Reducts of the Henson graphs with a constant

András Pongrácz[1]

Central European University,
Department of Mathematics and its Applications,
1051 Budapest, Hungary, Nádor street 9.

Abstract

Let (H_n, E) denote the Henson graph, the unique countable homogeneous graph whose age consists of all finite K_n-free graphs. In this note the reducts of the Henson graphs with a constant are determined up to first-order interdefinability. It is shown that up to first-order interdefinability $(H_3, E, 0)$ has 13 reducts and $(H_n, E, 0)$ has 16 reducts for $n \geq 4$.

Keywords: first-order, reduct, Ramsey, homogeneous, automorphism
2010 MSC: 03C10, 03C35, 03C40, 03C50, 20B27, 20B35

1. Introduction

For $n \geq 3$ we denote by (H_n, E) the unique countable homogeneous graph that embeds a finite graph A if and only if A is K_n-free, where K_n denotes the complete graph on n vertices. The graphs (H_n, E) were first constructed by C. W. Henson in [1]. A. H. Lachlan and R. Woodrow [2] have shown that apart from trivial examples, the random graph (R, E), the Henson graphs (H_n, E) and their complements are the only countably infinite homogeneous graphs. A countable structure Δ is *homogeneous* if every isomorphism between finite induced substructures extends to an automorphism of Δ. In particular, vertices of (H_n, E) are indistinguishable: for all $u, v \in (H_n, E)$, there exists an automorphism $\alpha \in \mathrm{Aut}(H_n, E)$ such that $\alpha(u) = v$. Hence, there is no ambiguity

Email address: `andras.pongracz@lix.polytechnique.fr` (András Pongrácz)
[1]The author has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013 Grant Agreement no. 257039).

in the notation $(H_n, E, 0)$: it denotes the structure that we obtain by adding a constant symbol 0 to the signature of (H_n, E) and interpret it as a vertex of (H_n, E). In this paper, we classify the structures that are first-order definable (without parameters) in $(H_n, E, 0)$, i.e., the *reducts* of $(H_n, E, 0)$.

The first result of this form is due to P. J. Cameron [3], who has shown that the dense linear order $(\mathbb{Q}, <)$ has five reducts up to *first-order interdefinability*. Two structures Γ and Δ are first-order interdefinable if Γ has a first-order definition (without parameters) in Δ and vice versa, i.e., if they are reducts of one another. S. Thomas [4] proved that the random graph (R, E) has five reducts up to first-order interdefinability, and determined the reducts of the random k-uniform hypergraph for all $k \geq 2$ in [5]. In [4] it was shown that the Henson graphs (H_n, E) have no proper non-trivial reducts, i.e.,

Theorem 1.1. *[Thomas] Every reduct of (H_n, E) is first-order interdefinable either with (H_n, E) itself or with $(H_n, =)$ for all $n \geq 3$.*

In [4] Thomas posed the following conjecture.

Conjecture 1. *Every countable homogeneous structure over a finite relational language has finitely many reducts up to first-order interdefinability.*

J. H. Bennett has shown that the conjecture holds for the random tournament in [6]. Recently, M. Junker and M. Ziegler [7] proved that $(\mathbb{Q}, <, 0)$ has 116 reducts up to first-order interdefinability.

The purpose of this paper is to verify Thomas' conjecture for $(H_n, E, 0)$ for all $n \geq 3$. Note that $(H_n, E, 0)$ is indeed first-order interdefinable with a structure that is homogeneous in a finite relational language (see Remark 2.1). There is an essential difference between the result for $n = 3$ and for $n \geq 4$. Up to first-order interdefinability $(H_3, E, 0)$ has 13 reducts, and $(H_n, E, 0)$ has 16 reducts for $n \geq 4$ (see Theorem 2.3). This characterisation is based on the Nešetřil-Rödl theorem in [8] and a method introduced by M. Bodirsky and M. Pinsker applied in [9, 10, 11, 12]. The current note is the first implementation of the Bodirsky-Pinsker method to obtain a new first-order characterisation of the reducts of a homogeneous structure.

2. The main result

2.1. Closed groups

Let D be a countable set. A relational structure $\Gamma = (D, (Q_j)_{j \in J})$ is a *reduct* of $\Delta = (D, (R_i)_{i \in I})$ if Q_j is first-order definable from the set of relations $\{R_i \mid i \in I\}$ for all $j \in J$. If Γ is a reduct of Δ, then clearly $\mathrm{Aut}(\Delta) \subseteq \mathrm{Aut}(\Gamma)$. If Δ is ω-categorical, then the converse also holds by (a consequence of) Ryll-Nardzewski's theorem (see in [13]). A countable structure is *ω-categorical* if it is the unique countable model of its first-order theory up to isomorphism. If Δ is a countable structure that is homogeneous in a finite relational language, then Δ is ω-categorical, thus Ryll-Nardzewski's theorem [13] establishes a Galois connection between reducts of Δ and subgroups of $\mathrm{Sym}(D)$ that contain $\mathrm{Aut}(\Delta)$. Throughout the paper, $\mathrm{Sym}(D)$ denotes the full symmetric group acting on D, i.e., the group of all permutations of D. This Galois connection is given by the operators Aut mapping reducts to their automorphism groups, and Inv mapping permutation groups $\mathrm{Aut}(\Delta) \subseteq G \subseteq \mathrm{Sym}(D)$ to the structure with all relations on D that are invariant under the action of G. Just like every Galois connection, this gives rise to a closure operator. In our case, a permutation group $\mathrm{Aut}(\Delta) \subseteq G \subseteq \mathrm{Sym}(D)$ is *closed* if $G = \mathrm{Aut}(\Gamma)$ for some reduct Γ of Δ. Equivalently, G is closed if whenever $\alpha \in \mathrm{Sym}(D)$ is such that for all finite $F \subseteq D$ there exists a $\gamma \in G$ with $\alpha \restriction_F = \gamma \restriction_F$, then $\alpha \in G$. Moreover, given two reducts Γ_1 and Γ_2 of Δ, Γ_1 is a reduct of Γ_2 if and only if $\mathrm{Aut}(\Gamma_2) \subseteq \mathrm{Aut}(\Gamma_1)$. In particular, Γ_1 and Γ_2 are first-order interdefinable if and only if $\mathrm{Aut}(\Gamma_1) = \mathrm{Aut}(\Gamma_2)$. Thus reducts of a countable, homogeneous structure Δ in a finite relational language up to first-order interdefinability can be understood via the characterisation of closed supergroups of $\mathrm{Aut}(\Delta)$ in $\mathrm{Sym}(D)$. By ordering reducts $\Gamma_1 \preceq \Gamma_2$ if and only if Γ_1 is a reduct of Γ_2, and factoring out by first-order interdefinability, we obtain a complete lattice on the equivalence classes. The strongest form of

Ryll-Nardzewski's theorem states that the lattice we obtain this way is anti-isomorphic to the lattice of closed groups $\mathrm{Aut}(\Delta) \subseteq G \subseteq \mathrm{Sym}(D)$ ordered by inclusion. The anti-isomorphism is given by the operators Aut and Inv that are inverses of each other. In Subsection 2.2, we show a picture of the lattice of closed supergroups of $\mathrm{Aut}(H_n, E, 0)$ (see Theorem 2.3). Hence, one can obtain the lattice of reducts of $(H_n, E, 0)$ up to first-order interdefinability by turning that picture upside-down.

2.2. Reduct classification

To present the main result of the paper, we need the following definitions.

Definition 2.1. *We denote by U_1 and U_2 the set of all neighbours and non-neighbours of 0 in $(H_n, E, 0)$, respectively. As an abuse of notation, we denote three formally different things by 0: the constant symbol 0, the vertex in (H_n, E) that is the interpretation of 0 and the unary relation that is interpreted as $\{0\}$.*

Using standard terminology, we say that a function $f : H_n \to H_n$ *preserves* a relation R on H_n if whenever a tuple is in R, its f-image is also in R. If f does not preserve R, then f *violates* R.

Definition 2.2. *Let $X_1, X_2 \subseteq H_n$ be disjoint sets, and let G_1, G_2 be permutation groups acting on X_1, X_2, respectively. Then $G_1 \times G_2$ denotes the group of all permutations $\alpha \in \mathrm{Sym}(H_n)$ such that $\alpha \restriction_{X_i} \in G_i$ for $i \in \{1, 2\}$, and α fixes $H_n \setminus (X_1 \cup X_2)$ pointwise. The group $(\mathrm{Sym}(X_1) \times \mathrm{Sym}(X_2)) \rtimes Z_2$ consists of the permutations in $\mathrm{Sym}(H_n)$ that either preserve X_1 and X_2 or flip X_1 and X_2, and fix $H_n \setminus (X_1 \cup X_2)$ pointwise. We denote by $\mathrm{Sym}(H_n \setminus \{0\})$ the group of all permutations in $\mathrm{Sym}(H_n)$ that fix 0.*

Theorem 2.3. *The closed supergroups of $\mathrm{Aut}(H_n, E, 0)$ in $\mathrm{Sym}(H_n)$ are*

1. $\mathrm{Aut}(H_n, E, 0)$
2. $\mathrm{Aut}(H_n, E)$
3. $\mathrm{Aut}(H_n, 0, E \restriction_{H_n \setminus \{0\}})$
4. $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$
5. $\mathrm{Sym}(U_1) \times \mathrm{Aut}(U_2, E)$
6. $\mathrm{Sym}(U_2) \times \mathrm{Aut}(U_1, E)$
7. $\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)$
8. $\mathrm{Sym}(U_1 \cup \{0\}) \times \mathrm{Aut}(U_2, E)$
9. $\mathrm{Sym}(U_2 \cup \{0\}) \times \mathrm{Aut}(U_1, E)$
10. $(\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)) \rtimes Z_2$
11. $\mathrm{Sym}(U_1 \cup \{0\}) \times \mathrm{Sym}(U_2)$
12. $\mathrm{Sym}(U_2 \cup \{0\}) \times \mathrm{Sym}(U_1)$
13. $\mathrm{Sym}(H_n \setminus \{0\})$
14. $(\mathrm{Sym}(U_1 \cup \{0\}) \times \mathrm{Sym}(U_2)) \rtimes Z_2$
15. $(\mathrm{Sym}(U_2 \cup \{0\}) \times \mathrm{Sym}(U_1)) \rtimes Z_2$
16. $\mathrm{Sym}(H_n)$

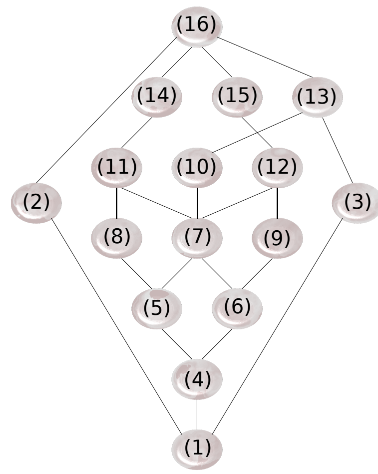

Figure 1: The lattice of closed groups

If $n \geq 4$, then all these groups are different, and $(H_n, E, 0)$ has 16 reducts up to first-order interdefinability. If $n = 3$, then three pairs of groups in the list are identified by the equation $\mathrm{Aut}(U_1, E) = \mathrm{Sym}(U_1)$, and $(H_3, E, 0)$ has 13 reducts up to first-order interdefinability.

One can also provide a description of all reducts of $(H_n, E, 0)$ up to first-order interdefinability by using the other side of the Galois connection: relational structures. In other words, we can construct a representative in every equivalence class of first-order interdefinability, i.e., structures corresponding to each automorphism group in the above list. We provide two examples, the rest of the cases are left to the reader. Item (5) is the automorphism group of the structure $(H_n, 0, U_1, U_2, E \restriction_{U_2})$. In case of item (10), let E' be a binary relation symbol. The interpretation of E' is a complete bipartite graph on the set $U_1 \cup U_2$ with bipartition (U_1, U_2). Then E' has a first-order definition in $(H_n, E, 0)$, and the automorphism group of $(H_n, 0, E')$ is $(\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)) \rtimes Z_2$.

It is easy to show that there is a representative in every equivalence class that is homogeneous in an at most binary relational language.

Remark 2.1. $(H_n, E, 0)$ *is first-order interdefinable with the relational structure* $(H_n, E, 0, U_1, U_2)$*, and the latter structure is homogeneous.*

By Remark 2.1, Theorem 2.3 is a special case of Thomas' conjecture.

3. Preliminaries

3.1. Ramsey theory

In [8] the following Ramsey-type theorem is shown. Note that throughout the paper $A \leq \Delta$ means that A is a substructure of Δ.

Theorem 3.1 (Nešetřil, Rödl). *Let* $n \geq 3$ *and* $r \geq 2$*. Then for all finite* K_n*-free graphs* A *there exists a finite* K_n*-free graph* B *such that if edges and non-edges of* B *are coloured with* r *colours, then there exists a copy* $A' \leq B$ *of* A *that is monochromatic, i.e., all edges have the same colour and all non-edges have the same colour.*

The class of ordered K_n-free graphs has an even stronger property, namely that it is a Ramsey class [14]. A class $\mathfrak{C}$ of finite structures is called a Ramsey class if for all $A, B \in \mathfrak{C}$ and $r \in \mathbb{N}$ there is a $C \in \mathfrak{C}$ such that if the copies of A in C are coloured with r colours, then there is a copy $B' \leq C$ isomorphic to B that is monochromatic. The class of finite structures that embed into a structure Δ is called the *age* of Δ, and it is denoted by $\mathrm{Age}(\Delta)$. A (homogeneous) structure Δ is a Ramsey structure if $\mathrm{Age}(\Delta)$ is a Ramsey class. If a class $\mathfrak{C}$ of finite structures is the age of some countable homogeneous structure, then this structure is uniquely determined up to isomorphism, and it is denoted by $\mathrm{Flim}(\mathfrak{C})$ (see [13]). A structure Δ is *ordered* if there exists a total order that is first-order definable in Δ.

Theorem 3.2 (Nešetřil, Rödl). *Let $\mathfrak{C}$ be the class of all finite ordered K_n-free graphs. Then $\mathfrak{C}$ is a Ramsey class for all $n \geq 3$. In particular,* $\mathrm{Flim}(\mathfrak{C})$ *is a homogeneous ordered Ramsey structure, i.e., given any $n \geq 3$, $r \geq 2$ and finite ordered K_n-free graphs A, B, there exists a finite ordered K_n-free graph C such that if the copies of A in C are coloured with r colours, then there is a monochromatic copy of B in C.*

Given a $c \in \Delta$ we denote by (Δ, c) the structure obtained by adding a constant symbol to the language of Δ interpreted as the element c. In [12] the following is shown.

Proposition 3.3 (Bodirsky, Pinsker, Tsankov). *Let Δ be a countable, homogeneous, ordered Ramsey structure, and let $c \in \Delta$. Then (Δ, c) is an ordered Ramsey structure.*

We need to generalise Theorem 3.1 for structures that we obtain by adding finitely many constants to a Henson graph.

Definition 3.4. *Let $k \in \mathbb{N}$. We call a class $\mathfrak{C}$ of finite structures a k-*Ramsey class *if for all $A, B \in \mathfrak{C}$ with $|A| \leq k$ and for all $r \in \mathbb{N}$ there exists a $C \in \mathfrak{C}$ such that if the copies of A in C are coloured with r colours, then there is a copy $B' \leq C$ isomorphic to B that is monochromatic. We call a (homogeneous) structure Δ a k-Ramsey structure if* $\mathrm{Age}(\Delta)$ *is a k-Ramsey class.*

Proposition 3.5. *Let $n \geq 3$, $t \geq 0$ and $c_1, \ldots, c_t \in H_n$. Then $(H_n, E, c_1, \ldots, c_t)$ is 2-Ramsey.*

Proof. Let $\{S_1, S_2, \ldots, S_k\}$ be a set containing exactly one copy of each at most 2-element structure in $\mathrm{Age}(H_n, E, c_1, \ldots, c_t)$ up to isomorphism. We show that for any $r \geq 2$, $S \in \{S_1, S_2, \ldots, S_k\}$ and $B \in \mathrm{Age}(H_n, E, c_1, \ldots, c_t)$ there exists a $C \in \mathrm{Age}(H_n, E, c_1, \ldots, c_t)$ such that if the copies of S in C are coloured with r colours, then there is a monochromatic copy of B in C.

According to Theorem 3.2 and Proposition 3.3 we can extend the language of $(H_n, E, c_1, \ldots, c_t)$ with a total order $\prec$ so that it becomes an ordered Ramsey structure. We claim that $B \leq (H_n, E, c_1, \ldots, c_t)$ has an ordered version $B^{\prec} \in \mathrm{Age}(H_n, E, \prec, c_1, \ldots, c_t)$ such that all ordered versions of the copies of S in $B^{\prec}$ are isomorphic to some $S^{\prec}$. If $|S| = 1$ or $|S| = 2$ and both vertices of S have the same 1-type in $(H_n, E, c_1, \ldots, c_t)$, i.e., they satisfy the same first-order formulas in $(H_n, E, c_1, \ldots, c_t)$, then S has only one ordered version in $\mathrm{Age}(H_n, E, \prec, c_1, \ldots, c_t)$ up to isomorphism. Thus we may assume that $|S| = 2$ and the two vertices of S have different 1-types t_1 and t_2 in $(H_n, E, c_1, \ldots, c_t)$. If all vertices of type t_1 are smaller than all vertices of type t_2 with respect to $\prec$, or vice versa, then the claim follows immediately. If this is not the case, then there exists an appropriate $B^{\prec} \leq \mathrm{Age}(H_n, E, \prec, c_1, \ldots, c_t)$ such that all vertices in $B^{\prec}$ that have type t_1 are smaller than those of type t_2. Hence, the claim follows. According to the Ramsey property of $(H_n, E, \prec, c_1, \ldots, c_t)$ there is a $C^{\prec}$ in $\mathrm{Age}(H_n, E, \prec, c_1, \ldots, c_t)$ such that if the copies of $S^{\prec}$ in $C^{\prec}$ are coloured with r colours, then there is a monochromatic copy of $B^{\prec}$ in $C^{\prec}$. The structure that we obtain by omitting $\prec$ from $C^{\prec}$ is an appropriate choice for C.

3.2. Closed monoids and canonical functions

Similarly to closed subgroups of $\mathrm{Sym}(D)$, it is possible to define closed submonoids of the monoid of all unary operations on D, i.e., D^D. The topology is the topology of pointwise convergence on D^D, so a unary function f is in the closure of a set of unary operations $S \subseteq D^D$ if and only if f can be interpolated on any finite subset of D by some function in S. In this case, we also say that S *generates* f. Note that there is a slight ambiguity between the notions of a closed group and a closed monoid, namely, a closed group G normally generates a lot of functions that are not in G. In particular, the monoid closure of $\mathrm{Aut}(H_n, E, 0)$ is the set of all self-embeddings of $(H_n, E, 0)$, that is, the set of all injective (but not necessarily surjective) unary operations on H_n that fix 0 and preserve the edge relation E and the non-edge relation N. The main idea of the general strategy introduced by M. Bodirsky and M. Pinsker in [9, 10, 11, 12] to investigate reducts of countable homogeneous structures Δ is to show that any function f together with the automorphism group $\mathrm{Aut}(\Delta)$ generates a so-called *canonical* function (see Proposition 3.7 for the precise statement we need).

Definition 3.6. *A function $g : \Delta \to \Gamma$ is canonical if whenever two tuples $\bar{x}, \bar{y} \in \Delta^n$ satisfy the same first-order formulas in Δ (that is, they have the same n-type), then the tuples $g(\bar{x})$ and $g(\bar{y})$ also have the same n-type in Γ. The behaviour of a canonical function g is the set of all type conditions satisfied by g, i.e., the collection of all pairs (s, t) where s and t are n-types of Δ and Γ, respectively, and whenever $\bar{x}$ has type s we have that $g(\bar{x})$ has type t.*

If Δ and Γ are arbitrary ω-categorical structures, then we might need all (or at least infinitely many) type conditions (s, t) in order to describe the behaviour of a canonical function $f : \Delta \to \Gamma$. This is not the case, however, if Δ and Γ are homogeneous in a finite relational language. The reason is that if Δ is homogeneous in a finite relational language with maximal arity m, then the type of a tuple is uniquely determined by the type of its m-element subtuples. In particular, as any structure that we obtain by adding finitely many constants to (H_n, E) is homogeneous in a binary relational language, we have the following.

Remark 3.1. *Let $s, t \geq 0$, and let $c_1, \ldots, c_s, d_1, \ldots, d_t \in H_n$. A function $g : (H_n, E, c_1, \ldots, c_s) \to (H_n, E, d_1, \ldots, d_t)$ is canonical if and only if the following two conditions hold.*

- *For any 1-element structure $S \in \mathrm{Age}(H_n, E, c_1, \ldots, c_s)$ there exists a 1-element structure $S' \in \mathrm{Age}(H_n, E, d_1, \ldots, d_t)$ such that the g-image of any copy of S is isomorphic to S'.*
- *For any 2-element structure $S \in \mathrm{Age}(H_n, E, c_1, \ldots, c_s)$ we have that whenever $S_1, S_2 \leq (H_n, E, c_1, \ldots, c_s)$ are copies of S, then $g(S_1) \in E \Leftrightarrow g(S_2) \in E$.*

Moreover, the behaviour of g is uniquely determined by the type conditions it satisfies for 1-element substructures and the set of isomorphism types of 2-element substructures that are mapped to edges by g.

In Remark 3.1 the structures $(H_n, E, c_1, \dots, c_s)$ and $(H_n, E, d_1, \dots, d_t)$ are to be understood as relational structures. The languages of these structures consist of all at most binary first-order definable relations. The main ideas of the following argument can be found in [9, Proposition 21]. As there are some subtle technical problems to work out in order to obtain what we need, we present the full proof.

Proposition 3.7. *Let $s, t \geq 0$, and let $c_1, \dots, c_s, d_1, \dots, d_t \in H_n$. Let $\Delta = (H_n, E, c_1, \dots, c_s)$ and $\Gamma = (H_n, E, d_1, \dots, d_t)$, and let $f : \Delta \to \Gamma$ be an injective function. Then there exists an injective function*

$$g \in \overline{\{\beta \circ f \circ \alpha \mid \alpha \in \mathrm{Aut}(\Delta), \beta \in \mathrm{Aut}(\Gamma)\}}$$

such that g is canonical as a function from Δ to Γ, and $g(c_i) = f(c_i)$ for all $i \in \{1, \dots, s\}$.

Proof. Let $A_i \leq \Delta$ for $i \in \mathbb{N}$ be such that $A_1 \subsetneq A_2 \subsetneq \cdots$ and $\bigcup A_i = \Delta$. Let $S_1, \dots, S_q$ be a set of representatives of the isomorphism types of at most 2-element substructures of Δ, and let $\{T_1, \dots, T_r\}$ consist of the symbols E, N, and the isomorphism types of 1-element substructures of Γ. According to Proposition 3.5 for all A_j there exists a $B_j \in \mathrm{Age}(\Delta)$ such that if the at most 2-element substructures of B_j are coloured with r colours, then there is a monochromatic copy A'_j of A_j in B_j. Let us choose an arbitrary[2] copy of B_j in Δ and colour its at most 2-element substructures by the symbol T_i corresponding to their f-image. Then for any $1 \leq m \leq q$ we have that all copies of S_m in the monochromatic A'_j have the same f-image up to isomorphism. This way we can assign a set of type conditions b_j to the at most 2-element substructures of A'_j for all $j \in \mathbb{N}$. As there are only finitely many possible set of such type conditions, there is a behaviour b that occurs infinitely many times in the sequence $(b_j)_{j \in \mathbb{N}}$. By thinning out the sequence $(A_j)_{j \in \mathbb{N}}$, we may assume that $b_j = b$ for all $j \in \mathbb{N}$.

[2]Later on, we will need some variants of this argument where we can make use of a less arbitrary choice of the copy of B_j.

Let $\alpha_j \in \mathrm{Aut}(\Delta)$ be such that $\alpha_j(A_j) = A'_j$. Then $f \circ \alpha_j$ modifies the at most 2-element substructures of A_j according to the type conditions in b. Hence, if $j < \ell$, then the mapping $(f \circ \alpha_\ell \circ (f \circ \alpha_j)^{-1}) \restriction_{f \circ \alpha_j(A_j)}$ preserves unary relations, E and N in Γ, and thus by homogeneity of Γ it extends to an automorphism $\beta_{j,\ell} \in \mathrm{Aut}(\Gamma)$. The sequence $\beta_{1,2}^{-1} \circ \cdots \circ \beta_{j,j+1}^{-1} \circ f \circ \alpha_{j+1}$ is convergent in the closed monoid of all injective functions, and it tends to a canonical function $h : \Delta \to \Gamma$ with behaviour b. The function $h \circ f^{-1} \restriction_{\{c_1,\ldots,c_s\}}$ preserves unary relations, E and N in Γ, and consequently, it extends to an automorphism $\beta \in \mathrm{Aut}(\Gamma)$. Thus $g = \beta^{-1} \circ h$ is the limit of the sequence $(\beta^{-1} \circ \beta_{1,2}^{-1} \circ \cdots \circ \beta_{j,j+1}^{-1}) \circ f \circ \alpha_{j+1}$, g agrees with f on the constants $\{c_1, \ldots, c_s\}$, and g is canonical from Δ to Γ.

3.3. Canonical functions and closed groups

Definition 3.8. *The U_i-part of a structure $A \leq (H_n, E, 0, U_1, U_2)$ is $U_i \cap A$ for $i \in \{1, 2\}$. The 0-part of a structure $A \leq (H_n, E, 0, U_1, U_2)$ is $\{0\} \cap A$. The intermediate pairs in a structure $A \leq (H_n, E, 0, U_1, U_2)$ are the 2-element substructures of A with one point in U_1 and one point in U_2. Intermediate edges and non-edges are the intermediate pairs constituting an edge and a non-edge, respectively.*

Definition 3.9. *Let $X, Y \subseteq H_n$ be disjoint sets. We say that a functions f eradicates edges (non-edges) on X if every pair of elements in X is mapped to a non-edge (edge) by f. Similarly, f eradicates edges (non-edges) between X and Y if every pair of elements in $X \times Y$ is mapped to a non-edge (edge) by f. If f eradicates edges (non-edges) between U_1 and U_2, then we say that f eradicates intermediate edges (non-edges). If f eradicates intermediate edges or non-edges, then we say that f eradicates intermediate pairs.*

The proof of Proposition 3.7 can be refined to show the following statement.

Lemma 3.10. *Let $\mathrm{Aut}(H_n, E, 0) \subseteq G$ be a closed group. Assume that for any $A \in \mathrm{Age}(H_n, E, 0)$ there exists a copy $A' \leq (H_n, E, 0)$ of A and a permutation $\pi_A \in G$ that eradicates intermediate pairs of A'. Then we have that*

$\mathrm{Aut}(U_1,E)\times\mathrm{Aut}(U_2,E)\subseteq G$. *In particular, if G generates a canonical function $g:(H_n,E,0)\to(H_n,E,0)$ that eradicates intermediate pairs, then G contains* $\mathrm{Aut}(U_1,E)\times\mathrm{Aut}(U_2,E)$.

Proof. By using the condition in the first part of the assertion, the copy of B_j in the proof of Proposition 3.7 can be chosen such that intermediate pairs of B_j are eradicated by some permutation in G. Thus after thinning out the sequence b_j in the proof of Proposition 3.7, the generated canonical function $g:(H_n,E,0)\to(H_n,E,0)$ eradicates intermediate pairs, and it is enough to prove the second part of the assertion.

To this end we have to show that any permutation $\alpha\in\mathrm{Aut}(U_1,E)\times\mathrm{Aut}(U_2,E)$ can be interpolated on any finite substructure of H_n by an element of G. Let $A\le(H_n,E,0)$ be finite and let $B=\alpha(A)$. Then A and B differ only in the intermediate pairs, i.e, $\alpha\restriction_A$ is a partial isomorphism of $(H_n,E,0)$ except that some intermediate edges might be mapped to non-edges and vice versa. There exist $\gamma_A,\gamma_B\in G$ such that $\gamma_A\restriction_A=g\restriction_A$ and $\gamma_B\restriction_B=g\restriction_B$. Thus $(\gamma_B\circ\alpha\circ\gamma_A^{-1})\restriction_{\gamma_A(A)}$ is a partial isomorphism of $(H_n,E,0)$, and consequently, it extends to some $\beta\in\mathrm{Aut}(H_n,E,0)$. Hence, $(\gamma_B^{-1}\circ\beta\circ\gamma_A)\restriction_A=\alpha\restriction_A$.

Lemma 3.11. *Let* $\mathrm{Aut}(U_1,E)\times\mathrm{Aut}(U_2,E)\subseteq G$ *be a closed group. Assume that G generates a canonical function $g:(H_n,E,0)\to(H_n,E)$ that eradicates edges on U_ℓ for some $\ell\in\{1,2\}$. Then G contains* $\mathrm{Aut}(U_m,E)\times\mathrm{Sym}(U_\ell)$ *with* $\{\ell,m\}=\{1,2\}$.

Proof. Let $\alpha\in\mathrm{Aut}(U_m,E)\times\mathrm{Sym}(U_\ell)$, and let $A,B\le(H_n,E,0)$ be such that $\alpha(A)=B$. Then A and B have isomorphic U_m-parts and 0-parts, and they have the same number of vertices in U_2. By applying some permutations in $\mathrm{Aut}(U_1,E)\times\mathrm{Aut}(U_2,E)$ we may assume that there are no intermediate edges in A and in B. Let $\gamma_A,\gamma_B\in G$ be such that γ_A and γ_B eradicate edges on the U_ℓ-part of A and B, respectively. Then $(\gamma_B\circ\alpha\circ\gamma_A^{-1})\restriction_{\gamma_A(A)}$ is a partial isomorphism of $(H_n,E,0)$, and thus it extends to some $\beta\in\mathrm{Aut}(H_n,E,0)$. Hence, $\gamma_B^{-1}\circ\beta\circ\gamma_A\in G$ agrees with α on A.

4. Closed supergroups of the automorphism group

4.1. Destroying structure on U_1 or U_2

Definition 4.1. *We denote by I_n the empty graph on n vertices.*

Lemma 4.2. *Let* $\mathrm{Aut}(H_n, E, 0) \subseteq G$ *be a closed group, and assume that G generates a canonical function $g : (H_n, E, 0) \to (H_n, E)$ that violates E or N on U_2. Then G contains* $\mathrm{Aut}(U_1, E) \times \mathrm{Sym}(U_2)$.

Proof. Since I_n embeds into U_2, g cannot violate N on U_2, as otherwise the image of g would contain a copy of K_n. Thus g eradicates edges on U_2. We show that G contains $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$.

According to Lemma 3.10 it is enough to prove that for any finite $A \leq (H_n, E, 0)$ there exists an element of G that maps all intermediate pairs of A to non-edges. Let us denote the vertices of A in U_1 by $X = \{x_1, \ldots, x_r\}$ and in U_2 by $Y = \{y_1, \ldots, y_s\}$. The intermediate edges of A are going to be deleted in r steps, i.e., with the composition of r permutations $\pi_1, \ldots, \pi_r$ in G. The i-th step is as follows. Assume that the intermediate pairs containing $x_1, \ldots, x_{i-1}$ are already mapped to non-edges by the permutation $\pi_{i-1} \circ \cdots \circ \pi_1$ such that $\pi_{i-1} \circ \cdots \circ \pi_1$ maps the elements of Y into U_2. Elements of X are not necessarily mapped into U_1. Let $v_1, v_2, \ldots, v_{i-1}$ be the images of $x_1, x_2, \ldots, x_{i-1}$, respectively. Let u_i be the image of x_i, and let $z_1, \ldots, z_s$ be the images of $y_1, \ldots, y_s$, respectively. We need a permutation $\pi_i \in G$ such that $\pi_i(v_j z_k) \in N$, $\pi_i(u_i z_k) \in N$ and $\pi_i(z_k) \in U_2$ for all j, k. Let $(\pi_{i-1} \circ \cdots \circ \pi_1)(A) = A'$.

We construct a structure $B \leq (H_n, E, 0)$ by using A'. The vertices $z_1, \ldots, z_s$ are replaced by the elements $\{z_{p,q} \mid 1 \leq p \leq s, 1 \leq q \leq n-1\}$ in U_2 such that $z_{p,1} = z_p$ for all $1 \leq p \leq s$. The new vertices are chosen such that for every vertex $w \in A'$ we have $w z_{p,q} \in E \Leftrightarrow w z_p \in E$ for all p, q, and similarly, $z_{p_1,q_1} z_{p_2,q_2} \in E \Leftrightarrow z_{p_1} z_{p_2} \in E$ for all p_1, p_2, q_1, q_2.

We need to verify that B is indeed in $\mathrm{Age}(H_n, E, 0)$. As $z_{p,q_1} z_{p,q_2} \in N$ for all p, q_1, q_2, a complete subgraph K of $B \cup \{0\}$ cannot contain two vertices of the form $\{z_{p,q_1}, z_{p,q_2}\}$. Thus all the vertices of the form $z_{p,q}$ in K have different first indices. The function that is identical on $K \cap (A' \cup \{0\})$ and maps $z_{p,q}$ to z_p for all vertices of the form $z_{p,q} \in K$ is an embedding into $A' \cup \{0\}$, thus K has at most $n-1$ elements.

Let $C = \{v_1, v_2, \ldots, v_{i-1}, z'_{1,1}, z'_{1,2}, \ldots, z'_{s,n-1}\} \leq (H_n, E, 0)$ be such that $v_j z'_{p,q} \in N$ for all j, p, q, $z'_{p_1,q_1} z'_{p_2,q_2} \in E \Leftrightarrow (p_1 = p_2) \wedge (q_1 \neq q_2)$ for all p_1, p_2, q_1, q_2 and $z'_{p,q} \in U_2$ for all p, q.

Let $D = \{v_1, v_2, \ldots, v_{i-1}, z_{1,1}, z_{1,2}, \ldots, z_{s,n-1}\}$ and let $f : D \to C$ be the function that fixes $v_1, \ldots, v_{i-1}$ and maps every element of the form $z_{p,q}$ to $z'_{p,q}$. There are permutations $\gamma_D, \gamma_C \in G$ such that $\gamma_D \restriction_D = g \restriction_D$ and $\gamma_C \restriction_C = g \restriction_C$. There exists a $\beta \in \mathrm{Aut}(H_n, E, 0)$ such that $(\gamma_C \circ f \circ \gamma_D^{-1}) \restriction_{\gamma_D(D)} = \beta \restriction_{\gamma_D(D)}$. Hence, $(\gamma_C^{-1} \circ \beta \circ \gamma_D) \restriction_D = f \restriction_D$, and $\delta = \gamma_C^{-1} \circ \beta \circ \gamma_D \in G$. For any $1 \leq p \leq s$ the vertex $\delta(u_i)$ cannot be connected to all vertices of the form $\delta(z_{p,q})$, as the vertices $\delta(z_{p,q})$, $1 \leq q \leq n-1$ induce a complete graph of size $n-1$. Thus for all $1 \leq p \leq s$ there exists a $1 \leq q(p) \leq n-1$ such that $\delta(u_i)\delta(z_{p,q(p)}) \in N$. Let $\mu \in \mathrm{Aut}(H_n, E, 0)$ be such that μ fixes every element of A' that is not of the form z_p, and $\mu(z_p) = z_{p,q(p)}$. Then $\pi_i = \delta \circ \mu$ is an appropriate choice. Thus G indeed contains $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$ by Lemma 3.10, and the statement follows from Lemma 3.11.

If we switch the roles of U_1 and U_2 in the proof of Lemma 4.2, then the proof fails when we construct C, as K_{n-1} cannot be embedded into U_1. Hence, the analogue version of Lemma 4.2 is somewhat more complicated to prove, and it requires an auxiliary lemma.

Lemma 4.3. *Let* $\mathrm{Aut}(H_n, E, 0) \subseteq G$ *be a closed group with* $n \geq 4$. *Assume that for any* K_n*-free graph* S *there exists a permutation* $\pi_S \in G$ *and a copy* S' *of* S *in* U_2 *such that* $\pi_S(S')$ *is* K_3*-free. Then* G *generates a canonical function* $g : (H_n, E, 0) \to (H_n, E)$ *that violates* E *on* U_2.

Proof. We slightly modify the argument in the proof of Proposition 3.7. The sequence $B_1, B_2, \ldots$ can be chosen so that the image of the U_2-part of B_j under some permutation in G is K_3-free for all $j \geq 1$. Then for large enough j the canonical behaviour assigned to A'_j cannot preserve E on U_2. Thus after thinning out the sequence we obtain a canonical function that violates E on U_2.

Lemma 4.4. *Let* $\mathrm{Aut}(H_n, E, 0) \subseteq G$ *be a closed group, and assume that* G *generates a canonical function* $g : (H_n, E, 0) \to (H_n, E)$ *that violates* E *or* N *on* U_1*. Then* G *contains* $\mathrm{Aut}(U_2, E) \times \mathrm{Sym}(U_1)$.

Proof. Just as in the proof of Lemma 4.2 we have that g eradicates edges on U_1. For $n = 3$, U_1 is an independent set, thus we may assume that $n \geq 4$. According to Lemma 3.11 it suffices to show that G contains $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$. According to Lemmas 4.3 and 4.2 we may assume that there is a K_n-free graph S such that the image of any copy of S in U_2 under any permutation in G contains a triangle. Throughout the proof we fix such a graph S.

The method is similar to that of the proof of Lemma 4.2. Let $A \leq (H_n, E, 0)$ be finite. Let us denote the vertices of A in U_1 by $X = \{x_1, \ldots, x_r\}$ and in U_2 by $Y = \{y_1, \ldots, y_s\}$. According to Lemma 3.10 it is enough to show that there exists an element of G such that the intermediate edges of A are mapped to non-edges. This permutation will be constructed in s steps. In the i-th step we construct a permutation π_i. Assume that the intermediate pairs of A containing $y_1, \ldots, y_{i-1}$ are mapped to non-edges by the permutation $\pi_{i-1} \circ \cdots \circ \pi_1$. Assume further that the permutation $\pi_{i-1} \circ \cdots \circ \pi_1$ maps X into U_1. Let us denote the images of the points $x_1, \ldots, x_r, y_1, \ldots, y_{i-1}, y_i$ under the permutation $\pi_{i-1} \circ \cdots \circ \pi_1$ by $z_1, \ldots, z_r, v_1, \ldots, v_{i-1}, u_i$, respectively. We need to find a permutation π_i such that $\pi_i(v_j z_k) \in N$, $\pi_i(u_i z_k) \in N$ and $\pi_i(z_k) \in U_1$ for all j, k. Let $A' = \{z_1, \ldots, z_r, v_1, \ldots, v_{i-1}, u_i\}$.

Let $B \leq (H_n, E, 0)$ be a finite structure whose U_2-part and 0-part are equal to those of A' such that there is a function $f_1 : A' \to B$ that is a partial isomorphism of $(H_n, E, 0)$ except that the U_1-part of B is an independent set, thus f_1 might violate E on U_1. There exist $\gamma_{A'}, \gamma_B \in G$ such that $\gamma_{A'} \restriction_{A'} = g \restriction_{A'}$ and $\gamma_B \restriction_B = g \restriction_B$. Then there is a $\beta \in \mathrm{Aut}(H_n, E, 0)$ such that $\gamma_B \circ f_1 \circ \gamma_{A'}^{-1} \restriction_{\gamma_{A'}(A')} = \beta \restriction_{\gamma_{A'}(A')}$. Hence, the function f_1 extends to $\gamma_B^{-1} \circ \beta \circ \gamma_{A'} \in G$, and thus we may assume that the U_1-part of A' is an independent set. In particular, the set $\{z_1, \dots, z_r\}$ induces an empty graph. If $u_i \in U_1$ then we are done, so we may assume that $u_i \notin U_1$. We may assume that $u_i \neq 0$, as otherwise we can replace y_i by $\mu(y_i)$ where $\mu \in \mathrm{Aut}(H_n, E, 0)$ fixes $A \setminus \{y_i\}$, and then $\pi_{i-1} \circ \cdots \circ \pi_1 \circ \mu$ maps y_i to a non-zero element while all the properties we assumed so far hold. Thus $u_i \in U_2$. Let us denote by r_E and r_N the number of vertices in $\{z_1, \dots, z_r\}$ that are connected and not connected to u_i, respectively. Let $C = \{v_1, \dots, v_{i-1}, u_i\}$.

We construct a finite $D \leq (H_n, E, 0)$. Let $C_1, \dots, C_{|S|} \leq (H_n, E, 0)$ be $|S|$ disjoint isomorphic copies of C. In the k-th copy the points are $v_1^k, \dots, v_{i-1}^k$ and u_i^k. Between two copies C_ℓ and C_m there are no edges, except that the set $\{u_j^k \mid 1 \leq k \leq |S|\}$ induces a graph in U_2 isomorphic to S. From now on we identify this set with S. Let $\{z_{p,q}^t \mid 1 \leq p \leq n-2, 1 \leq q \leq 6r_E + 2r_N, 1 \leq t \leq \binom{|S|}{3}\}$ be an independent set in U_1. We have $z_{p,q}^t v_m^k \in N$ for all p, q, t, m, k. Let us enumerate the 3-element subsets of S so that every 3-element subset of S has an index between 1 and $\binom{|S|}{3}$. Each subset is ordered according to the parameter k of the u_i^k. The vertex $z_{p,q}^t$ is connected to u_i^k if and only if either

- $1 \leq q \leq 2r_E$ and u_i^k is the second or third element in the 3-element subset of index t, or
- $2r_E + 1 \leq q \leq 4r_E$ and u_i^k is the third or first element in the 3-element subset of index t, or
- $4r_E + 1 \leq q \leq 6r_E$ and u_i^k is the first or second element in the 3-element subset of index t.

Let $D = C_1 \cup \cdots \cup C_{|S|} \cup \{z_{p,q}^t \mid 1 \leq p \leq n-2, 1 \leq q \leq 6r_E + 2r_N, 1 \leq t \leq \binom{|S|}{3}\}$. We show that there exists such a D, i.e., the above construction does not produce a copy of K_n in D or a copy of K_{n-1} in the U_1-part of D. Note that the U_1-part of D is empty, so it is enough to check that any complete subgraph K of D has at most $n-1$ vertices. If K contains a vertex of the form $z_{p,q}^t$, then $|K| \leq 3 < n$ as $z_{p,q}^t$ has degree 2 in D. Finally, if $K \subseteq C_1 \cup \cdots \cup C_{|S|}$, then

$|K| \leq n-1$ as C and S are K_n-free. Let $D' = \{v_j^k \mid 1 \leq j \leq i-1, 1 \leq k \leq |S|\}$ and $D'' = D \setminus S$.

Now we construct a finite $F \leq (H_n, E, 0)$. The underlying set of F is $D' \cup \{w_{p,q}^t \mid 1 \leq p \leq n-2, 1 \leq q \leq 6r_E + 2r_N, 1 \leq t \leq \binom{|S|}{3}\}$. We have $v_j^k w_{p,q}^t \in N$ for all j, k, p, q, t and $w_{p_1,q_1}^{t_1} w_{p_2,q_2}^{t_2} \in E \Leftrightarrow (q_1 = q_2) \wedge (t_1 = t_2) \wedge (p_1 \neq p_2)$. It is clear that such an F exists. Let $f_2 : D'' \to F$ be the function fixing D' pointwise and mapping $z_{p,q}^t$ to $w_{p,q}^t$ for all p, q, t.

Let $\gamma_F, \gamma_{D''} \in G$ be such that $\gamma_F \restriction_F = g \restriction_F$ and $\gamma_{D''} \restriction_{D''} = g \restriction_{D''}$. Then there is a $\delta \in \mathrm{Aut}(H_n, E, 0)$ such that $\gamma_F \circ f_2 \circ \gamma_{D''}^{-1} \restriction_{\gamma_{D''}(D'')} = \delta \restriction_{\gamma_{D''}(D'')}$. Hence, the partial map f_2 extends to $\rho = \gamma_F^{-1} \circ \delta \circ \gamma_{D''} \in G$.

According to the choice of S it has a 3-element subset whose image under ρ is a triangle. Without loss of generality we may assume that it is $\{u_i^1, u_i^2, u_i^3\}$ which has index $t = 1$. For a fixed $1 \leq q \leq 6r_E + 2r_N$ the vertices $\rho(w_{p,q}^1)$ for $1 \leq p \leq n-2$ induce a copy of K_{n-2} in U_1. Hence, for all q there are at least two points in $\{\rho(u_i^1), \rho(u_i^2), \rho(u_i^3)\}$ that are not connected to at least one of these $(n-2)$ points. For all $1 \leq q \leq 6r_E + 2r_N$ let us assign two such vertices from $\{\rho(u_i^1), \rho(u_i^2), \rho(u_i^3)\}$.

By a simple pigeonhole argument, there are at least two vertices in the set $\{\rho(u_i^1), \rho(u_i^2), \rho(u_i^3)\}$ that are assigned at least r_N times to numbers between $6r_E + 1$ and $6r_E + 2r_N$. Without loss of generality we may assume that $\rho(u_i^1)$ and $\rho(u_i^2)$ are such. Again, by a simple pigeonhole argument, $\rho(u_i^1)$ or $\rho(u_i^2)$ is assigned to at least r_E times to some $4r_E + 1 \leq q \leq 6r_E$. Without loss of generality we may assume that $\rho(u_i^1)$ is such. Thus there exist

- r_E numbers $q_1, \ldots, q_{r_E}$ such that $4r_E + 1 \leq q_1, \ldots, q_{r_E} \leq 6r_E$ and for some $1 \leq p(q_j) \leq n-2$ depending on q_j we have that $\rho(w_{p(q_j),q_j}^1)\rho(u_i^1) \in N$ for all $1 \leq j \leq r_E$, and

- r_N numbers $q'_1, \ldots, q'_{r_N}$ such that $6r_E + 1 \leq q'_1, \ldots, q'_{r_N} \leq 6r_E + 2r_N$ and for some $1 \leq p(q'_j) \leq n-2$ depending on q'_j we have that $\rho(z_{p(q'_j),q'_j}^1)\rho(u_i^1) \in N$ for all $1 \leq j \leq r_N$.

Let $A'' = \{\rho(z_{p(q_j),q_j}^1) \mid 1 \leq j \leq r_E\} \cup \{\rho(z_{p(q'_j),q'_j}^1) \mid 1 \leq j \leq r_N\} \cup \{v_j^1 \mid 1 \leq$

$j \leq i-1\} \cup \{u_i^1\}$. The function $f_3 : A' \to A''$ with

- $f_3(v_j) = v_j^1$ for all $1 \leq j \leq i-1$,
- $f_3(u_i) = u_i^1$,
- f_3 mapping the r_E vertices in A' of the form z_m connected to u_i to the r_E vertices of the form $z^1_{p(q_j),q_j}$ with $4r_E+1 \leq q_j \leq 6r_E$ such that $\rho(z^1_{p(q_j),q_j})$ is not connected to $\rho(u_i^1)$,
- f_3 mapping the r_N vertices in A' of the form z_m not connected to u_i to the r_N vertices of the form $z^1_{p(q'_j),q'_j}$ with $6r_E+1 \leq q'_j \leq 6r_E+2r_N$ such that $\rho(z^1_{p(q'_j),q'_j})$ is not connected to $\rho(u_i^1)$

is a partial isomorphism of $(H_n, E, 0)$. Let $\nu \in \mathrm{Aut}(H_n, E, 0)$ be an automorphism that extends f_3. Then $\pi_i = \rho \circ \nu$ is an appropriate choice.

In the upcoming proofs we use the following notations.

Definition 4.5. *Let $c_1, \ldots, c_k \in H_n \setminus \{0\}$. We denote by $U_{i_0 i_1 \ldots i_k}$ with $i_0 \in \{1,2\}$ and $i_j \in \{c_j, \not{c}_j\}$ the subset of H_n that consists of the vertices w such that*

- *w is connected to 0 iff $i_0 = 1$,*
- *for $j = 1, \ldots, k$ we have that w is connected to c_j iff $i_j = c_j$.*

E.g., for a vertex $0 \neq u \in H_n$, $U_{1\not{u}}$ is the set of elements in U_1 that are not connected to u.

Lemma 4.6. *Let $0 \neq u \in H_n$, and let $\mathrm{Aut}(H_n, E, 0) \subseteq G$ be a closed group. Let $g : (H_n, E, 0, u) \to (H_n, E)$ be a canonical function generated by G. Let $\{\ell, m\} = \{1, 2\}$ and assume that $g(U_\ell \setminus \{u\}) \subseteq U_\ell$. Then either G contains $\mathrm{Sym}(U_\ell) \times \mathrm{Aut}(U_m, E)$ or g preserves E and N on $U_\ell \setminus \{u\}$.*

Proof. Every finite set $A \subseteq U_\ell$ can be mapped into $U_{\ell\not{u}}$ by an automorphism of $\mathrm{Aut}(H_n, E, 0)$. Thus by Lemma 4.2 or Lemma 4.4 we may assume that g

preserves E and N on $U_{\ell \not{u}}$. According to the axioms of H_n, g cannot map non-edges between $U_{\ell u}$ and $U_{\ell \not{u}}$ to edges. Indeed, there is a copy of $K_{n+\ell-3}$ in $U_{\ell \not{u}}$ and a vertex not connected to any of these $n+\ell-3$ vertices in $U_{\ell u}$, and the g-image of these points would induce a copy of $K_{n+\ell-2}$ in U_ℓ. Assume that g eradicates edges between $U_{\ell u}$ and $U_{\ell \not{u}}$. Let A be a finite subset of U_ℓ. Then there is an automorphism $\alpha \in \mathrm{Aut}(H_n, E, 0)$ that maps a given element $x \in A$ into $U_{\ell u}$ and the rest of A into $U_{\ell \not{u}}$. Thus in $(g \circ \alpha)(A)$, the image of the given point x is isolated, and $g \circ \alpha$ preserves E and N on $A \setminus \{x\}$. By iterating such steps, A can be mapped to an independent set in U_ℓ, and the assertion follows from Lemma 4.2 or Lemma 4.4.

Thus we may assume that g preserves edges and non-edges between $U_{\ell u}$ and $U_{\ell \not{u}}$. According to the defining axioms of (H_n, E), g cannot map non-edges in $U_{\ell u}$ to edges. Thus g preserves N on $U_\ell \setminus \{u\}$. If g violates E on $U_{\ell u}$, then we can systematically delete all edges of a given finite $A \leq U_\ell$ by a composition of functions in $\{g\} \cup \mathrm{Aut}(H_n, E, 0)$, and we are done by Lemma 4.2 or Lemma 4.4.

Lemma 4.7. *Let $\mathrm{Aut}(H_n, E, 0) \subseteq G$ be a closed group, and assume that G generates a canonical function $g : (H_n, E, 0) \to (H_n, E, 0)$ that violates at least one of the relations U_1, U_2, $E\restriction_{U_1}$, $E\restriction_{U_2}$, $N\restriction_{U_1}$ and $N\restriction_{U_2}$. Then G contains $\mathrm{Aut}(H_n, E)$, or $\mathrm{Aut}(H_n, 0, E\restriction_{H_n\setminus\{0\}})$, or $\mathrm{Sym}(U_1) \times \mathrm{Aut}(U_2, E)$, or $\mathrm{Sym}(U_2) \times \mathrm{Aut}(U_1, E)$.*

Proof. By Lemmas 4.2 and 4.4 we may assume that g preserves E and N on U_1 and on U_2. In particular, g preserves U_2, since a copy of K_{n-1} in U_2 cannot be mapped into U_1 by g. Thus $g(U_1 \cup U_2) \subseteq U_2$, and in particular, U_1 and U_2 are in the same G-orbit, and every finite set in $H_n \setminus \{0\}$ can be mapped into U_2 by an appropriate permutation in G.

Assume that g eradicates intermediate edges. Let $u \in U_2$ and $\gamma \in G$ be such that $\gamma(u) \in U_1$. Let $h : (H_n, E, 0, u) \to (H_n, E, 0)$ be an injective canonical function provided by Proposition 3.7 with $f = \gamma$. By Lemma 4.2 we may assume that h preserves E and N on $U_{2\not{u}}$, and in particular, $h(U_{2\not{u}}) \subseteq U_2$. If $h(U_{2u}) \subseteq U_1$, then we are done by applying Lemma 4.6 to $g \circ h$. If $h(U_{2u}) \subseteq U_2$, then we may assume that h preserves E and N on $U_2 \setminus \{u\}$ by Lemma 4.6. Then $g \circ h$ preserves U_2, and it preserves edges and non-edges on U_2, except that $g \circ h(u)$ is an isolated point in $g \circ h(U_2)$. Thus by iterating functions in the set $\{g \circ h\} \cup \mathrm{Aut}(H_n, E, 0)$ any finite subset of U_2 can be mapped to an independent set.

As every finite subset of $H_n \setminus \{0\}$ can be mapped into U_2 by some permutation in G, we have that every finite set $A \subseteq H_n \setminus \{0\}$ can be mapped to an independent set in U_2 by G. If G preserves 0, then this implies that $\mathrm{Sym}(H_n \setminus \{0\}) \subseteq G$. If G violates 0, then every finite subset of H_n can be mapped to an independent set in U_2 by G, and thus $G = \mathrm{Sym}(H_n)$.

Hence, we may assume that g preserves intermediate edges and non-edges. As g preserves E and N on U_2, g cannot map non-edges between 0 and U_2 to edges, as it would contradict the defining axioms of H_n. Thus g preserves N between 0 and U_2.

Case 1. Assume that g maps edges between 0 and U_1 to non-edges. Then every pair that contains 0 is mapped to a non-edge by g. We show that $\mathrm{Aut}(H_n, 0, E \restriction_{H_n \setminus \{0\}})$ is contained in G. Let $\alpha \in \mathrm{Aut}(H_n, 0, E \restriction_{H_n \setminus \{0\}})$ and let A be a finite subset of H_n. Let $\alpha(A) = B$. Let γ_A and γ_B be permutations in G such that $\gamma_A \restriction_A = g \restriction_A$ and $\gamma_B \restriction_B = g \restriction_B$. Then $\gamma_B \circ \alpha \circ \gamma_A^{-1} \restriction_{\gamma_A(A)}$ is a partial isomorphism of $(H_n, E, 0)$ that extends to some automorphism $\beta \in \mathrm{Aut}(H_n, E, 0)$. Thus $\gamma_B^{-1} \circ \beta \circ \gamma_A \in G$ interpolates α on A.

Case 2. Assume that g maps edges between 0 and U_1 to edges. Then g preserves E and N. As $g(U_1) \subseteq U_2$, we have $g(0) \neq 0$. We prove that G contains $\mathrm{Aut}(H_n, E)$. Let $\alpha \in \mathrm{Aut}(H_n, E)$ and let A be a finite subset of H_n. Let $\alpha(A) = B$. Note that $g \circ g = g^2$ maps every finite subset of H_n into U_2. Let γ_A and γ_B be permutations in G such that $\gamma_A \restriction_A = g^2 \restriction_A$ and $\gamma_B \restriction_B = g^2 \restriction_B$. Then $\gamma_B \circ \alpha \circ \gamma_A^{-1} \restriction_{\gamma_A(A)}$ is a partial isomorphism of $(H_n, E, 0)$ that extends to some automorphism $\beta \in \mathrm{Aut}(H_n, E, 0)$. Thus $\gamma_B^{-1} \circ \beta \circ \gamma_A \in G$ interpolates α on A.

Lemma 4.8. *Let* $\mathrm{Aut}(H_n, E, 0) \subseteq G$ *be a closed group, and assume that* G *gen-*

erates a canonical function $g : (H_n, E, 0) \to (H_n, E, 0)$ that violates at least one of the relations U_1, U_2, $E \restriction_{U_1 \cup U_2}$ and $N \restriction_{U_1 \cup U_2}$. Then G contains $\mathrm{Aut}(H_n, E)$, or $\mathrm{Aut}(H_n, 0, E \restriction_{H_n \setminus \{0\}})$, or $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$.

Proof. According to Lemma 4.7 we may assume that g preserves U_1, U_2, and edges and non-edges on U_1 and on U_2. In particular, g cannot map intermediate non-edges to edges, as it would contradict the defining axioms of H_n. Thus g eradicates intermediate edges, and we are done by Lemma 3.10.

Lemma 4.9. *Let $\mathrm{Aut}(H_n, E, 0) \subseteq G$ be a closed group, and assume that the sets U_1 and U_2 are contained in the same G-orbit. Then G generates a canonical function $g : (H_n, E, 0) \to (H_n, E, 0)$ such that $g(U_1) \subseteq U_2$ or $g(U_2) \subseteq U_1$.*

Proof. Let $u \in U_2$ and $\gamma \in G$ be such that $\gamma(u) \in U_1$. Let $h : (H_n, E, 0, u) \to (H_n, E, 0)$ be an injective canonical function provided by Proposition 3.7 with $f = \gamma$. In particular, $h(u) \in U_1$.

There exists a canonical function $h' : (H_n, E, 0) \to (H_n, E, 0, u)$ in the monoid generated by $\mathrm{Aut}(H_n, E, 0)$, and consequently by G, such that $h'(0) = 0$ and $h'(U_\ell) \subseteq U_{\ell \not{u}}$ for all $\ell \in \{1, 2\}$. Hence, if $h(U_{2\not{u}}) \subseteq U_1$, then $g = h \circ h' : (H_n, E, 0) \to (H_n, E, 0)$ is canonical such that $g(U_2) \subseteq U_1$. Similarly, if $h(U_{1\not{u}}) \subseteq U_2$, then we obtain that G generates a canonical function $g : (H_n, E, 0) \to (H_n, E, 0)$ such that $g(U_1) \subseteq U_2$. Thus we may assume that $h(U_{2\not{u}}) \subseteq U_2$ and $h(U_{1\not{u}}) \subseteq U_1$.

Assume that $h(U_{2u}) \subseteq U_1$. Given any finite set $A \subseteq U_2$, there is an element of $\mathrm{Aut}(H_n, E, 0)$ that maps one element of A into U_{2u} and the rest of A into $U_{2\not{u}}$. By composing this automorphism with h, and another automorphism that maps the U_1-part of the image of A into $U_{1\not{u}}$, and then iterating such steps, we can construct a function generated by G that maps A into U_1. By following the proof of Proposition 3.7 we have that G generates a canonical function $g : (H_n, E, 0) \to (H_n, E, 0)$ such that $g(U_2) \subseteq U_1$. Thus we may assume that $h(U_{2u}) \subseteq U_2$. Similarly, we may assume that $h(U_{1u}) \subseteq U_1$. Hence, $h(U_1 \cup \{u\}) \subseteq U_1$ and $h(U_2 \setminus \{u\}) \subseteq U_2$.

Given any finite set $A \subseteq U_2$, there is an element of $\mathrm{Aut}(H_n, E, 0)$ that maps one element of A to u and the rest of A into $U_{2\not{u}}$. Hence, by composing functions in the set $\{h\} \cup \mathrm{Aut}(H_n, E, 0)$ any finite subset of U_2 can be mapped into U_1. Thus we can conclude that G generates a canonical function $g : (H_n, E, 0) \to (H_n, E, 0)$ such that $g(U_2) \subseteq U_1$ as in the previous case.

Lemma 4.10. *Let* $\mathrm{Aut}(H_n, E, 0) \subseteq G$ *be a closed group, and assume that the sets* U_1 *and* U_2 *are contained in the same* G*-orbit. Let* $\{\ell, m\} = \{1, 2\}$. *Then there exists a permutation* $\gamma \in G$ *and two vertices in* U_ℓ *such that* γ *maps both of these vertices into* U_m.

Proof. As $U_1 \cup U_2$ is contained in a G-orbit, there exists a permutation $\rho \in G$ and an element $x_1 \in U_1$ such that $\rho(x_1) \in U_2$. We may assume that no vertex in U_1 other than x_1 is mapped into U_2 by ρ, and at most one vertex in U_2 is mapped into U_1 by ρ, or else ρ or ρ^{-1} is a good choice for γ. Let $x_2 \in U_1$ and $\alpha \in \mathrm{Aut}(H_n, E, 0)$ be such that $\rho^{-1}(x_2) \in U_1$, $\alpha(x_2) = x_1$ and $\rho(\alpha(\rho(x_1))) \in U_2$. Then $\gamma = \rho \circ \alpha \circ \rho \in G$ maps x_1 and $\rho^{-1}(x_2)$ into U_2.

Lemma 4.11. *Let* $\mathrm{Aut}(H_n, E, 0) \subseteq G$ *be a closed group, and assume that the sets* U_1 *and* U_2 *are contained in the same* G*-orbit. Then* G *contains* $\mathrm{Aut}(H_n, E)$, *or* $\mathrm{Aut}(H_n, 0, E \restriction_{H_n \setminus \{0\}})$, *or* $\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)$.

Proof. By Lemmas 4.9 and 4.7 we have that G contains $\mathrm{Aut}(H_n, E)$, or $\mathrm{Aut}(H_n, 0, E \restriction_{H_n \setminus \{0\}})$, or $\mathrm{Sym}(U_1) \times \mathrm{Aut}(U_2, E)$ or $\mathrm{Sym}(U_2) \times \mathrm{Aut}(U_1, E)$. Assume that G contains $\mathrm{Sym}(U_\ell) \times \mathrm{Aut}(U_m, E)$ with $\{\ell, m\} = \{1, 2\}$.

According to Lemma 4.10 there exists a permutation $\gamma \in G$ and two vertices $u, v \in U_m$ such that $\gamma(u), \gamma(v) \in U_\ell$. The transposition $t_{\gamma(u)\gamma(v)}$ switching $\gamma(u)$ and $\gamma(v)$ is in G. Thus $t_{uv} = \gamma^{-1} \circ t_{\gamma(u)\gamma(v)} \circ \gamma$, the transposition switching u and v, is in G. Note that the Henson graphs and the complements of the Henson graphs are connected, except for (H_2, E), which is empty. Hence, by using a composition of elements in $\mathrm{Aut}(H_n, E, 0) \cup \{t_{uv}\}$, it is possible to switch a given pair of elements in U_m while fixing every other element in H_n. The transpositions in U_m together with $\mathrm{Sym}(U_\ell) \times \mathrm{Aut}(U_m, E)$ generate $\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)$.

4.2. Orbit systems and big groups

Lemma 4.12. *Let* $\mathrm{Aut}(H_n, E, 0) \subseteq G$ *be a closed group such that* $\{0\}$ *and* U_ℓ *are in the same orbit for some* $\ell \in \{1, 2\}$. *Then there exists an element* $u \in U_\ell$ *and a permutation* $\gamma \in G$ *such that* γ *switches* 0 *and* u.

Proof. Let $\{m\} = \{1, 2\} \setminus \{\ell\}$. Let $\rho \in G$ and $u \in U_\ell$ be such that $\rho(0) = u$. If the ρ-preimage v of 0 is also in U_ℓ, then there is a permutation $\alpha \in \mathrm{Aut}(H_n, E, 0)$ such that $\alpha(u) = v$. Thus $\gamma = \rho \circ \alpha$ switches 0 and u. Thus assume that $v \in U_m$. Let $w \neq v$ be in U_m, and let $\rho(w) = z$. If $z \in U_m$ then there are $\beta, \delta \in \mathrm{Aut}(H_n, E, 0)$ with $\beta(v) = z$ and δ switching v and w. Hence, $\gamma = \rho \circ \beta^{-1} \circ \rho \circ \delta \circ \rho^{-1} \circ \beta \circ \rho^{-1}$ switches 0 and u. Finally, if $z \in U_\ell$ then there exist $\mu, \nu \in \mathrm{Aut}(H_n, E, 0)$ with $\mu(v) = w$ and ν switching u and z. Thus $\gamma = \rho \circ \mu^{-1} \circ \rho^{-1} \circ \nu \circ \rho \circ \mu \circ \rho^{-1}$ switches 0 and u.

Lemma 4.13. *Let* $\mathrm{Aut}(H_n, E, 0) \subseteq G$ *be a closed group, and assume that the orbits of* G *are precisely* $U_\ell \cup \{0\}$ *and* U_m *with* $\{\ell, m\} = \{1, 2\}$. *Then* G *contains* $\mathrm{Sym}(U_\ell \cup \{0\}) \times \mathrm{Aut}(U_m, E)$.

Proof. First we show that G contains $\mathrm{Sym}(U_\ell) \times \mathrm{Aut}(U_m, E)$. According to Lemma 4.12 there exists a $\gamma \in G$ and a $u \in U_\ell$ such that γ switches 0 and u. By applying Proposition 3.7 with $f = \gamma$ we obtain that G generates a canonical function $g : (H_n, E, 0, u) \to (H_n, E, 0)$ that switches 0 and u.

Clearly $g(U_\ell \setminus \{u\} \subseteq U_\ell)$ and $g(U_m) \subseteq U_m$. By Lemma 4.6 we may assume that g preserves E and N on $U_\ell \setminus \{u\}$. Let $h : (H_n, E, 0) \to (H_n, E, 0)$ be a function in the closed monoid generated by $\mathrm{Aut}(H_n, E, 0)$, and consequently by G, such that $h(U_\ell) \subseteq U_{\ell\not\ell}$. Then the function $g_1 = g \circ h \circ g$ is generated by G, it maps $U_\ell \cup \{0\}$ into U_ℓ, $g_1(u)$ is an isolated vertex in $g_1(U_\ell \cup \{0\})$, and $g_1 \restriction_{U_\ell \cup \{0\}}$ preserves N. Thus the U_ℓ-part of any finite $A \leq (H_n, E, 0)$ can be mapped to an independent set by a composition of functions in $\{g_1\} \cup \mathrm{Aut}(H_n, E, 0)$. Hence, by one of Lemmas 4.2 and 4.4 we have that G contains $\mathrm{Sym}(U_\ell) \times \mathrm{Aut}(U_m, E)$.

The set $\{g\} \cup (\mathrm{Sym}(U_\ell) \times \mathrm{Aut}(U_m, E))$ generates a canonical function $g' : (H_n, E, 0) \to (H_n, E, 0)$ such that $g'(U_\ell \cup \{0\}) \subseteq U_\ell$ is an independent set with no intermediate edges. Let $\alpha \in \mathrm{Sym}(U_\ell \cup \{0\}) \times \mathrm{Aut}(U_m, E)$, and let $A, B \subseteq H_n$ be finite sets such that $\alpha(A) = B$. There exist $\gamma_A, \gamma_B \in G$ such that $\gamma_A \restriction_A = g' \restriction_A$ and $\gamma_B \restriction_B = g' \restriction_B$. Thus $(\gamma_B \circ \alpha \circ \gamma_A^{-1}) \restriction_{\gamma_A(A)}$ is a partial isomorphism of $(H_n, E, 0)$, and consequently, it extends to some $\beta \in \mathrm{Aut}(H_n, E, 0)$. Hence, $\gamma_B^{-1} \circ \beta \circ \gamma_A$ interpolates α on A.

4.3. Minimal groups above $\mathrm{Aut}(H_n, E, 0)$

Lemma 4.14. *Let* $\mathrm{Aut}(H_n, E, 0) \subsetneq G$ *be a closed group. Then* G *contains* $\mathrm{Aut}(H_n, E)$*, or* $\mathrm{Aut}(H_n, 0, E \restriction_{H_n \setminus \{0\}})$*, or* $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$*.*

Proof. By Lemmas 4.11 and 4.13 we may assume that the orbits of G are precisely $\{0\}$, U_1 and U_2. Then G does not violate any of the unary relations of $(H_n, E, 0)$, thus it violates E. As edges and non-edges containing 0 are preserved, an edge uv in $H_n \setminus \{0\}$ is mapped to a non-edge by some $\gamma \in G$. Let $g : (H_n, E, 0, u, v) \to (H_n, E, 0)$ be an injective canonical function provided by Proposition 3.7 with $f = \gamma$. In particular, $g(uv) \in N$.

By Lemmas 4.2 and 4.4 we may assume that g preserves E and N on $U_{i\not u\not v}$ for all $i \in \{1, 2\}$. It is clear that g preserves N on all the U_{ijk}, as all these sets contain a copy of I_n.

Let $\ell \in \{1, 2\}$ and X be one of the sets $\{u\}, \{v\}, U_{ijk}$ with $i \in \{1, 2\}, j \in \{u, \not u\}, k \in \{v, \not v\}$ such that if $\ell = 1$ then $X \subseteq U_1$. Then there is a copy K of $K_{n+\ell-3}$ in $U_{\ell\not u\not v}$ and a vertex w not connected to any point of K in X. Thus g preserves N between X and $U_{\ell\not u\not v}$, as otherwise $g(K \cup \{w\})$ is isomorphic to $K_{n+\ell-2}$, contradicting the defining axioms of $(H_n, E, 0)$. In particular, if $u \in U_1$, then N is preserved between u and $U_{1\not u\not v}$. We deal with the case $u \in U_2$ later.

For all $\ell \in \{1, 2\}, j \in \{u, \not u\}, k \in \{v, \not v\}$ and $A \subseteq U_\ell$ finite there is an $\alpha \in \mathrm{Aut}(H_n, E, 0)$ such that the α-image of A is in $U_{\ell\not u\not v}$ except for a given vertex which is mapped into $U_{\ell jk}$. Thus by Lemmas 4.2 and 4.4 we may assume that g preserves E between $U_{\ell jk}$ and $U_{\ell \not u \not v}$, as otherwise we can delete edges in the U_ℓ-part of A with a composition of functions in the set $\{g\} \cup \mathrm{Aut}(H_n, E, 0)$. Similarly, g preserves E on $U_{\ell jk}$, or we can eradicate edges by using automorphisms that map an edge of A into $U_{\ell jk}$ and all other vertices of A into $U_{\ell \not u \not v}$.

Let $\ell \in \{1,2\}, j_1, j_2 \in \{u, \not u\}, k_1, k_2 \in \{v, \not v\}$ be such that $\{\not u, \not v\} \neq \{j_1, k_1\} \neq \{j_2, k_2\} \neq \{\not u, \not v\}$, and assume that g violates N between $U_{\ell j_1 k_1}$ and $U_{\ell j_2 k_2}$. There exist $x \in U_{\ell j_1 k_1}$, $y \in U_{\ell j_2 k_2}$ and a copy K of $K_{n+\ell-4}$ in $U_{\ell \not u \not v}$ such that $xy \in N$ and all other pairs in $K \cup \{x, y\}$ are edges. Hence, $g(K \cup \{x, y\})$ is isomorphic to K_n, a contradiction. Thus we may assume that g preserves N on $U_\ell \setminus \{u, v\}$ for all $\ell \in \{1, 2\}$.

If g violates E on $U_\ell \setminus \{u, v\}$ for some $\ell \in \{1, 2\}$, then we can systematically delete edges in the U_ℓ-part of any finite $A \leq (H_n, E, 0)$, and then we are done by Lemmas 4.2 and 4.4. Thus we may assume that g preserves E and N on $U_\ell \setminus \{u, v\}$ for all $\ell \in \{1, 2\}$.

For all finite $A \leq (H_n, E, 0)$ there is a $\beta \in \mathrm{Aut}(H_n, E, 0)$ such that the β-image of the U_1-part of A is in $U_{1 \not u \not v}$ and the β-image of the U_2-part of A is in $U_{2 \not u \not v}$. Thus g preserves E between $U_{1 \not u \not v}$ and $U_{2 \not u \not v}$, as otherwise $g \circ \beta$ eradicates intermediate edges of A, and we are done by Lemma 3.10.

Let $j \in \{u, \not u\}$ and $k \in \{v, \not v\}$ be such that $\{j, k\} \neq \{\not u, \not v\}$. There exists an intermediate non-edge with one endpoint in $U_{1 \not u \not v}$ and the other in U_{2jk}, and $n-2$ additional points in $U_{2 \not u \not v}$ such that any pair of vertices in these n points other than the intermediate non-edge is in E. Thus g preserves N between $U_{1 \not u \not v}$ and U_{2jk}, as otherwise the g-image of these n vertices would induce a copy of K_n in (H_n, E).

Let $\{\ell, m\} = \{1, 2\}, j \in \{u, \not u\}$ and $k \in \{v, \not v\}$ be such that $\{j, k\} \neq \{\not u, \not v\}$ and there is an edge between $U_{\ell jk}$ and $U_{m \not u \not v}$. Let $A \leq (H_n, E, 0)$ be finite. If A contains an intermediate edge, then there exists a $\delta \in \mathrm{Aut}(H_n, E, 0)$ such that the δ-image of the U_ℓ-part of A is in $U_{\ell \not u \not v}$ except for an endpoint of a given intermediate edge in A which is in $U_{\ell jk}$, and the δ-image of the U_2-part of A is in $U_{m \not u \not v}$. Thus if g violates E between $U_{\ell jk}$ and $U_{m \not u \not v}$, then $g \circ \delta$ deletes an intermediate edge in A and it preserves intermediate non-edges of A. Hence, we may assume that g preserves E between $U_{\ell jk}$ and $U_{m\not u\not v}$, as otherwise we can systematically delete intermediate edges of A, and we are done by Lemma 3.10.

Assume that g violates N between $U_1 \setminus \{u, v\}$ and $U_2 \setminus \{u, v\}$. Let xy be an intermediate non-edge violated by g such that $x \in U_1 \setminus \{u, v\}$ and $y \in U_2 \setminus \{u, v\}$. Then there is a copy K of K_{n-2} in $U_{2\not u\not v}$ such that x and y are connected to all points in K. Then $g(K \cup \{x, y\})$ is isomorphic to K_n, a contradiction. Thus g preserves N on $H_n \setminus \{0, u, v\}$.

If g violates E between $U_1 \setminus \{u, v\}$ and $U_2 \setminus \{u, v\}$, then we can systematically delete intermediate edges of any finite $A \leq (H_n, E, 0)$, and we are done by Lemma 3.10. Thus we may assume that g preserves E and N on $H_n \setminus \{0, u, v\}$.

We have already seen that if $u \in U_1$, then N is preserved between u and $U_{1\not u\not v}$. Assume that $u \in U_2$. If g violates E between u and $U_{2u\not v}$, then we can systematically isolate every point in a given finite $A \subseteq U_2$ by mapping a given point of A to u by an automorphism of $(H_n, E, 0)$ and all other vertices of A into $U_{2u\not v} \cup U_{2\not u\not v}$, and then applying g. Thus we may assume that E is preserved by g between u and $U_{2u\not v}$, as otherwise we are done by Lemma 4.2. Let $x \in U_{1\not u\not v}$. There exist $n-2$ vertices in $U_{2u\not v}$ such that xu is the only non-edge in the graph induced by these $n-2$ vertices, x and u. Thus g cannot violate N between u and $U_{1\not u\not v}$, as it would contradict the defining axioms of (H_n, E). Hence, g preserves N between u and $U_{1\not u\not v}$, and similarly, g preserves N between v and $U_{1\not u\not v}$.

Let $\{\ell, m\} = \{1, 2\}$. Assume that g violates E between u and $U_{\ell u\not v}$. If $u \in U_\ell$, then we proceed as in the previous paragraph. Hence, we may assume that $u \in U_m$. Then for all finite $A \leq (H_n, E, 0)$ there is a $\mu \in \mathrm{Aut}(H_n, E, 0)$ such that $\mu(A) \subseteq \{0\} \cup (U_m \setminus \{v\}) \cup U_{\ell u\not v} \cup U_{\ell\not u\not v}$, and if A contains an intermediate edge, then μ maps its endpoint in U_m to u. Then $g \circ \mu$ deletes an intermediate edge of A, and preserves its intermediate non-edges. Thus by iterating such steps, we can eradicate intermediate edges of A, and we are done by Lemma 3.10. Hence, we may assume that g preserves E between u and $U_{\ell u\not v}$ for all $\{\ell\} \in \{1, 2\}$. Similarly, we may assume that g preserves E between v and $U_{\ell\not u v}$ for all $\{\ell\} \in \{1, 2\}$.

Assume that g violates N between u and $U_{\ell \not{u} v}$ for some $\{\ell\} \in \{1,2\}$. There exists an $x \in U_{\ell \not{u} v}$ and a copy K of K_{n-2} in $U_{2u\not{v}}$ such that x is connected to every vertex of K. Thus $g(K \cup \{u,x\})$ is isomorphic to K_n, a contradiction. Hence, g preserves N between u and $U_{\ell \not{u} v}$ for all $\{\ell\} \in \{1,2\}$. Similarly, g preserves N between v and $U_{\ell u \not{v}}$ for all $\{\ell\} \in \{1,2\}$.

Thus g preserves N. Let $A \leq (H_n, E, 0)$ be finite. If uv is an intermediate edge, then we can systematically delete intermediate edges of A, and we are done by Lemma 3.10. If uv is in U_ℓ for some $\{\ell\} \in \{1,2\}$, then we can systematically delete edges in U_ℓ, and we are done by Lemmas 4.2 and 4.4.

Lemma 4.15. *Let* $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E) \subsetneq G$ *be a closed group. Assume that the orbits of* G *are* $\{0\}$, U_1 *and* U_2. *Then* $G = \mathrm{Aut}(U_1, E) \times \mathrm{Sym}(U_2)$, *or* $G = \mathrm{Aut}(U_2, E) \times \mathrm{Sym}(U_1)$, *or* $G = \mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)$.

Proof. As G strictly contains $\mathrm{Aut}(U_1, E) \times \mathrm{Aut}(U_2, E)$, there is a $\gamma \in G$ that violates E on U_m for some $m \in \{1,2\}$. By Theorem 1.1, $\mathrm{Aut}(U_m, E)$ is a maximal closed subgroup in $\mathrm{Sym}(U_m)$, thus γ and $\mathrm{Aut}(U_m, E)$ generate every permutation of U_m. Hence, any finite subset of U_m can be mapped to an independent set in U_m by an element of G, and Lemmas 4.2 and 4.4 imply that G contains a group of the form $\mathrm{Aut}(U_\ell, E) \times \mathrm{Sym}(U_m)$ with $\{\ell, m\} = \{1,2\}$.

We may assume that the containment is strict, as otherwise we are done. Then the same argument as above yields that $\mathrm{Sym}(U_\ell) \times \mathrm{Aut}(U_m) \subseteq G$, and consequently, $\mathrm{Sym}(U_\ell) \times \mathrm{Sym}(U_m) \subseteq G$. As $\mathrm{Sym}(U_\ell) \times \mathrm{Sym}(U_m)$ is the biggest (closed) group with orbits $\{0\}$, U_1 and U_2, we have $G = \mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)$.

Lemma 4.16. *Let* $\mathrm{Aut}(H_n, 0, E \restriction_{H_n \setminus \{0\}}) \subsetneq G$ *be a closed group. Then* G *contains* $\mathrm{Sym}(H_n \setminus \{0\})$.

Proof. If G stabilises 0 then the restriction of the action of G to $H_n \setminus \{0\}$ is a closed group on $H_n \setminus \{0\}$ containing $\mathrm{Aut}(H_n \setminus \{0\}, E \restriction_{H_n \setminus \{0\}})$. Thus in this case we are done by Theorem 1.1.

By Lemma 4.12 we may assume that there exists a $\gamma \in G$ and an element $u \in U_2$ such that γ switches 0 and u. Let $g : (H_n, E, 0, u) \to (H_n, E, 0, u)$ be a canonical function provided by Proposition 3.7 with $f = \gamma$. We claim that every finite subset of H_n can be mapped to an independent set in U_2 by some element of G. If g does not preserve E or N on $U_{2\not{u}}$, then $\mathrm{Aut}(U_1, E) \times \mathrm{Sym}(U_2) \subseteq G$ by Lemma 4.2, and the claim follows by using compositions of functions in $\{\gamma\} \cup \mathrm{Aut}(H_n, 0, E \restriction_{H_n \setminus \{0\}}) \cup \mathrm{Sym}(U_2)$. So we may assume that g preserves E and N on $U_{2\not{u}}$, and consequently, $g(U_{2\not{u}}) \subseteq U_{2\not{u}}$. $\mathrm{Aut}(H_n, 0, E \restriction_{H_n \setminus \{0\}})$ generates a function h such that $h(0) = 0$ and $h(H_n \setminus \{0\}) \subseteq U_{2\not{u}}$. Let $g' = g \circ h \circ g$. Then $g'(u)$ is an isolated vertex in $g'(H_n)$. By iterating such steps, the claim follows. Hence, $G = \mathrm{Sym}(H_n)$.

4.4. Closed groups above $\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)$

The structure $(H_n, U_1, U_2, c_1, \ldots, c_s)$ is homogeneous in a unary relational language for all $c_1, \ldots, c_s \in H_n$. A unary function g from $(H_n, U_1, U_2, c_1, \ldots, c_s)$ to $(H_n, U_1, U_2, d_1, \ldots, d_t)$ is canonical if and only if for every 1-element structure $S \in \mathrm{Age}(H_n, U_1, U_2, c_1, \ldots, c_s)$ there exists a 1-element structure $S' \in \mathrm{Age}(H_n, U_1, U_2, d_1, \ldots, d_t)$ such that the g-image of any copy of S is isomorphic to S'. The behaviour of g is uniquely determined by the type conditions satisfied by such 1-element substructures. Moreover, we have the following analogue statement of Proposition 3.7.

Proposition 4.17. *Let $s, t \geq 0$, and let $c_1, \ldots, c_s, d_1, \ldots, d_t \in H_n$. Let $\Delta = (H_n, U_1, U_2, c_1, \ldots, c_s)$ and $\Gamma = (H_n, U_1, U_2, d_1, \ldots, d_t)$, and let $f : \Delta \to \Gamma$ be an injective function. Then there exists an injective function*

$$g \in \overline{\{\beta \circ f \circ \alpha \mid \alpha \in \mathrm{Aut}(\Delta), \beta \in \mathrm{Aut}(\Gamma)\}}$$

such that g is canonical as a function from Δ to Γ, and $g(c_i) = f(c_i)$ for all $i \in \{1, \ldots, s\}$.

Proposition 4.18. *Let $X \cup Y$ be a partition of a countably infinite set D such that X and Y are infinite. Let $\mathrm{Sym}(X) \times \mathrm{Sym}(Y) \subsetneq G$ be a closed group acting on D. Then either $G = (\mathrm{Sym}(X) \times \mathrm{Sym}(Y)) \rtimes Z_2$ or $G = \mathrm{Sym}(D)$.*

Proof. Note that the group $\mathrm{Sym}(X) \times \mathrm{Sym}(Y)$ is a maximal subgroup of $(\mathrm{Sym}(X) \times \mathrm{Sym}(Y)) \rtimes Z_2$. Hence, if $G \neq (\mathrm{Sym}(X) \times \mathrm{Sym}(Y)) \rtimes Z_2$, then there exists a $\gamma \in G \setminus (\mathrm{Sym}(X) \times \mathrm{Sym}(Y)) \rtimes Z_2$. We may assume that there are elements $u, v \in X$ such that $\gamma(u) = u$ and $\gamma(v) \in Y$. According to Proposition 4.17, G generates a function g that is canonical as a function from (D, X, Y, u, v) to (D, X, Y).

Case 1. Assume that $g(X \setminus \{u, v\}) \subseteq X$ and $g(Y) \subseteq Y$. Then any finite subset of D can be mapped into Y by some element of G, and thus $G = \mathrm{Sym}(D)$.

Case 2. Assume that $g(X \setminus \{u, v\}) \subseteq X$ and $g(Y) \subseteq X$. Then any finite subset of D can be mapped into X by some element of G, and $G = \mathrm{Sym}(D)$ follows.

Case 3. Assume that $g(X \setminus \{u, v\}) \subseteq Y$ and $g(Y) \subseteq Y$. Then any finite subset of D can be mapped into Y by some element of G, and thus $G = \mathrm{Sym}(D)$.

Case 4. Assume that $g(X \setminus \{u, v\}) \subseteq Y$ and $g(Y) \subseteq X$. Let α be the transposition that switches u and an element of X that is in the image of g. Then $g \circ \alpha \circ g$ preserves X and Y, except that it maps an element of Y into X. Then any finite subset of D can be mapped into X by some element of G, and thus $G = \mathrm{Sym}(D)$.

The following is well-known, the simple proof is left to the reader.

Proposition 4.19. *Let D be an infinite set. Then* $\mathrm{Sym}(D \setminus \{c\})$ *is a maximal subgroup of* $\mathrm{Sym}(D)$ *for any $c \in D$.*

Proposition 4.20. *Let* $\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2) \subsetneq G$ *be a closed group. Then G equals to one of the groups* $(\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)) \rtimes Z_2$, $\mathrm{Sym}(U_1 \cup \{0\}) \times \mathrm{Sym}(U_2)$, $\mathrm{Sym}(U_2 \cup \{0\}) \times \mathrm{Sym}(U_1)$, $\mathrm{Sym}(H_n \setminus \{0\})$, $(\mathrm{Sym}(U_1 \cup \{0\}) \times \mathrm{Sym}(U_2)) \rtimes Z_2$, $(\mathrm{Sym}(U_2 \cup \{0\}) \times \mathrm{Sym}(U_1)) \rtimes Z_2$, $\mathrm{Sym}(H_n)$.

Proof. First assume that 0 is a fixed point of G. Then $G \restriction_{U_1 \cup U_2}$ is a closed group acting on $U_1 \cup U_2$ that strictly contains $\mathrm{Sym}(U_1) \times \mathrm{Sym}(U_2)$, and thus we are done by Proposition 4.18. Hence, we may assume that 0 is not a fixed point of G. If G is not transitive, then we are done by Lemma 4.13 and Proposition 4.18.

Thus we may assume that G is transitive. By Lemma 4.12 there exists a $\gamma \in G$ and an element $u \in U_1$ such that γ switches 0 and u. Let $g : (H_n, 0, u, U_1, U_2) \to (H_n, 0, U_1, U_2)$ be a canonical function provided by Proposition 4.17 with $f = \gamma$.

Case 1. Assume that $g(U_1 \setminus \{u\}) \subseteq U_1$ and $g(U_2) \subseteq U_2$. Then we proceed as in the proof of Lemma 4.13 and obtain that $\mathrm{Sym}(U_1 \cup \{0\}) \times \mathrm{Sym}(U_2) \subseteq G$. The assertion follows from Proposition 4.18.

Case 2. Assume that $g(U_1 \setminus \{u\}) \subseteq U_1$ and $g(U_2) \subseteq U_1$. Then any finite subset of H_n can be mapped into U_1 by some element of G, and $G = \mathrm{Sym}(H_n)$ follows.

Case 3. Assume that $g(U_1 \setminus \{u\}) \subseteq U_2$ and $g(U_2) \subseteq U_2$. Then any finite subset of H_n can be mapped into U_2 by some element of G, and $G = \mathrm{Sym}(H_n)$ follows.

Case 4. Assume that $g(U_1 \setminus \{u\}) \subseteq U_2$ and $g(U_2) \subseteq U_1$. Then there exists a permutation $\pi \in G$ such that π switches 0 and u, $\pi(U_1 \setminus \{u\}) = U_2$ and $\pi(U_2) = U_1 \setminus \{u\}$. Let t_{uv} be the transposition that switches u and v for some $u \neq v \in U_1$. Then $\rho = \pi \circ t_{uv} \circ \pi$ is a transposition that switches 0 with an element in U_2. It is obvious that ρ and $\mathrm{Sym}(U_2)$ generate every permutation in $\mathrm{Sym}(U_2 \cup \{0\})$, and thus G contains $\mathrm{Sym}(U_2 \cup \{0\}) \times \mathrm{Sym}(U_1)$. The assertion follows from Proposition 4.18.

5. Characterisation of the reducts

We are ready to prove the main theorem of the paper.

Proof of Theorem 2.3. Let $\mathrm{Aut}(H_n, E, 0) \subseteq G$. If the orbits of G are $\{0\}$, U_1 and U_2, then we are done by Lemmas 4.14 and 4.15. If U_1 and U_2 are contained in the same G-orbit, then the assertion follows from Lemmas 4.11, 4.16, Theorem 1.1 and Propositions 4.20 and 4.19. Thus we may assume that the orbits of G are precisely $U_\ell \cup \{0\}$ and U_m with $\{\ell, m\} = \{1, 2\}$. Then G contains $\mathrm{Sym}(U_\ell \cup \{0\}) \times \mathrm{Aut}(U_m, E)$ by Lemma 4.13. If the containment is strict, then some $\gamma \in G$ violates E on U_m. By Theorem 1.1 we have that any finite subset of U_m can be mapped to an independent set in U_m, and then

$G = \mathrm{Sym}(U_\ell \cup \{0\}) \times \mathrm{Sym}(U_m)$ by our assumption and one of Lemmas 4.2 and 4.4.

6. Acknowledgements

The author is indebted to the anonymous referee for significantly improving the presentation of the paper, and to Csaba Szabó for his useful comments.

References

[1] C. W. Henson, A family of countable homogeneous graphs, Pacific Journal of Mathematics 38 (1971) 69–83.

[2] A. H. Lachlan, R. Woodrow, Countable ultrahomogeneous undirected graphs, Transactions of the American Mathematical Society 262 (1) (1980) 51–94.

[3] P. J. Cameron, Transitivity of permutation groups on unordered sets, Mathematische Zeitschrift 148 (1976) 127–139.

[4] S. Thomas, Reducts of the random graph, Journal of Symbolic Logic 56 (1) (1991) 176–181.

[5] S. Thomas, Reducts of random hypergraphs, Annals of Pure and Applied Logic 80 (2) (1996) 165–193.

[6] J. H. Bennett, The reducts of some infinite homogeneous graphs and tournaments, Ph.D. thesis, Rutgers university (1997).

[7] M. Junker, M. Ziegler, The 116 reducts of $(\mathbb{Q}, <, a)$, Journal of Symbolic Logic 73 (3) (2008) 861–884.

[8] J. Nešetřil, V. Rödl, Partitions of finite relational and set systems, Journal of Combinatorial Theory, Series A 22 (3) (1977) 289–312.

[9] M. Bodirsky, M. Pinsker, Reducts of Ramsey structures, in: Model Theoretic Methods in Finite Combinatorics, Vol. 558 of Contemporary Mathematics, American Mathematical Society, 2011, pp. 489–519.

[10] M. Bodirsky, M. Pinsker, Schaefer's theorem for graphs, in: Proceedings of STOC'11, 2011, pp. 655–664.

[11] M. Bodirsky, M. Pinsker, Minimal functions on the random graph, Israel Journal of Mathematics 200 (1) (2014) 251–296.

[12] M. Bodirsky, M. Pinsker, T. Tsankov, Decidability of definability, Journal of Symbolic Logic 78 (4) (2013) 1036–1054.

[13] W. Hodges, A shorter model theory, Cambridge University Press, Cambridge, 1997.

[14] J. Nešetřil, V. Rödl, Ramsey classes of set systems, Journal of Combinatorial Theory, Series A 34 (2) (1983) 183–201.