(\frac{e^{2k}+e^{-2k}}{4}+\frac{1}{2}\right)=$$

$$t(a_1(t)-ia_2(t))e^{-2ip(k)}\left(\frac{e^{2k}+e^{-2k}}{4}-\frac{1}{2}\right)+te^{-2ic(t)-2ip(k)}\left(\frac{3e^{-2k}-e^{2k}}{4}+\frac{1}{2}\right), \quad (8)$$

$$it^2(a_1(t)-ia_2(t))(r_1(k)-ir_2(k))\left(\frac{e^{2k}-e^{-2k}}{4}\right)+it^2e^{-2ic(t)-2ip(k)}\left(\frac{e^{-2k}+1}{2}\right)=$$

$$it^2(a_1(t)-ia_2(t))e^{-2ip(k)}\left(\frac{1-e^{-2k}}{2}\right)+it^2e^{-2ic(t)}(r_1(k)-ir_2(k))\left(\frac{e^{2k}+e^{-2k}}{2}+1\right). \quad (9)$$

Equation (7) yields that $a_1(t)-ia_2(t)=e^{-2ic(t)}$, $b_1(t)=b_2(t)=0$. Putting this into (8) we obtain $r_1(k)-ir_2(k)=\frac{2e^{-2k-2ip(k)}}{e^{2k}+1}$. Applying these for equation (9) we get that $it^2(e^{2k}-2e^{-2k}-1)=0$ holds for all $t,k\in\mathbb{R}$. This contradiction proves the assertion. □

Lemma 15. *Let L be a 3-dimensional connected topological loop such that the group G_l of L is locally isomorphic to the group $PSL_2(\mathbb{C})$. Then G_l is a proper subgroup of $Mult(L)$.*

Proof. By Lemma 7 we may assume that L is simply connected. Since $\dim(L)=3$ the stabilizer H_l of $e\in L$ in G_l is locally isomorphic either to the the group $PSL_2(\mathbb{R})$ or to $SO_3(\mathbb{R})$. As $SL_2(\mathbb{C})/SL_2(\mathbb{R})$ is homeomorphic to $S^2\times\mathbb{R}$ and $SU_2(\mathbb{C})$ contains central elements $\neq 1$ of $SL_2(\mathbb{C})$ we have $G_l=PSL_2(\mathbb{R})$ and H_l is the subgroup $SO_3(\mathbb{R})$. Suppose that $G_l=Mult(L)$. Then H_l coincides with the group $Inn(L)\cong SO_3(\mathbb{R})$ (cf. Lemma 8) and it can be chosen as the group of matrices $\left\{ \pm \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},\ a,b \in \mathbb{C}, a\bar{a}+b\bar{b}=1 \right\}$. The elements of L can be parameterized by the matrices

$$\mu_r(u,z) = \begin{pmatrix} \exp[(ri-1)u] & 0 \\ z & \exp[(1-ri)u] \end{pmatrix},$$

where $u \in \mathbb{R}$, $z \in \mathbb{C}$ and $r \in \mathbb{R}$ is fixed. For each $r \in \mathbb{R}$ the continuous section $\sigma_r : L \to G_l$ can be written as

$$\begin{aligned} \sigma_r : \mu_r(u,z) &\mapsto \pm\mu_r(u,z) \begin{pmatrix} a_1(u,z)+ia_2(u,z) & b_1(u,z)+ib_2(u,z) \\ -b_1(u,z)+ib_2(u,z) & a_1(u,z)-ia_2(u,z) \end{pmatrix} \\ &= \pm\mu_r(u,z) g(a_1(u,z), a_2(u,z), b_1(u,z), b_2(u,z)), \end{aligned} \tag{10}$$

such that $a_1(u,z), a_2(u,z), b_1(u,z), b_2(u,z) : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ are continuous functions with $a_1(0,0)=1, a_2(0,0)=b_1(0,0)=b_2(0,0)=0$ and for all $u \in \mathbb{R}$, $z \in \mathbb{C}$ one has $a_1(u,z)^2+a_2(u,z)^2+b_1(u,z)^2+b_2(u,z)^2=1$. For each $r \in \mathbb{R}$ the loop multiplication $\circ$ can be expressed by the matrix multiplication as

$$\mu_r(u_1,z_1) \circ \mu_r(u_2,z_2) = \mu_r(u_1,z_1)\mu_r(u',z'), \tag{11}$$

where $\mu_r(u',z')$ is given by the relation

$$g(a_1(u,z), a_2(u,z), b_1(u,z), b_2(u,z))\mu_r(u_2,z_2) = \pm\mu_r(u',z') \begin{pmatrix} c_1+ic_2 & d_1+id_2 \\ -d_1+id_2 & c_1-ic_2 \end{pmatrix}$$

with suitable real numbers c_1, c_2, d_1, d_2 such that $c_1^2+c_2^2+d_1^2+d_2^2=1$. The matrix $\mu_r(u',z')$ is well defined because the covering map $SL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$ is an isomorphism on the 3-dimensional solvable subgroup of the given Iwasawa decomposition of $SL_2(\mathbb{C})$. Multiplying the matrices on both sides of (12) and using the relation $(c_1+ic_2)(c_1-ic_2)-(d_1+id_2)(-d_1+id_2)=1$ we obtain the equation

$$\begin{aligned} \exp(-2u') = {} & [\exp(2u_2)+z_2\overline{z_2}-\exp(-2u_2)](b_1(u_1,z_1)^2+b_2(u_1,z_1)^2)+\exp(-2u_2)+ \\ & \overline{z_2}\exp[(ri-1)u_2](a_1(u_1,z_1)+ia_2(u_1,z_1))(b_1(u_1,z_1)-ib_2(u_1,z_1))+ \\ & z_2\exp[(-ri-1)u_2](a_1(u_1,z_1)-ia_2(u_1,z_1))(b_1(u_1,z_1)+ib_2(u_1,z_1)). \end{aligned} \tag{12}$$

Since $G_l = Mult(L)$ it follows from Lemma 8, that the mapping $f(\mu_r(u,z)) : \mu_r(x,y) \mapsto \lambda^{-1}_{\mu_r(u,z)}\lambda_{\mu_r(x,y)}\mu_r(u,z) : L \to L$ is an element of the stabilizer H_l. Hence there exists a matrix $E = g(e_1(u,z), e_2(u,z), g_1(u,z), g_2(u,z)) \in H_l$ with suitable functions $e_1(u,z), e_2(u,z), g_1(u,z), g_2(u,z) : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ such that $e_1^2(u,z)+e_2^2(u,z)+g_1^2(u,z)+g_2^2(u,z)=1$ satisfying

$$\lambda^{-1}_{\mu_r(u,z)}\lambda_{\mu_r(x,y)}\mu_r(u,z) = E\mu_r(x,y)(\pm\rho),$$

where $\rho \in H_l$. This condition is equivalent to

$$\mu_r(u,z)^{-1}\mu_r(x,y)g(a_1(x,y), a_2(x,y), b_1(x,y), b_2(x,y))\mu_r(u,z) =$$

$$g(a_1(u,z),a_2(u,z),b_1(u,z),b_2(u,z))E\mu_r(x,y)(\pm\rho). \tag{13}$$

Applying relation (12) to both sides of (13) we see that there exist $\mu_r(u',z')$ and $\mu_r(x',y')$ with $u',x'\in\mathbb{R}$ such that equation (13) obtain the form

$$\mu_r(x-u,-z\exp[(ri-1)x]+y\exp[(ri-1)u])\mu_r(u',z')=\mu_r(x',y').$$

Then one has $x'=u'-u+x$. Applying relation (12) we get

$$\exp(-2u')=[\exp(2u)+z\bar{z}-\exp(-2u)](b_1(x,y)^2+b_2(x,y)^2)+\exp(-2u)+$$
$$\bar{z}\exp[(ri-1)u](a_1(x,y)+ia_2(x,y))(b_1(x,y)-ib_2(x,y))+z\exp[(-ri-1)u](a_1(x,y)-ia_2(x,y))(b_1(x,y)+ib_2(x,y)) \tag{14}$$

and

$$\exp(-2x')=[\exp(2x)+y\bar{y}-\exp(-2x)](l_1(u,z)^2+l_2(u,z)^2)+\exp(-2x)+$$
$$\bar{y}\exp[(ri-1)x](k_1(u,z)+ik_2(u,z))(l_1(u,z)-il_2(u,z))+y\exp[(-ri-1)x](k_1(u,z)-ik_2(u,z))(l_1(u,z)+il_2(u,z)), \tag{15}$$

where for all $u\in\mathbb{R}$, $z\in\mathbb{C}$ the continuous functions $k_1(u,z)$, $k_2(u,z)$, $l_1(u,z)$, $l_2(u,z):\mathbb{R}\times\mathbb{C}\to\mathbb{R}$ are given by

$$g(k_1(u,z),k_2(u,z),l_1(u,z),l_2(u,z))=g(a_1(u,z),a_2(u,z),b_1(u,z),b_2(u,z))E.$$

As $\exp(2x')=\exp(2(u'-u+x))$ we obtain

$$[\exp(4x)+y\bar{y}\exp(2x)-1](l_1(u,z)^2+l_2(u,z)^2)+$$
$$\bar{y}\exp[(ri+1)x](k_1(u,z)+ik_2(u,z))(l_1(u,z)-il_2(u,z))+$$
$$y\exp[(-ri+1)x](k_1(u,z)-ik_2(u,z))(l_1(u,z)+il_2(u,z))=$$
$$[\exp(4u)+z\bar{z}\exp(2u)-1](b_1(x,y)^2+b_2(x,y)^2)+$$
$$\bar{z}\exp[(ri+1)u](a_1(x,y)+ia_2(x,y))(b_1(x,y)-ib_2(x,y))+$$
$$z\exp[(-ri+1)u](a_1(x,y)-ia_2(x,y))(b_1(x,y)+ib_2(x,y)).$$

For fixed $u\in\mathbb{R}$, $z\in\mathbb{C}$ the right hand side of (16) is a bounded function of $x\in\mathbb{R}$ and $y\in\mathbb{C}$ so is the left hand side. Hence it follows that $l_1(u,z)=l_2(u,z)=0$ for all $u\in\mathbb{R}$, $z\in\mathbb{C}$. Therefore the right hand side of (16) is constant 0. From this we obtain that $b_1(x,y)=b_2(x,y)=0$ for all $x\in\mathbb{R}$ and $y\in\mathbb{C}$. Hence the image of the section $\sigma_r:L\to G_l$ in (10) has the form $\sigma_r(L)=$

$$\left\{\pm\begin{pmatrix}\exp[(ri-1)u] & 0\\ z & \exp[(1-ri)u]\end{pmatrix}\begin{pmatrix}a_1(u,z)+ia_2(u,z) & 0\\ 0 & a_1(u,z)-ia_2(u,z)\end{pmatrix}, u\in\mathbb{R}, z\in\mathbb{C}\right\}$$

such that $a_1(u,z),a_2(u,z):\mathbb{R}\times\mathbb{C}\to\mathbb{R}$ are continuous functions with $a_1(0,0)=1, a_2(0,0)=0$ and $a_1(u,z)^2+a_2(u,z)^2=1$ for all $u\in\mathbb{R}$, $z\in\mathbb{C}$. Since the image $\sigma_r(L)$ does not generate the group $PSL_2(\mathbb{C})$ we have a contradiction and the assertion is proved. □

3. Proofs of the theorems

Proof of Theorem 2. By Theorem 18.18 in [19] and Theorem 1 in [9] if there exists a connected topological proper loop L having a quasi-simple

Lie group G as its multiplication group, then L has dimension at least 3. According to Lemma 5 the Lie algebra $\mathbf{inn(L)}$ of the inner mapping group $Inn(L)$ of L is a maximal subalgebra $\mathbf{h}$ of the Lie algebra $\mathbf{g}$ of G with the property $\dim(\mathbf{g}) - \dim(\mathbf{h}) \geq 3$. By Lemma 7 we may assume that L is simply connected, furthermore if $\dim(L) = 3$, then L is homeomorphic either to the sphere S^3 or to $\mathbb{R}^3$. According to Proposition 3.2 in [10] we have the following possibilities: If L is homeomorphic to the 3-sphere S^3, then $Mult(L)$ with $\dim(Mult(L)) \leq 8$ is either $SL_2(\mathbb{C})$ or $PSU_3(\mathbb{C}, 1)$ or the universal covering of $SL_3(\mathbb{R})$. If L is homeomorphic to $\mathbb{R}^3$, then $Mult(L)$ is the group $PSL_2(\mathbb{C})$.

I) Let G be locally isomorphic to the group $PSL_2(\mathbb{C})$.

We may choose as basis of $\mathbf{g} = \mathfrak{sl}_2(\mathbb{C})$ the set $\{e_1, e_2, e_3, \mathrm{i}e_1, \mathrm{i}e_2, \mathrm{i}e_3\}$, where $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ form a real basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ with the Lie algebra multiplication: $[e_1, e_2] = 2e_3$, $[e_1, e_3] = 2e_2$, $[e_3, e_2] = 2e_1$. Every maximal Lie algebra $\mathbf{h}$ of $\mathbf{g} = \mathfrak{sl}_2(\mathbb{C})$ with $\dim(\mathbf{g}) - \dim(\mathbf{h}) \geq 3$ has (up to conjugation) one of the following forms (cf. Theorem 15 in [16], p. 129, or Corollary 2.16 in [3], p. 277):

$$\mathbf{h_1} \cong \mathfrak{sl}_2(\mathbb{R}) = \langle \mathrm{i}e_1, e_2, \mathrm{i}e_3 \rangle, \quad \mathbf{h_2} \cong \mathfrak{so}_3(\mathbb{R}) = \langle e_3, \mathrm{i}e_1, \mathrm{i}e_2 \rangle.$$

Since $SL_2(\mathbb{C})/SL_2(\mathbb{R})$ is homeomorphic to $S^2 \times \mathbb{R}$ it follows that $\mathbf{inn(L)}$ coincides with $\mathbf{h_2} = \mathfrak{so}_3(\mathbb{R})$ (cf. Lemma 6). As $Co_{Mult(L)}(Inn(L)) = 1$ and $SU_2(\mathbb{C})$ contains central elements $\neq 1$ of $SL_2(\mathbb{C})$, we get for the pair $(Mult(L), Inn(L)) = (PSL_2(\mathbb{C}), SO_3(\mathbb{R}))$ and L is homeomorphic to $\mathbb{R}^3$. Since L is a proper loop the group G_l of L is a subgroup of $Mult(L)$ with dimension ≥ 4. By Corollary 2.16 of [3], p. 277, G_l is conjugate either to the simply connected Lie group $K = \left\{ \pm \begin{pmatrix} z & 0 \\ u & z^{-1} \end{pmatrix}, z, u \in \mathbb{C}, z \neq 0 \right\}$ or to $PSL_2(\mathbb{C})$.

In the first case every loop is isomorphic to a loop $L_{t(k)}$ given by Theorem 11 and the set Λ of the left translations of $L_{t(k)}$ is determined by (1). An arbitrary left transversal T_r of the group $Inn(L) = SO_3(\mathbb{R})$ in the group $Mult(L) = PSL_2(\mathbb{C})$ has the form (10), where r is an arbitrary fixed real number. By Lemma 4 a) the group $PSL_2(\mathbb{C})$ is isomorphic to the multiplication group $Mult(L_{t(k)})$ of the loop $L_{t(k)}$ precisely if the set $\{a^{-1}b^{-1}ab; a \in \Lambda, b \in T_r\}$ is contained in $SO_3(\mathbb{R})$ and the set $\{\Lambda, T_r\}$ generates the group $PSL_2(\mathbb{C})$. The products $a^{-1}b^{-1}ab$ with $a \in \left\{ \begin{pmatrix} 1 & 0 \\ l_2 i & 1 \end{pmatrix}, l_2 \in \mathbb{R} \right\}$, $b \in T_r$ are elements of $SO_3(\mathbb{R})$ if the equations

$$2\cos(2ru)(b_2(u,z)a_1(u,z) + b_1(u,z)a_2(u,z)) + 2\sin(2ru)(b_1(u,z)a_1(u,z) - b_2(u,z)a_2(u,z)) =$$

$$l_2((b_2(u,z)^2 - b_1(u,z)^2)\cos(2ru) + 2b_1(u,z)b_2(u,z)\sin(2ru)), \tag{16}$$

$$(b_1(u,z)^2 - b_2(u,z)^2)\sin(2ru) = -2b_1(u,z)b_2(u,z)\cos(2ru) \tag{17}$$

are satisfied for all $l_2, u \in \mathbb{R}$, $z \in \mathbb{C}$. As for all $l_2 \neq 0$ equation (16) holds it follows that

$$(b_2(u,z)^2 - b_1(u,z)^2)\cos(2ru) = -2b_1(u,z)b_2(u,z)\sin(2ru). \tag{18}$$

Multiplying equation (17) with $\cos(2ru)$, equation (18) with $\sin(2ru)$ and adding the obtained equations we get $0 = 2b_1(u,z)b_2(u,z)$. Putting this into equations (17) and (18) it follows that $b_1(u,z) = b_2(u,z) = 0$ for all $u \in \mathbb{R}$, $z \in \mathbb{C}$. Hence for all $r \in \mathbb{R}$ the set T_r has the form $T_r =$

$$\left\{ \pm \begin{pmatrix} \exp[(ri-1)u] & 0 \\ z & \exp[(1-ri)u] \end{pmatrix} \begin{pmatrix} a_1(u,z) + ia_2(u,z) & 0 \\ 0 & a_1(u,z) - ia_2(u,z) \end{pmatrix}, u \in \mathbb{R}, z \in \mathbb{C} \right\}, \tag{19}$$

with the continuous functions $a_1(u,z)$, $a_2(u,z) : (\mathbb{R}, \mathbb{C}) \to \mathbb{R}$, $a_2(0,0) = 0$, $a_1(0,0) = 1$ and $a_1(u,z)^2 + a_2(u,z)^2 = 1$. But then the set $\{\Lambda, T_r\}$ does not generate the group $PSL_2(\mathbb{C})$. Hence the case $(Mult(L), G_l) = (PSL_2(\mathbb{C}), K)$ cannot occur. Lemma 15 excludes the case $Mult(L) = G_l = PSL_2(\mathbb{C})$. Therefore a Lie group locally isomorphic to $PSL_2(\mathbb{C})$ is not the multiplication group of a connected topological loop.

II. Let G be locally isomorphic to the group $SL_3(\mathbb{R})$.
The Lie algebra $\mathbf{g} = \mathfrak{sl}_3(\mathbb{R})$ is isomorphic to the Lie algebra of matrices

$$(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + \lambda_5 e_5 + \lambda_6 e_6 + \lambda_7 e_7 + \lambda_8 e_8) \mapsto$$

$$\begin{pmatrix} \lambda_5 & \lambda_6 & \lambda_2 \\ \lambda_7 & \lambda_8 & \lambda_1 \\ \lambda_4 & \lambda_3 & -\lambda_5 - \lambda_8 \end{pmatrix}; \lambda_i \in \mathbb{R}, i = 1, \cdots, 8.$$

If there exists a connected topological proper loop L with $\dim(L) \geq 3$ such that the group $Mult(L)$ of L is locally isomorphic to $SL_3(\mathbb{R})$, then the Lie algebra $\mathbf{inn(L)}$ of the inner mapping group $Inn(L)$ of L is a maximal subalgebra $\mathbf{h}$ of $\mathbf{g} = \mathfrak{sl}_3(\mathbb{R})$ with the property $\dim(\mathbf{g}) - \dim(\mathbf{h}) \geq 3$ (cf. Lemma 5). Using the classification in [16], pp. 288-289 and [15], p. 384, we get that $\mathbf{h}$ has (up to conjugation) one of the following forms:

$$\mathbf{h}_1 = \langle e_1 - e_3, e_2 - e_4, e_7 - e_6 \rangle \cong \mathfrak{so}_3(\mathbb{R}), \ \mathbf{h}_2 = \langle e_1 - e_3, e_2 + e_4, e_7 + e_6 \rangle \cong \mathfrak{sl}_2(\mathbb{R}).$$

Let $\widetilde{SL_3(\mathbb{R})}$ be the universal covering group of $SL_3(\mathbb{R})$. The factor space $\widetilde{SL_3(\mathbb{R})}/SL_2(\mathbb{R})$ is homeomorphic to $S^2 \times \mathbb{R}^3$. Hence $\mathbf{inn(L)}$ is the Lie algebra $\mathbf{h_1} = \mathfrak{so}_3(\mathbb{R})$ (cf. Lemma 6). Since every maximal compact subgroup K of $\widetilde{SL_3(\mathbb{R})}$ contains central elements $\neq 1$ of $\widetilde{SL_3(\mathbb{R})}$ and $Co_{Mult(L)}Inn(L) = 1$ we obtain $(Mult(L), Inn(L)) = (SL_3(\mathbb{R}), SO_3(\mathbb{R}))$ and L is homeomorphic to $\mathbb{R}^5$. Then the inner mapping group $Inn(L)$ of L is isomorphic to $SO_3(\mathbb{R})$. Every element of $SO_3(\mathbb{R})$ can be represented by the matrix

$$g(t,u,z) := \begin{pmatrix} \cos t \ \cos u - \sin t \ \cos z \ \sin u & \cos t \ \sin u + \sin t \ \cos z \ \cos u & \sin t \ \sin z \\ -\sin t \ \cos u - \cos t \ \cos z \ \sin u & -\sin t \ \sin u + \cos t \ \cos z \ \cos u & \cos t \ \sin z \\ \sin z \ \sin u & -\sin z \ \cos u & \cos z \end{pmatrix}, \tag{20}$$

where $t, u \in [0, 2\pi]$ and $z \in [0, \pi]$. The elements of L can be written as $\mu(a_i) = \begin{pmatrix} a_1 & a_3 & a_5 \\ 0 & a_2 & a_4 \\ 0 & 0 & a_1^{-1}a_2^{-1} \end{pmatrix}$, $a_1, a_2 > 0$, $a_3, a_4, a_5 \in \mathbb{R}$. Arbitrary left transversals to the group $Inn(L)$ in $Mult(L)$ are

$$A = \{\mu(a_i) g(t(a_i), u(a_i), z(a_i)), a_1, a_2 > 0, a_3, a_4, a_5 \in \mathbb{R}\} \text{ and}$$

$$B = \{\mu(b_i)g(r(b_i), p(b_i), v(b_i)), b_1, b_2 > 0, b_3, b_4, b_5 \in \mathbb{R}\},$$

where $t(a_i)$, $u(a_i)$, $r(b_i)$, $p(b_i) : \mathbb{R}_+^2 \times \mathbb{R}^3 \to [0, 2\pi]$, $z(a_i)$, $v(b_i) : \mathbb{R}_+^2 \times \mathbb{R}^3 \to [0, \pi]$ are continuous functions such that $t(1,1,0,0,0) = u(1,1,0,0,0) = z(1,1,0,0,0) = 0 = r(1,1,0,0,0) = p(1,1,0,0,0) = v(1,1,0,0,0)$. The group $G = SL_3(\mathbb{R})$ is isomorphic to the multiplication group $Mult(L)$ of L precisely if for all $a \in A$ and $b \in B$ one has $a^{-1}b^{-1}ab \in Inn(L)$ and the set $\{A, B\}$ generates the group G (cf. Lemma 3). The product $a^{-1}b^{-1}ab$ with

$$a = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} g(t(a), u(a), z(a)) \in A \text{ and } b = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} g(r(b), p(b), v(b)) \in B,$$

where $t(a) := t(a,1,0,0,0)$, $u(a) := u(a,1,0,0,0)$, $z(a) := z(a,1,0,0,0)$, $r(b) := r(1,1,0,0,b)$, $p(b) := p(1,1,0,0,b)$, $v(b) := v(1,1,0,0,b)$, $a > 0$, $b \in \mathbb{R}$, is contained in $Inn(L)$ precisely if there exist elements

$$m(a,b) = g(w_1(a,b), w_2(a,b), w_3(a,b)) \in Inn(L)$$

with $w_i(a,b) := w_i(a,1,0,0,b)$, $i = 1,2,3$, such that for all $a > 0$ and $b \in \mathbb{R}$ the matrix equation

$$ab = bam \tag{21}$$

holds. Comparing the $(3,1)$- and the $(3,2)$-entries on both sides of (21) we obtain in both cases that only one term contains b. As b is independent variable in both expressions its coefficient must be 0. Hence we get $\frac{b \sin v(b) \sin p(b)}{a} = \frac{b \sin v(b) \cos p(b)}{a} = 0$ for all $a > 0$ and $b \in \mathbb{R}$, or equivalently $\sin v(b) \sin p(b) = \sin v(b) \cos p(b) = 0$ for all $b \in \mathbb{R}$. Since $\sin p(b) = \cos p(b) \neq 0$ the function $v(b)$ must be the constant function 0. Applying this at the comparison of the $(3,1)$-entries on both sides of (21) we have for all $a > 0$, $b \in \mathbb{R}$ that

$$\begin{gathered}\sin z(a) \sin u(a)(\cos r(b) \cos p(b) - \sin r(b) \sin p(b)) + \sin z(a) \cos u(a)(\sin r(b) \cos p(b) + \cos r(b) \sin p(b)) = \\ \sin z(a) \sin u(a)(\cos w_1(a,b) \cos w_2(a,b) - \sin w_1(a,b) \cos w_3(a,b) \sin w_2(a,b)) + \\ \sin z(a) \cos u(a)(\sin w_1(a,b) \cos w_2(a,b) + \cos w_1(a,b) \cos w_3(a,b) \sin w_2(a,b)) + \cos z(a) \sin w_3(a,b) \sin w_2(a,b).\end{gathered} \tag{22}$$

Comparing the $(3,2)$-entries on both sides of (21) one gets for all $a > 0$, $b \in \mathbb{R}$ that

$$\begin{gathered}\sin z(a) \sin u(a)(\cos r(b) \sin p(b) + \sin r(b) \cos p(b)) - \sin z(a) \cos u(a)(\cos r(b) \cos p(b) - \sin r(b) \sin p(b)) = \\ \sin z(a) \sin u(a)(\cos w_1(a,b) \sin w_2(a,b) + \sin w_1(a,b) \cos w_3(a,b) \cos w_2(a,b)) - \\ \sin z(a) \cos u(a)(\cos w_1(a,b) \cos w_3(a,b) \cos w_2(a,b) - \sin w_1(a,b) \sin w_2(a,b)) - \cos z(a) \sin w_3(a,b) \cos w_2(a,b).\end{gathered} \tag{23}$$

As $\sin z(a) \sin u(a)$, $\sin z(a) \cos u(a)$, $\cos z(a)$ are linearly independent one has from (22) and from (23) the equation

$$\sin w_3(a,b) \sin w_2(a,b) = 0 = \sin w_3(a,b) \cos w_2(a,b).$$

This yields $w_3(a,b) = 0$ for all $a > 0$, $b \in \mathbb{R}$. Putting this into (21) and comparing the $(1,3)$-entries on both sides of (21) one has

$$ab(\cos t(a) \cos u(a) - \sin t(a) \cos z(a) \sin u(a)) =$$

$$a \sin t(a) \sin z(a)(\cos r(b) \cos p(b) - \sin r(b) \sin p(b) - 1)+$$

$$\cos t(a) \sin z(a)(\cos r(b) \sin p(b) + \sin r(b) \cos p(b)) + \frac{b}{a} \cos z(a) \tag{24}$$

for all $a > 0$, $b \in \mathbb{R}$. As ab, a, $\frac{b}{a}$ are independent variables their coefficients are equal on both sides of (24). Therefore for all $a > 0$, $b \in \mathbb{R}$ one has $\frac{b}{a} \cos z(a) = 0$. This is the case precisely if $\cos z(a) = 0$ for all $a > 0$, or equivalently $z(a) := z(a, 1, 0, 0, 0) \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$. But $z(1, 1, 0, 0, 0) = 0$ which is a contradiction. Hence there does not exist any connected topological loop having a Lie group locally isomorphic to $SL_3(\mathbb{R})$ as its multiplication group.

III. Finally we consider the Lie groups G locally isomorphic to $PSU_3(\mathbb{C}, 1)$. The Lie algebra $\mathbf{g} = \mathfrak{su}_3(\mathbb{C}, 1)$ can be treated as the Lie algebra of matrices

$$(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + \lambda_5 e_5 + \lambda_6 e_6 + \lambda_7 e_7 + \lambda_8 e_8) \mapsto$$

$$\begin{pmatrix} -\lambda_1 \mathrm{i} & -\lambda_2 - \lambda_3 \mathrm{i} & \lambda_4 + \lambda_5 \mathrm{i} \\ \lambda_2 - \lambda_3 \mathrm{i} & \lambda_1 \mathrm{i} + \lambda_6 \mathrm{i} & \lambda_7 + \lambda_8 \mathrm{i} \\ \lambda_4 - \lambda_5 \mathrm{i} & \lambda_7 - \lambda_8 \mathrm{i} & -\lambda_6 \mathrm{i} \end{pmatrix}; \lambda_j \in \mathbb{R}, j = 1, \cdots, 8.$$

The maximal subgroups of $PSU_3(\mathbb{C}, 1)$ are determined in Lemma 12. The Lie algebras $\mathbf{h}_i$, $i = 1, 2, 3, 4$ of the Lie groups H_i listed there are given by

$$\mathbf{h}_1 = \langle e_1, e_2, e_3, e_6 \rangle, \ \mathbf{h}_2 = \langle e_1 - \frac{1}{2} e_6, e_8, e_4 - e_3, e_5 + e_2, e_6 + e_7 \rangle, \tag{25}$$

$$\mathbf{h}_3 = \langle e_1, e_6, e_7, e_8 \rangle, \ \mathbf{h}_4 = \langle e_2, e_4, e_7 \rangle. \tag{26}$$

According to Lemmata 13 and 14 it remains to treat the case $Mult(L) = PSU_3(\mathbb{C}, 1)$ and $Inn(L) = H_2$, where H_2 is given by Lemma 12. Then L is homeomorphic to S^3. If the group G_l is a proper subgroup of $Mult(L)$, then we have $\dim(G_l) = 4$ or 5. The elements of L can be represented as elements of the group $Spin_3(\mathbb{R})$. Then G_l contains a subgroup isomorphic to $Spin_3(\mathbb{R})$ and hence it is the group H_1 given in Lemma 12 (1). The stabilizer H_l of $e \in L$ in G_l is the intersection $G_l \cap Inn(L)$, which is a group isomorphic to $SO_2(\mathbb{R})$. By Proposition 10 this case cannot occur. Hence we have to concentrate us only to the case $Mult(L) = G_l = PSU_3(\mathbb{C}, 1)$. Then the stabilizer H_l of $e \in L$ in G_l coincides with the group $Inn(L) = H_2$ given in Lemma 12 (2) (cf. Lemma 8). Hence every element of H_l can be written as $g(z_1, z_2, t, u, k) =$

$$\begin{pmatrix} 1 & iz_1 - z_2 & z_1 + iz_2 \\ z_1 i + z_2 & 1 + it - \frac{z_1^2 + z_2^2}{2} & t + \frac{z_1^2 + z_2^2}{2} \\ z_1 - iz_2 & t + \frac{z_1^2 + z_2^2}{2} & 1 - it + \frac{z_1^2 + z_2^2}{2} \end{pmatrix} \begin{pmatrix} e^{-ik} & 0 & 0 \\ 0 & \frac{1}{2}\left(e^{-u} + e^{u}\right) e^{\frac{1}{2}ik} & \frac{1}{2}\left(e^{u} - e^{-u}\right) ie^{\frac{1}{2}ik} \\ 0 & \frac{1}{2}\left(e^{-u} - e^{u}\right) ie^{\frac{1}{2}ik} & \frac{1}{2}\left(e^{-u} + e^{u}\right) e^{\frac{1}{2}ik} \end{pmatrix},$$

where $z_1, z_2, t, u, k \in \mathbb{R}$. Moreover, the elements of L can be parameterized by the matrices $\mu(a, b) = \begin{pmatrix} a_1 + ia_2 & b_1 + ib_2 & 0 \\ -b_1 + ib_2 & a_1 - ia_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where $a = a_1 + ia_2$,

$b = b_1 + ib_2$ with $a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1$. A continuous section $\sigma : L \to G_l$ can be given by

$$\sigma : \mu(a,b) \mapsto \mu(a,b) g(z_1(a,b), z_2(a,b), t(a,b), u(a,b), k(a,b)) =$$

$$\mu(a,b) \begin{pmatrix} 1 & iz_1(a,b) - z_2(a,b) & z_1(a,b) + iz_2(a,b) \\ z_1(a,b)i + z_2(a,b) & 1 + it(a,b) - \frac{z_1(a,b)^2 + z_2(a,b)^2}{2} & t(a,b) + \frac{z_1(a,b)^2 + z_2(a,b)^2}{2} \\ z_1(a,b) - iz_2(a,b) & t(a,b) + \frac{z_1(a,b)^2 + z_2(a,b)^2}{2} & 1 - it(a,b) + \frac{z_1(a,b)^2 + z_2(a,b)^2}{2} \end{pmatrix} \cdot$$

$$\begin{pmatrix} e^{-ik(a,b)} & 0 & 0 \\ 0 & \frac{1}{2}\left(e^{-u(a,b)} + e^{u(a,b)}\right) e^{\frac{1}{2}ik(a,b)} & \frac{1}{2}\left(e^{u(a,b)} - e^{-u(a,b)}\right) ie^{\frac{1}{2}ik(a,b)} \\ 0 & \frac{1}{2}\left(e^{-u(a,b)} - e^{u(a,b)}\right) ie^{\frac{1}{2}ik(a,b)} & \frac{1}{2}\left(e^{-u(a,b)} + e^{u(a,b)}\right) e^{\frac{1}{2}ik(a,b)} \end{pmatrix} \tag{27}$$

such that the continuous functions $z_1(a,b)$, $z_2(a,b)$, $t(a,b)$, $u(a,b)$, $k(a,b)$: $Spin_3(\mathbb{R}) \to \mathbb{R}$ satisfy $z_1(1,0) = z_2(1,0) = t(1,0) = k(1,0) = u(1,0) = 0$.

The condition $G_l = Mult(L)$ is satisfied if and only if for all $\mu(c,d) \in L$ the map $f(\mu(c,d)) : \mu(a,b) \mapsto \lambda_{\mu(c,d)}^{-1} \lambda_{\mu(a,b)} \mu(c,d) : L \to L$ is an element of H_l (cf. Lemma 8). For $a = a_1 + ia_2 = 0$ this means that there are continuous functions $x_i(c,d), y(c,d), s(c,d), h(c,d) : Spin_3(\mathbb{R}) \to \mathbb{R}$, $i = 1,2$, with $c\bar{c} + d\bar{d} = 1$, $x_i(1,0) = y(1,0) = s(1,0) = h(1,0) = 0$ such that the matrix equation

$$\mu(c,d)^{-1} \sigma(\mu(0,b)) \mu(c,d) = g(z_1(c,d), z_2(c,d), t(c,d), u(c,d), k(c,d)) \cdot$$
$$g(x_1(c,d), x_2(c,d), y(c,d), s(c,d), h(c,d)) \mu(0,b) g(w_1, w_2, m, p, q) \tag{28}$$

holds for all $b, c, d \in \mathbb{C}$, with $b\bar{b} = 1 = c\bar{c} + d\bar{d}$ for a suitable element $g(w_1, w_2, m, p, q) \in H_l$. Let $g(p_1(c,d), p_2(c,d), n(c,d), g(c,d), j(c,d))$ be the product

$$g(z_1(c,d), z_2(c,d), t(c,d), u(c,d), k(c,d)) \cdot g(x_1(c,d), x_2(c,d), y(c,d), s(c,d), h(c,d))$$

with continuous functions $p_i(c,d), n(c,d), g(c,d), j(c,d) : Spin_3(\mathbb{R}) \to \mathbb{R}$, $i = 1,2$, with $p_i(1,0) = n(1,0) = g(1,0) = j(1,0) = 0$. The $(1,1)$-entry on the left hand side of the matrix equation (28) is

$$(-b_1 + ib_2)\{(-d_1 - id_2)(c_1 + ic_2)e^{-ik(b)} + (d_1^2 + d_2^2)(z_1(b)i - z_2(b))e^{\frac{1}{2}ik(b) - u(b)}\} +$$
$$(b_1 + ib_2)\{(c_1^2 + c_2^2)(z_1(b)i + z_2(b))e^{-ik(b)} + it(b)(c_1 - ic_2)(-d_1 + id_2)e^{-u(b) + \frac{1}{2}ik(b)} +$$
$$\frac{1}{2}(c_1 - ic_2)(-d_1 + id_2)e^{\frac{1}{2}ik(b)}[(e^{-u(b)} + e^{u(b)})(1 - \frac{1}{2}(z_1^2(b) + z_2^2(b))) + \frac{1}{2}i(z_1^2(b) + z_2^2(b))(e^{-u(b)} - e^{u(b)})]\}. \tag{29}$$

The $(1,1)$-entry on the right hand side of the matrix equation (28) is

$$e^{-ip}\{e^{\frac{1}{2}ig(c,d) - j(c,d)}[(ip_1(c,d) - p_2(c,d))(-b_1 + ib_2) + (p_1(c,d) + ip_2(c,d))(w_1 - iw_2)] +$$
$$e^{-ig(c,d)}(b_1 + ib_2)(w_1 i + w_2)\}. \tag{30}$$

For all $c_1, c_2, d_1, d_2, b_1, b_2 \in \mathbb{R}$ with $c_1^2 + c_2^2 + d_1^2 + d_2^2 = 1 = b_1^2 + b_2^2$ the expression (29) coincides with (30). The term

$$e^{\frac{1}{2}ig(c,d) - j(c,d) - ip}(p_1(c,d) + ip_2(c,d))(w_1 - iw_2) = 0$$

because it does not depend on $b = b_1 + ib_2$. Hence one of the following cases holds:

$$(a) \;\; p_1(c,d) = p_2(c,d) = 0 \;\; \text{or} \;\; (b) \; w_1 = w_2 = 0.$$

In case (a) one has

$$e^{-ig(c,d)-ip}(b_1+ib_2)(w_1i+w_2)=$$

$$(-b_1+ib_2)\{(-d_1-id_2)(c_1+ic_2)e^{-ik(b)}+(d_1^2+d_2^2)(z_1(b)i-z_2(b))e^{\frac{1}{2}ik(b)-u(b)}\}+$$

$$(b_1+ib_2)\{(c_1^2+c_2^2)(z_1(b)i+z_2(b))e^{-ik(b)}+it(b)(c_1-ic_2)(-d_1+id_2)e^{-u(b)+\frac{1}{2}ik(b)}+$$

$$\frac{1}{2}(c_1-ic_2)(-d_1+id_2)e^{\frac{1}{2}ik(b)}[(e^{-u(b)}+e^{u(b)})(1-\frac{1}{2}(z_1^2(b)+z_2^2(b)))+\frac{1}{2}i(z_1^2(b)+z_2^2(b))(e^{-u(b)}-e^{u(b)})]\}. \quad (31)$$

As the right hand side of (31) is a polynomial in $c_1, c_2, d_1, d_2 \in \mathbb{R}$ the left hand side is also a polynomial. This yields that $p = -g(c,d)$. Since the left hand side of (31) depends linearly on b_1 and b_2, this holds also for the right hand side. Therefore we obtain $t(b) = 0 = z_1(b) = z_2(b)$. Using these relations in (31) and putting there $b_1 = 1$, $b_2 = 0$ we obtain

$$w_1i+w_2=(d_1+id_2)(c_1+ic_2)e^{-ik(b)}+\frac{1}{2}(c_1-ic_2)(-d_1+id_2)e^{\frac{1}{2}ik(b)}(e^{-u(b)}+e^{u(b)}), \quad (32)$$

putting there $b_1 = 0$, $b_2 = 1$ we get

$$w_1i+w_2=(-d_1-id_2)(c_1+ic_2)e^{-ik(b)}+\frac{1}{2}(c_1-ic_2)(-d_1+id_2)e^{\frac{1}{2}ik(b)}(e^{-u(b)}+e^{u(b)}). \quad (33)$$

Comparing (32) and (33) the equation $2(d_1+id_2)(c_1+ic_2)e^{-ik(b)} = 0$ is satisfied for all $c_1, c_2, d_1, d_2 \in \mathbb{R}, b = b_1+ib_2 \in \mathbb{C}$ with $c_1^2+c_2^2+d_1^2+d_2^2 = 1 = b_1^2+b_2^2$. This is a contradiction.

In case (b) we get from (29) and (30) that

$$e^{\frac{1}{2}ig(c,d)-j(c,d)-ip}(ip_1(c,d)-p_2(c,d))(-b_1+ib_2)=$$

$$(-b_1+ib_2)\{(-d_1-id_2)(c_1+ic_2)e^{-ik(b)}+(d_1^2+d_2^2)(z_1(b)i-z_2(b))e^{\frac{1}{2}ik(b)-u(b)}\}+$$

$$(b_1+ib_2)\{(c_1^2+c_2^2)(z_1(b)i+z_2(b))e^{-ik(b)}+it(b)(c_1-ic_2)(-d_1+id_2)e^{-u(b)+\frac{1}{2}ik(b)}+$$

$$\frac{1}{2}(c_1-ic_2)(-d_1+id_2)e^{\frac{1}{2}ik(b)}[(e^{-u(b)}+e^{u(b)})(1-\frac{1}{2}(z_1^2(b)+z_2^2(b)))+\frac{1}{2}i(z_1^2(b)+z_2^2(b))(e^{-u(b)}-e^{u(b)})]\}. \quad (34)$$

The same argument as in case (a) gives that $t(b) = 0 = z_1(b) = z_2(b)$; moreover, $j(c,d) = 0$, $p = \frac{1}{2}g(c,d)$. Using these relations in (34) and putting $b_1 = 1$ and $b_2 = 0$ into (34) we have

$$ip_1(c,d)-p_2(c,d)=(-d_1-id_2)(c_1+ic_2)e^{-ik(b)}-\frac{1}{2}(c_1-ic_2)(-d_1+id_2)e^{\frac{1}{2}ik(b)}(e^{-u(b)}+e^{u(b)}). \quad (35)$$

As the left hand side of (35) does not depend on $b = b_1+ib_2$, also the right hand side is independent of b. This yields $k(b) = 0 = u(b)$. Therefore the range of the section σ defined by (27) is a subgroup $\cong Spin_3(\mathbb{R})$ of G_l. Hence $\sigma(L)$ does not generate G_l and a group locally isomorphic to $PSU_3(\mathbb{C},1)$ is not the multiplication group of a connected topological proper loop. This proves Theorem 2. □

Proof of Theorem 1. By Lemmata 9 and 7 the loop L is quasi-simple and homeomorphic either to S^3 or to $\mathbb{R}^3$. If $\dim(Mult(L)) \geq 8$, then according to Proposition 3.2 in [10] L is homeomorphic to S^3 and $Mult(L)$ is either $SO_0(1,4)$ or $Sp_4(\mathbb{R})$ or $SL_4(\mathbb{R})$.

The 4-dimensional locally compact connected quasifields Q, which coordinatize non-desarguesian topological translation planes and have the field $\mathbb{C}$ as their kernel, are classified by [14]. The multiplicative loops Q^* of Q are the direct products of $\mathbb{R}$ and a loop L homeomorphic to S^3 such that its multiplication group is the group $SL_4(\mathbb{R})$ (cf [8], Section 3).

In order to prove Theorem 1 we have to exclude the Lie groups $SO_0(1,4)$ and $Sp_4(\mathbb{R})$.

I. Let G be the group $Sp_4(\mathbb{R})$. This group consists of the real (4×4)-matrices $A = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}$ satisfying $K_1^T K_3 = K_3^T K_1$, $K_2^T K_4 = K_4^T K_2$, $K_1^T K_4 - K_3^T K_2 = I_2$ with respect to a symplectic basis $\{e_i\}, i = 1,2,3,4$, such that $\langle e_i, e_j \rangle = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$ (cf. Exercise 4, p. 138, in [20]).

If there is a connected topological proper loop L homeomorphic to S^3 and having the group $Sp_4(\mathbb{R})$ as the group $Mult(L)$, then using Section 2.5 and Example 1, pp. 358-359, in [20] the subgroup $Inn(L)$ of L may be written in the form

$$H = \{g(a_1, a_2, b_1, c_1, c_2, c_4, d_1, d_4); a_1, a_2, b_1, c_1, c_2, c_4, d_1 \in \mathbb{R}, d_4 > 0, a_1 d_1 - c_1 b_1 = 1\} \text{ with}$$

$$g(a_1, a_2, b_1, c_1, c_2, c_4, d_1, d_4) = \begin{pmatrix} a_1 & a_2 & b_1 & 0 \\ 0 & \frac{1}{d_4} & 0 & 0 \\ c_1 & c_2 & d_1 & 0 \\ (a_1 c_2 - c_1 a_2) d_4 & c_4 & (c_2 b_1 - a_2 d_1) d_4 & d_4 \end{pmatrix}. \tag{36}$$

We prove that there does not exist left transversals A and B to the group H given by (36) in $G = Sp_4(\mathbb{R})$ such that for all $a \in A$ and $b \in B$ one has $a^{-1}b^{-1}ab \in H$ and the set $\{A, B\}$ generates G (cf. Lemma 3). An arbitrary left transversal to the group H in G is

$$A_{f_i} = \left\{ \begin{pmatrix} 1 & 0 & 0 & k_1 \\ -k_3 & 1 & k_1 & k_2 \\ 0 & 0 & 1 & k_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} g(f_i(k_j)), i = 1, \cdots, 8, k_j \in \mathbb{R}, j = 1,2,3 \right\}$$

such that for $i \in \{1, \cdots, 7\}$, $f_i(k_j) : \mathbb{R}^3 \to \mathbb{R}$, $f_8(k_j) : \mathbb{R}^3 \to \mathbb{R}_+$ are continuous functions with $f_i(0,0,0) = 1$, if $i \in \{1,7,8\}$, $f_i(0,0,0) = 0$, if $i \in \{2,3,4,5,6\}$. For

$$a = \begin{pmatrix} 1 & 0 & 0 & k \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} g(f_i(k)) \in A_{f_i}, \quad b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -l & 1 & 0 & 0 \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{pmatrix} g(\tilde{f}_i(l)) \in A_{\tilde{f}_i}, \ l, k \in \mathbb{R},$$

where $f_i(k) := f_i(k,0,0)$, $\tilde{f}_i(l) := \tilde{f}_i(0,0,l)$ with $f_1(k)f_7(k) - f_3(k)f_4(k) = 1 = \tilde{f}_1(l)\tilde{f}_7(l) - \tilde{f}_3(l)\tilde{f}_4(l)$, the product $a^{-1}b^{-1}ab$ is contained in H if and only if there are elements $m := m(k,l) = g(w_i(k,0,l)) \in H$, $w_1(k,0,l)w_7(k,0,l) - w_3(k,0,l)w_4(k,0,l) = 1$ such that for all $k, l \in \mathbb{R}$ the matrix equation

$$ab = bam \tag{37}$$

holds. Comparing the $(1,4)$-entries, respectively the $(2,4)$-entries on both sides of the matrix equation (37) we obtain for all $k,l \in \mathbb{R}$

$$kf_8(k)[\tilde{f}_1(l)w_8(k,0,l) - l\tilde{f}_8(l)(f_5(k)f_3(k) - f_2(k)f_7(k)) - \tilde{f}_8(l)] = lf_3(k)\tilde{f}_8(l), \qquad (38)$$

respectively

$$kl(f_7(k)\tilde{f}_8(l) + \tilde{f}_1(l)f_8(k)w_8(k,0,l)) = 0. \qquad (39)$$

From (39) one gets $\tilde{f}_1(l) = -\frac{f_7(k)\tilde{f}_8(l)}{f_8(k)w_8(k,l)}$. Putting this into (38) we have

$$-k(f_8(k) + f_7(k)) = lf_3(k) + klf_8(k)(f_5(k)f_3(k) - f_2(k)f_7(k)). \qquad (40)$$

As k and l are independent variables we obtain $k(f_8(k) + f_7(k)) = 0$ for all $k \in \mathbb{R}$, and therefore $f_8(k) = -f_7(k)$ for all $k \in \mathbb{R}$. Since $f_7(0) = f_8(0) = 1$ we get a contradiction to the fact that the functions f_7 and f_8 are continuous. This contradiction shows that the group $Sp_4(\mathbb{R})$ is not the multiplication group of a connected topological loop homeomorphic to S^3.

II. Now we deal with the Lie group $SO_0(1,4)$. This group consists of the real quadratic matrices A such that $\det(A) = 1$ and one has $A^T I_{1,4} A = I_{1,4}$, where $I_{1,4} = \mathrm{diag}(-1,1,1,1,1)$. The geometry determined by the action of the subgroup $SO_0(1,4) \subset O(1,4)$ on S^3 is the Möbius geometry (cf. Sections 2.6 and 2.7, [20]). The 3-dimensional Möbius space S^3 is understood to be the hypersphere Q for the scalar product $\langle , \rangle$, defined by (3) in [20], p. 243, in the 3-dimensional real projective space. Applying Proposition 1, p. 314, and Lemma 6.19, p. 280, in [20] for the pseudo-orthogonal basis $(\mathbf{e}_i)$, $i = 0,1,2,3$, given by (60), p. 280, in [20] we get the following: If there exists a connected topological loop L homeomorphic to S^3 and having the group $SO_0(1,4)$ as the multiplication group $Mult(L)$, then the inner mapping group $Inn(L)$ of L is the group

$$H = \left\{ h(\lambda,b,C) = \begin{pmatrix} \frac{\lambda^{-1}}{2} + \frac{\lambda}{2} + \frac{\langle b,b\rangle}{4\lambda} & \frac{\lambda^{-1}b^T C}{\sqrt{2}} & -\frac{\lambda^{-1}}{2} + \frac{\lambda}{2} + \frac{\langle b,b\rangle}{4\lambda} \\ \frac{b}{\sqrt{2}} & C & \frac{b}{\sqrt{2}} \\ -\frac{\lambda^{-1}}{2} + \frac{\lambda}{2} - \frac{\langle b,b\rangle}{4\lambda} & -\frac{\lambda^{-1}b^T C}{\sqrt{2}} & \frac{\lambda^{-1}}{2} + \frac{\lambda}{2} - \frac{\langle b,b\rangle}{4\lambda} \end{pmatrix} \right\}, \qquad (41)$$

where $b \in \mathbb{R}^3$, $\lambda > 0$, $C := g(t,u,z) \in SO_3(\mathbb{R})$ is represented by the matrix given by (20). The elements of L can be represented as

$$\mu(a_i) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a_1 & -a_2 & -a_3 & -a_4 \\ 0 & a_2 & a_1 & -a_4 & a_3 \\ 0 & a_3 & a_4 & a_1 & -a_2 \\ 0 & a_4 & -a_3 & a_2 & a_1 \end{pmatrix}, a_i \in \mathbb{R} \text{ such that } a_1^2+a_2^2+a_3^2+a_4^2 = 1.$$

The sets

$$A = \{\mu(a_i)h(\lambda(a_i), b_1(a_i), b_2(a_i), b_3(a_i), g(t(a_i), u(a_i), z(a_i))), a_1, a_2, a_3, a_4 \in \mathbb{R}\} \text{ and}$$

$$B = \{\mu(d_i)h(\kappa(d_i), c_1(d_i), c_2(d_i), c_3(d_i), g(p(d_i), r(d_i), v(d_i))), d_1, d_2, d_3, d_4 \in \mathbb{R}\},$$

where $\lambda(a_i) \neq \kappa(d_i) : \mathbb{R}^4 \to \mathbb{R}_+$, $b_i(a_i) \neq c_i(d_i) : \mathbb{R}^4 \to \mathbb{R}$, $i = 1, 2, 3$, $t(a_i) \neq p(d_i)$ $u(a_i) \neq r(d_i) : \mathbb{R}^4 \to [0, 2\pi]$, $z(a_i) \neq v(d_i) : \mathbb{R}^4 \to [0, \pi]$ are continuous functions with $\lambda(1,0,0,0) = 1 = \kappa(1,0,0,0)$, $b_i(1,0,0,0) = c_i(1,0,0,0) = 0 = t(1,0,0,0) = u(1,0,0,0) = p(1,0,0,0) = r(1,0,0,0) = z(1,0,0,0) = v(1,0,0,0)$, are two different left transversals to $Inn(L) = H$ given by (41) in the group $G = SO_0(1,4)$. The group G is not isomorphic to the multiplication group $Mult(L)$ of L if not for all $a \in A$ and $b \in B$ one has $a^{-1}b^{-1}ab \in Inn(L)$ (cf. Lemma 3). The elements

$$a = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a_1 & -a_2 & 0 & 0 \\ 0 & a_2 & a_1 & 0 & 0 \\ 0 & 0 & 0 & a_1 & -a_2 \\ 0 & 0 & 0 & a_2 & a_1 \end{pmatrix} h(\lambda(a_i), b_1(a_i), b_2(a_i), b_3(a_i), g(t(a_i), u(a_i), z(a_i))),$$

with $\lambda(a_i) := \lambda(a_1, a_2, 0, 0)$, $b_i(a_i) := b_i(a_1, a_2, 0, 0)$, $i = 1, 2, 3$,

$$g(t(a_i), u(a_i), z(a_i)) := g(t(a_1, a_2, 0, 0), u(a_1, a_2, 0, 0), z(a_1, a_2, 0, 0)),$$

are in A whereas the element

$$b = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix} h(\kappa, c_1, c_2, c_3, g(p, r, v)),$$

with $g(p, r, v) := g(p(0,0,1,0), r(0,0,1,0), v(0,0,1,0))$, $\kappa := \kappa(0,0,1,0)$, $c_i := c_i(0,0,1,0)$, $i = 1, 2, 3$, lies in B. The product $a^{-1}b^{-1}ab$ is contained in $Inn(L)$ precisely if there are elements $m(a_1, a_2, 1) := m(a_1, a_2, 1, 0) =$

$$g(\gamma(a_1, a_2, 1, 0), f_i(a_1, a_2, 1, 0), g(w_1(a_1, a_2, 1), w_2(a_1, a_2, 1), w_3(a_1, a_2, 1))) \in Inn(L)$$

such that for all $a_1, a_2 \in \mathbb{R}$, $a_1^2 + a_2^2 = 1$, the matrix equation

$$ab = bam \tag{42}$$

holds. Taking the difference of the $(4,1)$- and $(4,5)$-entries on both sides of the matrix equation (42) we obtain

$$-2\sqrt{2}\kappa\gamma(a_i)c_1(1 - a_1) + 2\lambda(a_i)a_1(2\sin z(a_i)\cos u(a_i) + \sqrt{2}b_3(a_i)) =$$

$$a_2(2\lambda^2(a_i) - 2 - (b_1^2(a_i) + b_2^2(a_i) + b_3^2(a_i)) + 4\kappa\gamma(a_i)\sin r \sin v - 2\sqrt{2}b_3(a_i)\sin z(a_i)\cos u(a_i))+$$

$$2\sqrt{2}a_2 b_1(a_i)(\sin t(a_i)\cos z(a_i)\cos u(a_i) + \cos t(a_i)\sin u(a_i))+$$

$$2\sqrt{2}a_2 b_2(a_i)(\cos t(a_i)\cos z(a_i)\cos u(a_i) - \sin t(a_i)\sin u(a_i))$$

for all $a_1, a_2 \in \mathbb{R}$, $a_1^2 + a_2^2 = 1$. As a_1, a_2 are independent variables and $\kappa > 0$, $\gamma(a_i) > 0$, $\lambda(a_i) > 0$ it follows that

$$c_1 = 0, \ \ b_3(a_i) = -\sqrt{2}\sin z(a_i)\cos u(a_i) \text{ and} \tag{43}$$

$$2\lambda^2(a_i) - 2 - (b_1^2(a_i) + b_2^2(a_i) - 2\sin^2 z(a_i)\cos^2 u(a_i)) + 4\kappa\gamma(a_i)\sin r \sin v+$$

$$2\sqrt{2}b_1(a_i)(\sin t(a_i)\cos z(a_i)\cos u(a_i) + \cos t(a_i)\sin u(a_i))+$$

$$2\sqrt{2}b_2(a_i)(\cos t(a_i)\cos z(a_i)\cos u(a_i) - \sin t(a_i)\sin u(a_i)) = 0. \tag{44}$$

Using (43) and taking the difference of the $(5,1)$- and $(5,5)$-entries on both sides of (42) we have for all a_1, $a_2 \in \mathbb{R}$, $a_1^2 + a_2^2 = 1$, that

$$2\sqrt{2}\kappa\gamma(a_i)c_2(1 - a_1) + a_1(2 - 2\lambda^2(a_i) + b_1^2(a_i) + b_2^2(a_i) - 2\sin^2 z(a_i)\cos^2 u(a_i)) -$$

$$2\sqrt{2}a_1 b_1(a_i)(\sin t(a_i)\cos z(a_i)\cos u(a_i) + \cos t(a_i)\sin u(a_i)) -$$

$$2\sqrt{2}a_1 b_2(a_i)(\cos t(a_i)\cos z(a_i)\cos u(a_i) - \sin t(a_i)\sin u(a_i)) = -2\cos r\sin v a_2.$$

As all κ, $\gamma(a_i)$, $\lambda(a_i)$ are greater than 0 and a_1, a_2 are independent variables from this we get

$$c_2 = 0,\ \sin v\cos r = 0 \text{ and} \tag{45}$$

$$(2 - 2\lambda^2(a_i) + b_1^2(a_i) + b_2^2(a_i) - 2\sin^2 z(a_i)\cos^2 u(a_i)) -$$

$$2\sqrt{2}b_1(a_i)(\sin t(a_i)\cos z(a_i)\cos u(a_i) + \cos t(a_i)\sin u(a_i)) -$$

$$2\sqrt{2}b_2(a_i)(\cos t(a_i)\cos z(a_i)\cos u(a_i) - \sin t(a_i)\sin u(a_i)) = 0.$$

Comparing (44) and (46) one has $\sin r\sin v = 0$. Together with (45) we get $\sin v = 0$. We applying (43), as well as $\sin v = 0$ and $c_2 = 0$ in the matrix equation (42). Then the difference of the $(3,1)$- and $(3,5)$-entries on both sides of (42) yields for all a_1, $a_2 \in \mathbb{R}$, $a_1^2 + a_2^2 = 1$, that

$$2\sqrt{2}a_2\left[\gamma(a_i)c_3\cos v + \lambda(a_i)b_1(a_i) - \frac{\sin t(a_i)\cos z(a_i)\cos u(a_i) + \cos t(a_i)\sin u(a_i)}{\sqrt{2}}\right] =$$

$$-\gamma(a_i)(2-2\kappa^2+c_3^2)+a_1[\gamma(a_i)(2+2\kappa^2-c_3^2)-2\lambda(a_i)(\sqrt{2}b_2(a_i)+2\sin t(a_i)\sin u(a_i)-2\cos t(a_i)\cos z(a_i)\cos u(a_i))].$$

Since a_1, a_2 are independent variables one obtains

$$2 + c_3^2 = 2\kappa^2 \quad \text{and} \tag{46}$$

$$2\gamma(a_i) = \lambda(a_i)(\sqrt{2}b_2(a_i) + 2\sin t(a_i)\sin u(a_i) - 2\cos t(a_i)\cos z(a_i)\cos u(a_i)). \tag{47}$$

Taking the difference of the $(2,1)$- and $(2,5)$-entries on both sides of (42) we get for all a_1, $a_2 \in \mathbb{R}$, $a_1^2 + a_2^2 = 1$, that

$$-\sqrt{2}\kappa\gamma(a_i)c_3(1 - a_1) + a_1[\lambda(a_i)(-\sqrt{2}b_1(a_i) + 2\sin t(a_i)\cos z(a_i)\cos u(a_i) + 2\cos t(a_i)\sin u(a_i))] =$$

$$a_2[2\kappa\gamma(a_i)\cos v + \lambda(a_i)(-\sqrt{2}b_2(a_i) - 2\sin t(a_i)\sin u(a_i) + 2\cos t(a_i)\cos z(a_i)\cos u(a_i))].$$

As a_1, a_2 are independent variables and $\kappa > 0$, $\gamma(a_i) > 0$, $\lambda(a_i) > 0$ from this it follows

$$c_3 = 0,\ b_1(a_i) = \sqrt{2}(\sin t(a_i)\cos z(a_i)\cos u(a_i) + \cos t(a_i)\sin u(a_i)) \text{ and} \tag{48}$$

$$b_2(a_i) = \sqrt{2}\frac{\kappa\gamma(a_i)}{\lambda(a_i)}\cos v - \sqrt{2}(\sin t(a_i)\sin u(a_i) - \cos t(a_i)\cos z(a_i)\cos u(a_i)). \tag{49}$$

As $c_3 = 0$ from (46) one gets $\kappa = 1$. Putting (49) into (47) we obtain $2\gamma(a_i)(1-\cos v) = 0$ hence $\cos v = 1$ and

$$b_2(a_i) = \sqrt{2}\left[\frac{\gamma(a_i)}{\lambda(a_i)} - \sin t(a_i)\sin u(a_i) + \cos t(a_i)\cos z(a_i)\cos u(a_i)\right]. \quad (50)$$

Using (43), (45), (48), (50), equation (46) reduces to

$$1 - \lambda^2(a_i) + \frac{\gamma(a_i)}{\lambda(a_i)^2} = \cos^2 z(a_i). \quad (51)$$

Furthermore for the difference of the $(1,1)$- and $(1,5)$-entries on both sides of (42) one obtains $4\gamma(a_i) = 2\lambda^2(a_i) + 2\frac{\gamma^2(a_i)}{\lambda^2(a_i)}$. As $\gamma(a_i) > 0$, $\lambda(a_i) > 0$ we have $\gamma(a_i) := \lambda^2(a_i)$. Putting this into (51) for all $a_1, a_2 \in \mathbb{R}$, $a_1^2+a_2^2 = 1$ one gets $\cos^2 z(a_i) = 1$. This yields that $\cos z(a_i) = 1$ because $z(1,0,0,0) = 0$. Therefore it follows that

$$b_3 = 0,\ b_1(a_i) = \sqrt{2}(\sin t(a_i)\cos u(a_i) + \cos t(a_i)\sin u(a_i)) \text{ and}$$
$$b_2(a_i) = \sqrt{2}(\lambda(a_i) - \sin t(a_i)\sin u(a_i) + \cos t(a_i)\cos u(a_i)).$$

Comparing the $(2,4)$-entries on both sides of the matrix equation (42) we have

$$-a_1(\cos t(a_i)\cos u(a_i) - \sin t(a_i)\sin u(a_i)) - a_2(\sin t(a_i)\cos u(a_i) + \cos t(a_i)\sin u(a_i)) =$$
$$-a_1\cos w_3(a_1,a_2,1) + \frac{\sqrt{2}}{2\lambda^2(a_i)}a_2\sin w_3(a_1,a_2,1)\cos w_2(a_1,a_2,1) +$$
$$\frac{\sqrt{2}}{2\lambda(a_i)}a_2[f_3(a_1,a_2,1) + \sin w_3(a_1,a_2,1)\sin w_2(a_1,a_2,1)(\sin t(a_i)\cos u(a_i) + \cos t(a_i)\sin u(a_i))] +$$
$$\frac{\sqrt{2}}{2\lambda(a_i)}a_2\cos w_2(a_1,a_2,1)\sin w_3(a_1,a_2,1)(\cos t(a_i)\cos u(a_i) - \sin t(a_i)\sin u(a_i)). \quad (52)$$

Comparing the $(3,4)$-entries on both sides of the matrix equation (42) one obtains

$$-a_2(\cos t(a_i)\cos u(a_i) - \sin t(a_i)\sin u(a_i)) + a_1(\sin t(a_i)\cos u(a_i) + \cos t(a_i)\sin u(a_i)) =$$
$$a_2\cos w_3(a_1,a_2,1) + \frac{\sqrt{2}}{2\lambda^2(a_i)}a_1\sin w_3(a_1,a_2,1)\cos w_2(a_1,a_2,1) +$$
$$\frac{\sqrt{2}}{2\lambda(a_i)}a_1[f_3(a_1,a_2,1) + \sin w_3(a_1,a_2,1)\sin w_2(a_1,a_2,1)(\sin t(a_i)\cos u(a_i) + \cos t(a_i)\sin u(a_i))] +$$
$$\frac{\sqrt{2}}{2\lambda(a_i)}a_1\sin w_3(a_1,a_2,1)\cos w_2(a_1,a_2,1)(\cos t(a_i)\cos u(a_i) - \sin t(a_i)\sin u(a_i)). \quad (53)$$

As a_1 and a_2 are independent variables we get from (52) and (53) that

$$\cos t(a_i)\cos u(a_i) - \sin t(a_i)\sin u(a_i) = \cos w_3(a_1,a_2,1) = \sin t(a_i)\sin u(a_i) - \cos t(a_i)\cos u(a_i).$$

Therefore $\cos t(a_i)\cos u(a_i) = \sin t(a_i)\sin u(a_i)$ for all $a_1, a_2 \in \mathbb{R}$, $a_1^2 + a_2^2 = 1$. As $t(1,0,0,0) = u(1,0,0,0) = 0$ we get a contradiction. This proves that the group $SO_0(1,4)$ is not the multiplication group of a connected topological loop homeomorphic to S^3. □

Acknowledgement.
This paper was supported by János Bolyai Research Fellowship and the European Union's Seventh Framework Programme (FP7/2007-2013) under grant agreement no. 317721, no. 318202.

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