Journal for Geometry and Graphics
Volume 11 (2007), No. 1, 27–38.

Generalized Rabl Mappings and Apollonius-Type Problems

Sándor Bácsó, Zoltán Szilasi

Faculty of Informatics, Debrecen University
H-4010 Debrecen, P.O. Box 12, Hungary
email: bacsos@inf.unideb.hu, szilasi@math.klte.hu

Abstract. In this paper we introduce the n-dimensional generalization of the point $\mapsto$ circle mapping defined and studied by K. Rabl [13] and show that our generalization can be used effectively to solve three-dimensional Apollonius-type problems. To carry out the constructions in practice, we use on the one hand a simple computer algorithm via Maple, on the other hand some tools of projective and descriptive geometry, including Maurin's projection of the four-space. In the two-dimensional case we present a simple direct method of construction based on elementary projective geometry.

Key Words: cyclography, Rabl mapping, Maurin projection, Apollonius-type problems

MSC: 51N05, 51N15

1. Introduction

1.1. Basic conventions

The scene of our considerations is the real projective space $\mathbb{P}^n(\mathbb{R})$, in particular $\mathbb{P}^3(\mathbb{R})$ and $\mathbb{P}^2(\mathbb{R})$. We interpret $\mathbb{P}^n(\mathbb{R})$ as the augmented Euclidean affine space $\mathbb{E}^n$, so we shall freely use typical Euclidean concepts in our formulations. For calculations we assume once and for all that an orthonormal affine frame $(O; E_1, \dots, E_n)$ of $\mathbb{E}^n$ is fixed, where $O \in \mathbb{E}^n$ is a so-called origin, and the vectors $\overrightarrow{OE_1}, \dots, \overrightarrow{OE_n}$ form an orthonormal basis of the underlying Euclidean vector space of $\mathbb{E}^n$. If we write $P(p_1, \dots, p_n)$, then it means that P is a point of $\mathbb{E}^n$ with Cartesian coordinates $p_1, \dots, p_n$, i.e., $\overrightarrow{OP} = \sum_{i=1}^n p_i \overrightarrow{OE_i}$.

Equations of figures will always be Cartesian equations with respect to $(0; E_1, \dots, E_n)$. We endow $\mathbb{E}^n$ with the orientation represented by the basis $(\overrightarrow{OE_1}, \dots, \overrightarrow{OE_n})$, and with the metric d given by $d(P,Q) := \|\overrightarrow{PQ}\|$. Sometimes we interpret $\mathbb{E}^{n-1}$ as a subspace of $\mathbb{E}^n$ via the inclusion

$$P(p_1, \dots, p_{n-1}) \;\mapsto\; \widetilde{P}(p_1, \dots, p_{n-1}, 0).$$

ISSN 1433-8157/\$ 2.50

We note finally that the usual differential calculus is available in $\mathbb{E}^n$ (see e.g. [1, 2.7.7]). The i-th partial derivative of a function $f: \mathbb{E}^n \to \mathbb{R}$, i.e., its derivative in the direction of $\overrightarrow{0E_i}$ will be denoted by $D_i f$ $(i \in \{1, \dots, n\})$.

1.2. Preliminaries

A *cyclographic mapping* is a kind of nonlinear mapping that sends the points of $\mathbb{E}^3$ to directed circles of $\mathbb{E}^2$ such that the centre of an image circle is the orthogonal projection of the given point on the plane, the length of its radius is the distance of the point and the plane, and the circle is positively (resp. negatively) oriented according to whether the point is in the positive (resp. negative) half-space determined by $\mathbb{E}^2$ in $\mathbb{E}^3$ with respect to the fixed orientation. We agree that the image of each point of the plane $\mathbb{E}^2$ is itself. Obviously, the image of a proper point of the space can be obtained as the intersection of the target plane with a rectangular cone of revolution whose vertex is the given point and whose axis is perpendicular to the plane. Such a cone is said to be a *C-cone*. It can easily be shown (see e.g. [12]) that the image circle of a point lying on the surface of a C-cone and different to the vertex is tangent to the image of the vertex.

A higher dimensional generalization of cyclographic mappings is due to L. Gyarmathi [7]. To assign $(n-2)$-spheres in $\mathbb{E}^{n-1}$ to the points of $\mathbb{E}^n$ $(n \geq 3)$ he applied the so-called Γ-*cones* which can be defined in the same way as the C-cones. He gained results analogous to the classical ones, and, applying them, he presented a method to find an $(n-1)$-dimensional sphere tangent to $n+1$ given spheres in $\mathbb{E}^n$. It is remarkable that to carry out effectively the construction in the three dimensional case he used Maurin's projection of $\mathbb{E}^4$ ([8, 11]).

An ingenious modification of classical cyclographic mapping was proposed by K. Rabl [13]. Instead of using C-cones he applied paraboloids of revolution with axis perpendicular to the image plane and containing the given point as vertex. Then the image of a point is a circle again, but the radius of the circle is the square root of the distance between the point and the image plane. The image of a linear range is a pencil of circles. It may be proved that the envelope of this family of circles is a parabola. This observation is the key to solve the problem: *how to construct a parabola with double contact to given circles?*

Thus we arrive at the fascinating field of Apollonius-type problems. As a stimulating guide in this direction as well as for further generalizations of cyclography we refer to T. Schwarcz's paper [14].

2. Generalized Rabl mappings

Definition 2.1 By a *generalized Rabl mapping* we mean the mapping which assigns to each point $P(p_1, \dots, p_n)$ of $\mathbb{E}^n$ $(n \geq 3)$ the intersection of the paraboloid of revolution given by the Cartesian equation

$$\sum_{i=1}^{n-1} (x_i - p_i)^2 + x_n - p_n = 0$$

and the hyperplane of equation $x_n = 0$.

Remarks: (i) Obviously, the image of a point $P(p_1, \dots, p_n)$ under the generalized Rabl mapping is the $(n-2)$-sphere in $\mathbb{E}^{n-1}$ of equation

$$\sum_{i=1}^{n-1} (x_i - p_i)^2 = p_n.$$

The paraboloid of revolution applied in the construction intersects the remaining coordinate hyperplanes in $(n-1)$-dimensional paraboloids of revolution.

(ii) For the sake of simplicity we shall frequently speak of 'the cyclographic image of a point' rather than 'the image of a point under a generalized Rabl mapping'. This is an abuse of language, but leads to no confusion in our context.

Proposition 2.2 *Let l be a line in $\mathbb{E}^n$ given parametrically by*

$$x_1 = \ldots = x_{n-2} = 0, \quad x_{n-1} = t, \quad x_n = \gamma t \quad (\gamma \neq 0).$$

The envelope of the family of the $(n-2)$-spheres formed by the cyclographic images of the points of l is the paraboloid of revolution in $\mathbb{E}^{n-1}$ described by the equations

$$\sum_{i=1}^{n-2} x_i^2 - \gamma x_{n-1} - \frac{\gamma^2}{4} = 0.$$

Proof: The paraboloids of revolution used in the construction form a family of surfaces given by

$$\sum_{i=1}^{n-2} x_i^2 + (x_{n-1} - t)^2 + x_n - \gamma t = 0,$$

therefore the cyclographic images of the points of l satisfy the equation

$$\sum_{i=1}^{n-2} x_i^2 + (x_{n-1} - t)^2 - \gamma t = 0.$$

Let

$$F(x_1, \ldots, x_{n-1}, t) := \sum_{i=1}^{n-2} x_i^2 + (x_{n-1} - t)^2 - \gamma t.$$

The envelope of the family of hypersurfaces in $\mathbb{E}^{n-1}$ given by $F(x_1, \ldots, x_{n-1}, t) = 0$ has to satisfy the relation

$$D_n F(x_1, \ldots, x_{n-1}, t) = 0.$$

From this we obtain that

$$t = x_{n-1} + \frac{\gamma}{2}.$$

So our candidate for the envelope of the family is the paraboloid of revolution in $\mathbb{E}^{n-1}$ whose equation is

$$\sum_{i=1}^{n-2} x_i^2 - \gamma x_{n-1} - \frac{\gamma^2}{4} = 0.$$

Since $D_n(D_n F) = 2 \neq 0$, it is indeed the desired envelope. □

Remark: The basic facts concerning the envelope of a family of surfaces in $\mathbb{E}^3$ can be found e.g. in L.P. Eisenhart's classical book [3]. The generalization to higher dimensional hypersurfaces is immediate.

Proposition 2.3 *Let $\mathcal{P}$ be a parabola in $\mathbb{E}^n$ given parametrically by*

$$x_1 = \ldots = x_{n-2} = 0, \quad x_{n-1} = t, \quad x_n = \alpha + \beta t^2,$$

where $\beta \notin \{0, 1\}$. The envelope of the family of $(n-2)$-spheres in $\mathbb{E}^{n-1}$ formed by the cyclographic images of the points of $\mathcal{P}$ is the quadratic surface of equation

$$\sum_{i=1}^{n-1} (x_i)^2 - \frac{\beta}{\beta - 1} x_{n-1}^2 - \alpha = 0.$$

Proof: Now the paraboloids of revolution used in the construction of the image $(n-2)$-spheres satisfy the equation

$$\sum_{i=1}^{n-2} (x_i)^2 + (x_{n-1} - t)^2 + x_n - \alpha - \beta t^2 = 0,$$

so for the family of the cyclographic images we get the equation

$$\sum_{i=1}^{n-2} x_i^2 + (x_{n-1} - t)^2 - \alpha - \beta t^2 = 0.$$

Arguing as above, the vanishing of the n-th partial derivative of the function given by the left-hand side yields $t = \frac{x_{n-1}}{1 - \beta}$. Substituting this expression of t into the first relation, we get the equation of the desired envelope. □

3. Applications to Apollonius-type problems

In our forthcoming considerations we shall systematically use the following technique: 'to solve a three-dimensional problem, transform it into a four-dimensional problem'. The key observation is the following

Proposition 3.1 *Let two non-concentric spheres S_1 and S_2 be given in $\mathbb{E}^3$, and let $\boldsymbol{P}$ be a paraboloid of revolution tangent to both S_1 and S_2. Interpret the spheres as the images of two points of $\mathbb{E}^4$ under the generalized Rabl mapping. If l is the line through these points then $\boldsymbol{P}$ is the envelope of the family of spheres formed by the cyclographic images of the points of l.*

Proof: Let the equations of S_1 and S_2 be

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r_1^2 \quad \text{and} \quad (x - d)^2 + (y - e)^2 + (z - f)^2 = r_2^2,$$

respectively. By our assumption, $(a, b, c) \neq (d, e, f)$. S_1 is the cyclographic image of the point $P_1(a, b, c, r_1^2)$, while S_2 is the cyclographic image of $P_2(d, e, f, r_2^2)$; then $l = \overleftrightarrow{P_1 P_2}$. Now we find a parametric representation for l, and, as above, determine the desired envelope. Omitting the straightforward calculation, we get a paraboloid of revolution given by the equation

$$(x - a - p(d - a))^2 + (y - b - p(e - b))^2 + (z - c - p(f - c))^2 - r_1^2 - p^2(r_2^2 - r_1^2) = 0,$$

where

$$p := -\frac{1}{2} \frac{2(x - a)(a - d) + 2(y - b)(b - e) + 2(z - c)(c - f) + r_1^2 - r_2^2}{(a - d)^2 + (b - c)^2 + (c - f)^2}. \qquad \square$$

Practical methods for the construction of the envelope

(A) *Using computer:* Having the equation of the envelope $\boldsymbol{P}$, its axonometric representation can easily be carried out using e.g. MAPLE: we get a simple algorithm which provides the representation plotted against the centres and the radii of the given spheres. Fig. 1 has been made in this way.

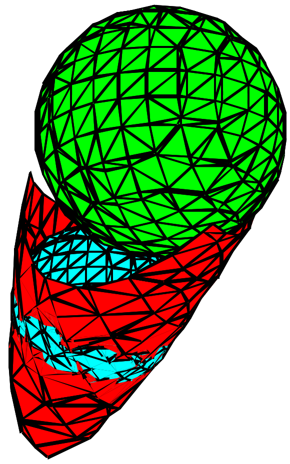

Figure 1: The paraboloid $\boldsymbol{P}$ enveloping the images of the points of a line l

(B) *Using Maurin's projection:* We start with the representation of the given spheres S_1, S_2 in a Monge system. Next, by Maurin's projection of $\mathbb{E}^4$ [11] we determine the points A, B in $\mathbb{E}^4$ whose images under the generalized Rabl mapping are S_1 and S_2. It turns out from our previous calculations (see also [13]) that the axis of the described paraboloid $\boldsymbol{P}$ is just the orthogonal projection of $\overleftrightarrow{AB}$ in $\mathbb{E}^3$, its focus is the trace of $\overleftrightarrow{AB}$, and the parameter of $\boldsymbol{P}$ is $\frac{1}{2}\tan\alpha$, where α is the angle between $\overleftrightarrow{AB}$ and its orthogonal projection. Thus the data which completely describe $\boldsymbol{P}$ can be represented by the Maurin projection, and it is also true that the first and second projections of $\boldsymbol{P}$ are the same in the Maurin system as in the Monge system.

Fig. 2 displays the construction when the line $l := \overleftrightarrow{O_1O_2}$ through the centres of S_1 and S_2 is parallel to the planes of projections and S_2 is a unit sphere. Reconstruction of points of $\mathbb{E}^4$ was made by revolving the third plane of projection onto the second one. By Proposition 2.2, the distance between a point to be found and the hyperplane determined by the first two planes of projection is the square of the radius of the sphere obtained as the cyclographic image of the point. Thus, with the help of the revolved image, the trace of the line l can be determined, and, as it follows from our previous remark, it is the focus of the desired paraboloid. In this way the first and second projections of the focus can be obtained. Since the above mentioned angle α appears in the revolved image, we can also determine the parameter of $\boldsymbol{P}$ as well as the first and second projection of its directrix plane. These data determine the desired paraboloid uniquely.

Fig. 2 shows the contour of $\boldsymbol{P}$. In its construction we used the fact that, due also to the special arrangement, the directrix plane of $\boldsymbol{P}$ is a profile plane, and, furthermore, the contour appears in its true magnitude. Thus the parameters of the contour parabolas are the same as the parameter of $\boldsymbol{P}$. $\boldsymbol{P}$ is tangent to S_1 and S_2 along parallel circles, both of them lying in a projecting plane as a consequence of the special arrangement again. To find these circles, we contructed both on the first and on the second plane of projection parabolas which have double contact with two circles. It is known (see [13]) that if a circle has double contact with a parabola, then the distance between the orthogonal projection of a tangency point on the axis of the parabola and the centre of the circle is just the parameter of the parabola. This observation makes immediate the construction of the desired points of double contact. Both the first and the second projection of the parallel circles appear as segments connecting the tangency points.

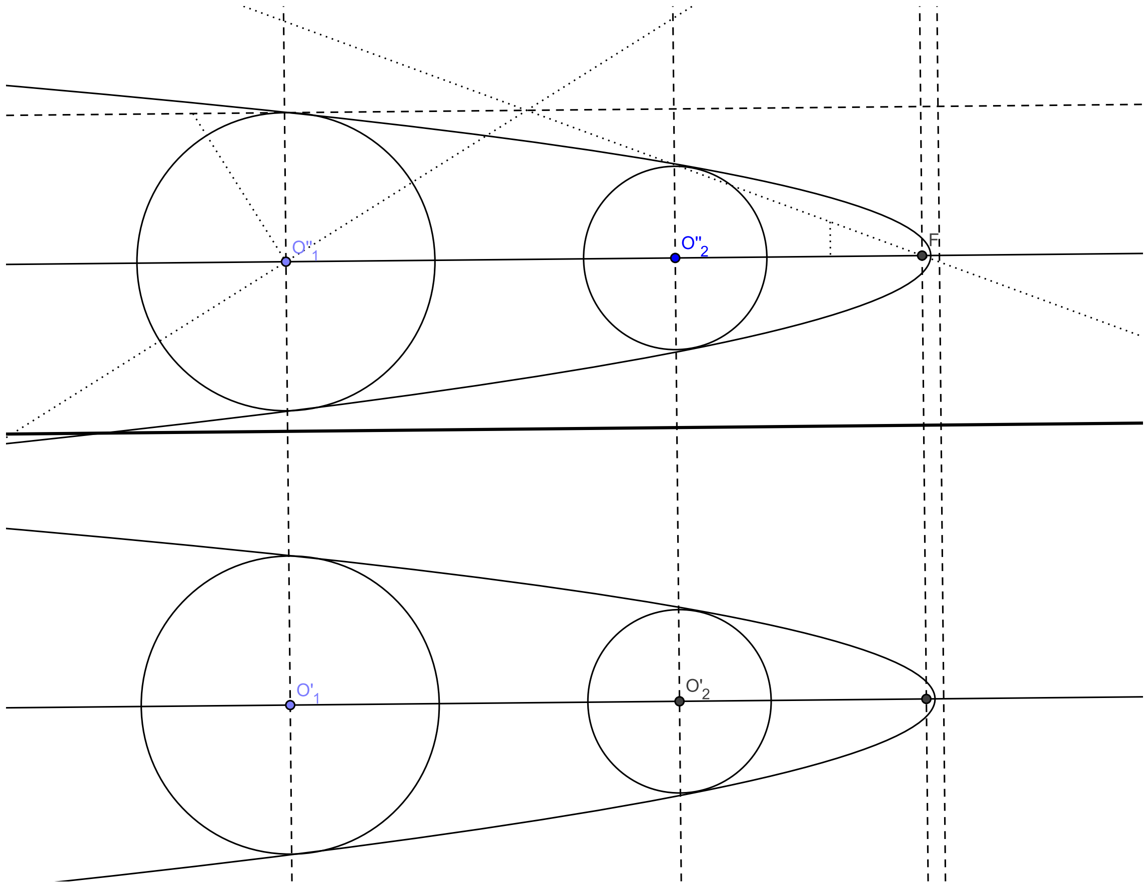

Figure 2: Top and front view of the image of a line l in $\mathbb{E}^4$ under the Rabl mapping

Proposition 3.2 *Let S_1, S_2, S_3 be spheres in $\mathbb{E}^3$ with pairwise distinct centres. Suppose S_1, S_2 and S_3 are the cyclographic images of the point A, B and C in $\mathbb{E}^4$, respectively. Let $\mathcal{P}$ be the parabola passing through these points and with axis perpendicular to the line of the centres of the spheres. The envelope of the cyclographic images of the points of $\mathcal{P}$ is a quadratic surface which is tangent to the given spheres along circles.*

Proof: We may assume without loss of generality that the centres of the spheres S_1, S_2, S_3 are the points $O_1(0,0,a)$, $O_2(0,0,b)$, $O_3(0,0,c)$, respectively. Then their equations take the following simple forms:

$$\begin{aligned} S_1: &\quad x^2+y^2+(z-a)^2=r_1^2,\\ S_2: &\quad x^2+y^2+(z-b)^2=r_2^2,\\ S_3: &\quad x^2+y^2+(z-c)^2=r_3^2. \end{aligned}$$

S_1, S_2 and S_3 are the cyclographic images of the points

$$A(0,0,a,r_1^2), \quad B(0,0,b,r_2^2), \quad C(0,0,c,r_3^2)$$

in $\mathbb{E}^4$. Let the vector $\boldsymbol{v}$ in the underlying vector space of $\mathbb{E}^4$ represent a direction parallel to the plane passing through the points A, B, C and perpendicular to the line connecting the centres of the spheres. Since a parabola is uniquely determined by three of its points, the direction of its axis, and by the ideal line (as a tangent line) of its plane, there is a unique parabola $\mathcal{P}$ containing the points A, B, C and having $\boldsymbol{v}$ as a direction vector of its axis. $\mathcal{P}$ can be parametrized as follows:

$$x_1 = x_2 = 0, \quad x_3 = t, \quad x_4 = k + lt + mt^2.$$

Here the real numbers k, l, m have to satisfy the system of linear equations

$$\begin{aligned} k + al + a^2m &= r_1^2, \\ k + bl + b^2m &= r_2^2, \\ k + cl + c^2m &= r_3^2. \end{aligned}$$

By Cramer's rule this system has a unique solution since the determinant of its matrix is the Vandermonde determinant $V(a,b,c)$ generated by the pairwise distinct real numbers a, b, c. Now Proposition 2.3 assures that the envelope of the family of the cyclographic images of the points of $\mathcal{P}$ is indeed a quadratic surface which is obviously tangent to the given spheres S_1, S_2 and S_3. □

Lemma 1 *Let an ellipse of equation*

$$\frac{x^2}{a} + \frac{by^2}{a(b-1)} = 1$$

be given in $\boldsymbol{E}^2$. *If a circle has double contact with the ellipse, then the distance between the orthogonal projections of the tangency points on the major axis and the centre of the circle is* $|k_1(b-1)|$, *where* $|k_1|$ *is the distance between the centre of the ellipse and the centre of the circle.*

Proof: Let $E(x_0, y_0)$ be a tangency point. The tangent line at E to the ellipse is the polar of E, and its equation is

$$\frac{x_0}{a}x + \frac{by_0}{a(b-1)}y = 1,$$

therefore the equation of the normal line through E is

$$\frac{by_0}{b-1}x - x_0y = \frac{bx_0y_0}{b-1} - x_0y_0.$$

The normal line meets the x-axis at the centre $K(k_1, k_2)$ of the circle having double contact with the ellipse, so k_1 has to satisfy the relation

$$by_0k_1 = bx_0y_0 - (b-1)x_0y_0,$$

whence

$$k_1 = \frac{x_0}{b}.$$

Since the orthogonal projection of E on the major axis is the point $E'(x_0, 0)$, the distance between E' and K is

$$d(E', K) = |x_0 - k_1| = |k_1(b-1)|.$$

The geometric meaning of $|k_1|$ is obvious because the centre of the ellipse is the origin. □

Construction of tangent quadratic surface to three given spheres in $\mathbb{E}^3$

As in our previous construction, there is a simple computer algorithm to find an axonometric representation of the configuration.

Now we describe a more sophisticated and much more elegant method, based on techniques of projective and descriptive geometry:
Keeping the notation of Proposition 3.2 and its proof as a starting step we represent the given spheres in a Monge system. Our goal is *to represent a quadratic surface tangent to every sphere*. For simplicity we assume that the centres O_1, O_2, O_3 lie on the same line l, and l is parallel to all of the planes of projection.

Carrying out effectively the construction, we chose the radii of the spheres in such a way that the desired tangent quadratic surface became an *ellipsoid of revolution*; the other cases can be handled similarly. The second projection of this ellipsoid is an ellipse. We are going to explain how to construct it.

We need the points A, B, C in $\mathbb{E}^4$ whose cyclographic images are the given spheres S_1, S_2, S_3, respectively. To represent them, we applied again Maurin's projection. Using the technique of revolving as in (B) above, we revolved the third plane of projection of the Maurin system onto the second one. Fixing a unit segment, the distances between the points A, B, C and the plane of projection are the squares of the radii of S_1, S_2, S_3, resp., so the revolved points (P_1), (P_2), (P_3) can easily be constructed.

Now we are in a position to represent the (suitably oriented) parabola $\mathcal{P}$ described in Proposition 3.2: (P_1), (P_2), (P_3) are three points of the revolved parabola $(\mathcal{P})$, and the ordering lines give the direction of the axis of $(\mathcal{P})$. The intersection of the revolved parabola

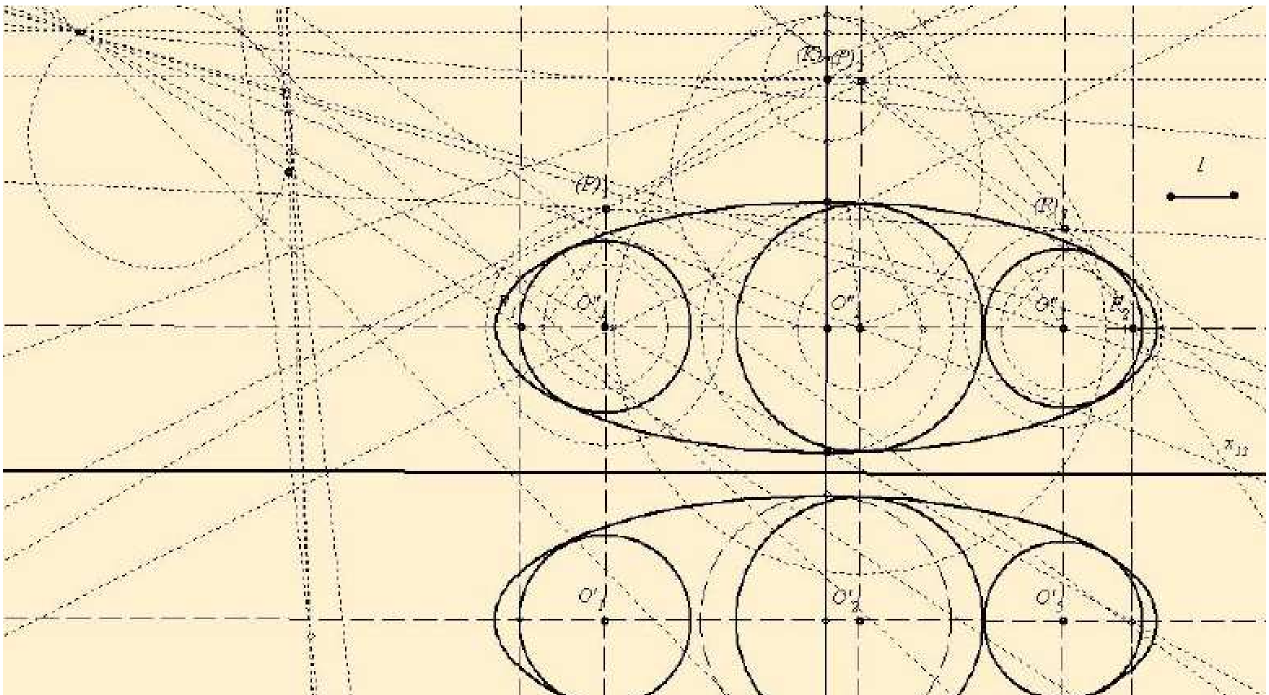

Figure 3: Constructing the enveloping quadratic surface
of three given spheres with aligned centers

and the second projection l'' of l (the line of the centres of S_1, S_2 and S_3) gives the trace of the parabola $\mathcal{P}$ on the second plane of projection. As it was shown by Rabl, the points so obtained are just the *foci of the intersection of the desired ellipsoid of revolution and the second plane of projection.*

To find the intersection of the revolved parabola $(\mathcal{P})$ and the line l'' we used Steiner's construction. Projecting the points of $(\mathcal{P})$ from the ideal point represented by the direction of the axis of $(\mathcal{P})$ and from the point (P_1) we get two projectively related pencils. Under this projectivity the ideal line, as the tangent line of $(\mathcal{P})$ at a known point, corresponds to the line passing through the centres of the projections. Thus, intersecting three corresponding elements of the two pencils with the line l'' we obtain two projectively related linear ranges with common carrier line. The double points of this projectivity are the desired foci. The perpendicular bisector of the segment of foci gives the straight line containing the minor axis of the ellipse. The intersection of this line and $(\mathcal{P})$ can be constructed using Pascal's theorem, so we get the vertex (K) of $(\mathcal{P})$. The distance between K and the second projection of O_2 (the centre of S_2) is the length of the minor axis, so we have enough information to represent the second projection of the desired ellipsoid. Due to the special arrangement chosen, this ellipse is congruent to the first projection of the ellipsoid. Thus we achieved our goal: the representation of the tangent quadratic surface in a Maurin system, and hence in a Monge system as well.

Finally we sketch how to find the contact parallel circles. We construct both on the first and on the second plane of projection ellipses with double contact to the desired circles. This construction can be carried out easily using Lemma 1. The segments connecting the corresponding tangency points represent the circles on both planes of projection.

4. An elementary method of construction in the plane

We shall utilize the following nice observation, which can be found in J.L.S. Hatton's classical text [9].

Proposition 4.1 *Given three conics $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$. Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ have double contact with $\mathcal{C}_3$. Let A, B, C, D be the (proper or ideal) points of intersection of $\mathcal{C}_1$ and $\mathcal{C}_2$, P and P' the tangency points of $\mathcal{C}_1$ and $\mathcal{C}_3$, Q and Q' the tangency points of $\mathcal{C}_2$ and $\mathcal{C}_3$. Then the straight lines $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, $\overleftrightarrow{PP'}$ and $\overleftrightarrow{QQ'}$ are concurrent and form a harmonic quadruple.*

Lemma 2 *Suppose a conic $\mathcal{C}$ has double contact with the circles c_1 and c_2. Let the tangency points of $\mathcal{C}$ and c_1 be P and P', the tangency points of $\mathcal{C}$ and c_2 be Q and Q'. Then the straight lines $\overleftrightarrow{PP'}$ and $\overleftrightarrow{QQ'}$ are parallel, and the radical axis of c_1 and c_2 is their midparallel.*

Proof: Let the points of intersection of c_1 and c_2 be A, B, C, D. Then two of these points, e.g. A and B, are the absolute points (considered $\mathbb{P}^2(\mathbb{R})$ as a subspace of the complex projective plane). The remaining (real or imaginary) points C and D lie on the radical axis of c_1 and c_2. Thus $\overleftrightarrow{AB}$ is the ideal line of the plane, and the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ is represented by the pencil of all lines perpendicular to the line of the centres of c_1 and c_2. Now Proposition 4.1 implies that the point of intersection of $\overleftrightarrow{PP'}$ and $\overleftrightarrow{QQ'}$ is the ideal point of the radical axis of c_1 and c_2, and the lines $\overleftrightarrow{PP'}$, $\overleftrightarrow{CD}$, $\overleftrightarrow{QQ'}$, and $\overleftrightarrow{AB}$ form a harmonic quadruple. Since the ideal line is a member of the quadruple we conclude the statement. □

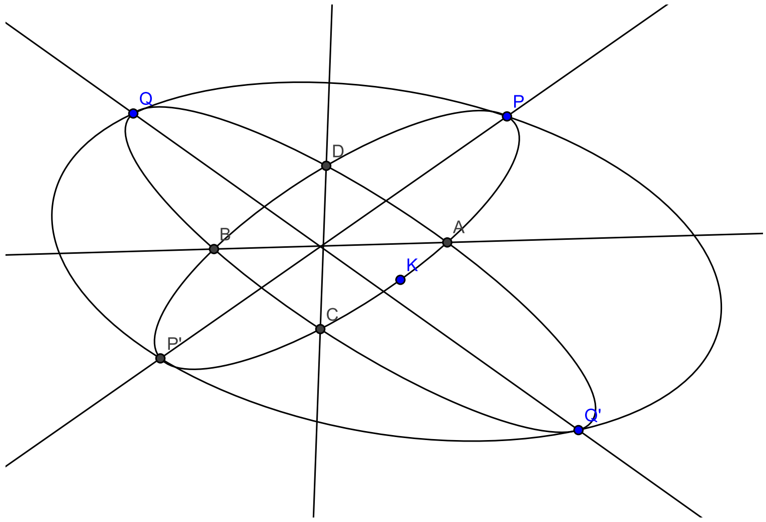

Figure 4: The straight lines $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, $\overleftrightarrow{PP'}$, and $\overleftrightarrow{QQ'}$ are concurrent and form a harmonic quadruple

Example 4.3: Given three circles c_1, c_2, c_3 with pairwise distinct collinear centres. Construct a conic $\mathcal{C}$ which has double contact with each of the three circles.

Construction: We use the following notation (see Fig. 5): l is the line passing through the centres of c_1, c_2, c_3;

$$\left.\begin{array}{c} P, P' \\ Q, Q' \\ R, R' \end{array}\right\} \text{ are the tangency points of } \left\{\begin{array}{l} c_1 \text{ and } C, \\ c_2 \text{ and } C, \\ c_3 \text{ and } C; \end{array}\right.$$

U, Y, Z are the points of intersection $\overleftrightarrow{PP'}$ and l, $\overleftrightarrow{QQ'}$ and l, $\overleftrightarrow{RR'}$ and l, resp., and H, I, J the points of intersection of l and the axis of c_1 and c_2, c_2 and c_3, c_3 and c_1, respectively.

Then by Lemma 2 H, I and J are the midpoints of U and Y, Y and Z, Z and U, resp. Thus with the arrangement displayed in Fig. 5 we have

$$d(U, H) = d(H, I) = d(H, J) = d(I, J).$$

Using this we can easily construct first the points U, Y, next the points P, P' and Q, Q'. The construction of R and R' is similar. Thus we obtain six points of the desired conic (ellipse in Fig. 5).

Example 4.4: Given two circles c_1 and c_2 with centres O_1 and O_2, resp. Construct a parabola $\mathcal{P}$ which has double contact with c_1 and c_2.

Construction: Let the tangency points of c_1 and $\mathcal{C}$ be P and P', and let $\mathcal{C}$ be tangent to c_2 at Q and Q'. Using the abbreviation $l := \overleftrightarrow{O_1O_2}$, suppose that l intersects the radical axis of c_1 and c_2 at H, the line$\overleftrightarrow{PP'}$ at U, and the line $\overleftrightarrow{QQ'}$ at Y.

By Lemma 2 segments $\overline{HU}$ and $\overline{HY}$ are congruent. The length of both $\overline{O_1U}$ and $\overline{O_2Y}$ is equal to the parameter p of $\mathcal{P}$, therefore we have

$$d(O_1, H) = d(O_2, H)$$

or

$$d(O_1, H) = d(H, U) + p \quad \text{and} \quad d(O_2, H) = d(H, U) - p.$$

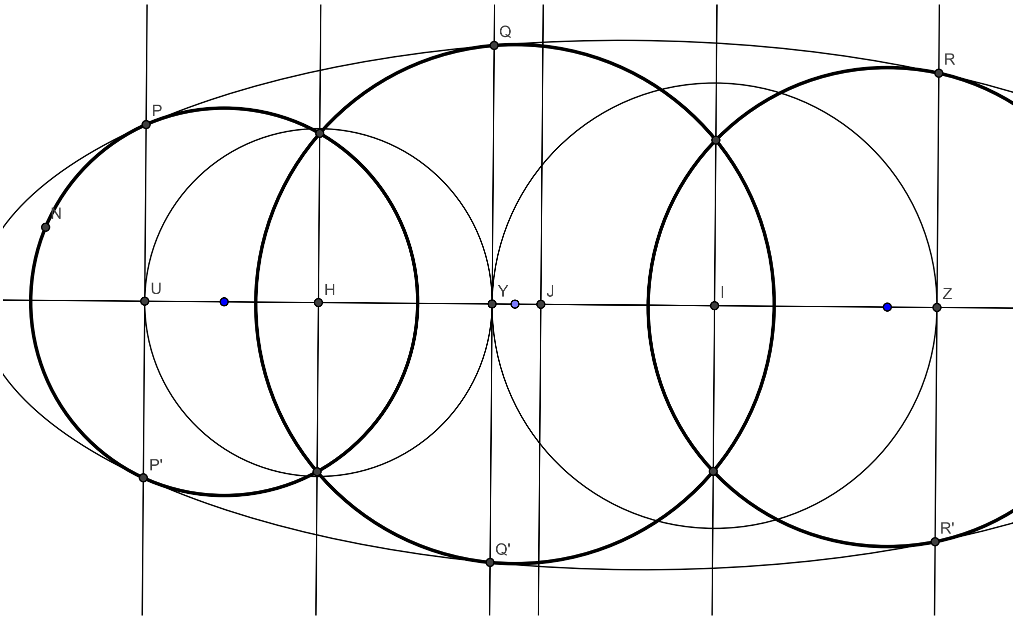

Figure 5: A conic enveloping three circles (Example 4.3)

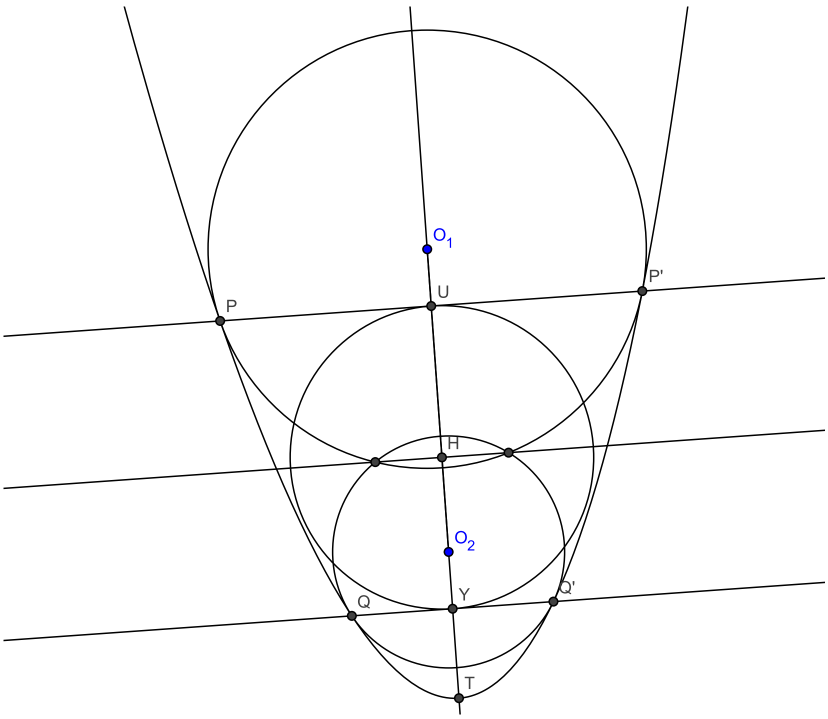

Figure 6: A parabola enveloping three circles (Example 4.4)

In the first case c_1 and c_2 are congruent, and we have only degenerate solutions. In the second case

$$d(H,U) = \frac{1}{2}\left(d(O_1,H) + d(O_2,H)\right),$$

so $\overline{HX}$ as well as the tangency points can be constructed immediately. It follows from Proposition 4.1 that the line l represents the ideal point of $\mathcal{P}$, so we have obtained five points of the desired parabola. Using Pascal's theorem, its vertex can also be easily constructed.

References

[1] M. Berger: *Geometry I.* Springer 1987.

[2] H. Brauner: *Eine Verallgemeinerung der Zyklographie.* Arch. Math. **9**, 470–480 (1958).

[3] L.P. Eisenhart: *An introduction to differential geometry with the use of tensor calculus.* Princeton University Press 1940.

[4] W. Fiedler: *Cyklographie, oder Konstruction der Aufgaben über Kreise und Kugeln und elementare Geometrie der Kreis- und Kugel-Systeme.* Leipzig 1882.

[5] K. Fladt: *Die nichteuklidische Zyklographie und ebene Inversionsgeometrie (Geometrie von Laguerre, Lie und Möbius) I.* Arch. Math. **7**, 391–398 (1956).

[6] K. Fladt: *Die nichteuklidische Zyklographie und ebene Inversionsgeometrie (Geometrie von Laguerre, Lie und Möbius) II.* Arch. Math. **7**, 399–405 (1956).

[7] L. Gyarmathi: *Konstruktive Lösung der Apollonius-Aufgabe im n-dimensionalen Raum durch Benützung einer Erweiterung der zyklographischen Abbildung auf mehrdimensionale Räume.* Publicationes Matematicae **1**, 123–128 (1949).

[8] L. Gyarmathi: *Die Anwendung der projizierenden Raumelemente bei der Maurinschen Abbildung des vierdimensionalen linearen Raumes* [Hungarian]. Matematikai Lapok **5**, 145–150 (1954).

[9] J.L.S. Hatton: *The Principles of Projective Geometry Applied to the Straight Line and Conic.* Cambridge University Press 1913.

[10] B. Kerékjártó: *Les fondements de la géométrie. Tome 2. Géométrie projective.* Gauthier-Villars – Akadémiai Kiadó, Paris – Budapest 1966.

[11] J. Maurin: *Géométrie descriptive a quatre dimensions.* Paris 1948.

[12] E. Müller, J. Krames: *Vorlesungen über Darstellende Geometrie, II: Zyklographie.* Verlag Deuticke, 1929.

[13] K. Rabl: *Eine bemerkungwerte Abbildung der Punkte des Raumes auf die Kreise einer Ebene.* Elem. Math. **29**, 6–12 (1974).

[14] T. Schwarcz: *On two generalizations of cyclographic projection* [Hungarian]. Acta Academiae Pedagogicae Nyíregyháziensis **11**/c, 57–75 (1988).

[15] G. Weiss: *$(n, 2)$-Axonometries and the Contour of Hyperspheres.* J. Geometry Graphics **1**, 157–167 (1997).

Received August 8, 2006; final form March 13, 2007