Results in Mathematics

Multiplicative loops of quasifields having complex numbers as kernel

--Manuscript Draft--

Manuscript Number:	RIMA-D-17-00256
Full Title:	Multiplicative loops of quasifields having complex numbers as kernel
Article Type:	Original research
Keywords:	multiplicative loops of locally compact quasifields; semifields; sections in Lie groups; translation planes; automorphism groups
Corresponding Author:	Agota Figula Debreceni Egyetem HUNGARY
Corresponding Author Secondary Information:	
Corresponding Author's Institution:	Debreceni Egyetem
Corresponding Author's Secondary Institution:	
First Author:	Giovanni Falcone
First Author Secondary Information:	
Order of Authors:	Giovanni Falcone
	Agota Figula
	Karl Strambach
Order of Authors Secondary Information:	
Funding Information:	H2020 European Research Council (317721) — Not applicable
Abstract:	We determine the multiplicative loops of locally compact connected 4-dimensional quasifields Q having the field of complex numbers as their kernel. In particular, we turn our attention to multiplicative loops which have either a normal subloop of dimension one or which contain a subgroup isomorphic to $Spin_3(R)$. Although the 4-dimensional semifields Q are known, their multiplicative loops have interesting Lie groups generated by left or right translations. We determine explicitly the quasifields Q which coordinatize locally compact translation planes of dimension 8 admitting an at least 16-dimensional Lie group as automorphism group.

Noname manuscript No.
(will be inserted by the editor)

Multiplicative loops of quasifields having complex numbers as kernel

Giovanni Falcone · Ágota Figula · Karl Strambach

Received: date / Accepted: date

Abstract We determine the multiplicative loops of locally compact connected 4-dimensional quasifields Q having the field of complex numbers as their kernel. In particular, we turn our attention to multiplicative loops which have either a normal subloop of dimension one or which contain a subgroup isomorphic to $Spin_3(\mathbb{R})$. Although the 4-dimensional semifields Q are known, their multiplicative loops have interesting Lie groups generated by left or right translations. We determine explicitly the quasifields Q which coordinatize locally compact translation planes of dimension 8 admitting an at least 16-dimensional Lie group as automorphism group.

Keywords Multiplicative loops of locally compact quasifields · semifields · sections in Lie groups · translation planes · automorphism groups

1 Introduction

Since the seventies of the last century the locally compact connected topological non-desarguesian translation planes became a popular subject of geometrical research ([2]-[7], [8], [12]-[14], [17], [21], [24]). These planes are coordinatized

In accordance with Karl's will, we cordially dedicate this paper to the 75th birthday of our common friend Heinrich Wefelscheid

G. Falcone
Dipartimento di Matematica e Informatica, via Archirafi 34, 90123 Palermo, Italy
E-mail: giovanni.falcone@unipa.it

Á. Figula
University of Debrecen, Institute of Mathematics, P.O.Box 400, H-4002, Debrecen, Hungary
Tel.: +36-52-512900-22721
Fax: E-mail: figula@science.unideb.hu

K. Strambach
Universität Erlangen-Nürnberg, Department Mathematik, Cauerstrasse 11, D-91058 Erlangen, Germany
E-mail: stramba@math.fau.de

by locally compact quasifields Q having either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers as their kernel (cf. [11], IX.5.5 Theorem, p. 323). The classification of topological translation planes $\mathcal{A}$ was accomplished by reconstructing the spreads corresponding to $\mathcal{A}$. In this way D. Betten determined all 4-dimensional planes having an at least 7-dimensional automorphism group ([2]-[7], [21]) and H. Hähl classified the 8-dimensional topological translation planes admitting an at least 16-dimensional automorphism group and coordinatizing by quasifields having the field $\mathbb{C}$ as their kernels ([8], [12]-[14]). Using another tool N. Knarr determined the 8-dimensional planes coordinatizing by semifields having the field $\mathbb{C}$ as their kernel (cf. [17], Section 6).

However only few results are known on the loop theoretical characterizations of the multiplicative structure of locally compact quasifields. As the first step in this direction the algebraic structure of the multiplicative loops of topological quasifields having dimension 2 over their kernel $\mathbb{R}$ was described in [9]. After this the question naturally arises: How we can determine the algebraic structure of the multiplicative loop Q^* of a quasifield Q having dimension 2 over its kernel $\mathbb{C}$? This paper is devoted to answer this question. In that case the topological dimension of Q is 4.

Before our investigation P. T. Nagy and K. Strambach proved that the group G topologically generated by the left translations of the 2-dimensional proper multiplicative loops Q^* is the connected component of $GL_2(\mathbb{R})$ but the group topologically generated by the right translations of Q^* has infinite dimension (cf. [18], Section 29, p. 345). In contrast to this we find that the group G topologically generated by the left translations of the multiplicative loops Q^* of 4-dimensional quasifields having the field $\mathbb{C}$ as their kernel is one of the following groups: $Spin_3(\mathbb{R}) \times \mathbb{R}$, $Spin_3(\mathbb{R}) \times \mathbb{C}$, $SL_2(\mathbb{C}) \times \mathbb{R}$, $GL_2(\mathbb{C})$. The classification of Hähl and Knarr shows that all these Lie groups are realized as the group generated by the left translations of a multiplicative loop Q^*. In particular, if G is the group $Spin_3(\mathbb{R}) \times \mathbb{R}$, then Q^* is associative and Q is either a proper Kalscheuer's near field or the skewfield of quaternions. We note that any locally compact 2-dimensional near field is the field of complex numbers ([11], XI.12.2 Proposition, p. 348). In Section 3 we determine the continuous sharply transitive sections corresponding to the loops Q^* and study their properties. The images of these sections $\sigma : G/H \to G$, where H is the stabilizer of the identity of a loop Q^*, are the spreads corresponding to the planes $\mathcal{A}$ coordinatizing by Q. Using the images of these sections we prove that if a loop Q^* contains a 1-dimensional normal subloop, then Q^* is a central extension of the group $\mathbb{R}$ by a loop homeomorphic to S^3 (cf. Theorem 6).

The multiplicative loops Q^* of locally compact left quasifields Q having the field $\mathbb{C}$ as their kernels such that the set of the left translations of Q^* is the product $\mathcal{T}\mathcal{K}$, where $\mathcal{T}$ is the set of the left translations of a 3-dimensional compact loop and $\mathcal{K}$ is the set of the left translations of Q^* corresponding to the 1-dimensional connected subgroup of the kernel of Q consisting of real elements, form an important subclass of loops, that we call decomposable loops. Namely, if Q^* has a normal subgroup of the form $N = \{(u, 0), u > 0\}$

or if it contains the group $Spin_3(\mathbb{R})$, then Q^* is decomposable (cf. Theorem 9 and Proposition 10).

Although P. Plaumann and K. Strambach showed that any locally compact 2-dimensional semifield is the field of complex numbers (cf. [23]) in Knarr's classification (cf. Theorem 6.6. in [17], p. 83) there are two families of 4-dimensional proper semifields having the field $\mathbb{C}$ as their kernels. The multiplicative loops Q^* of these semifields are direct products of $\mathbb{R}$ and a compact loop K homeomorphic to the 3-sphere (cf. Proposition 11). For K one obtains two classes of loops having the group $SL_4(\mathbb{R})$ as the group Γ topologically generated by all translations of K. These 3-dimensional compact loops are till now the only known examples such that Γ is a (finite dimensional) Lie group. Hence the group generated by all translations of the multiplicative loops Q^* of 4-dimensional proper semifields are Lie groups in contrary to the 2-dimensional proper quasifields. Also the group topologically generated by the left translations of K has a remarkable structure: it is isomorphic to the group of complex (2×2)-matrices the determinants of which have absolute value 1.

In Section 6 we use Hähl's classification to determine in our framework the multiplicative loops Q^* of the quasifields Q which coordinatize the 8-dimensional locally compact translation planes $\mathcal{A}$ with an automorphism group of dimension at least 16 such that the kernel of Q is isomorphic to the field $\mathbb{C}$. There are three classes of these quasifields Q.

The first class consists of quasifields Q such that the automorphism group of the 8-dimensional locally compact translation planes coordinatizing by Q contains a subgroup $\mathcal{G}$ isomorphic to $Spin_4(\mathbb{R})$ in the stabilizer of an affine point and $\mathcal{G}$ acts as $SO_4(\mathbb{R})$ on the line L_∞ of infinity (cf. [12]). The multiplicative loops Q^* of Q have the group $Spin_3(\mathbb{R}) \times \mathbb{C}$ as the group generated by their left translations and contain the group $Spin_3(\mathbb{R})$ (cf. Proposition 12).

In the second class are the quasifields Q such that the automorphism group of the 8-dimensional locally compact translation planes coordinatizing by Q has a subgroup $\mathcal{G}$ (locally) isomorphic to $SL_2(\mathbb{C})$ (cf. [13]). The multiplicative loops Q^* of Q are central extensions of the group $\mathbb{R}$ by a loop homeomorphic to S^3 and they have the group $SL_2(\mathbb{C}) \times \mathbb{R}$ as the group generated by their left translations (cf. Proposition 13).

The third class consists of quasifields Q such that the automorphism group A of the 8-dimensional locally compact translation planes coordinatizing by Q contains a subgroup $\mathcal{G}$ isomorphic to $Spin_3(\mathbb{R})$ in the stabilizer of an affine point such that $\mathcal{G}$ acts as $SO_3(\mathbb{R})$ on the line L_∞ of infinity and has fixed point on L_∞. Moreover, the stabilizer of an affine point in A contains a 3-dimensional closed subgroup N of shears with fixed axis such that $\mathcal{G}$ normalizes N (cf. [14]). The multiplicative loops Q^* of Q have the group $GL_2(\mathbb{C})$ as the group generated by their left translations (cf. Proposition 15).

2 Preliminaries

A binary system $(L, \cdot)$ is called a *loop* if there exists an element $1 \in L$ such that $x = 1 \cdot x = x \cdot 1$ holds for all $x \in L$ and for any given $a, b \in L$ the equations $a \cdot y = b$ and $x \cdot a = b$ have unique solutions which are denoted by $y = a \backslash b$ and $x = b/a$. A loop L is *proper* if it is not a group. If $(L, \cdot)$ is a loop and $K \subset L$ is such that $(K, \cdot)$ is a loop, then $(K, \cdot)$ is called a *subloop* of $(L, \cdot)$.
Given two loops $(L_1, \circ)$ and $(L_2, *)$, a map $\alpha : L_1 \to L_2$ such that $\alpha(x) * \alpha(y) = \alpha(x \circ y)$ holds for all $x, y \in L_1$ is called a homomorphism. Two loops $(L_1, \circ)$ and $(L_2, *)$ are called isomorphic if there exists a bijective homomorphism $\alpha : L_1 \to L_2$. The kernel of a homomorphism $\alpha : L_1 \to L_2$ from a loop L_1 to a loop L_2 is a *normal* subloop N of L_1, i.e. a subloop of L_1 such that

$$x \cdot N = N \cdot x,\ (x \cdot N) \cdot y = x \cdot (N \cdot y),\ (N \cdot x) \cdot y = N \cdot (x \cdot y) \tag{1}$$

hold for all $x, y \in L_1$. A loop L is called simple if $\{1\}$ and L are its only normal subloops.
For all $a \in L$, the left translations $\lambda_a : L \to L, x \mapsto a \cdot x$ (as well as the right translations $\rho_a : L \to L, x \mapsto x \cdot a$) are bijections of L and the loop L can be identified with a transversal of the group G generated by the left translations, modulo the stabilizer H of the identity.
A loop L is called *topological*, if it is a topological space and the binary operations $(a, b) \mapsto a \cdot b$, $(a, b) \mapsto b/a$, $(a, b) \mapsto a \backslash b : L \times L \to L$ are continuous. Then the left and right translations of L are homeomorphisms of L. We call a connected topological loop *quasi-simple* if it contains no normal subloop of positive dimension.
Every topological connected loop L having a Lie group G as the group topologically generated by the left translations of L corresponds to a sharply transitive continuous section $\sigma : G/H \to G$, where $G/H = \{xH | x \in G\}$ consists of the left cosets of the stabilizer H of $1 \in L$ such that $\sigma(H) = 1_G$ and $\sigma(G/H)$ generates G. The section σ is sharply transitive if the image $\sigma(G/H)$ acts sharply transitively on G/H, which means that to any xH, yH there exists precisely one $z \in \sigma(G/H)$ with $zxH = yH$ (cf. [18], Sections 1.2, 1.3).

A (left) quasifield is an algebraic structure $(Q, +, \cdot)$ such that $(Q, +)$ is an abelian group with neutral element 0, $(Q \setminus \{0\}, \cdot)$ is a loop, $0 \cdot x = x \cdot 0 = 0$, and between these operations the (left) distributive law $x \cdot (y + z) = x \cdot y + x \cdot z$ holds. If for any given $a, b, c \in Q$ the equation $a \cdot x + b \cdot x = c$ with $a + b \neq 0$ has precisely one solution, then Q is called *planar*. A translation plane is an affine plane with transitive group of translations. The translation planes may be coordinatized by planar quasifields (cf. [22], Kap. 8).

The kernel K_r of a (left) quasifield Q is a skewfield defined by

$$(x + y) \cdot k = x \cdot k + y \cdot k \text{ and } (x \cdot y) \cdot k = x \cdot (y \cdot k) \text{ for all } x, y \in Q, k \in K_r.$$

The *center* Z of Q is the set $\{z \in K_r |\ z \cdot x = x \cdot z \text{ for all } x \in Q\}$.

If Q is a semifield, that is, if in Q also the right distributive law holds, then Q may be consider also as a (right) quasifield and its kernel K_l is given by

$$k \cdot (x + y) = k \cdot x + k \cdot y \text{ and } k \cdot (x \cdot y) = (k \cdot x) \cdot y \text{ for all } x, y \in Q, k \in K_l.$$

The (left) quasifield Q is a right vector space over K_r and for all $a \in Q$ the map $\lambda_a : Q \to Q, x \mapsto a \cdot x$ is K_r-linear. According to [16], Theorem 7.3, p. 160, every quasifield having finite dimension over its kernel is planar.
A locally compact connected topological quasifield is a locally compact connected topological space Q such that $(Q, +)$ is a topological group, $(Q \setminus \{0\}, \cdot)$ is a topological loop, the multiplication $\cdot : Q \times Q \to Q$ is continuous and the mappings $\lambda_a : x \mapsto a \cdot x$ and $\rho_a : x \mapsto x \cdot a$ with $0 \neq a \in Q$ are homeomorphisms of Q. Moreover one has (cf. [11], pp. 322-323) either $K_r = \mathbb{R}$ and the dimension of Q over $\mathbb{R}$ is $1, 2, 4$ or 8, or $K_r = \mathbb{C}$ and the dimension of Q over $\mathbb{C}$ is 1 or 2, or Q is the skewfield of quaternions. We treat multiplicative loops of locally compact connected topological quasifields Q having dimension 2 over the field $\mathbb{C}$ and coordinatizing non-desarguesian 8-dimensional topological translation planes. Then $(Q, +)$ is the vector group $\mathbb{C}^2$ and the multiplicative loop $Q^* = (Q \setminus \{0\}, \cdot)$ is a topological loop homeomorphic to $\mathbb{R} \times S^3$, where S^3 is the 3-sphere.
A *near field* is a quasifield with associative multiplication. Every locally compact connected near field is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or to the near field $\mathbb{H}_r = (\mathbb{H}, +, \circ)$ obtained from the skewfield $(\mathbb{H}, +, \cdot)$ of quaternions by modifying the multiplication $\cdot$ with $x \circ y = x \cdot \varphi(x)^{-1} \cdot y \cdot \varphi(x)$, where $\varphi(x) = \exp(ir \log |x|)$, for some $r \in \mathbb{R}$. The near fields $\mathbb{H}_r$, $r \neq 0$, are called proper Kalscheuer's near fields. The multiplicative group of the near fields $\mathbb{H}_r$ is $Spin_3(\mathbb{R}) \times \mathbb{R}$.

3 Multiplicative loops of topological quasifields having dimension 2 over their kernel $\mathbb{C}$

Assume that Q is a right vector space over $\mathbb{C}$ with the scalar multiplication induced by $\mathbb{C}^*$. Let e_1 be the identity element of the multiplicative loop Q^* of Q which is the generator of the kernel K_r as vector space and let $B = \{e_1, e_2\}$ be a basis of Q over K_r. Once we fix B, we identify Q with the vector space of pairs $(x, y)^t \in \mathbb{C}^2$ and K_r with the subspace of pairs $(x, 0)^t$. The element $(1, 0)^t$ is the identity element of the multiplicative loop Q^* of Q. Then the set Λ_Q of all left translations of Q is a spread set $\mathcal{M}$ of the vector space Q (cf. Proposition 1.14 in [17], p. 12). Moreover, the set $\mathcal{M}$ consists of matrices $C(\alpha, \beta, \gamma, \delta) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \alpha, \beta, \gamma, \delta \in \mathbb{C}$. By Section 1.2 in [17], pp. 10-14, the vectors $(\alpha, \gamma)^t$ consists of all vectors of Q. Hence if $(\alpha, \gamma)^t$ is an element of Q, then there exists a unique matrix of $\mathcal{M} = \Lambda_Q$ having $(\alpha, \gamma)^t$ as the first column (cf. [9], p. 2595).
As $K_r = \mathbb{C}$ the group G topologically generated by the left translations of Q^* is a connected closed subgroup of $GL_2(\mathbb{C})$ (cf. [18], p. 345). Since the loop Q^* is homeomorphic to $S^3 \times \mathbb{R}$ the group G acts transitively on the sphere S^3 and it contains a 4-dimensional subgroup $S = Spin_3(\mathbb{R}) \times \mathbb{R}$. Since $Spin_3(\mathbb{R})$ is a maximal compact subgroup of $SL_2(\mathbb{C})$ it follows that for $G \neq Spin_3(\mathbb{R}) \times \mathbb{R}$ the group G is isomorphic either to the group $Spin_3(\mathbb{R}) \times \mathbb{C}$ or to $SL_2(\mathbb{C}) \times \mathbb{R}$ or to $GL_2(\mathbb{C})$ (cf. [26], p. 24).

Proposition 1 *Let Q^* be the multiplicative loop for a locally compact connected topological 4-dimensional quasifield Q having the field $\mathbb{C}$ as its kernel. If Q^* contains the normal subgroup $Spin_3(\mathbb{R})$, then Q is either a Kalscheuer's near field or the group G topologically generated by the left translations of Q^* is the group $Spin_3(\mathbb{R}) \times \mathbb{C}$.*

Proof By Lemma 1.7, p. 19, in [18], the left translations of a normal subloop of Q^* generate a normal subgroup of the group G topologically generated by all left translations of Q^*. The group topologically generated by the left translations of the normal subgroup $Spin_3(\mathbb{R})$ of Q^* is isomorphic to $Spin_3(\mathbb{R})$. But this group is normal in the group $G = Spin_3(\mathbb{R}) \times \mathbb{R}$ or in $G = Spin_3(\mathbb{R}) \times \mathbb{C}$. In the first case Q^* is the group $Spin_3(\mathbb{R}) \times \mathbb{R}$, or equivalently Q is a Kalscheuer's near field.

Assume that the loop Q^* is proper. As $\dim(Q^*) = 4$ and the stabilizer H of the identity element of Q^* in G does not contain any non-trivial normal subgroup of G we may assume that H is the subgroup

$$H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & e^{is} \end{pmatrix}, s \in \mathbb{R} \right\} \tag{2}$$

if G is $Spin_3(\mathbb{R}) \times \mathbb{C}$ or H has the form

$$H = \left\{ \begin{pmatrix} k & l \\ 0 & k^{-1} \end{pmatrix}, k > 0, l \in \mathbb{C} \right\} \tag{3}$$

if G is $SL_2(\mathbb{C}) \times \mathbb{R}$ or H is the subgroup

$$H = \left\{ \begin{pmatrix} k & l \\ 0 & k^{-1}e^{is} \end{pmatrix}, k > 0, l \in \mathbb{C}, s \in \mathbb{R} \right\} \tag{4}$$

if G is $GL_2(\mathbb{C})$. The elements g of G have a unique decomposition as the product

$$g = \begin{pmatrix} ux & -u\overline{y} \\ uy & u\overline{x} \end{pmatrix} h$$

with $x, y \in \mathbb{C},\ x\overline{x} + y\overline{y} = 1,\ 0 < u \in \mathbb{R}, h \in H$. Hence the loop Q^* corresponds to a continuous sharply transitive section of the form $\sigma : G/H \to G$;

$$\begin{pmatrix} ux & -u\overline{y} \\ uy & u\overline{x} \end{pmatrix} H \mapsto \begin{pmatrix} ux & -u\overline{y} \\ uy & u\overline{x} \end{pmatrix} \begin{pmatrix} a(u,x,y) & b(u,x,y) \\ 0 & a^{-1}(u,x,y)e^{ic(u,x,y)} \end{pmatrix} = M_{(u,x,y)} \tag{5}$$

such that $a(u,x,y)$, respectively $b(u,x,y)$, respectively $c(u,x,y)$ are continuous functions with positive, respectively complex, respectively real values. If G is isomorphic to $GL_2(\mathbb{C})$, then one has $a(1,1,0) = 1$, $b(1,1,0) = 0$ and $c(1,1,0) = 0$. If G is isomorphic to $SL_2(\mathbb{C}) \times \mathbb{R}$, then one has $a(1,1,0) = 1$, $b(1,1,0) = 0$ and $c(u,x,y)$ is the constant function 0. If G is isomorphic to $Spin_3(\mathbb{R}) \times \mathbb{C}$, then the function $a(u,x,y)$ is the constant function 1, the function $b(u,x,y)$ is the constant function 0 and $c(1,1,0) = 0$. The section σ given by (5) is sharply transitive precisely if for all given (u_1,x_1,y_1), (u_2,x_2,y_2) in $\mathbb{R}_{>0} \times Spin_3(\mathbb{R})$

there exists precisely one $(u, x, y) \in \mathbb{R}_{>0} \times Spin_3(\mathbb{R})$ and suitable $k > 0$, $l \in \mathbb{C}$, $s \in \mathbb{R}$ such that the matrix equation

$$\begin{pmatrix} ux & -u\overline{y} \\ uy & u\overline{x} \end{pmatrix} \begin{pmatrix} a(u,x,y) & b(u,x,y) \\ 0 & a^{-1}(u,x,y)e^{ic(u,x,y)} \end{pmatrix} \begin{pmatrix} u_1x_1 & -u_1\overline{y_1} \\ u_1y_1 & u_1\overline{x_1} \end{pmatrix} = \begin{pmatrix} u_2x_2 & -u_2\overline{y_2} \\ u_2y_2 & u_2\overline{x_2} \end{pmatrix} \begin{pmatrix} k & l \\ 0 & k^{-1}e^{is} \end{pmatrix} \tag{6}$$

is satisfied. As the determinant of the matrices on both sides of (6) are equal we get that $u = u_1^{-1}u_2$, $c(u, x, y) = s$. Using this, system (6) reduces to

$$\begin{pmatrix} x & -\overline{y} \\ y & \overline{x} \end{pmatrix} \begin{pmatrix} a(u_1^{-1}u_2,x,y) & b(u_1^{-1}u_2,x,y) \\ 0 & a^{-1}(u_1^{-1}u_2,x,y)e^{is} \end{pmatrix} = \begin{pmatrix} x_2 & -\overline{y_2} \\ y_2 & \overline{x_2} \end{pmatrix} \begin{pmatrix} k & l \\ 0 & k^{-1}e^{is} \end{pmatrix} \begin{pmatrix} \overline{x_1} & \overline{y_1} \\ -y_1 & x_1 \end{pmatrix}. \tag{7}$$

Comparing in both sides of matrix equation (7) the elements in the first column we obtain

$$xa(u_1^{-1}u_2, x, y) = x_2\overline{x_1}k - y_1x_2l + y_1\overline{y_2}k^{-1}e^{is}, \tag{8}$$

$$ya(u_1^{-1}u_2, x, y) = y_2\overline{x_1}k - y_1y_2l - y_1\overline{x_2}k^{-1}e^{is}. \tag{9}$$

Taking of both equation the complex conjugation we have

$$\overline{x}a(u_1^{-1}u_2, x, y) = \overline{x_2}x_1k - \overline{y_1x_2l} + \overline{y_1}y_2k^{-1}e^{-is}, \tag{10}$$

$$\overline{y}a(u_1^{-1}u_2, x, y) = \overline{y_2}x_1k - \overline{y_1y_2l} - \overline{y_1}x_2k^{-1}e^{-is}. \tag{11}$$

Multiplying (8) with (10) and (9) with (11) and adding the obtained equations we get

$$a(u_1^{-1}u_2, x, y) = \sqrt{k^2x_1\overline{x_1} - k\overline{ly_1x_1} - kly_1x_1 + y_1\overline{y_1}(l\overline{l} + k^{-2})}, \tag{12}$$

$$x = \frac{x_2\overline{x_1}k - y_1x_2l + y_1\overline{y_2}k^{-1}e^{is}}{\sqrt{k^2x_1\overline{x_1} - k\overline{ly_1x_1} - kly_1x_1 + y_1\overline{y_1}(l\overline{l} + k^{-2})}}, \tag{13}$$

$$y = \frac{y_2\overline{x_1}k - y_1y_2l - y_1\overline{x_2}k^{-1}e^{is}}{\sqrt{k^2x_1\overline{x_1} - k\overline{ly_1x_1} - kly_1x_1 + y_1\overline{y_1}(l\overline{l} + k^{-2})}}. \tag{14}$$

Comparing in both sides of matrix equation (7) the elements in the second column we obtain

$$xb(u_1^{-1}u_2, x, y) - \overline{y}a^{-1}(u_1^{-1}u_2, x, y)e^{is} = x_2\overline{y_1}k + x_1x_2l - x_1\overline{y_2}k^{-1}e^{is}, \tag{15}$$

$$yb(u_1^{-1}u_2, x, y) + \overline{x}a^{-1}(u_1^{-1}u_2, x, y)e^{is} = y_2\overline{y_1}k + x_1y_2l + x_1\overline{x_2}k^{-1}e^{is}. \tag{16}$$

Multiplying equation (15) with $\overline{x}$ and (16) with $\overline{y}$ and adding the obtained equations we have

$$b(u_1^{-1}u_2, x, y) = \frac{x_1\overline{y_1}k^2 - \overline{y_1}^2\overline{l}k + x_1^2lk - x_1\overline{y_1}(l\overline{l} + k^{-2})}{\sqrt{k^2x_1\overline{x_1} - k\overline{ly_1x_1} - kly_1x_1 + y_1\overline{y_1}(l\overline{l} + k^{-2})}}. \tag{17}$$

For a continuous sharply transitive section σ given by (5) the following holds: $u = u_1^{-1}u_2 > 0$, x, y are given by (13), (14) and for all fixed $u = u_1^{-1}u_2 > 0$

functions $a(u_1^{-1}u_2, x, y)$, $b(u_1^{-1}u_2, x, y)$ are given by (12), (17) and function $c(u_1^{-1}u_2, x, y)$ is the constant function s. Moreover, any continuous sharply transitive section σ given by (5) has this form with suitable $k > 0$, $l \in \mathbb{C}$, $s \in \mathbb{R}$. As (13), (14) show x, y depend on $(x_1, y_1, x_2, y_2) \in S^3 \times S^3$. Hence the continuous strictly positive function $a(u, x, y)$, the continuous complex function $b(u, x, y)$ and the continuous real function $c(u, x, y)$ are defined on the set $\mathbb{R}_{>0} \times S^3 \times S^3$.

According to Theorem 1.11 in [18], p. 21, if L_1 and L_2 are isomorphic loops such that they have the same group G of left translations and the same stabilizer H of $e \in L$ and $e' \in L'$, then there is an automorphism of G leaving H invariant and mapping the left translations of L onto the left translations of L'. Each automorphism $\neq$ id of the group $G = GL_2(\mathbb{C})$, respectively $SL_2(\mathbb{C}) \times \mathbb{R}$, respectively $Spin_3(\mathbb{R}) \times \mathbb{C}$ leaving the subgroup H given by (4), respectively (3), respectively (2) invariant are conjugations with an element $\neq 1$ of H. Let Q_1^* and Q_2^* be two multiplicative loops Q^* for locally compact connected topological 4-dimensional quasifields Q having the field $\mathbb{C}$ as their kernel such that Q_1^* and Q_2^* have the same group G of left translations and the same stabilizer H of $e_i \in Q_i^*$, $i = 1, 2$. The loops Q_1^* and Q_2^* are isomorphic if there exists an element $h \in H$ such that $\sigma_1(G/H) = h^{-1}\sigma_2(G/H)h$.

As Q is a left quasifield, any $(s, z)^t \in Q^*$ induces a linear transformation $M_{(u,x,y)} \in \sigma(G/H)$. More precisely

$$\begin{pmatrix} s \\ z \end{pmatrix} \cdot \begin{pmatrix} w \\ v \end{pmatrix} = M_{(u,x,y)} \begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} ux & -u\overline{y} \\ uy & u\overline{x} \end{pmatrix} \begin{pmatrix} a(u,x,y) & b(u,x,y) \\ 0 & a^{-1}(u,x,y)e^{ic(u,x,y)} \end{pmatrix} \begin{pmatrix} w \\ v \end{pmatrix},$$

where $s = uxa(u, x, y)$, $z = uya(u, x, y)$.

The kernel K_r of Q consists of $(0, 0)^t$, $(s, 0)^t$, $s \in \mathbb{C} \setminus \{0\}$, such that the matrix representation of the left translation with the element $(s, 0)^t$, $s \in \mathbb{C} \setminus \{0\}$ has the form

$$M_{(u,x,0)} = \begin{pmatrix} uxa(u,x,0) & uxb(u,x,0) \\ 0 & u\bar{x}a^{-1}(u,x,0)e^{ic(u,x,0)} \end{pmatrix} \tag{18}$$

with $s = uxa(u, x, 0)$, $x\overline{x} = 1$. The left translation $\lambda_{(1,0)^t}$ with the identity element $(1, 0)^t$ of Q^* is the identity matrix. As s and x have the unique polar representation $|s|e^{i\alpha}$, $e^{i\beta}$ we get that $|s| = ua(u, x, 0)$ and $\alpha = \beta$. The left translation with the element $(r, 0)^t \in K_r$, $r > 0$, has the form

$$M_{(u,1,0)} = \begin{pmatrix} ua(u,1,0) & ub(u,1,0) \\ 0 & ua^{-1}(u,1,0)e^{ic(u,1,0)} \end{pmatrix}. \tag{19}$$

Since $K_r = \mathbb{C}$ the set $K_r \setminus \{(0, 0)\}$ is a commutative subgroup S of Q^*. Therefore the group topologically generated by all left translations of the elements of $K_r \setminus \{(0, 0)\}$ is isomorphic to the multiplicative group $SO_2(\mathbb{R}) \times \mathbb{R}$ of the field $\mathbb{C}$. Hence one has $M_{(u,x,0)}M_{(v,z,0)} = M_{(uv,xz,0)} = M_{(v,z,0)}M_{(u,x,0)}$ for all $u, v > 0$, $x, z \in \mathbb{C}$, $x\bar{x} = 1$, $z\bar{z} = 1$, and therefore the conditions $c(uv, xz, 0) = c(u, x, 0) + c(v, z, 0)$, $a(uv, xz, 0) = a(u, x, 0)a(v, z, 0)$ and

$$b(uv, xz, 0) = a(u, x, 0)b(v, z, 0) + \frac{1}{z^2}b(u, x, 0)a^{-1}(v, z, 0)e^{ic(v,z,0)} =$$

$$a(v,z,0)b(u,x,0)+\frac{1}{x^2}b(v,z,0)a^{-1}(u,x,0)e^{ic(u,x,0)}$$

are satisfied. The subgroup S is never normal in the loop Q^* since otherwise the factor loop Q^*/S would be homeomorphic to S^2 and there does not exist a multiplication with identity on the 2-sphere ([1]).

Proposition 2 *The involution* $(-1,0)^t$ *of the multiplicative loop* Q^* *of a locally compact* 4*-dimensional connected topological quasifield* Q *having the field* $\mathbb{C}$ *as its kernel is contained in the centre of* Q^*.

Proof The left translation belonging to the involution $(-1,0)^t$ has the form $\begin{pmatrix} -1 & -b(1,-1,0) \\ 0 & -e^{ic(1,-1,0)} \end{pmatrix}$, i.e. $a(u,-1,0)=1$ since $f(u)=-ua(u,-1,0)$, $u>0$ is strictly monotone (cf. [9], p. 2596). As

$$\lambda_{(-1,0)^t}\lambda_{(-1,0)^t}=\begin{pmatrix} 1 & b(1,-1,0)(1+e^{ic(1,-1,0)}) \\ 0 & e^{2ic(1,-1,0)} \end{pmatrix}$$

is the left translation of the identity element $(1,0)^t\in Q^*$ we obtain that $b(1,-1,0)=0=c(1,-1,0)$. Hence $(-1,0)^t$ is in the centre of Q^*.

Proposition 3 *Let*

$$\begin{pmatrix} uk & -u\bar{l} \\ ul & u\bar{k} \end{pmatrix} H \mapsto \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}\begin{pmatrix} k & -\bar{l} \\ l & \bar{k} \end{pmatrix}\begin{pmatrix} a(1,k,l) & b(1,k,l) \\ 0 & a^{-1}(1,k,l)e^{ic(1,k,l)} \end{pmatrix}, u>0, k,l\in\mathbb{C}, k\bar{k}+l\bar{l}=1 \tag{20}$$

be a section belonging to a multiplicative loop Q^* *of a locally compact* 4*-dimensional connected topological quasifield* Q *having the field* $\mathbb{C}$ *as its kernel. Then* Q^* *contains a* 3*-dimensional compact subloop.*

Proof The image of section (20), seen as a set of (4×4)-real matrices, acts sharply transitively on the point set $\mathbb{R}^4\setminus\{(0,0,0,0)^t\}$. Since the real matrices corresponding to the subgroup $\left\{\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, u>0\right\}$ consist of homotheties fixing the point $(0,0,0,0)^t$ it leaves any line through $(0,0,0,0)^t$ fixed. Hence the subset of (4×4)-real matrices corresponding to

$$\mathcal{T}=\left\{\begin{pmatrix} k & -\bar{l} \\ l & \bar{k} \end{pmatrix}\begin{pmatrix} a(1,k,l) & b(1,k,l) \\ 0 & a^{-1}(1,k,l)e^{ic(1,k,l)} \end{pmatrix}, k,l\in\mathbb{C}, k\bar{k}+l\bar{l}=1\right\} \tag{21}$$

acts sharply transitively on the oriented lines through $(0,0,0,0)^t$. Therefore $\mathcal{T}$ corresponds to a 3-dimensional compact loop.

Proposition 4 *Let* Q *be a* 4*-dimensional locally compact connected topological quasifield having the field* $\mathbb{C}$ *as its kernel* K_r. *If the multiplicative loop* Q^* *of* Q *has a* 1*-dimensional connected normal subloop* N^*, *then* N^* *is a group isomorphic to* $\mathbb{R}$ *and has the form* $N_k^*=\{(e^{s+iks},0)^t; s\in\mathbb{R}\}\subset K_r$ *with some real constant* k. *The group* N_k *topologically generated by the left translations of* N_k^* *is* $N_k=\left\{\begin{pmatrix} e^{s+iks} & 0 \\ 0 & e^{s+iks} \end{pmatrix}, s\in\mathbb{R}\right\}$.

Proof By Lemma 1.7, p. 19, in [18], the left translations of a normal subloop N^* of Q^* generate a normal subgroup N of the group $G = Spin_3(\mathbb{R}) \times \mathbb{C}$ or $G = SL_2(\mathbb{C}) \times \mathbb{R}$ or $G = GL_2(\mathbb{C})$ topologically generated by all left translations of Q^*. If N^* is homeomorphic to the 1-sphere S^1, then the factor loop Q^*/N^* is defined on a topological product of spaces having as a factor the 2-sphere. Since there does not exist a multiplication with identity on the 2-sphere (cf. [1]) we have that N^* is homeomorphic to $\mathbb{R}$. If N^* would be a proper loop, then the group topologically generated by its left translations is the universal covering $\widetilde{PSL_2(\mathbb{R})}$ of $PSL_2(\mathbb{R})$ (cf. [18], Section 18, p. 235). But $\widetilde{PSL_2(\mathbb{R})}$ is not a subgroup of G. Hence $N^* \cong N$ is isomorphic to $\mathbb{R}$. Since N is normal in G it is a subgroup of $\mathbb{C}^* = \left\{ \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix},\ z \in \mathbb{C} \setminus \{0\} \right\}$ isomorphic to $\mathbb{R}$. Then N has the form $N_k = \left\{ \begin{pmatrix} e^{s+iks} & 0 \\ 0 & e^{s+iks} \end{pmatrix},\ s \in \mathbb{R} \right\}$, where $k \in \mathbb{R}$ is a fixed constant. According to (18) the group N_k is contained in the set $\mathcal{K}$ of the left translations corresponding to the kernel K_r such that $u = e^s$, $x = e^{iks}$, $a(u,x,0) = 1$, $b(u,x,0) = 0$, $c(u,x,0) = 2ks$. Hence the normal subgroup N^* of Q^* has the form $N_k^* = \{(e^{s+iks}, 0)^t,\ s \in \mathbb{R}\}$, $k \in \mathbb{R}$.

Proposition 5 *Let Q be a 4-dimensional locally compact connected topological quasifield having the field $\mathbb{C}$ as its kernel K_r. The multiplicative loop Q^* of Q has a 1-dimensional connected normal subgroup $N_k^* = \{(e^{s+iks}, 0)^t; s \in \mathbb{R}\}$, $k \in \mathbb{R}$, as in Proposition 4 if and only if for all $u > 0$, $(x,y) \in \mathbb{C}^2$, $x\bar{x} + y\bar{y} = 1$, one has $a(u, e^{iks}, 0) = 1$, $b(u, e^{iks}, 0) = 0$, $c(u, e^{iks}, 0) = 2ks$, $a(u,m,n) = a(1,x,y)$, $b(u,m,n) = b(1,x,y)$, $c(u,m,n) = c(1,x,y) + 2ks$, where $m = e^{iks}x$, $n = e^{iks}y$.*

Proof The set $\mathcal{K}_{N_k^*} = \{M_{(e^s, e^{iks}, 0)}, s \in \mathbb{R}\}$ of the left translations of Q^* corresponding to the subgroup $N_k^* = \{(e^{s+iks}, 0)^t; s \in \mathbb{R}\}$, $k \in \mathbb{R}$, of the kernel K_r of Q has the form N_k given in Proposition 4. The conditions $a(u, e^{iks}, 0) = 1$, $b(u, e^{iks}, 0) = 0$, $c(u, e^{iks}, 0) = 2ks$ are proved in the proof of Proposition 4. Now we find the necessary and sufficient conditions under which N_k^* is normal in Q^*. According to (5) the element

$$\begin{pmatrix} ux & -u\bar{y} \\ uy & u\bar{x} \end{pmatrix} \begin{pmatrix} a(u,x,y) & b(u,x,y) \\ 0 & a^{-1}(u,x,y)e^{ic(u,x,y)} \end{pmatrix}$$

belongs to the left translation of $(uxa(u,x,y), uya(u,x,y))^t$, $u > 0$, $x, y \in \mathbb{C}$, $x\bar{x} + y\bar{y} = 1$. For all elements $q_1 := (x,y)^t$ of S^3, $q_2 := (v,w)^t$ of Q^* the condition $(N_k^* \cdot q_1) \cdot q_2 = N_k^* \cdot (q_1 \cdot q_2)$ of (1) is satisfied if and only if we have

$$\left[\begin{pmatrix} e^{s+iks} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \right] \cdot \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} e^{s'+iks'} \\ 0 \end{pmatrix} \cdot \left[\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} v \\ w \end{pmatrix} \right]$$

for all $x, y \in \mathbb{C}$, $x\bar{x} + y\bar{y} = 1$, $(v,w) \in \mathbb{C}^2 \setminus \{(0,0)\}$ with suitable $s, s' \in \mathbb{R}$. This is the case precisely if one has

$$\begin{pmatrix} ua(u,m,n)mv + ub(u,m,n)mw - ua^{-1}(u,m,n)w\bar{n}e^{ic(u,m,n)} \\ ua(u,m,n)nv + ub(u,m,n)nw + ua^{-1}(u,m,n)w\bar{m}e^{ic(u,m,n)} \end{pmatrix} =$$

$$\begin{pmatrix} e^{s'+iks'}(a(1,x,y)xv+b(1,x,y)xw-a^{-1}(1,x,y)w\bar{y}e^{ic(1,x,y)}) \\ e^{s'+iks'}(a(1,x,y)yv+b(1,x,y)yw+a^{-1}(1,x,y)w\bar{x}e^{ic(1,x,y)}) \end{pmatrix},$$

where $e^{s+iks}x = ua(u,m,n)m$, $e^{s+iks}y = ua(u,m,n)n$, $m\bar{m}+n\bar{n}=1$. Since $e^{2s}(x\bar{x}+y\bar{y}) = u^2a^2(u,m,n)(m\bar{m}+n\bar{n})$ one has $e^s = ua(u,m,n)$, $m = e^{iks}x$, $n = e^{iks}y$. Using this for all $x,y\in\mathbb{C}$, $x\bar{x}+y\bar{y}=1$, $(v,w)\in\mathbb{C}^2\setminus\{(0,0)\}$ we have

$$[x(va(u,m,n)+wb(u,m,n))e^{iks}-\bar{y}wa^{-1}(u,m,n)e^{ic(u,m,n)-iks}]\cdot$$
$$[y(va(1,x,y)+wb(1,x,y))+\bar{x}wa^{-1}(1,x,y)e^{ic(1,x,y)}] =$$
$$[y(va(u,m,n)+wb(u,m,n))e^{iks}+\bar{x}wa^{-1}(u,m,n)e^{ic(u,m,n)-iks}]\cdot$$
$$[x(va(1,x,y)+wb(1,x,y))-\bar{y}wa^{-1}(1,x,y)e^{ic(1,x,y)}].$$

The last equation holds if and only if for all $(v,w)\in\mathbb{C}^2\setminus\{(0,0)\}$ one has

$$(a(u,m,n)a^{-1}(1,x,y)e^{ic(1,x,y)}e^{iks}-a^{-1}(u,m,n)a(1,x,y)e^{ic(u,m,n)}e^{-iks})(x\bar{x}+y\bar{y})vw$$
$$+(b(u,m,n)a^{-1}(1,x,y)e^{ic(1,x,y)}e^{iks}-a^{-1}(u,m,n)b(1,x,y)e^{ic(u,m,n)}e^{-iks})(x\bar{x}+y\bar{y})w^2=0.$$

Hence we have

$$a(u,m,n)a^{-1}(1,x,y)e^{ic(1,x,y)}e^{iks}-a^{-1}(u,m,n)a(1,x,y)e^{ic(u,m,n)}e^{-iks}=0,$$
$$b(u,m,n)a^{-1}(1,x,y)e^{ic(1,x,y)}e^{iks}-a^{-1}(u,m,n)b(1,x,y)e^{ic(u,m,n)}e^{-iks}=0.$$

Multiplying both equations with $e^{-ic(1,x,y)}e^{-iks}$ one has

$$a(u,m,n)a^{-1}(1,x,y)-a^{-1}(u,m,n)a(1,x,y)e^{ic(u,m,n)-ic(1,x,y)-2iks}=0,$$
$$b(u,m,n)a^{-1}(1,x,y)-a^{-1}(u,m,n)b(1,x,y)e^{ic(u,m,n)-ic(1,x,y)-2iks}=0.$$

As $a(u,m,n)$ is positive for all $u>0$, $m=e^{iks}x$, $n=e^{iks}y$, we obtain $c(u,m,n)=c(1,x,y)+2ks$ and hence $a(u,m,n)=a(1,x,y)$, $b(u,m,n)=b(1,x,y)$ for all $u>0$, $x,y\in\mathbb{C}$, $x\bar{x}+y\bar{y}=1$. For all elements $q_1:=(x,y)^t$ of S^3, $q_2:=(v,w)^t$ of Q^* the condition $(q_1\cdot N_k^*)\cdot q_2 = q_1\cdot(N_k^*\cdot q_2)$ of (1) is satisfied if and only if for all $x,y\in\mathbb{C}$, $x\bar{x}+y\bar{y}=1$, $(v,w)\in\mathbb{C}^2\setminus\{(0,0)\}$ one has

$$\left[\begin{pmatrix} x \\ y\end{pmatrix}\cdot\begin{pmatrix} e^{s+iks} \\ 0\end{pmatrix}\right]\cdot\begin{pmatrix} v \\ w\end{pmatrix} = \begin{pmatrix} x \\ y\end{pmatrix}\cdot\left[\begin{pmatrix} e^{s'+iks'} \\ 0\end{pmatrix}\cdot\begin{pmatrix} v \\ w\end{pmatrix}\right] \quad \text{or}$$

$$\begin{pmatrix} e^{s+iks}a(1,x,y)x & e^{s+iks}(xb(1,x,y)-\bar{y}a^{-1}(1,x,y)e^{ic(1,x,y)}) \\ e^{s+iks}a(1,x,y)y & e^{s+iks}(yb(1,x,y)+\bar{x}a^{-1}(1,x,y)e^{ic(1,x,y)}) \end{pmatrix}\begin{pmatrix} v \\ w\end{pmatrix} =$$

$$\begin{pmatrix} a(1,x,y)x & b(1,x,y)x-a^{-1}(1,x,y)\bar{y}e^{ic(1,x,y)} \\ a(1,x,y)y & b(1,x,y)y+a^{-1}(1,x,y)\bar{x}e^{ic(1,x,y)} \end{pmatrix}\begin{pmatrix} e^{s'+iks'}v \\ e^{s'+iks'}w\end{pmatrix}$$

for suitable $s,s'\in\mathbb{R}$. This is equivalent to

$$\begin{pmatrix} ua(u,m,n)mv+ub(u,m,n)mw-ua^{-1}(u,m,n)w\bar{n}e^{ic(u,m,n)} \\ ua(u,m,n)nv+ub(u,m,n)nw+ua^{-1}(u,m,n)w\bar{m}e^{ic(u,m,n)} \end{pmatrix} =$$

$$\begin{pmatrix} ve^{s'+iks'}a(1,x,y)x+e^{s'+iks'}w[b(1,x,y)x-a^{-1}(1,x,y)\bar{y}e^{ic(1,x,y)}] \\ ve^{s'+iks'}a(1,x,y)y+e^{s'+iks'}w[b(1,x,y)y+a^{-1}(1,x,y)\bar{x}e^{ic(1,x,y)}] \end{pmatrix} \tag{22}$$

with $ua(u,m,n)m = xa(1,x,y)e^{s+iks}$, $ua(u,m,n)n = ya(1,x,y)e^{s+iks}$. The last two equations yield that $ua(u,m,n)=e^sa(1,x,y)$, $m=xe^{iks}$, $n=ye^{iks}$. Using this a direct computation yields that equality (22) is true if and only if $a(u,m,n)=a(1,x,y)$, $c(u,m,n)=c(1,x,y)+2ks$ and $b(u,m,n)=b(1,x,y)$ for all $u>0$, $x,y\in\mathbb{C}$, $x\bar{x}+y\bar{y}=1$. This proves the assertion.

Theorem 6 *Let Q be a 4-dimensional locally compact connected topological quasifield having the field $\mathbb{C}$ as its kernel K_r. If the multiplicative loop Q^* of Q has a 1-dimensional connected normal subloop, then it is a central extension of a normal subgroup N_k^*, $k \in \mathbb{R}$, isomorphic to $\mathbb{R}$ by a loop homeomorphic to S^3.*

Proof According to Proposition 4 the only possibility for a connected normal subloop of dimension 1 is a group N_k^*, $k \in \mathbb{R}$. N_k^* is a central subgroup of Q^* since the set $\mathcal{K}_{N_k^*}$ of the left translations of N_k^* consists of diagonal matrices of the group G generated by all left translations of Q^*. The intersection of a compact subloop of Q^* with N_k^* is 1 (cf. Proposition 3 and Proposition 4). Hence Q^* is a central extension as in the assertion.

Corollary 7 *Let Q be a 4-dimensional locally compact connected topological quasifield having the field $\mathbb{C}$ as its kernel K_r. The multiplicative loop Q^* of Q has the 1-dimensional normal subgroup $N^* = \{(e^s, 0)^t; s \in \mathbb{R}\} = \{(u, 0)^t; u > 0\}$, if and only if for all $u > 0$, $(x, y) \in \mathbb{C}^2$, $x\bar{x} + y\bar{y} = 1$, one has $a(u, 1, 0) = 1$, $b(u, 1, 0) = 0$, $c(u, 1, 0) = 0$, $a(u, x, y) = a(1, x, y)$, $b(u, x, y) = b(1, x, y)$, $c(u, x, y) = c(1, x, y)$. In this case the set $\mathcal{K} = \{M(u, 1, 0), u > 0\}$ of the left translations of Q^* belonging to the subgroup $\{(u, 0)^t, u > 0\}$ of the kernel K_r of Q is $\left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, u > 0 \right\}$. Then the set Λ_{Q^*} of all left translations of Q^* can be written into the form*

$$\left\{ \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \begin{pmatrix} ua(1, x, y) & ub(1, x, y) \\ 0 & ua^{-1}(1, x, y)e^{ic(1,x,y)} \end{pmatrix}, u > 0, x, y \in \mathbb{C}, x\bar{x} + y\bar{y} = 1 \right\}. \tag{23}$$

4 Decomposable multiplicative loops of 4-dimensional quasifields

Definition 1 *We call the multiplicative loop Q^* of a locally compact connected topological 4-dimensional quasifield Q having $K_r = \mathbb{C}$ as its kernel decomposable, if the set of all left translations of Q^* is a product $\mathcal{T}\mathcal{K}$, where $\mathcal{T}$ is the set of all left translations of a 3-dimensional compact loop of form (21) and $\mathcal{K}$ is the set of all left translations of Q^* belonging to the subgroup $\{M(u, 1, 0), u > 0\} \cong \mathbb{R}$ of the kernel K_r of Q of form (19).*

If the loop Q^* is decomposable, then it contains a compact subloop of form (21). Then one has

$$\begin{pmatrix} k & -\bar{l} \\ l & \bar{k} \end{pmatrix} \begin{pmatrix} a(1, k, l) & b(1, k, l) \\ 0 & a^{-1}(1, k, l)e^{ic(1,k,l)} \end{pmatrix} \left[\begin{pmatrix} ua(u, 1, 0) & ub(u, 1, 0) \\ 0 & ua^{-1}(u, 1, 0)e^{ic(u,1,0)} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] =$$

$$= \begin{pmatrix} uk & -u\bar{l} \\ ul & u\bar{k} \end{pmatrix} \begin{pmatrix} a(u, k, l) & b(u, k, l) \\ 0 & a^{-1}(u, k, l)e^{ic(u,k,l)} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{24}$$

Equation (24) yields that $a(u, k, l) = a(1, k, l)a(u, 1, 0)$.
Now we give sufficient and necessary conditions for the loop Q^* to be decomposable.

Proposition 8 *The multiplicative loop Q^* of a locally compact connected topological 4-dimensional quasifield Q with $K_r = \mathbb{C}$ is decomposable if and only if one has $a(u,k,l) = a(1,k,l)a(u,1,0)$, $c(u,k,l) = c(u,1,0) + c(1,k,l)$ and $b(u,k,l) = a(1,k,l)b(u,1,0) + a^{-1}(u,1,0)e^{ic(u,1,0)}b(1,k,l)$ for all $u > 0$, $k,l \in \mathbb{C}$, $k\overline{k} + l\overline{l} = 1$.*

Proof The point $(x,y)^t$ is the image of the point $(1,0)^t$ under a suitable linear mapping $M_{(u,k,l)}$ and the set $\{M_{(u,k,l)}; u > 0, k,l \in \mathbb{C}, k\overline{k}+l\overline{l} = 1\}$ acts sharply transitively on Q^*. The matrix equation

$$\begin{pmatrix} k & -\bar{l} \\ l & \bar{k} \end{pmatrix} \begin{pmatrix} a(1,k,l) & b(1,k,l) \\ 0 & a^{-1}(1,k,l)e^{ic(1,k,l)} \end{pmatrix} \left[\begin{pmatrix} ua(u,1,0) & ub(u,1,0) \\ 0 & ua^{-1}(u,1,0)e^{ic(u,1,0)} \end{pmatrix} \begin{pmatrix} sxa(s,x,y) \\ sya(s,x,y) \end{pmatrix} \right] =$$

$$= \begin{pmatrix} uk & -u\bar{l} \\ ul & u\bar{k} \end{pmatrix} \begin{pmatrix} a(u,k,l) & b(u,k,l) \\ 0 & a^{-1}(u,k,l)e^{ic(u,k,l)} \end{pmatrix} \begin{pmatrix} sxa(s,x,y) \\ sya(s,x,y) \end{pmatrix} \quad (25)$$

holds precisely if the identities of the assertion are satisfied.

Theorem 9 *If the multiplicative loop Q^* of a locally compact connected topological 4-dimensional quasifield Q with $K_r = \mathbb{C}$ has the 1-dimensional normal subgroup $N^* = \{(u,0)^t; u > 0\}$, then Q^* is decomposable.*

Proof By Corollary 7 the loop Q^* has a 1-dimensional normal subloop $N^* = \{(u,0)^t; u > 0\}$ if and only if for all $u > 0$, $k,l \in \mathbb{C}$, $k\overline{k} + l\overline{l} = 1$ one has $a(u,1,0) = 1$, $b(u,1,0) = 0 = c(u,1,0)$, $a(u,k,l) = a(1,k,l)$, $b(u,k,l) = b(1,k,l)$ and $c(u,k,l) = c(1,k,l)$. Therefore the identities given in the assertion of Proposition 8 are satisfied.

Proposition 10 *The set Λ_{Q^*} of all left translations of the multiplicative loop Q^* for a locally compact connected topological 4-dimensional quasifield Q having the field $\mathbb{C}$ as its kernel contains the group $Spin_3(\mathbb{R})$ if and only if Λ_{Q^*} has the form*

$$\Lambda_{Q^*} = \left\{ \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \begin{pmatrix} ua(u,1,0) & ub(u,1,0) \\ 0 & ua^{-1}(u,1,0)e^{ic(u,1,0)} \end{pmatrix}, u > 0, x,y \in \mathbb{C}, x\bar{x} + y\bar{y} = 1 \right\}, \quad (26)$$

where $a(u,1,0)$, $b(u,1,0)$, $c(u,1,0)$ are continuous functions, $a(u,1,0) > 0$ and $ua(u,1,0)$ is strictly monotone. In this case Q^* is decomposable.

Proof If the set Λ_{Q^*} contains the group $Spin_3(\mathbb{R})$, then for each fixed $u > 0$ the function $a(u,x,y)$ is constant with value 1 and the functions $c(u,x,y)$, $b(u,x,y)$ are constants with value 0. So the functions $a(u,x,y) = a(u,1,0)$, $b(u,x,y) = b(u,1,0)$, $c(u,x,y) = c(u,1,0)$ do not depend on the variables x, y. Hence the identities in Proposition 8 are satisfied and the set Λ_{Q^*} has the form $\mathcal{TK}$ as in the assertion.

Each positive number r has precisely one representation as $ua(u,1,0)$ if and only if the function $ua(u,1,0)$ with $a(u,1,0) > 0$ is strictly monotone in $u > 0$. If $ua(u,1,0)$ is a strictly monotone continuous function, then for arbitrary continuous functions $a(u,1,0)$, $b(u,1,0)$, $c(u,1,0)$ with $a(u,1,0) > 0$ the set given by (26) is the set Λ_{Q^*} of all left translations of the multiplicative loop Q^* of a locally compact quasifield Q having $K_r = \mathbb{C}$ as its kernel such that Λ_{Q^*} contains the group $Spin_3(\mathbb{R})$. Hence Q^* is decomposable and the assertion is proved.

If the function $a(u,1,0)$ in (26) of Proposition 10 is differentiable, then for every $u>0$ the derivative $a(u,1,0)+ua'(u,1,0)$ is either always positive or negative equivalently $[ln(a(u,1,0))]'$ is always greater or smaller, then $-u^{-1}$.

5 Semifields with complex kernel

Knarr in [17], Section 6, has been determined the 4-dimensional semifields Q having the field $\mathbb{C}$ as their kernel. The product $(x_1,x_2)^t*(y_1,y_2)^t$, $(x_1,x_2)^t$, $(y_1,y_2)^t\in\mathbb{C}^2\setminus\{(0,0)\}$ of the multiplicative loop Q^* of Q is given either by

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}*\begin{pmatrix}y_1\\y_2\end{pmatrix}=\lambda_{(x_1,x_2)}\begin{pmatrix}y_1\\y_2\end{pmatrix}=\begin{pmatrix}x_1&-e^{i\delta}\bar{x_2}\\x_2&\bar{x_1}\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix},\ 0<\delta<\pi \tag{27}$$

or by

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}*\begin{pmatrix}y_1\\y_2\end{pmatrix}=\lambda_{(x_1,x_2)}\begin{pmatrix}y_1\\y_2\end{pmatrix}=\begin{pmatrix}x_1&-c\bar{x_2}-x_2\\x_2&\bar{x_1}+r\bar{x_2}\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}, \tag{28}$$

where $r\geq 0$ and $c=c_1+ic_2\in\mathbb{C}$, $c_2\geq 0$ are constants such that for all $u\in\mathbb{R}$ one has $0<P_{r,c}(u)=u^4+(2\mathrm{Re}\ c-r^2)u^2-2ru+|c|^2-1$ (cf. [17], p. 83), $\bar{z}$ is the complex conjugate of $z\in\mathbb{C}$.
Both kernels K_r and K_l of the quasifields Q_δ, $0<\delta<\pi$ corresponding to the multiplication (27) are $K_r=K_l=\{(k,0),\ k\in\mathbb{C}\}$ and the centre of Q_δ is $Z=\{(k,0),\ k\in\mathbb{R}\}$. Hence Q_δ are real algebras which are called Rees algebras ([18], Section 29.2, pp. 346-348). The kernel K_r of the quasifield $Q_{(r,c)}$ corresponding to the multiplication (28) is $K_r=\{(k,0),k\in\mathbb{C}\}$ whereas the kernel $K_l=\{(k,0),k\in\mathbb{R}\}$ is isomorphic to $\mathbb{R}$ and coincides with the centre Z of $Q_{(r,c)}$.

Proposition 11 *Let Q be a 4-dimensional semifield having $K_r=\mathbb{C}$ as its kernel and coordinatizing an 8-dimensional locally compact non-desarguesian translation plane $\mathcal{A}$. Then the set Λ_{Q^*} of all left translations of the multiplicative loop Q^* has form (23) defined as follows:*
a) If Q^*_δ is given by (27), then one has

$$a(1,x,y)=\frac{1}{\sqrt[4]{(x\bar{x})^2+2x\bar{x}y\bar{y}\cos\delta+(y\bar{y})^2}},$$

$$e^{ic(1,x,y)}=\frac{x\bar{x}+y\bar{y}e^{i\delta}}{\sqrt[2]{(x\bar{x})^2+2x\bar{x}y\bar{y}\cos\delta+(y\bar{y})^2}},$$

$$b(1,x,y)=\frac{\bar{x}\bar{y}(1-e^{i\delta})}{\sqrt[4]{(x\bar{x})^2+2x\bar{x}y\bar{y}\cos\delta+(y\bar{y})^2}}$$

with

$$u=\sqrt[4]{(x_1\bar{x_1})^2+2x_1\bar{x_1}x_2\bar{x_2}\cos\delta+(x_2\bar{x_2})^2},$$

$$x=\frac{x_1}{\sqrt[2]{x_1\bar{x_1}+x_2\bar{x_2}}},\ y=\frac{x_2}{\sqrt[2]{x_1\bar{x_1}+x_2\bar{x_2}}}.$$

The semifields $Q_\delta, 0 < \delta < \pi$ coordinatize a one-parameter family of planes $\mathcal{A}_\delta$. The multiplicative loop Q_δ^* is the direct product of the group $\mathbb{R}$ and a loop L_δ diffeomorphic to S^3 and having the multiplication

$$\begin{pmatrix} x_1 & -e^{i\delta}\bar{x_2} \\ x_2 & \bar{x_1} \end{pmatrix} \circ \begin{pmatrix} y_1 & -e^{i\delta}\bar{y_2} \\ y_2 & \bar{y_1} \end{pmatrix} = \begin{pmatrix} z_1 & -e^{i\delta}\bar{z_2} \\ z_2 & \bar{z_1} \end{pmatrix},$$

where $z_1 = x_1y_1 - e^{i\delta}\bar{x_2}y_2$, $z_2 = x_2y_1 + \bar{x_1}y_2$, $|det(\lambda_{(x_1,x_2)})| = |det(\lambda_{(y_1,y_2)})| = 1 = |det(\lambda_{(z_1,z_2)})|$. The loop Q_δ^* is decomposable. The group generated by all left translations of Q_δ^* is the group $GL_2(\mathbb{C})$. The group generated by all left translations of L_δ and the group generated by all right translations of L_δ is the group of complex (2×2)-matrices the determinants of which have absolute value 1. The group generated by all translations of L_δ is the group $SL_4(\mathbb{R})$.

b) If $Q^*_{(r,c)}$ is given by (28), then one has

$$a(1,x,y) = \frac{1}{\sqrt[4]{(x\bar{x} + rx\bar{y} + cy\bar{y} + y^2)(x\bar{x} + ry\bar{x} + \bar{c}y\bar{y} + \bar{y}^2)}}$$

$$e^{ic(1,x,y)} = \frac{x\bar{x} + rx\bar{y} + cy\bar{y} + y^2}{\sqrt[2]{(x\bar{x} + rx\bar{y} + cy\bar{y} + y^2)(x\bar{x} + ry\bar{x} + \bar{c}y\bar{y} + \bar{y}^2)}}$$

$$b(1,x,y) = \frac{(1-c)\bar{x}\bar{y} - \bar{x}y + r\bar{y}^2}{\sqrt[4]{(x\bar{x} + rx\bar{y} + cy\bar{y} + y^2)(x\bar{x} + ry\bar{x} + \bar{c}y\bar{y} + \bar{y}^2)}}$$

with

$$u = \sqrt[4]{(x_1\bar{x_1} + rx_1\bar{x_2} + cx_2\bar{x_2} + x_2^2)(x_1\bar{x_1} + rx_2\bar{x_1} + \bar{c}x_2\bar{x_2} + \bar{x_2}^2)},$$

$$x = \frac{x_1}{\sqrt[2]{x_1\bar{x_1} + x_2\bar{x_2}}},\ y = \frac{x_2}{\sqrt[2]{x_1\bar{x_1} + x_2\bar{x_2}}}.$$

The multiplicative loop $Q^*_{(r,c)}$ is the direct product of the group $\mathbb{R}$ and a loop $L_{(r,c)}$ diffeomorphic to S^3 and having the multiplication

$$\begin{pmatrix} x_1 & -c\bar{x_2} - x_2 \\ x_2 & \bar{x_1} + r\bar{x_2} \end{pmatrix} \circ \begin{pmatrix} y_1 & -c\bar{y_2} - y_2 \\ y_2 & \bar{y_1} + r\bar{y_2} \end{pmatrix} = \begin{pmatrix} z_1 & -c\bar{z_2} - z_2 \\ z_2 & \bar{z_1} + r\bar{z_2} \end{pmatrix},$$

where $z_1 = x_1y_1 - c\bar{x_2}y_2 - x_2y_2$, $z_2 = x_2y_1 + \bar{x_1}y_2 + r\bar{x_2}y_2$, $|det(\lambda_{(x_1,x_2)})| = |det(\lambda_{(y_1,y_2)})| = 1 = |det(\lambda_{(z_1,z_2)})|$. The loop $Q^*_{(r,c)}$ is decomposable. The group generated by all left translations of $Q^*_{(r,c)}$ is the group $GL_2(\mathbb{C})$. The group generated by all left translations of $L_{(r,c)}$ is the group of complex (2×2)-matrices the determinants of which have absolute value 1. The group generated by all right translations of $L_{(r,c)}$ and the group generated by all translations of $L_{(r,c)}$ is the group $SL_4(\mathbb{R})$.

Proof We denote by Q_1^*, respectively by Q_2^* the multiplicative loop Q_δ^*, respectively $Q_{(r,c)}^*$. Let $G_{Q_i^*}$, $i = 1, 2$, be the group topologically generated by the left translations of Q_i^*. The group $G_{Q_1^*}$ is the group $GL_2(\mathbb{C})$ (cf. [18], p. 346). We prove that $G_{Q_2^*}$ is $GL_2(\mathbb{C})$ too. Since the group $\det(G_{Q_2^*})$ coincides with $\mathbb{C}^*$ the group $G_{Q_2^*}$ is isomorphic either to the group $Spin_3(\mathbb{R}) \times \mathbb{C}$ or to $GL_2(\mathbb{C})$. For any matrix M in $Spin_3(\mathbb{R}) \times \mathbb{C}$ one has $M \cdot \bar{M}^t \in \mathbb{R} \cdot I$, where I is the identity matrix. But if $x_2 \neq 0$ we have

$$\begin{pmatrix} x_1 & -c\bar{x_2} - x_2 \\ x_2 & \bar{x_1} + r\bar{x_2} \end{pmatrix} \begin{pmatrix} \bar{x_1} & \bar{x_2} \\ -\bar{c}x_2 - \bar{x_2} & x_1 + rx_2 \end{pmatrix} =$$

$$\begin{pmatrix} |x_1|^2 + (1 + |c|^2)|x_2|^2 + c\bar{x_2}^2 + \bar{c}x_2^2 & (1-c)x_1\bar{x_2} - x_2x_1 - rx_2^2 - crx_2\bar{x_2} \\ (1-\bar{c})\bar{x_1}x_2 - \bar{x_2}\bar{x_1} - r\bar{x_2}^2 - \bar{c}rx_2\bar{x_2} & |x_1|^2 + |x_2|^2 + r(\bar{x_2}x_1 + x_2\bar{x_1}) \end{pmatrix} \notin \mathbb{R} \cdot I.$$

Hence $G_{Q_2^*}$ is the group $GL_2(\mathbb{C})$. In both cases Q_i^*, $i = 1, 2$, has a central subgroup $Z_0^* = \{(k, 0), k > 0\} \cong \mathbb{R}$ and the factor loop Q_i^*/Z_0^* is homeomorphic to S^3. Hence the group $G_{Q_i^*/Z_0^*}$ topologically generated by the left translations of Q_i^*/Z_0^* acts transitively on S^3. Since $G_{Q_i^*}$ is the group $GL_2(\mathbb{C})$ the group $G_{Q_i^*/Z_0^*}$ is the group $GL_2(\mathbb{C})/Z_0 \cong SL_2(\mathbb{C}) \times SO_2(\mathbb{R})$, where Z_0 is the group of the left translations by the elements of Z_0^*. Hence any maximal compact subgroup of $G_{Q_i^*/Z_0^*}$ is isomorphic to the direct product $SU_2(\mathbb{C}) \times SO_2(\mathbb{R})$.

Let S_δ be the set of matrices $\lambda_{(x_1,x_2)} = \begin{pmatrix} x_1 & -e^{i\delta}\bar{x_2} \\ x_2 & \bar{x_1} \end{pmatrix}$ with $|\det(\lambda_{(x_1,x_2)})| = 1$. Then S_δ topologically generates the group Δ_1 of complex matrices A with $|\det(A)| = 1$ because S_δ contains non-compact elements and the map $S_\delta \to S_\delta Z_0/Z_0$ is bijective. The product $\circ : S_\delta \times S_\delta \to S_\delta$ given by

$$\begin{pmatrix} x_1 & -e^{i\delta}\bar{x_2} \\ x_2 & \bar{x_1} \end{pmatrix} \circ \begin{pmatrix} y_1 & -e^{i\delta}\bar{y_2} \\ y_2 & \bar{y_1} \end{pmatrix} = \begin{pmatrix} z_1 & -e^{i\delta}\bar{z_2} \\ z_2 & \bar{z_1} \end{pmatrix},$$

where $z_1 = x_1y_1 - e^{i\delta}\bar{x_2}y_2$, $z_2 = x_2y_1 + \bar{x_1}y_2$ yields a loop L_δ diffeomorphic to S^3 since the set S_δ is a system of representatives with respect to the subgroup $\left\{ \begin{pmatrix} k & l \\ 0 & k^{-1}e^{is} \end{pmatrix}, k > 0, l \in \mathbb{C}, s \in \mathbb{R} \right\}$ in the group Δ_1. It follows that the multiplicative loop Q_δ^* of Q_δ is isomorphic to the direct product of $\mathbb{R}$ and L_δ. By Theorem 9 the loop Q_δ^* is decomposable. The group Σ_δ topologically generated by the right translations of L_δ is conjugate to the group Δ_1 in $SL_4(\mathbb{R})$ whereas the group topologically generated by all translations of L_δ is the group $SL_4(\mathbb{R})$ (cf. [18], Section 29, pp. 347-348).

Let $S_{(r,c)}$ be the set of matrices

$$\lambda_{(x_1,x_2)} = \begin{pmatrix} x_1 & -c\bar{x_2} - x_2 \\ x_2 & \bar{x_1} + r\bar{x_2} \end{pmatrix}, \ |\det(\lambda_{(x_1,x_2)})| = 1.$$

Since $S_{(r,c)}$ contains non-compact matrices and the map $S_{(r,c)} \to S_{(r,c)}Z_0/Z_0$ is bijective, the set $S_{(r,c)}$ topologically generates the group Δ_2 which is again the group of complex matrices A with $|\det(A)| = 1$. The product $\circ : S_{(r,c)} \times S_{(r,c)} \to S_{(r,c)}$ given by

$$\begin{pmatrix} x_1 & -c\bar{x_2} - x_2 \\ x_2 & \bar{x_1} + r\bar{x_2} \end{pmatrix} \circ \begin{pmatrix} y_1 & -c\bar{y_2} - y_2 \\ y_2 & \bar{y_1} + r\bar{y_2} \end{pmatrix} = \begin{pmatrix} z_1 & -c\bar{z_2} - z_2 \\ z_2 & \bar{z_1} + r\bar{z_2} \end{pmatrix},$$

where $z_1 = x_1y_1 - c\bar{x}_2y_2 - x_2y_2$, $z_2 = x_2y_1 + \bar{x}_1y_2 + r\bar{x}_2y_2$, yields a loop $L_{(r,c)}$ diffeomorphic to S^3 because $L_{(r,c)}$ is a system of representatives with respect to the subgroup $\left\{ \begin{pmatrix} k & l \\ 0 & k^{-1}e^{is} \end{pmatrix}, k > 0, l \in \mathbb{C}, s \in \mathbb{R} \right\}$ in the group Δ_2. Hence the multiplicative loop $Q^*_{(r,c)}$ of $Q_{(r,c)}$ is isomorphic to the direct product of $\mathbb{R}$ and $L_{(r,c)}$. By Theorem 9 the loop $Q^*_{(r,c)}$ is decomposable.
The right translations $\rho_{(a,b)}$ of $L_{(r,c)}$ with $a = a_1 + ia_2$, $b = b_1 + ib_2$ are represented in the group $SL_4(\mathbb{R})$ as

$$\rho_{(a,b)}(x,y) = (x_1, x_2, y_1, y_2) \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ -a_2 & a_1 & b_2 & -b_1 \\ -c_1b_1 + c_2b_2 - b_1 & -c_2b_1 - c_1b_2 - b_2 & a_1 + rb_1 & a_2 + rb_2 \\ -c_2b_1 - c_1b_2 + b_2 & c_1b_1 - c_2b_2 - b_1 & -a_2 + rb_2 & a_1 - rb_1 \end{pmatrix}, \tag{29}$$

where $\det(\rho_{(a,b)}) = 1$, $x = x_1 + ix_2$, $y = y_1 + iy_2$ and the parameters $c = c_1 + ic_2$, $c_2 \geq 0$, $r \geq 0$ satisfy the inequality in (28). Since the loop $L_{(r,c)}$ is diffeomorphic to S^3 the non-compact connected Lie group $\Sigma_{(r,c)}$ topologically generated by the right translations $\rho_{(a,b)}$ of $L_{(r,c)}$ acts transitively on S^3 and a subgroup of $SL_4(\mathbb{R})$. The quasi-simple connected non-compact Lie groups G which act transitively on S^3 are: $SL_2(\mathbb{C})$, $SO_5(\mathbb{R},1)$, $SU_3(\mathbb{C},1)$, the universal covering of $SL_3(\mathbb{R})$, $Sp_4(\mathbb{R})$, $SL_4(\mathbb{R})$ (cf. Table 2.3 in [20], p. 400, in the terminology of [25]). The real representation of the group $GL_2(\mathbb{C})$ with respect to the basis $(1,0)^t$, $(i,0)^t$, $(0,1)^t$, $(0,i)^t$ has the form $G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $\det(G) \neq 0$ such that A, B, C, D are real (2×2)-matrices of the shape $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, $x^2 + y^2 > 0$. Therefore the group $\Sigma_{(r,c)}$ has no representation in the group $SL_2(\mathbb{C})$. The groups $SO_5(\mathbb{R},1)$, $SU_3(\mathbb{C},1)$ have no linear representation in $SL_4(\mathbb{R})$ (see [24], pp. 623-624). The universal covering of $SL_3(\mathbb{R})$ has no linear representation. If $\Sigma_{(r,c)}$ were isomorphic to the group $Sp_4(\mathbb{R})$, then there would exist a skew-symmetric matrix W such that $A^TWA = W$ for all matrices A in (29). For the matrix

$$A = \begin{pmatrix} 0 & 0 & d & d \\ 0 & 0 & d & -d \\ -c_1d + c_2d - d & -c_2d - c_1d - d & rd & rd \\ -c_2d - c_1d + d & c_1d - c_2d - d & rd & -rd \end{pmatrix},$$

with $0 < c_1^2 + c_2^2 - 1 = \frac{1}{4d^4}$, this is possible only if $W = 0$. Therefore the group $\Sigma_{(r,c)}$ as well as the group topologically generated by all translations of $L_{(r,c)}$ are isomorphic to $\mathrm{SL}_4(\mathbb{R})$.
According to Corollary 7 the sets $\Lambda_{Q^*_\delta}$, $\Lambda_{Q^*_{(r,c)}}$ has form (23). The determinant of a matrix $M_{(u,x,y)}$ in (23) is $u^2e^{ic(1,x,y)} \in \mathbb{C}$. Since $M_{(u,x,y)}$ coincides with $\lambda_{(x_1,x_2)}$ given by (27), respectively (28) we obtain $u = \sqrt[4]{|\det(\lambda_{(x_1,x_2)})|^2}$, $e^{ic(1,x,y)} = \frac{\det(\lambda_{(x_1,x_2)})}{u^2}$. Since one has $x_1 = uxa(1,x,y), x_2 = uya(1,x,y)$ we obtain $x_1\bar{x_1} + x_2\bar{x_2} = u^2a^2(1,x,y)$. As the second column of $\lambda_{(x_1,x_2)}$ is

$$(uxb(1,x,y) - u\bar{y}a^{-1}(1,x,y)e^{ic(1,x,y)}, uyb(1,x,y) + u\bar{x}a^{-1}(1,x,y)e^{ic(1,x,y)})^t$$

one has $b(1, x, y) = \frac{1}{u}\left(\bar{x}m_{12} + \bar{y}m_{22}\right)$, where m_{12}, m_{22} are the entries of the second column of $\lambda_{(x_1,x_2)}$. A straightforward computation yields that for the loop Q_δ^* one has

$$a(1, x, y) = \frac{\sqrt[2]{x_1\bar{x_1} + x_2\bar{x_2}}}{\sqrt[4]{(x_1\bar{x_1})^2 + 2x_1\bar{x_1}x_2\bar{x_2}\cos\delta + (x_2\bar{x_2})^2}},$$

$$e^{ic(1,x,y)} = \frac{x_1\bar{x_1} + x_2\bar{x_2}e^{i\delta}}{\sqrt[2]{(x_1\bar{x_1})^2 + 2x_1\bar{x_1}x_2\bar{x_2}\cos\delta + (x_2\bar{x_2})^2}},$$

$$b(1, x, y) = \frac{\bar{x_1}\bar{x_2}(1 - e^{i\delta})}{\sqrt[4]{(x_1\bar{x_1})^2 + 2x_1\bar{x_1}x_2\bar{x_2}\cos\delta + (x_2\bar{x_2})^2}\,\sqrt[2]{x_1\bar{x_1} + x_2\bar{x_2}}},$$

$$u = \sqrt[4]{(x_1\bar{x_1})^2 + 2x_1\bar{x_1}x_2\bar{x_2}\cos\delta + (x_2\bar{x_2})^2},$$

and for the loop $Q_{(r,c)}^*$ we have

$$a(1, x, y) = \frac{\sqrt[2]{x_1\bar{x_1} + x_2\bar{x_2}}}{\sqrt[4]{(x_1\bar{x_1} + rx_1\bar{x_2} + cx_2\bar{x_2} + x_2^2)(x_1\bar{x_1} + rx_2\bar{x_1} + \bar{c}x_2\bar{x_2} + \bar{x_2}^2)}}$$

$$e^{ic(1,x,y)} = \frac{x_1\bar{x_1} + rx_1\bar{x_2} + cx_2\bar{x_2} + x_2^2}{\sqrt[2]{(x_1\bar{x_1} + rx_1\bar{x_2} + cx_2\bar{x_2} + x_2^2)(x_1\bar{x_1} + rx_2\bar{x_1} + \bar{c}x_2\bar{x_2} + \bar{x_2}^2)}}$$

$$b(1, x, y) = \frac{(1 - c)\bar{x_1}\bar{x_2} - \bar{x_1}x_2 + r\bar{x_2}^2}{\sqrt[4]{(x_1\bar{x_1} + rx_1\bar{x_2} + cx_2\bar{x_2} + x_2^2)(x_1\bar{x_1} + rx_2\bar{x_1} + \bar{c}x_2\bar{x_2} + \bar{x_2}^2)}\,\sqrt[2]{x_1\bar{x_1} + x_2\bar{x_2}}},$$

$$u = \sqrt[4]{(x_1\bar{x_1} + rx_1\bar{x_2} + cx_2\bar{x_2} + x_2^2)(x_1\bar{x_1} + rx_2\bar{x_1} + \bar{c}x_2\bar{x_2} + \bar{x_2}^2)},$$

and for both loops one has

$$x = \frac{x_1}{\sqrt[2]{x_1\bar{x_1} + x_2\bar{x_2}}},\ y = \frac{x_2}{\sqrt[2]{x_1\bar{x_1} + x_2\bar{x_2}}}.$$

From this it follows that the form of the functions $a(1, x, y)$, $b(1, x, y)$, $e^{ic(1,x,y)}$ in both cases is given as in the assertion.

6 Quasifields with complex kernel and large automorphism groups

The 8-dimensional locally compact translation planes $\mathcal{A}$ with an automorphism group of dimension at least 16 such that the kernel of the quasifield Q coordinatizing $\mathcal{A}$ is isomorphic to the field $\mathbb{C}$ are determined by H. Hähl in [12], [13], [14] (cf. [8], Section 7). There are three types of such quasifields Q which are not semifields. Now we want to describe the multiplicative loops Q^* of Q.

Type 1: Let $\varphi : \mathbb{R}_{>0} \to Spin_3(\mathbb{R})$, $\varphi(1) = 1$ be a continuous mapping. Let $\mathbb{H} = (\mathbb{R}^4, +, \cdot)$ be the skewfield of quaternions. Then $\mathbb{H}_\varphi = (\mathbb{R}^4, +, \circ)$ with the multiplication $\circ$ defined by $0 \circ x = 0$ and for $m \neq 0$

$$m \circ x = m \cdot x^{\varphi(|m|)} = m \cdot \varphi(|m|)^{-1} \cdot x \cdot \varphi(|m|) \tag{30}$$

yields a 4-dimensional topological quasifield. The kernel K_r of $\mathbb{H}_\varphi$ is isomorphic to the field $\mathbb{C}$ if and only if $\varphi(\mathbb{R}_{>0})$ lies in a subfield of $\mathbb{H}$ isomorphic to $\mathbb{C}$ (cf. [12], pp. 234-238). Any subfield of $\mathbb{H}$ isomorphic to $\mathbb{C}$ is a 2-dimensional real vector space V_u with basis $\{1,u\}$, where $u = u_1 i + u_2 j + u_3 k$ is a pure quaternion of norm 1, i.e. $\bar{u} = -u$ and $u_1^2 + u_2^2 + u_3^2 = 1$. Let $V_i = \langle 1, i\rangle = \{r_1 + ir_2, r_1, r_2 \in \mathbb{R}\}$ be the subfield $\mathbb{C}$ of $\mathbb{H}$. There exists an automorphism β of $\mathbb{H}$ with the form $x \mapsto b^{-1}xb$, $b \in \mathbb{H}$, $|b| = 1$, such that $\beta(V_i) = \beta(\mathbb{C}) = V_u$. One has

$$\beta(m)\circ\beta(x) = \beta(m)\varphi(|m|)^{-1}b^{-1}xb\varphi(|m|) = \beta(m)\beta(b\varphi(|m|)^{-1}b^{-1}xb\varphi(|m|)b^{-1})$$

$$= \beta(m)\beta(x^{b\varphi(|m|)b^{-1}}) = \beta(mx^{\psi(|m|)}),$$

where $\psi = \beta^{-1} \circ \varphi$. Since $\varphi(\mathbb{R}_{>0}) \subset V_u$ and $\beta(\mathbb{C}) = V_u$ one has $\psi(\mathbb{R}_{>0}) \subset \mathbb{C}$. Hence the quasifields $\mathbb{H}_\psi$ with kernel $\mathbb{C}$, $\mathbb{H}_\varphi$ with kernel V_u are isomorphic with respect to $\beta : \mathbb{H}_\psi \to \mathbb{H}_\varphi$. From now we assume that $\varphi(\mathbb{R}_{>0}) \subset \mathbb{C}$ and the kernel K_r of $\mathbb{H}_\varphi$ is $\mathbb{C}$. As $\mathbb{H}$ is a right vector space over $\mathbb{C}$ one can choose as a basis of this vector space $\{1, j\}$. Let $m = m_1 + jm_2$ with $m_1, m_2 \in \mathbb{C}$ be an arbitrary quaternion. Since $\varphi(\mathbb{R}_{>0}) \subset \mathbb{C} \cap Spin_3(\mathbb{R})$ we have $\varphi(|m|)^{-1} = \overline{\varphi(|m|)}$ and $\varphi(|m|)^{-1}j = j\varphi(|m|)$. Then one has $j^{\varphi(|m|)} = j(\varphi(|m|))^2$ and hence

$$m \circ j = (m_1 + jm_2)j^{\varphi(|m|)} = (m_1 + jm_2)j(\varphi(|m|))^2 =$$

$$-\overline{m_2}(\varphi(|m|))^2 + j\overline{m_1}(\varphi(|m|))^2.$$

Then the multiplicative loop $\mathbb{H}_\varphi^*$ of $\mathbb{H}_\varphi$ is given by the multiplication:

$$m \circ x = M_{(m_1,m_2)}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} m_1 & -\overline{m_2}(\varphi(|m|))^2 \\ m_2 & \overline{m_1}(\varphi(|m|))^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{31}$$

where $x = x_1 + jx_2$, $x_1, x_2 \in \mathbb{C}$ and $\varphi : \mathbb{R}_{>0} \to \mathbb{C} \cap Spin_3(\mathbb{R})$, $\varphi(1) = 1$ is a continuous mapping.

Proposition 12 *The group $G_{\mathbb{H}_\varphi^*}$ topologically generated by all left translations of the loop $\mathbb{H}_\varphi^*$ is the group $Spin_3(\mathbb{R}) \times \mathbb{C}$. The set $\Lambda_{\mathbb{H}_\varphi^*}$ of all left translations of $\mathbb{H}_\varphi^*$ has the form (26), where*

$$a(u,1,0) = 1,\ b(u,1,0) = 0,\ e^{ic(u,1,0)} = (\varphi(|m|))^2,$$

with

$$u = |m|,\ x = \frac{m_1}{|m|},\ y = \frac{m_2}{|m|}.$$

The loop $\mathbb{H}_\varphi^*$ is decomposable and $\Lambda_{\mathbb{H}_\varphi^*}$ contains the group $Spin_3(\mathbb{R})$. The centre Z of $\mathbb{H}_\varphi^*$ is $\{1,-1\}$.

Proof Since the determinant of $M_{(m_1,m_2)}$ in (31) is $(m_1\overline{m_1} + m_2\overline{m_2})(\varphi(|m|))^2$ the group $\det(G_{\mathbb{H}_\varphi^*})$ coincides with $\mathbb{C}^*$. Moreover, for every matrix $M_{(m_1,m_2)}$ one has $M_{(m_1,m_2)} \cdot \overline{M}^t_{(m_1,m_2)} = (m_1\overline{m_1} + m_2\overline{m_2}) \cdot I$, where I is the identity matrix. Hence the group $G_{\mathbb{H}_\varphi^*}$ is the group $Spin_3(\mathbb{R}) \times \mathbb{C}$. According to (5) the determinant of the matrix $M_{(u,x,y)}$ is $u^2 e^{ic(u,x,y)} \in \mathbb{C}$. Since $M_{(u,x,y)}$ coincides with $M_{(m_1,m_2)}$ given by (31) we obtain $u = \sqrt[4]{|\det(M_{(m_1,m_2)})|^2} = |m|$ and $e^{ic(u,x,y)} = \frac{\det(M_{(m_1,m_2)})}{u^2} = (\varphi(|m|))^2$. As $G_{\mathbb{H}^*_\varphi} = Spin_3(\mathbb{R}) \times \mathbb{C}$ one has $a(u,x,y) = 1$, $b(u,x,y) = 0$ (cf. p. 6). Since $uxa(u,x,y) = |m|x = m_1$ and $uya(u,x,y) = |m|y = m_2$ we obtain the form of the variables x and y as in the assertion. As $e^{ic(u,x,y)} = e^{ic(u,1,0)} = (\varphi(|m|))^2$ the function c depends only on the variable $u = |m|$. Therefore the set $\Lambda_{\mathbb{H}^*_\varphi}$ of all left translations of $\mathbb{H}^*_\varphi$ has form (26) with functions $a(u,1,0)$, $b(u,1,0)$, $c(u,1,0)$ given by the assertion. By Proposition 10 the loop $\mathbb{H}^*_\varphi$ is decomposable and $\Lambda_{\mathbb{H}^*_\varphi}$ contains the group $Spin_3(\mathbb{R})$. As $|-1| = 1$ the centre Z of $\mathbb{H}^*_\varphi$ is $\{1,-1\}$.

Type 2:
Let $\varrho : S^1 \to \{il, l \in \mathbb{R}\}$ be a continuous non-constant function having pure imaginary values, $\rho : S^1 \to (1,\infty); z \mapsto \sqrt{1+|\varrho(z)|^2}$ and

$$B_z = \begin{pmatrix} \varrho(z)z & -\sqrt{1+|\varrho(z)|^2}z \\ \sqrt{1+|\varrho(z)|^2}z & -\varrho(z)z \end{pmatrix}.$$

The multiplication of the quasifield Q is given by formula (2) in [13], (p. 87):

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + rB_z \begin{pmatrix} \bar{x_1} \\ \bar{x_2} \end{pmatrix}, \tag{32}$$

where rz, $(r \geq 0, z \in S^1)$ and $c \in \mathbb{C}$ are uniquely determined by $a_1, a_2 \in \mathbb{C}$ such that for $a_2 \neq 0$ one has $z = \frac{a_2}{|a_2|}$, $r = \frac{|a_2|}{\rho\left(\frac{a_2}{|a_2|}\right)}$, $c = a_1 - \frac{a_2}{\rho\left(\frac{a_2}{|a_2|}\right)} \cdot \varrho\left(\frac{a_2}{|a_2|}\right)$ and for $a_2 = 0$ we have $rz = 0$, $c = a_1$. Hence (32) can be written into the form:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1x_1 - a_2\overline{x_2} + \frac{a_2\varrho(\frac{a_2}{|a_2|})}{\sqrt{1+|\varrho(\frac{a_2}{|a_2|})|^2}}(\overline{x_1} - x_1) \\ a_1x_2 + a_2\overline{x_1} - \frac{a_2\varrho(\frac{a_2}{|a_2|})}{\sqrt{1+|\varrho(\frac{a_2}{|a_2|})|^2}}(x_2 - \overline{x_2}) \end{pmatrix}.$$

It follows that the kernel K_1 of the quasifield Q_ϱ is $K_1 = \{(x,y)^t, x,y \in \mathbb{R}\}$ isomorphic to $\mathbb{C}$. The scalar multiplication in the right vector space $Q_\varrho = \mathbb{C}^2$ over K_1 is given in the following way: if $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in Q_\varrho$, $k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \in K_1$, then one has $a \circ k = \begin{pmatrix} a_1k_1 - a_2k_2 \\ a_1k_2 + a_2k_1 \end{pmatrix}$. Using this the left translation map $\lambda_{(a_1,a_2)}$ is K_1-linear. Applying the coordinate change $T : \mathbb{C}^2 \to \mathbb{C}^2, (r+si, u+vi)^t \mapsto (r+ui, s+vi)^t$ the kernel K_1 transforms to $K_2 = (x+iy,0)^t, x,y \in \mathbb{R}\} = \{(z,0)^t, z \in \mathbb{C}\}$. Since Q_ϱ has dimension 2 over the kernel K_2 we can identify the elements $(m_1,m_2)^t = (m_{11}+im_{12}, m_{21}+im_{22})^t \in \mathbb{C}^2$ of Q_ϱ with the vector $T(m_1,m_2)^t = (m_{11}+im_{21}, m_{12}+im_{22})^t \in \mathbb{C}^2$ and the multiplication of the loop Q^*_ϱ can be represented as follows:

$$\begin{pmatrix} m_{11}+im_{12} \\ m_{21}+im_{22} \end{pmatrix} \circ \begin{pmatrix} x_{11}+ix_{12} \\ x_{21}+ix_{22} \end{pmatrix} = T^{-1}\left(M_{(m_1,m_2)}T\begin{pmatrix} x_{11}+ix_{12} \\ x_{21}+ix_{22} \end{pmatrix}\right) =$$

$$T^{-1}\left(\begin{pmatrix} m_{11}+im_{21} & -m_{12}+\frac{2m_{21}Im(\varrho(\frac{m_2}{|m_2|}))}{\sqrt{1+(Im\varrho(\frac{m_2}{|m_2|}))^2}}+im_{22} \\ m_{12}+im_{22} & m_{11}+\frac{2m_{22}Im(\varrho(\frac{m_2}{|m_2|}))}{\sqrt{1+(Im\varrho(\frac{m_2}{|m_2|}))^2}}-im_{21} \end{pmatrix}\begin{pmatrix} x_{11}+ix_{21} \\ x_{12}+ix_{22} \end{pmatrix}\right). \tag{33}$$

Proposition 13 *The set $\Lambda_{Q_\varrho^*}$ of all left translations of the multiplicative loop Q_ϱ^* of Q_ϱ given by (33) corresponds to form (23) defined by:*

$$a(1,x,y)=\frac{1}{\sqrt[2]{1+\frac{2Im(\bar{x}y)Im\left(\varrho\left(\frac{Im(x)+iIm(y)}{\sqrt[2]{Im(x)^2+Im(y)^2}}\right)\right)}{\sqrt[2]{1+Im\left(\varrho\left(\frac{Im(x)+iIm(y)}{\sqrt[2]{Im(x)^2+Im(y)^2}}\right)\right)^2}}}},\quad c(1,x,y)=0,$$

$$b(1,x,y)=\frac{2Im\left(\varrho\left(\frac{Im(x)+iIm(y)}{\sqrt[2]{Im(x)^2+Im(y)^2}}\right)\right)(Im(x)Re(x)+Im(y)Re(y)-i(Im(x)^2+Im(y)^2))}{\sqrt[2]{1+Im\left(\varrho\left(\frac{Im(x)+iIm(y)}{\sqrt[2]{Im(x)^2+Im(y)^2}}\right)\right)^2+2Im(\bar{x}y)Im\left(\varrho\left(\frac{Im(x)+iIm(y)}{\sqrt[2]{Im(x)^2+Im(y)^2}}\right)\right)}}$$

with

$$u=\sqrt{m_{11}^2+m_{12}^2+m_{21}^2+m_{22}^2+\frac{2(m_{11}m_{22}-m_{12}m_{21})Im(\varrho(\frac{m_2}{|m_2|}))}{\sqrt{1+Im(\varrho(\frac{m_2}{|m_2|}))^2}}},$$

$$x=\frac{m_{11}+im_{21}}{\sqrt{m_{11}^2+m_{12}^2+m_{21}^2+m_{22}^2}},\quad y=\frac{m_{12}+im_{22}}{\sqrt{m_{11}^2+m_{12}^2+m_{21}^2+m_{22}^2}}.$$

The group topologically generated by the left translations of Q_ϱ^* is the group $SL_2(\mathbb{C})\times\mathbb{R}$. The loop Q_ϱ^* is decomposable and it is a central extension of the connected component $Z_0^*=\{(c,0)^t, c>0\}\cong\mathbb{R}$ of the centre $Z^*=\{(c,0)^t, c\in\mathbb{R}\setminus\{0\}\}$ of Q_ϱ^* by a 3-dimensional loop homeomorphic to S^3.

Proof The determinant of the complex matrix $M_{(m_1,m_2)}$ given by (33) is the real number $m_{11}^2+m_{12}^2+m_{21}^2+m_{22}^2+\frac{2m_{11}m_{22}Im(\varrho(\frac{m_2}{|m_2|}))}{\sqrt{1+Im(\varrho(\frac{m_2}{|m_2|}))^2}}-\frac{2m_{12}m_{21}Im(\varrho(\frac{m_2}{|m_2|}))}{\sqrt{1+Im(\varrho(\frac{m_2}{|m_2|}))^2}}$. Hence the group G topologically generated by the left translations of the loop Q_ϱ^* is the group $SL_2(\mathbb{C})\times\mathbb{R}$. The loop Q_ϱ^* has a central subgroup $Z^*=\{(c,0)^t, c\in\mathbb{R}\setminus\{0\}\}$. By Theorem 6 the loop Q_ϱ^* is a central extension of $Z_0^*=\{(c,0)^t, c>0\}$ isomorphic to $\mathbb{R}$ by a 3-dimensional loop homeomorphic to S^3. Hence the loop Q_ϱ^* is decomposable (cf. Theorem 9). Since $G=SL_2(\mathbb{C})\times\mathbb{R}$ according to Corollary 7 the set $\Lambda_{Q_\varrho^*}$ corresponds to form (23) with $c(1,x,y)=0$. Moreover the determinant of a matrix $M_{(u,x,y)}$ in the set (23) is $u^2\in\mathbb{R}$. Since $M_{(u,x,y)}$ coincides with $M_{(m_1,m_2)}$ given by (33) we obtain $u=\sqrt{\det(M_{(m_1,m_2)})}$. Since one has $m_{11}+im_{21}=uxa(u,x,y)$, $m_{12}+im_{22}=uya(u,x,y)$ we obtain $m_{11}^2+m_{12}^2+m_{21}^2+m_{22}^2=u^2a^2(u,x,y)$. As the second column of $M_{(m_1,m_2)}$ is $(uxb(u,x,y)-u\bar{y}a^{-1}(u,x,y), uyb(u,x,y)+u\bar{x}a^{-1}(u,x,y))^t$ one has $b(u,x,y)=\frac{1}{u}(\bar{x}M_{12}+\bar{y}M_{22})$, where M_{12}, M_{22} are the entries of the second column of $M_{(m_1,m_2)}$. A straightforward computation yields that

$$a(1,x,y)=\frac{\sqrt{m_{11}^2+m_{12}^2+m_{21}^2+m_{22}^2}}{u},\ b(1,x,y)=\frac{2Im(\varrho(\frac{m_2}{|m_2|}))(m_{21}m_{11}+m_{22}m_{12}-i(m_{21}^2+m_{22}^2))}{u\sqrt{1+Im(\varrho(\frac{m_2}{|m_2|}))^2}\sqrt{m_{11}^2+m_{12}^2+m_{21}^2+m_{22}^2}},$$

$$u = \sqrt{m_{11}^2 + m_{12}^2 + m_{21}^2 + m_{22}^2 + \frac{2(m_{11}m_{22} - m_{12}m_{21})Im(\varrho(\frac{m_2}{|m_2|}))}{\sqrt{1 + Im(\varrho(\frac{m_2}{|m_2|}))^2}}},$$

$$x = \frac{m_{11} + im_{21}}{\sqrt{m_{11}^2 + m_{12}^2 + m_{21}^2 + m_{22}^2}}, \quad y = \frac{m_{12} + im_{22}}{\sqrt{m_{11}^2 + m_{12}^2 + m_{21}^2 + m_{22}^2}}.$$

Hence the functions $a(1,x,y)$, $b(1,x,y)$ have the form as in the assertion.

The loop Q^*_ϱ is a central extension of the group $Z^*_0 = \mathbb{R}$ by a loop L^*_ϱ homeomorphic to S^3. By [19], pp. 761-762, the loop Q^*_ϱ is isomorphic to a loop $L(Id, h)$ realized on $\mathcal{S} \times \mathbb{R}$ and given by the multiplication $(\tau, t) * (\kappa, s) = (\tau\kappa, h(\tau, \kappa)ts)$, where $\mathcal{S}$ is a loop on S^3 isomorphic to L^*_ϱ. By Proposition 12.1 in [15], p. 225, the cohomotopy group $\pi^m(S^n)$ is exactly the homotopy group $\pi_n(S^m)$. Moreover, one has $\pi_1(S^3) = \pi_3(S^1) = 0$ (see Examples in [15], p. 109). Hence the function $h : S^3 \times S^3 \to \mathbb{R}$ has the property $h(\tau, 1) = h(1, \tau) = 1$. This means that h is the constant function 1 if the domain of h is S^3. The continuous section σ corresponding to the loop Q^*_ϱ is determined by the continuous functions $a(1,x,y) : S^3 \times S^3 \to \mathbb{R}_{>0}$, $b(1,x,y) : S^3 \times S^3 \to \mathbb{C}$ such that x as well as y depend on $(x_1, y_1, x_2, y_2) \in S^3 \times S^3$ (cf. formulas (13), (14) on p. 7). As $G_{Q^*_\varrho}$ is the group $SL_2(\mathbb{C}) \times \mathbb{R}$ the group $G_{Q^*_\varrho/Z^*_0}$ topologically generated by the left translations of Q^*_ϱ/Z^*_0 is isomorphic to the group $SL_2(\mathbb{C})$. In Proposition 1 in [10] we claimed: There is no almost topological proper loop L homeomorphic to S^3 such that the group G topologically generated by the left translations of L is isomorphic to the group $SL_2(\mathbb{C})$. According to Proposition 13 and the above discussion the claim of Proposition 1 is true under the additional condition that the domain of the continuous real function $f(x,y)$ and that of the continuous complex function $g(x,y)$ of the section σ_r, $r \in \mathbb{R}$, corresponding to L is S^3.

Type 3:
Let $\mathbb{H}$ be the skewfield of quaternions, $\mathbb{P}$ be the subspace of pure quaternions $\{ix + jy + kz; x, y, z \in \mathbb{R}\}$ and let $h = h_1 + ih_2 \in \mathbb{C}$ be a fixed element with $h_1 > 0$ and $|h| = 1$. Let $\Phi \subset \mathbb{H}$ be a closed subset homeomorphic to $\mathbb{R}$ which is the image of a continuous section $\varphi : \mathbb{H}/(\mathbb{P} \cdot h) \to \mathbb{H}; a \mapsto \varphi(a)$ such that $\varphi(0) = 0$, $\varphi(1) = 1$, i.e. for every $a \in \mathbb{H}$ there exists precisely one $\varphi(a) \in \Phi$ with $a + \mathbb{P} \cdot h = \varphi(a) + \mathbb{P} \cdot h$. Every $a \in \mathbb{H}$ has a unique decomposition as $a = \varphi(a) + p(a) \cdot h$, where $p : \mathbb{H} \to \mathbb{P}; a \mapsto (a - \varphi(a)) \cdot h^{-1}$ (cf. [14], p. 303-304). The multiplication of the quasifield $Q_{h,\Phi}$ is given by formula $(**)$ in [14] (p. 304):

$$a \circ x = p(a) \cdot x \cdot h + x \cdot \varphi(a), \tag{34}$$

where $a, x \in \mathbb{H}$, $\cdot$ is the multiplication in $\mathbb{H}$.

Proposition 14 *For all $h \neq 1$ the kernel K_r of $Q_{h,\Phi}$ is the field $\mathbb{C}$ of complex numbers and the multiplication of the loop $Q^*_{h,\Phi}$ is given by:*

$$\begin{pmatrix} a_1 + ia_2 \\ a_3 + ia_4 \end{pmatrix} \circ \begin{pmatrix} x_1 + ix_2 \\ x_3 + ix_4 \end{pmatrix} = M_{(a_1,a_2,a_3,a_4)} \begin{pmatrix} x_1 + ix_2 \\ x_3 + ix_4 \end{pmatrix} =$$

$$\begin{pmatrix} a_1 + ia_2 & (h_1 + ih_2)^2(ia_4 - a_3) \\ a_3 + ia_4 & 2\varphi_1(a) - a_1 + i(a_2 - \frac{2(\varphi_1(a)-a_1)h_1}{h_2}) \end{pmatrix} \begin{pmatrix} x_1 + ix_2 \\ x_3 + ix_4 \end{pmatrix}, \tag{35}$$

where $\varphi_1 : \mathbb{H} \to \mathbb{R}$ is a continuous function with $\varphi_1(0) = 0$, $\varphi_1(1) = 1$ such that $\varphi_1(a) + i\left(a_2 + \frac{(a_1-\varphi_1(a))h_1}{h_2}\right) \in \Phi$.

If $h = 1$, then every quasifield $Q_{1,\Phi'}$ having a 2-dimensional subfield of $\mathbb{H}$ isomorphic to the field $\mathbb{C}$ of complex numbers as its kernel is isomorphic to the quasifield $Q_{1,\Phi}$ such that the kernel K_r of $Q_{1,\Phi}$ is the field $\mathbb{C}$. Then the multiplication of the loop $Q^*_{1,\Phi}$ is given by:

$$\begin{pmatrix} a_1 + ia_2 \\ a_3 + ia_4 \end{pmatrix} \circ \begin{pmatrix} x_1 + ix_2 \\ x_3 + ix_4 \end{pmatrix} = M_{(a_1,a_2,a_3,a_4)} \begin{pmatrix} x_1 + ix_2 \\ x_3 + ix_4 \end{pmatrix} = \begin{pmatrix} a_1 + ia_2 & -a_3 + ia_4 \\ a_3 + ia_4 & a_1 - i(a_2 - 2b(a_1)) \end{pmatrix} \begin{pmatrix} x_1 + ix_2 \\ x_3 + ix_4 \end{pmatrix}, \tag{36}$$

where $b(a_1) \in \mathbb{R}$ is given such that for all $a \in \mathbb{H}$ one has $\varphi(a) = a_1 + ib(a_1) \in \Phi$, where $\varphi : \mathbb{H}/\mathbb{P} \to \Phi$ is a continuous section such that $\varphi(0) = 0$, $\varphi(1) = 1$ but not all $b(a_1)$ is 0.

Proof Assume that $h \neq 1$. According to [14], pp. 314-315, the kernel K_r of $Q_{h,\Phi}$ is isomorphic to the field of complex numbers if and only if h and Φ are in the same 2-dimensional subspace V with $1 \in V$. Since $h \in \mathbb{C}$ one has $\Phi \subset V = \{r_1 + ir_2, r_1, r_2 \in \mathbb{R}\} = \mathbb{C}$ and $K_r = \mathbb{C}$. Let $a = a_1 + ia_2 + j(a_3 + ia_4) = a_1 + ia_2 + ja_3 - ka_4$, $a_i \in \mathbb{R}$, $i = 1, ..., 4$, be an arbitrary quaternion. Since $\mathbb{P} \cdot h$ is the subspace $\{-xh_2 + ixh_1 + j(yh_1 + zh_2) + k(zh_1 - yh_2), x, y, z \in \mathbb{R}\}$ for all $a \in \mathbb{H}$ one has $\varphi(a) = \varphi_1(a) + i\varphi_2(a) \in \mathbb{C}$ such that $a_1 - \varphi_1(a) = -xh_2$ and $a_2 - \varphi_2(a) = xh_1$ for some $x \in \mathbb{R}$. Hence we obtain $\frac{a_1-\varphi_1(a)}{h_2} = \frac{\varphi_2(a)-a_2}{h_1}$ and $\varphi(a) = \varphi_1(a) + i\left(a_2 + \frac{(a_1-\varphi_1(a))h_1}{h_2}\right) \in \Phi$, $p(a) = i\frac{(\varphi_1(a)-a_1)}{h_2} + j(a_3h_1 + a_4h_2) + k(a_3h_2 - a_4h_1)$. Using this, a straightforward computation yields the multiplication of the loop $Q^*_{h,\Phi}$ given by (35) in the assertion.

Now we assume $h = 1$. The kernel K_r of $Q_{1,\Phi}$ is a 2-dimensional subfield of $\mathbb{H}$ isomorphic to $\mathbb{C}$. Hence K_r is a subspace $V_u = \{r_1 + r_2u, r_1, r_2 \in \mathbb{R}\}$ with a suitable pure quaternion u of norm 1. Let $Q_{1,\Phi}$, respectively $Q_{1,\Phi'}$ be two quasifields having $V_i = \mathbb{C}$, respectively V_u as its kernel. According to [14], p. 315, the automorphism $\beta : \mathbb{H} \to \mathbb{H}, x \mapsto b^{-1}xb$, $b \in \mathbb{H}$, $|b| = 1$ with $b^{-1}ib = u$ induces an isomorphism of $Q_{1,\Phi}$ onto $Q_{1,\Phi'}$ with $b^{-1}\Phi b = \Phi'$. Hence we may assume that $\Phi' \subset \mathbb{C}$ and the kernel K_r of $Q_{1,\Phi'}$ is $\mathbb{C} = \{r_1 + ir_2, r_1, r_2 \in \mathbb{R}\}$. Since for all $a \in \mathbb{H}$ one has $\varphi(a) \in \Phi \subset \mathbb{C}$ we obtain that $a - \varphi(a)$ is in $\mathbb{P}$ precisely if $\varphi(a) = a_1 + ib(a_1) \in \Phi$, $b(a_1) \in \mathbb{R}$ such that $\varphi(0) = 0$, $\varphi(1) = 1$ but not all $b(a_1)$ is 0. Hence one has $p(a) = i(a_2 - b(a_1)) + ja_3 - ka_4$ and the multiplication of $Q^*_{1,\Phi}$ is given by (36).

Proposition 15 *For all $h \in \mathbb{C}$ with $|h| = 1$ and $Re(h) > 0$ the group $G_{Q^*_{h,\Phi}}$ topologically generated by the left translations of the multiplicative loop $Q^*_{h,\Phi}$ is the group $GL_2(\mathbb{C})$. The centre Z of $Q^*_{h,\Phi}$ is discrete.*

a) For all $h \in \mathbb{C} \setminus \{1\}$, the set $\Lambda_{Q^*_{h,\Phi}}$ of all left translations of the loop $Q^*_{h,\Phi}$ given by (35) is the range of the section (5) defined by:

$$a(u, k, l) = \frac{\sqrt[2]{a_1^2 + a_2^2 + a_3^2 + a_4^2}}{u},$$

$$e^{ic(u,k,l)}=\frac{\left((a_3^2+a_4^2)(h_1+ih_2)^2-a_1^2-a_2^2+2\varphi_1(a)(a_1+ia_2)+\frac{2h_1(a_2-ia_1)(\varphi_1(a)-a_1)}{h_2}\right)}{u^2},$$

$$b(u,k,l)=\frac{2(h_1^2+ih_1h_2)((a_2a_4-a_3a_1)+i(a_1a_4+a_2a_3))+2\varphi_1(a)(a_3-ia_4)-\frac{2(a_4+ia_3)(\varphi_1(a)-a_1)h_1}{h_2}}{u\sqrt[2]{a_1^2+a_2^2+a_3^2+a_4^2}}$$

with

$$k=\frac{a_1+ia_2}{\sqrt[2]{a_1^2+a_2^2+a_3^2+a_4^2}},\quad l=\frac{a_3+ia_4}{\sqrt[2]{a_1^2+a_2^2+a_3^2+a_4^2}},\quad u=$$

$$\sqrt[4]{\left((a_3^2+a_4^2)(2h_1^2-1)-a_1^2-a_2^2+2\varphi_1(a)a_1+\frac{2h_1a_2(\varphi_1(a)-a_1)}{h_2}\right)^2+4\left((a_3^2+a_4^2)h_1h_2+\frac{h_1a_1(a_1-\varphi_1(a))}{h_2}+\varphi_1(a)a_2\right)^2}.$$

b) If $h=1$, then the set $\Lambda_{Q^*_{1,\Phi}}$ of all left translations of the multiplicative loop $Q^*_{1,\Phi}$ given by (36) is the range of the section (5) defined by:

$$a(u,k,l)=\frac{\sqrt[2]{a_1^2+a_2^2+a_3^2+a_4^2}}{u},\quad b(u,k,l)=\frac{2b(a_1)(a_4+ia_3)}{u\sqrt[2]{a_1^2+a_2^2+a_3^2+a_4^2}}$$

$$e^{ic(u,k,l)}=\frac{a_1^2+a_2^2+a_3^2+a_4^2+2b(a_1)(ia_1-a_2)}{u^2},$$

with

$$u=\sqrt[4]{(a_1^2+a_2^2+a_3^2+a_4^2-2b(a_1)a_2)^2+4b(a_1)^2a_1^2},\quad k=\frac{a_1+ia_2}{\sqrt[2]{a_1^2+a_2^2+a_3^2+a_4^2}},\quad l=\frac{a_3+ia_4}{\sqrt[2]{a_1^2+a_2^2+a_3^2+a_4^2}}.$$

Proof Since the determinant of $M_{(a_1,a_2,a_3,a_4)}$ given by (35), respectively by (36) is $-(a_1^2+a_2^2+a_3^2+a_4^2)+2(a_3^2+a_4^2)h_1^2+2\varphi_1(a)a_1+\frac{2a_2h_1(\varphi_1(a)-a_1)}{h_2}+i(2(a_3^2+a_4^2)h_1h_2+2\varphi_1(a)a_2-\frac{2a_1h_1(\varphi_1(a)-a_1)}{h_2})$, respectively $a_1^2+a_2^2+a_3^2+a_4^2+2b(a_1)(ia_1-a_2)$ the groups $\det(G_{Q^*_{h,\Phi}})$ and $\det(G_{Q^*_{1,\Phi}})$ coincide with $\mathbb{C}^*$. Hence the groups $G_{Q^*_{h,\Phi}}$, $G_{Q^*_{1,\Phi}}$ are isomorphic either to the group $Spin_3(\mathbb{R})\times\mathbb{C}$ or to $GL_2(\mathbb{C})$. For any matrix M in $Spin_3(\mathbb{R})\times\mathbb{C}$ one has $M\cdot\overline{M}^t\in\mathbb{R}\cdot I$, where I is the identity matrix. But a straightforward computation gives that the matrix $M_{(a_1,a_2,a_3,a_4)}$ given by (35) does not have this property. Hence $G_{Q^*_{h,\Phi}}$ is the group $GL_2(\mathbb{C})$. If $b(a_1)\neq 0$, then we have

$$\begin{pmatrix} a_1+ia_2 & -a_3+ia_4 \\ a_3+ia_4 & a_1-i(a_2-2b(a_1)) \end{pmatrix}\begin{pmatrix} a_1-ia_2 & a_3-ia_4 \\ -a_3-ia_4 & a_1+i(a_2-2b(a_1)) \end{pmatrix}=$$

$$\begin{pmatrix} a_1^2+a_2^2+a_3^2+a_4^2 & 2b(a_1)(a_4+ia_3) \\ 2b(a_1)(a_4+ia_3) & a_1^2+a_2^2+a_3^2+a_4^2-4b(a_1)(a_2-b(a_1)) \end{pmatrix}\notin\mathbb{R}\cdot I.$$

Hence also $G_{Q^*_{1,\Phi}}$ is the group $GL_2(\mathbb{C})$.

If the centre $Z<K_r$ of $Q_{h,\Phi}$, respectively of $Q_{1,\Phi}$ would be contain the field $\mathbb{R}$ of real numbers, then for all $r\in\mathbb{R}$ and $m\in\mathbb{H}$ one has $r\circ m=m\circ r=rm$, where the last multiplication is the scalar multiplication with $r\in\mathbb{R}$. As $r=p(r)\cdot h+\varphi(r)$ and $r\circ m=p(r)\cdot m\cdot h+m\cdot\varphi(r)=rm$ it follows that the image Φ of the section φ is the field $\mathbb{R}$ (cf. [14], p. 309) and hence $p(r)=(r-\varphi(r))h^{-1}=0$. But then K_r would be the field of real numbers or the quasifield is the skewfield of quaternions (cf. [14], p. 314) which is a contradiction. Hence the centre Z of $Q^*_{h,\Phi}$ as well as of $Q^*_{1,\Phi}$ is discrete.

According to (5) the determinant of the matrix $M_{(u,k,l)}$ is $u^2e^{ic(u,k,l)}\in\mathbb{C}$. Since $M_{(u,k,l)}$ coincides with $M_{(a_1,a_2,a_3,a_4)}$ given by (35), respectively by (36) we obtain $u=\sqrt[4]{|\det(M_{(a_1,a_2,a_3,a_4)})|^2}$ and $e^{ic(u,k,l)}=\frac{\det(M_{(a_1,a_2,a_3,a_4)})}{u^2}$. Since one has $a_1 + ia_2 = kua(u,k,l)$, $a_3 + ia_4 = lua(u,k,l)$ we obtain $a_1^2 + a_2^2 = u^2 k\bar{k}a^2(u,k,l)$ and $a_3^2 + a_4^2 = u^2 l\bar{l}a^2(u,k,l)$. Therefore we have $a(u,k,l) = \frac{\sqrt[2]{a_1^2+a_2^2+a_3^2+a_4^2}}{u}$. Since the second column of $M_{(a_1,a_2,a_3,a_4)}$ is

$$(ukb(u,k,l) - u\bar{l}a^{-1}(u,k,l)e^{ic(u,k,l)}, ulb(u,k,l) + u\bar{k}a^{-1}(u,k,l)e^{ic(u,k,l)})^t$$

one has $b(u,k,l) = \frac{1}{u}\left(\bar{k}M_{12} + \bar{l}M_{22}\right)$, where M_{12}, M_{22} are the entries of the second column of $M_{(a_1,a_2,a_3,a_4)}$. A straightforward computation yields the form of the functions $a(u,k,l)$, $b(u,k,l)$, $e^{ic(u,k,l)}$ and $u > 0$, $k,l \in \mathbb{C}$, $k\bar{k} + l\bar{l} = 1$ as given in assertion a), respectively b).

Acknowledgements This paper was supported by the European Union's Seventh Framework Programme (FP7/2007-2013) under grant agreements no. 317721.

References

1. Adams, J.F.: *On the non-existence of elements of Hopf invariant one.* Ann of Math. **72**, 20-104 (1960)
2. Betten, D.: 4-*dimensionale Translationsebenen.* Math. Z. **128**, 129-151 (1972)
3. Betten, D.: 4-*dimensionale Translationsebenen mit* 8-*dimensionaler Kollineationsgruppe.* Geom. Ded. **2**, 327-339 (1973)
4. Betten, D.: 4-*dimensionale Translationsebenen mit irreduzibler Kollineationsgruppe.* Arch. Math. **24**, 552-560 (1973)
5. Betten, D.: *4-dimensionale Translationsebenen mit genau einer Fixrichtung.* Deom. Ded. **3**, 405-440 (1975)
6. Betten, D.: *4-dimensionale Translationsebenen mit 7-dimensionaler Kollineationsgruppe.* J. Reine Angew. Math. **285**, 126-148 (1976)
7. Betten, D.: *4-dimensionale Translationsebenen mit kommutativer Standgruppe.* Math. Z. **154**, 125-141 (1977)
8. Boekholt, S.: *Zur Klassifikation actdimensionaler kompakter Ebenen mit mindestens 16-dimensionaler Automorphismengruppe.* Univ. Stuttgart, Fakultät Mathematik, 160 S. (2000).
9. Falcone, G., Figula, Á., Strambach, K.: *Multiplicative loops of* 2-*dimensional topological quasifields.* Communications in Algebra. **44**, 2592-2620 (2016)
10. Figula, Á., Strambach, K.: *Loops on spheres having a compact-free inner mapping group.* Monatsh. Math. **156**, 123-140 (2009)
11. Grundhöfer, T., Salzmann, H.: *Locally compact double loops and ternary fields.* Quasigroups and loops: theory and applications, Sigma Ser. Pure Math. **8**, 313-355 (1990)
12. Hähl, H.: *Achtdimensionale lokalkompakte Translationsebenen mit großen Streckungsgruppen*, Arch. Math. **34**, 231-242 (1980)
13. Hähl, H.: *Achtdimensionale lokalkompakte Translationsebenen mit zu* $SL_2(\mathbb{C})$ *isomorphen Kollineationsgruppen.* J. Reine und Angew. Math. **330**, 76-92 (1982)
14. Hähl, H.: *Achtdimensionale lokalkompakte Translationsebenen mit mindestens* 17-*dimensionaler Kollineationsgruppe*, Geom. Dedicata **21**, 299-340 (1986)
15. Hu, Sz.-T. *Homotopy Theory*, Academic Press, New York, London, (1959)
16. Hughes, D. R., Piper, F. C.: *Projective Planes*, Springer, Berlin (1973)
17. Knarr, N.: *Translation Planes*, Springer, Berlin (1995)
18. Nagy, P. T., Strambach, K.: *Loops in group theory and Lie theory*, de Gruyter, Berlin (2002)
19. Nagy, P. T., Strambach, K.: *Schreier Loops*, Czechoslovak Math. Journal **58**, 759-786 (2008)
20. Onishchik, A. L., Sulanke, R.: *Projective and Cayley-Klein Geometries*, Springer, Berlin (2006)

21. Ortleb, S.: *A new family of locally compact* 4*-dimensional translation planes admitting a* 7*-dimensional collineation group*, Adv. Geom. **9**, 1-12 (2009)
22. Pickert, G.: *Projektive Ebenen*, Springer, Berlin (1955)
23. Plaumann, P., Strambach, K.: *Zweidimensionale Quasialgebren mit Nullteilern*, Aequationes Math. **15**, 249-264 (1977)
24. Salzmann, H., Betten, D., Grundhöfer, T., Hähl, H., Löwen, R., Stroppel, M. *Compact projective planes*, de Gruyter, Berlin (1995)
25. Tits, J.: *Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen*, Springer, Berlin (1967)
26. Völklein, H.: *Transitivitätsfragen bei linearen Liegruppen.* Archiv Math. **36**, 23-34 (1981)