COLLEGE GEOMETRY

I. **General computational skills** Numbers, polynomials, functions. Equations in one variable, quadratic equations, inequalities.

II. **Elementary geometry** Right triangles, Pythagorean and related theorems. Trigonometry in right triangles. The extension of trigonometric expressions. General triangles, lines and circles in a triangle, sine and cosine rules. Quadrilaterals, sum of the interior angles, area. Special quadrilaterals. Polygons, decomposition into triangles, sum of the interior angles, area. Regular polygons. Circles, tangent and bitangent segments. The area of a circle, tangential and cyclic quadrilaterals. Geometrical transformations, isometries and similarity transformations.

III. **Coordinate geometry** The analytic model of Euclidean geometry. Distance between points in the coordinate plane. Equation of lines (slope-intersect form) and circles. Parallelism and perpendicularity. Distance of a point from a line. Intersections (line-line, line-circle, circle-circle). Tangent lines to a circle from an external point. Conics (ellipse, hyperbole, parabole). Coordinate geometry on the sphere: longitudes and latitudes.

References

[1] Zs. Juhász, Teach Yourself Mathematics, Studium '96, Debrecen, 1998.

[2] L. Kozma and Cs. Vincze, College Geometry, http://math.unideb.hu/media/nagy-abris//Collegegeom-main-1.pdf

1. General computational skills

1. Without calculator find the values of

$$8^{\frac{2}{3}}\cdot 2^{-2},\quad 7778^2-2223^2,\quad \frac{437^2-363^2}{537^2-463^2},\quad \sqrt{5-2\sqrt{6}}+\sqrt{3-2\sqrt{2}}+\sqrt{7-2\sqrt{12}},$$

$$\left(1+\frac{1}{2}\right)\cdot\left(1+\frac{1}{3}\right)\cdot\left(1+\frac{1}{4}\right)\cdot\ldots\cdot\left(1+\frac{1}{100}\right).$$

Which number is the bigger?

$$297\cdot 299 \text{ or } 298^2,\quad 345^2 \text{ or } 342\cdot 348,\quad \sqrt{101}-\sqrt{100} \text{ or } \frac{1}{20}.$$

Prove that

$$\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6}}}}<3.$$

2. Solve the equations:

$$3x-\frac{4}{3}=\frac{5}{12},\quad x^2-x-6=0,\quad x^3+6x^2-4x-24=0,\quad \frac{1}{x^2-9}+\frac{1}{x-3}=\frac{48}{(x-3)(x+38)}.$$

3. Sketch the graphs of the following functions:

$$f(x)=2x-1,\quad f(x)=\frac{1}{2}x^2-x+\frac{3}{2},\quad f(x)=2+\frac{1}{x-1}.$$

4. Find the domains of the functions

$$f(x)=\frac{2x-1}{x^2-x},\quad f(x)=\sqrt{5-x},\quad f(x)=\sqrt{(x-3)(5-x)},\quad f(x)=\frac{1}{x+3},\quad f(x)=\sqrt{2x+4},$$

$$f(x)=\sqrt{(x-2)(x+3)}.$$

5. Solve the following systems of equations:

$$\underbrace{3x-7y=66,\quad 2x-9y=-8} \text{ and } \underbrace{x^2-y=46,\quad x^2y=147}.$$

6. Find all integer roots of the equations $2x^3 + 11x^2 - 7x - 6 = 0$ and $x^3 - x^2 - 4x + 4 = 0$. Perform the divisions

$$(4x^5 + 2x - 1) : (x - 1) =? \text{ and } (x^4 - 3x^2 + 2x - 1) : (x - 1) =?$$

7. Express the numbers $\ln\sqrt{3}$ and $\ln\frac{1}{81}$ in terms of $\ln 3$ and solve the following equations

$$2^x 3^{x+2} = 54, \quad \ln\big(x(x-2)\big) = 0, \quad 3^x 2^{x+2} = 24, \quad \ln\big(x(x+2)\big) = 0.$$

8. Prove that $\log_2 3$ and $\sqrt{2}$ are irrational numbers.
9. Find an estimation for the values of $\sqrt{2}$ and $\log_2 3$ by the dictionary method.

2. Elementary geometry

1. In a right triangle the longest side AB is 5. The leg BC is 3.

a Calculate the missing leg and the area of the triangle.
b What is the radius of the circumscribed circle?
c What is the radius of the inscribed circle?
d What is the tangent of the angle at A?
e Compute the height belonging to the hypothenuse.

2. The area of a right angle triangle is 30, the sum of the legs is 17. Calculate the sides of the triangle.
3. The radius of the circumscribed circle around a right triangle is 5, one of the legs is 6.

a Calculate the missing leg of the triangle.
b What is the area of the triangle?
c What is the tangent of the angle at A?

4. The sides of a triangle are $a = 5$, $b = 12$ and $c = 13$. Calculate the angle opposite to the side c.
5. Find the missing quantities in each row of the following table

a	b	c	h	p	q
		12		3	
				4	16
				6	9
6					9
6	8				

where a and b denote the legs of a right triangle, c is the hypothenuse, h is the height belonging to c and h divides c into the parts p and q.
6. Three sides of a triangle are given: 13, 13 and 24.

a Calculate the heights and the area of the triangle.
b Calculate the biggest angle of the triangle.
c Calculate the radius of the inscribed circle of the triangle.

7. Three sides of a triangle are given: 10, 10 and 16.

a Calculate the heights and the area of the triangle.
b Calculate the biggest angle of the triangle.
c Calculate the radius of the inscribed circle of the triangle.

8. Find the possible values of $\cos x$ if $\sin x = \frac{1}{2}$. If $0 < \alpha < 90°$ and $\cos\alpha = \frac{5}{13}$ then $\sin\alpha =?$
9. Two sides of a triangle are $a = 8$ and $b = 6$, the angle α (opposite to the side a) is 45°. Calculate the missing side. Find the area of the triangle.
10. Two sides of a triangle are $a = 6$ and $b = 3$, the angle α (opposite to the side a) is 60°. Calculate the missing side and angles. Find the area of the triangle.
11. Two sides of a triangle and the angle enclosed by them are given: $a = 3$, $b = 4$ and $\gamma = 60°$.

a Find the missing side.
b Calculate the area of the triangle.
c Calculate the radius of the circumscribed circle.

12. Three sides of a triangle are given: $a = 3$, $b = 4$ and c=$\sqrt{13}$.

a Find the angle γ opposite to the side c.
b Calculate the area of the triangle.

13. Find the missing quantities in each row of the following table.

a	b	c	α	β	γ	Area	R	r
12	20				40°			
12				60°	40°			
	20		110°		40°			
	13.4	18.5	110°					
24	25	30						
19	12	9						
8	10	20						
	20	25	60°					
8	10					40		
8	10						5	
			75°	25°	80°			1

Warning. Observe that the cosine rule gives impossible values in case of $a = 8$, $b = 10$ and $c = 20$ (cf. triangle inequalities).

14. In a symmetrical trapezoid the length of the shorter parallel base is 10. The common length of the legs is 5 and the height is 4. What is the area of the trapezoid?

15. Three sides in a symmetrical trapezoid are of length 10, the fourth side has length 20. Calculate the area of the trapezoid.

16. In a symmetrical trapezoid the inclination angle of the diagonal to the longer parallel base is 45°, the length of the diagonal is 10. What is the area of the trapezium?

17. The longest base of a symmetrical trapezoid is 20, the length of the legs is 5, the height is 3.

a Calculate the area of the trapezoid.
b Calculate the angles of the trapezoid.

18. The longest base of a symmetrical trapezoid is 12, the length of the legs is 5, the height is 4.

a Calculate the area of the trapezoid.
b Calculate the angles of the trapezoid.

19. The longer diagonal of a rhombus is given: 12, and one of the angle of the rhombus is 60°

a Calculate the area of the rhombus.
b Calculate the length of the sides of the rhombus.

20. The shorter diagonal of a rhombus is given: 12, and one of the angle of the rhombus is 60°

a Calculate the area of the rhombus.
b Calculate the length of the sides of the rhombus.

21. The perimeter of a rhombus is 40, its area is 96. What are the angles, sides, and diagonals of the rhombus?

22. Calculate the common length of the sides of a regular 10-gon inscribed in the unit circle.

23. The length of the side of a regular 6-gon is given: 8.

a Calculate the angles of the polygon.
b What is the length of the shorter diagonal?
c What is the area of the polygon?

24. The length of the side of a regular 6-gon is given: 12.

a Calculate the angles of the polygon.
b What is the length of the shorter diagonal?
c What is the area of the polygon?

25. The length of the side of a regular 8-gon is given: 6. Calculate the area of the polygon.

26. Calculate the area of the bright part in Figure 1.

27. Let a circle with radius 2 be given. The distance between the point P and the center of the circle is 4. Calculate the common length of the tangent segments from P to the given circle and find the length of the shorter arc along the circle between the contact points A and B.

28. Let a circle with radius 3 be given. The distance between the point P and the center of the circle is 6. Calculate the common length of the tangent segments from P to the given circle and find the length of the shorter arc along the circle between the contact points A and B.

29. Two circles of radius 10 intersect each other. The distance of the their centers is 16. Calculate the area of the common part of the two circles!

30. Two circles of radius 5 intersect each other. The distance of the their centers is 8. Calculate the area of the common part of the two circles!

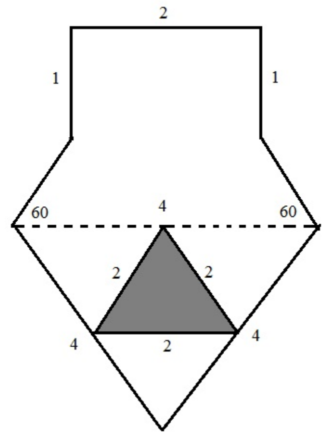

FIGURE 1. Excercise 26.

31. Find an estimation for the value of π by computing the area A_n of the inscribed regular 2^n - gon in the unit circle, $n = 2, 3, 4, 5$ and so on.
32. Can you give a recursive formula for A_{n+1}?
Hint.Using that

$$\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$$

we have:

$$\sin^2 x = \frac{1 - \cos(2x)}{2}.$$

Substituting $x = \alpha/2$

$$\sin^2(\alpha/2) = \frac{1 - \cos^2(\alpha)}{2} = \frac{1 - \sqrt{1 - \sin^2 \alpha}}{2}$$

provided that α is an acute angle. Note that $\alpha_{n+1} = \alpha_n/2$.

3. Coordinate Geometry

1. Compute the distance between the points $A(4, 1)$ and $B(1, -1)$. Find the coordinates of the midpoint of the segment AB.
2. Let the points $A(-3, 2)$, $B(2, 3)$ and $C(1, -2)$ be given. Compute the angle at A in the triangle ABC. Find the area of the triangle.
Hint.

- compute the sides of the triangle,
- use the cosine rule

$$BC^2 = AB^2 + AC^2 - 2 \cdot AB \cdot AC \cdot \cos\angle A$$

  to calculate the angle at A.

3. Let the points $A(-1, 1)$, $B(1, 3)$ and $C(3, 2)$ be given. Find the coordinates of the missing point D of the parallelogram $ABCD$. How many solutions we have?
4. Let the points $A(4, 1)$, $B(1, -1)$ and $C(1, 2)$ be given. Find the equation of the line

- passing through the points A and B,
- passing through the point C and parallel to the line AB,
- passing through the point C and perpendicular to the line AB.

Find the distance between the point C and the line AB.
5. Find the equation of the line passing through the point $C(3, 2)$ and

- parallel to $y = -3x + 5$
- perpendicular to $y = \frac{2}{3}x + 46$.

6. Find the common point of the lines $y = -2x + 3$ and $y = 3x - 8$.
7. Find the coordinates of the common point(s) of the lines passing through the points $A(4, 1)$, $B(1, -1)$ and $A'(-1, 2)$, $B'(3, 1)$.
8. Find the coordinates of the common point(s) of the circle $(x - 1)^2 + (y + 1)^2 = 16$ and the line $y = 2x - 3$.
9. Find the common points of the line $y = \frac{1}{2}x + 1$ and the circle $(x - 1)^2 + (y + 1)^2 = 25$.

10. Find the equation of the tangent lines to the circle $(x-1)^2+(y+1)^2=16$ which are parallel to $y=2x-3$.
Hint.

- determine the center of the circle: $C(1,-1)$,
- write the equation of the line passing through C and perpendicular to $y=2x-3$,
- the common points of the line in the previous step and the given circle are the contact points of the tangent lines.

11. Find the equation of the tangent line to the circle $(x-1)^2+(y+1)^2=25$ at the point $P(4,3)$.
Hint.

- determine the center of the circle: $C(1,-1)$,
- write the equation of the line passing through C and P,
- write the equation of the line passing through P and perpendicular to the line CP.

12. Find the common points of the circles $(x+1)^2+(y-1)^2=5$ and $x^2+y^2=1$
13. Find the equation of the tangent lines to the circle $(x-1)^2+(y+1)^2=16$ from the point $P(3,5)$.
Hint.

- determine the center of the circle: $C(1,-1)$,
- calculate the coordinates of the midpoint of the segment CP: $M(2,2)$.
- calculate the distance between C and M: $\sqrt{10}$,
- the common points of the circles $(x-1)^2+(y+1)^2=16$ and $(x-2)^2+(y-2)^2=10$ will be the contact points of the tangent lines.

14. Let the points $A(-1,1)$, $B(1,3)$ and $C(3,2)$ be given. Compute the coordinates of the center of the circumscribed circle, the orthocenter and the barycenter. Are they lying on the same line? Find the equation of the circumscribed circle of the triangle ABC.
15. Explain why the following method is working well.
Excercise: find the common points of the circles $(x-1)^2+(y+1)^2=16$ and $(x-2)^2+(y-2)^2=10$.

Solution. (displacement-solution-displacement) We are going to find the common points of the circles

$$(x+1)^2+(y+3)^2=16$$
$$x^2+y^2=10.$$

Taking the difference we have

$$x+3y=-2.$$

Therefore

$$(2+3y)^2+y^2=10$$
$$5y^2+6y-3=0$$
$$y_1=\frac{-3+2\sqrt{6}}{5},\quad y_2=\frac{-3-2\sqrt{6}}{5}$$

and, consequently,

$$x_1=-(2+3y_1),\quad x_2=-(2+3y_2).$$

The solutions of the original problem are

$$A(x_1+2,y_1+2)\ \text{ and }\ B(x_2+2,y_2+2)$$

16. Suppose that a point is moving from $A(0,10)$ towards the origin along the second coordinate axis with constant speed 5 m/s. At the same time another one is moving from $B(10,0)$ towards the origin along the first coordinate axis with constant speed 3 m/s. Find t_0 to minimize the distance between the moving points provided that they simultaneously started.

17. Suppose that a point is moving from $A(0,1)$ towards the point $C(3,5)$ along the straight line with constant speed 5 m/s. At the same time another one is moving from $B(1,0)$ towards the point $C(3,5)$ along the straight line with constant speed 3 m/s. Find t_0 to minimize the distance between the moving points provided that they simultaneously started.

The following excercises have been posed in the previous section too (see Elementary geometry, excercises 19 and 29). Find coordinate geometric solutions of the problems!

18. The longer diagonal of a rhombus is given: 12, and one of the angle of the rhombus is 60°

19. Two circles of radius 10 intersect each other. The distance of the their centers is 16. Calculate the area of the common part of the two circles!
20. Using that Budapest(47 N, 19 E) and Sydney(34 S, 151 E) find the spherical distance between the cities.