EXISTENTIAL REDUCTS OF THE BINARY BRANCHING SEMILINEAR ORDER AND THE THOMAS CONJECTURE

ANDRÁS PONGRÁCZ

Abstract. There exists, up to isomorphism, a unique countable existentially closed semilinear order, denoted by $(\mathbb{S}_2; \leq)$, whose reducts up to first-order interdefinability were characterized. We refine that result to obtain a classification up to existential interdefinability. It is shown that the techniques used to achieve this improvement can be applied in general to structures with an injective function that collapses types. This common phenomenon was a major technical difficulty before in problems about the classification of reducts of some structure Δ, but the present paper demonstrates how its solution can be reduced to the analysis of structures simpler than Δ.

Keywords:
Semilinear order, reduct, model-completeness, permutation group, self-embedding monoid, existential interdefinability

1. Introduction

A partial order $(P; \leq)$ is called *semilinear* if for all $a, b \in P$ there exists a $c \in P$ such that $a \leq c$ and $b \leq c$, and for every $a \in P$ the set $\{b \in P : a \leq b\}$ is a chain, that is, contains no incomparable pair of elements. Finite semilinear orders are closely related to rooted trees: the transitive closure of a rooted tree (viewed as a directed graph with the edges oriented towards the root) is a semilinear order, and the transitive reduction of any finite semilinear order is a rooted tree.

It was shown in [3] that there is a unique countable, dense, unbounded, nice, and binary branching semilinear order without joins, which is denoted by $(\mathbb{S}_2; \leq)$; see also [15, 14]. It is existentially closed in the class of all countable semilinear orders, and this property serves as an alternative definition of $(\mathbb{S}_2; \leq)$. Since all these properties of $(\mathbb{S}_2; \leq)$ can be expressed by first-order sentences, it follows that $(\mathbb{S}_2; \leq)$ is ω-*categorical*: it is, up to isomorphism, the unique countable model of its first-order theory. It also follows from general principles that the first-order theory T of $(\mathbb{S}_2; \leq)$ is *model complete*, that is, embeddings between models of T preserve all first-order formulas, and that T is the *model companion* of the theory of semilinear orders, i.e., is model complete and has the same universal consequences; again, we refer to [16] (Theorem 8.3.6). Droste proved that $(\mathbb{S}_2; \leq)$ is the unique countably infinite, non-linear, 3 set–homogeneous semilinear order (see Theorem 6.22 of [13]). It is not a homogeneous structure though, but first-order interdefinable with a homogeneous structure in a finite relational

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language, a property that is stronger than ω-categoricity. We write $x < y$ for $(x \leq y \wedge x \neq y)$ and $x\|y$ for $\neg(x \leq y) \wedge \neg(y \leq x)$, that is, for incomparability with respect to $\leq$. Then a 3-tuple (a, b, c) is in the C-relation if it is an antichain, i.e., $a\|b\|c\|a$, and there is a d such that $a\|d$ and $b, c < d$. In the terminology of rooted trees, this means that the a, b, c are the leaves of a binary tree whose root cuts it into two branches: one containing a, and the other containing b and c (together with a common upper bound of b and c); see Figure 1. Then $(\mathbb{S}_2, \leq, C)$ is homogeneous; cf. [3] for more details.

A *reduct* of a relational structure Δ is a relational structure Γ with the same domain as Δ such that every relation of Γ has a first-order definition over Δ without parameters (this slightly non-standard definition is common practice, see e.g. [24, 25, 17]). Thomas conjectured that every countable homogeneous structure in a finite relational language has finitely many reducts up to first-order interdefinability. This conjecture has been confirmed for various fundamental homogeneous structures, with particular activity in recent years [12, 24, 25, 1, 17, 21, 20, 10, 19, 4]. One of the main results of the paper [3] is that the conjecture holds for $(\mathbb{S}_2, \leq, C)$: it has three reducts up to first-order interdefinability, namely $(\mathbb{S}_2, \leq, C)$, $(\mathbb{S}_2, B)$ and $(\mathbb{S}_2, =)$ where B is the ternary relation

$$B(x, y, z) \leftrightarrow (x < y < z) \vee (z < y < x) \vee (x < y \wedge y\|z) \vee (z < y \wedge y\|x) \ .$$

By the theorem of Ryll-Nardzewski (see, e.g., Corollary 7.3.3. in Hodges [16]), two ω-categorical structures are first-order interdefinable if and only if they have the same automorphisms. The result about the reducts of $(\mathbb{S}_2; \leq)$ up to first-order interdefinability is equivalent to the statement that there are precisely three subgroups of $\mathrm{Sym}(\mathbb{S}_2)$ that contain the automorphism group of $(\mathbb{S}_2; \leq)$ and that are closed in $\mathrm{Sym}(\mathbb{S}_2)$ with respect to the *topology of point-wise convergence*, i.e., the subspace topology on $\mathrm{Sym}(\mathbb{S}_2)$ of the product topology on $(\mathbb{S}_2)^{\mathbb{S}_2}$ where $\mathbb{S}_2$ is taken to be discrete. The Ryll-Nardzewski theorem has several generalizations: reducts of an ω-categorical Δ up to existential positive interdefinability are in a one-to-one correspondence with the closed supermonoids of $\mathrm{End}(\Delta)$, i.e., the endomorphism monoids of reducts of Δ, and reducts of Δ up to primitive positive interdefinability are in a one-to-one correspondence with the closed superclones of $\mathrm{Pol}(\Delta)$, i.e., the polymorphism clones of reducts of Δ [2]. The main focus of the current paper is a more exotic generalization of the same notion. The reducts of an ω-categorical Δ up to existential interdefinability are in a one-to-one correspondence with the self-embedding monoids containing the self-embedding monoid $\mathrm{Emb}(\Delta)$ of Δ. These monoids are not as easy to describe as endomorphism monoids or polymorphism clones: the condition that a monoid be closed in the point-wise convergence topology is equivalent to the monoid being the endomorphism monoid of some structure, but those are seldom self-embedding monoids of structures. This is the main challenge why self-embedding monoids are hard to work with, and why there are few results classifying reducts of a structure up to existential interdefinability [11, 4, 6]. In the present paper, we show a method that can be applied to prove such classification results. The technique is illustrated on the semiliniar order $(\mathbb{S}_2; \leq)$, cf. Theorem 2.7. Then we prove some general results using recent improvements in structural Ramsey theory [26, 23, 18], to show that it has a potential to be applied to a broad class of structures, and to obtain results towards Thomas' conjecture; cf. Proposition 3.2, Corollary 3.3, Theorem 4.1, Corollary 4.3 and Proposition 4.6. There is no known example to a homogeneous structure in a finite relational language that has infinitely many reducts up to existential interdefinability. We note that the simplest countable structure $(\mathbb{N}, =)$ has infinitely many reducts up to existential positive (and by extension, also primitive positive) interdefinability. The current paper indicates that in some sense, reducts could be easier to handle up to existential interdefinability than the seemingly simpler problem of classification up to first-order interdefinability, and that the existential variant of the Thomas conjecture deserves more attention. In all verified instances of the Thomas conjecture, an ordered Ramsey expansion of the structure is made use of. The most effective method that builds on the existence of such an expansion was introduced in [8, 7] and was first applied to achieve a new classification result in [21, 20]. The main idea is to find the minimal closed supermonoids above $\mathrm{End}(\Delta)$ via *canonical functions*, and determine which of those can be generated by groups. The drawback of this approach is that it can be labor-intensive, and that it is often unclear what groups are generated by some canonical functions. It is still unknown whether there are only a finite number of minimal closed supergroups of the automorphisms group of a structure that satisfies the condition of the Thomas conjecture. In contrast, by using a simple argument based on canonical functions, it is clear that (provided the existence of a Ramsey expansion, which is widely believed to be the case for all homogeneous relational structures in a finite language) there are only finitely many minimal self-embedding supermonoids, closed (endomorphism) supermonoids and closed (polymorphisms) superclones. This is yet another argument why self-embedding supermonoids could be preferable to closed (automorphism) supergroups.

The improvement of the classification of reducts of $(\mathbb{S}_2; \leq)$ up to first-order interdefinability to existential interdefinability presented in this paper does not require the labor-intensive application of canonical functions, and in fact, the existence of a Ramsey-type expansion is not needed in any of the arguments; see Section 3. A technical element of the proof is a *zig-zag argument* to show that the classification of self-embedding supermonoids over the semilinear order containing a type-collapsing injection can be reduced to a classification problem over a simpler structure. We present a general inductive technique in Section 4, and prove that if the zig-zag argument never fails, then the first-order and existential Thomas conjecture (for ordered Ramsey structures) are equivalent.

2. The universal C-relation and the semilinear order

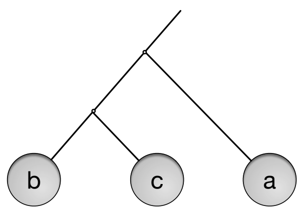


Figure 1. C(a,bc)

The following proposition, already mentioned in the introduction, is of central importance in understanding reducts up to existential interdefinability.

Proposition 2.1 (Proposition 3.4.7 in [2])**.** *For every ω-catgorical structure Δ,*

(1) *a relation S has an existential positive definition in Δ iff S is preserved by* $\mathrm{End}(\Delta)$*;*
(2) *a relation S has an existential definition in Δ iff S is preserved by* $\mathrm{Emb}(\Delta)$.

This yields the generalization of the Ryll-Nardziewski theorem: reducts of Δ up to existential interdefinability are in a one-to-one correspondence with the self-embedding supermonoids of Δ. As the next theorem shows, there is only one proper reducts of the universal C-relation $(\mathbb{L}, C)$ up to existential interdefinability: the structure $(\mathbb{L}, C)$ plays an important role as it was already observed in [3], since it is the C-structure on any maximal antichain in $(\mathbb{S}_2, \leq, C)$. The 4-ary relation Q is defined from C by the formula

$$Q(xy, uv) \leftrightarrow (C(x, uv) \wedge C(y, uv)) \vee (C(u, xy) \wedge C(v, xy))$$

The missing commas in between xy and uv indicates the symmetries of this relation: by transposing the first of second pairs, or by exchanging the two pairs of entries in a tuple in Q yields a tuple in Q. The same convention is used in the C-relation: it is invariant under switching the second pair of coordinates.

Using Proposition 2.1 together with a Ramsey analysis, in [4] the following classification was obtained.

Theorem 2.2 (Corollary 3 in [4])**.** *Let Γ be a reduct of the universal C-relation $(\mathbb{L}, C)$. Then* $\mathrm{Emb}(\Gamma)$ *is one of the following three monoids:* $\mathrm{Emb}(\mathbb{L}, C) = \overline{\mathrm{Aut}(\mathbb{L}, C)}$, $\mathrm{Emb}(\mathbb{L}, Q) = \overline{\mathrm{Aut}(\mathbb{L}, Q)}$, *or* $\mathrm{Emb}(\mathbb{L}, =) = \overline{\mathrm{Aut}(\mathbb{L}, =)}$. *In particular, $(\mathbb{L}, C)$ has three reducts up to existential interdefinability, namely $(\mathbb{L}, C)$, $(\mathbb{L}, Q)$, and $(\mathbb{L}, =)$, and all of these structures are model complete.*

We do not need the whole network of technical lemmas proved in [3], only the following assertions.

Proposition 2.3. *Let Γ be a reduct of $(\mathbb{S}_2; \leq)$. Then one of the following holds.*

(1) $\mathrm{End}(\Gamma)$ *contains a flat or a thin function.*
(2) $\mathrm{End}(\Gamma) = \overline{\mathrm{Aut}(\mathbb{S}_2; \leq)}$.
(3) $\mathrm{End}(\Gamma) = \overline{\mathrm{Aut}(\mathbb{S}_2; B)}$.

Lemma 2.4. *Let Γ be a reduct of $(\mathbb{S}_2; \leq)$ with a thin self-embedding. Then Γ is isomorphic to a reduct of $(\mathbb{Q}; <)$.*

The ternary relation R in $(\mathbb{S}_2, \leq C)$ was defined in [3]:

$$R(x, yz) \leftrightarrow C(x, yz) \vee (y < x \wedge z < x) \vee (x \| z \wedge x \| y \wedge (z < y \vee y < z))$$

Lemma 2.5. *Let Γ be a reduct of $(\mathbb{S}_2; \leq)$ with a flat self-embedding. Then Γ is isomorphic to a reduct of $(\mathbb{Q}; <)$, or it has a flat self-embedding that preserves R.*

Lemma 2.6. *Let Γ be a reduct of $(\mathbb{S}_2; \leq)$ which is isomorphic to a reduct of $(\mathbb{Q}; <)$. Then Γ is existentially interdefinable with $(\mathbb{S}_2; =)$.*

Just as R is the full preimage of C under a flat self-embedding $f \in \mathrm{Emb}(\mathbb{S}_2, R)$, P is the full preimage of Q under f, see [4]; namely $P(xy, uv)$ is defined by

$$(R(x, uv) \wedge R(y, uv)) \vee (R(u, xy) \wedge R(v, xy))$$

Then $\mathrm{Emb}(\mathbb{S}_2, P)$ contains all rerootings (see [3]), that is $\mathrm{Emb}(\mathbb{S}_2, B) \subseteq \mathrm{Emb}(\mathbb{S}_2, P)$. We are ready to present the assertion about the reducts of $(\mathbb{S}_2, \leq, C)$ up to existential interdefinability; cf. Figure 2.

Theorem 2.7. *The structure $(\mathbb{S}_2, \leq, C)$ has five reducts up to existential interdefinability with self-embedding monoids* $\mathrm{Emb}(\mathbb{S}_2, \leq, C), \mathrm{Emb}(\mathbb{S}_2, B), \mathrm{Emb}(\mathbb{S}_2, R), \mathrm{Emb}(\mathbb{S}_2, P), \mathrm{Emb}(\mathbb{S}_2, =)$.

The next section is dedicated to the proof of Theorem 2.7. It turns out that the section of the lattice of self-embedding monoids above $\mathrm{Emb}(\mathbb{S}_2, R)$ is in some sense induced by the lattice of self-embedding monoids above $\mathrm{Emb}(\mathbb{L}, C)$. That is, the three monoids in Theorem 2.2 correspond to the three monoids above $\mathrm{Emb}(\mathbb{S}_2, R)$ in Figure 2.

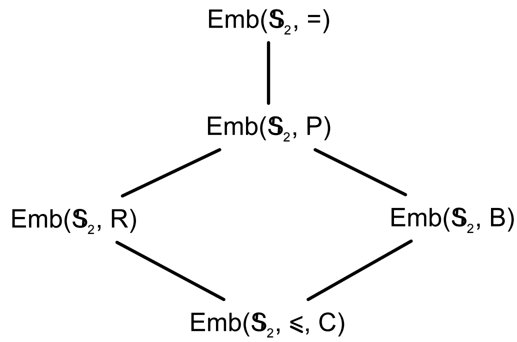


FIGURE 2. Reducts of $(\mathbb{S}_2, \leq, C)$ up to existential interdefinability

3. Existential reducts of the semilinear order

Lemma 3.1. *Let $\mathcal{M}$ be a self-embedding monoid such that* $\mathrm{Emb}(\mathbb{S}_2; \leq) \subseteq \mathcal{M} \subseteq \mathbb{S}_2^{\mathbb{S}_2}$ *and* $\mathcal{M} \notin \{\mathrm{Emb}(\mathbb{S}_2; \leq), \mathrm{Emb}(\mathbb{S}_2; B), \mathrm{Emb}(\mathbb{S}_2; =)\}$. *Then $\mathcal{M}$ contains a flat function* $f \in \mathrm{Emb}(\mathbb{S}_2; R)$.

Proof. There is a reduct Σ of $(\mathbb{S}_2; \leq)$ such that $\mathcal{M} = \mathrm{Emb}(\Sigma) = \mathrm{End}(\Sigma')$, where Σ' is obtained from Σ by adding the negation of all relations in Σ to the language. According to Proposition 2.3, $\mathcal{M}$ contains a flat or a thin function. It cannot contain a thin function by Lemmas 2.4 and 2.6, thus $\mathcal{M}$ contains a flat function. Then by Lemma 2.5, $\mathcal{M}$ contains a flat function $f \in \mathrm{Emb}(\mathbb{S}_2; R)$. ■

We now present a novel approach that makes it possible to reduce the classification of self-embedding monoids over $\mathrm{Emb}(\mathbb{S}_2; R)$ to the classification of self-embedding supermonoids of $\mathrm{Emb}(\mathbb{L}, C)$, cf. Theorem 2.2.

Proposition 3.2. *Let Δ be an ω-categorical structure with underlying set D and let $\mathcal{M}$ be a self-embedding monoid such that* $\mathrm{Emb}(\Delta) \subseteq \mathcal{M} \subseteq D^D$. *Let Γ be a reduct of Δ, and let $\Delta' := \Gamma\restriction_{D'}$ for some $D' \subseteq D$. Assume that*

- *Δ' is homogeneous,*
- *any partial isomorphism of Δ' extends to a function in $\mathcal{M}$, and*
- *there exists a $g \in \mathcal{M} \cap \mathrm{Emb}(\Gamma)$ whose image is contained in D'.*

Then $\mathrm{Emb}(\Gamma) \subseteq \mathcal{M}$.

Proof. Let T be an injective first-order definable relation in Δ such that both T and its complement are preserved by $\mathcal{M}$. We show that T has a quantifier-free definition in Γ. To this end, it is enough to prove that if two injective finite tuples a, b have the same quantifier-free type in Γ, then $T(a) \leftrightarrow T(b)$. As $g \in \mathrm{Emb}(\Gamma)$ with the image contained in D', the tuples $g(a)$ and $g(b)$ have the same quantifier-free type in Δ'. The partial isomorphism mapping

$g(a)$ to $g(b)$ extends to a function $h \in \mathcal{M}$ such that $h(g(a)) = g(b)$. As $g, h \in \mathcal{M}$ and $\mathcal{M}$ preserves both T and its complement, we obtain $T(a) \leftrightarrow T(b)$. ■

Corollary 3.3. *Let Δ be an ω-categorical structure with underlying set D and let Γ be a reduct of Δ. Let $\Delta' := \Gamma \restriction_{D'}$ for some $D' \subseteq D$. Assume that Δ' is homogeneous, and that two finite tuples in D' have the same type in Δ' iff they have the same type in Δ. Let $g \in \mathrm{Emb}(\Gamma)$ be a function with image contained in D'. Then the smallest self-embedding monoid containing $\mathrm{Aut}(\Delta)$ and g is $\mathrm{Emb}(\Gamma)$.*

Lemma 3.4. *Let $\mathcal{M}$ be a self-embedding monoid such that $\mathrm{Emb}(\mathbb{S}_2; \leq) \subseteq \mathcal{M} \subseteq \mathbb{S}_2^{\mathbb{S}_2}$ and $\mathcal{M} \notin \{\mathrm{Emb}(\mathbb{S}_2; \leq), \mathrm{Emb}(\mathbb{S}_2; B), \mathrm{Emb}(\mathbb{S}_2; =)\}$. Then $\mathrm{Emb}(\mathbb{S}_2; R) \subseteq \mathcal{M}$.*

Proof. According to Lemma 3.1, $\mathcal{M}$ contains a flat function $f \in \mathrm{Emb}(\mathbb{S}_2; R)$. Let L be a maximal antichain containing the image of f. Then the conditions of Corollary 3.3 apply with $\Delta = (\mathbb{S}_2; \leq)$, $\Gamma = (\mathbb{S}_2; R)$, $D' = L$, $\Delta' = (L; C)$, $g = f$. Hence, $\mathrm{Aut}(\mathbb{S}_2; \leq)$ together with f generate $\mathrm{Emb}(\mathbb{S}_2; R)$ as a self-embedding monoid. ■

Let $R_0 = C$, $R_1(a, b, c) \leftrightarrow (a \| b \wedge c < b)$, $R_2(a, b, c) \leftrightarrow (b \| c \wedge b < a \wedge c < a)$, $R_3(a, b, c) \leftrightarrow (b < c < a \wedge c < b < a)$. Note that $R_0 \cup R_1 \cup R_2 \cup R_3 = R$ is the full preimage of C under f. We say that a function h preserves a given n-ary relation on a set of n-tuples S if whenever a tuple in S is in the given relation, then so is its h-image. A monoid $\mathcal{N}$ preserves a relation on S if every function in $\mathcal{N}$ does.

Lemma 3.5. *If R is preserved on $R_0 \cup R_1 \cup R_2$ by a monoid $\mathrm{Emb}(\mathbb{S}_2; R) \subseteq \mathcal{M}$, then R is preserved by $\mathcal{M}$.*

Proof. Seeking for a contradiction, let $h \in \mathcal{M}$ preserve R on $R_0 \cup R_1 \cup R_2$ and violate R on R_3. By composing h with a flat function $f \in \mathrm{Emb}(\mathbb{S}_2; R)$ if necessary, we may assume that R_0, R_1, R_2 are mapped to C by h. To show the claim, we consider the 7-element substructure in Figure 3 of the semilinear order, and assume that h violates R on $\{a, u, w\}$.

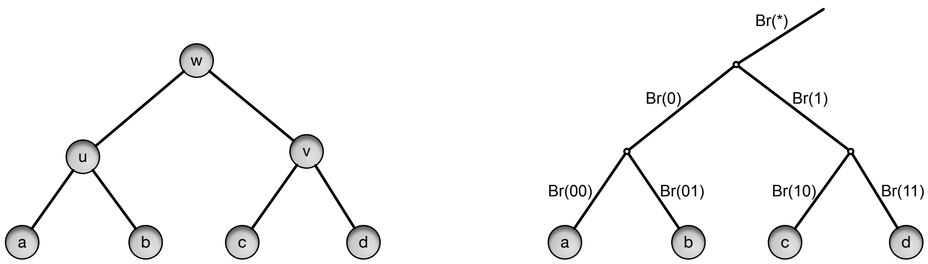


FIGURE 3. The 7-element substructure in the proof of Lemma 3.5

By applying an automorphism in $\mathrm{Aut}(\mathbb{S}_2; \leq)$, we may assume that the four points a, b, c, d are fixed. Since we have $R_2(u, a, b)$ and $R_1(c, a, u)$, the point $h(u)$ is attached to the branch segment $Br(0)$. Similarly, $h(v)$ is attached to the branch segment $Br(1)$. But then wherever $h(w)$ is attached to, R cannot be preserved on $(w, u, v) \in R_2$ and violated on $\{a, u, w\}$ by h at the same time, a contradiction. ■

Given a function h (or monoid $\mathcal{N}$), we can define a digraph on the quantifier-free n-types in the language $\{R\}$ for any $n \in \mathbb{N}$: there is a directed edge $(\underline{s}, \underline{t})$ in the graph iff h (or some function in $\mathcal{N}$) maps a representative of $\underline{s}$ to a tuple in $\underline{t}$. A relation is preserved by a monoid iff there is no out-edge from the set of types in the relation. However, a relation and its complement is preserved by a monoid iff there is no out-edge from or in-edge to the set of types in the relation.

In the next proof, we refer to a special type of antichains as *combs*, or more precisely, n-combs if it has n leaves. This is the proof where the zig-zag argument mentioned in the introduction appears: we walk back-and-forth on edges of the digraph induced by a monoid to find a path between any two types indistinguishable by the relation R.

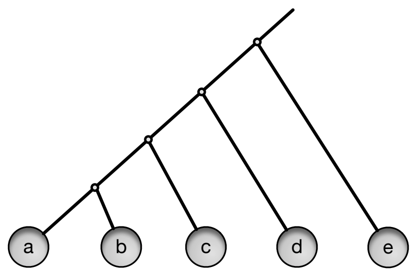


FIGURE 4. 5-comb

Lemma 3.6. *Let $\mathcal{M}$ be a self-embedding monoid such that* $\text{Emb}(\mathbb{S}_2; R) \subseteq \mathcal{M} \subseteq {\mathbb{S}_2}^{\mathbb{S}_2}$*. Assume that $\mathcal{M}$ preserves R on R_0. Then $\mathcal{M} = \text{Emb}(\mathbb{S}_2; R)$.*

Proof. We assume indirectly that there is a function $h \in \mathcal{M}$ that violates R. The assertion of the lemma follows if we show that the digraph on the quantifier-free n-types corresponding to $\mathcal{M}$ is weakly connected: in that case, $\mathcal{M}$ should be the set of all injective functions in ${\mathbb{S}_2}^{\mathbb{S}_2}$, and thus would not preserve R on R_0.

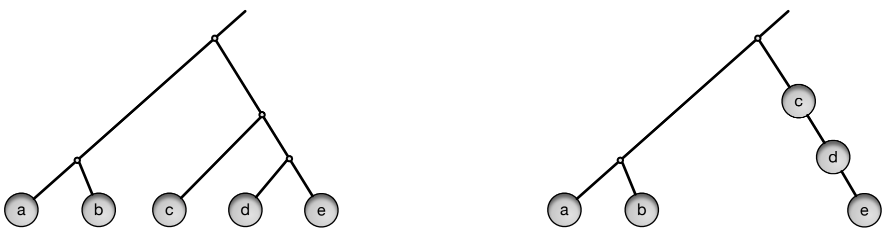
For the extent of this proof, we call two n-types $\underline{s}$ and $\underline{t}$ equivalent if by permuting the entries of a representative in $\underline{s}$ by some $\sigma \in S_n$ we obtain a tuple in $\underline{t}$; this equivalence relation is denoted by $\sim$. If $\underline{s} = \underline{t}$, we call the set of such $\sigma \in S_n$ the symmetries of $\underline{s}$. We say that $\underline{s}$ and $\underline{t}$ are connected up to equivalence if there are $\underline{s}'$ and $\underline{t}'$ in the same weakly connected component such that $\underline{s} \sim \underline{s}'$ and $\underline{t} \sim \underline{t}'$.

The strategy of the proof is to show by induction on n that all n-types are connected up to equivalence and that equivalent types are in the same weakly connected component. This clearly holds for $n = 1, 2, 3$. So let $n \geq 4$ and assume that the assertion holds for smaller n.

According to Lemma 3.5 a function $h \in \mathcal{M}$ violates R on $R_1 \cup R_2$. First assume that R is violated on R_1.

Given an arbitrary n-type $\underline{t}$, we can represent it as a C-relation on the leaves of a binary tree. This tree has a root, cutting it into two main branches. First, we show that we can replace the tree structure on these two branches so that both of them are combs. Let $\underline{t}'$ be the type we obtain by replacing the tree structure to a comb on the first branch. We represent the n-type so that the leaves on the two branches are contained in two incomparable branches of $(\mathbb{S}_2, \leq, C)$. By the induction hypothesis, there is a path in the weak component of the first branch of $\underline{t}$ and that of $\underline{t}'$. Each edge in this path is witnessed by an embedding that preserves the C-relation. Hence, by keeping the rest of $\underline{t}$ in the incomparable branch with the same C-structure, we can create a path of n-tuples between $\underline{t}$ and $\underline{t}'$. The other branch can be dealt with analogously.

As $n \geq 4$, one of these combs has at least two leaves, say in the second branch. Pick the two top leaves in the comb. Replace the tree structure on the second branch by deleting these two leaves and introducing two comparable points inside the second branch that are bigger than all the leaves in this branch. This modification has no effect on the R-structure.



Find an isomorphic copy of this tree in $(\mathbb{S}_2, \leq, C)$. By homogeneity of $(\mathbb{S}_2, \leq, C)$, we may assume that the triple containing the two newly introduced points in the second branch and the top leave in the first branch is a representative of R_1 where h violates R. We may also assume that all the leaves of this tree are fixed by h. A simple case distinction on the possible positions of the image of the two newly introduced points shows that the branch containing the vertices of the original first branch increases in size. By iterating this procedure, we find a comb in the weak connected component of $\underline{t}$. In such an n-comb, the top leaf is the smaller branch. Hence we can restart the whole process on this n-type, and obtain a new n-comb with a different top leaf. Each of these n-combs brings a copy of S_{n-1} with itself: however we permute the $n-1$ vertices in the bigger branch, we stay inside the same weak connected component by the induction hypothesis. As S_{n-1} is a maximal subgroup in S_n, and we have at least two different copies of these, any permutation of the entries of a type in the weak component of $\underline{t}$ yields a type in the same component. Thus any tuple $\underline{t}$ is in the same component with any permutation of the n-comb, thus all types are weakly connected.

Hence, we may assume that R is preserved on R_1; then it must be violated on R_2 by Lemma 3.5. Just like before, it is enough to focus on n-types where both branches are combs. Assume that there are at least three vertices a, b, c on one main branch of such an n-type $\underline{t}$. Pick the lowest one a on such a branch; see the illustration on Figure 5.

We claim that by deleting it, and replacing it with a new vertex that has its own main branch, we obtain a tuple $\underline{t}'$ in the same weak component. In order to show this claim, let us

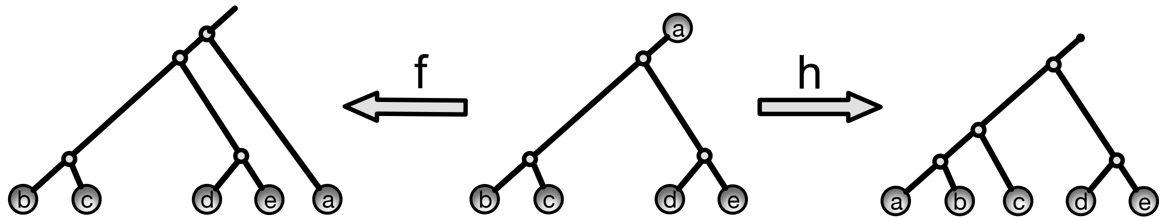
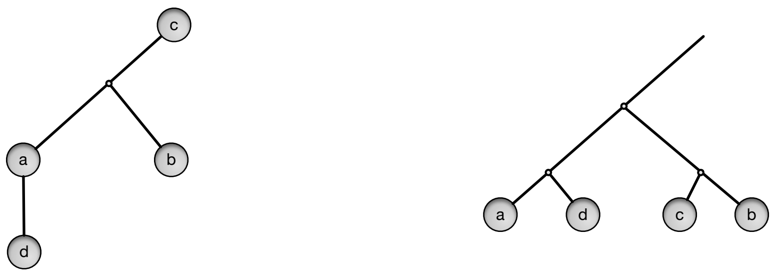


FIGURE 5. Moving a low-positioned leaf to a new branch

represent the same R-structure by putting the vertex on top of the tree rather than as a leaf of a new branch. This new vertex together with b, c form a tuple in R_2. We may assume that it is violated by h, as before, and that h fixes all the leaves. Then the top vertex can only be moved between b and c; either way, we obtain a tuple equivalent to $\underline{t}$.

If $\underline{t}$ is the 4-comb, then this argument shows that there is a 3-cycle on the entries leaving the tuple in the same weak component. We show that the two different R-structures on the leaves of a tree with four leaves are in the same component up to equivalence. To this end, pick a triple a, b, c in R_2 that is violated by h, and assume that h fixes the two leaves b, c. We may assume that a is moved below the branch with c on it. Let d be a vertex smaller that b.



As R_1 is preserved by h, the h-image of $\{a, b, c, d\}$ are the four leaves of a tree both of whose branches have two leaves. However, the R-structure of $\{a, b, c, d\}$ is the same as that of the leaves of a 4-comb. The symmetries of the h-image form a subgroup of S_4 isomorphic to the dihedral group D_4, a maximal subgroup which does not contain a 3-cycle. Together with the earlier observation, that the weak component is preserved by a 3-cycle, we have that there are representatives of both leaf structures in the same weak component, and it is invariant under all permutations of S_4. Hence, all 4-tuples are in the same weak component, that is, the assertion holds for $n = 4$. We proceed by induction for $n \geq 5$, and pick an n-type $\underline{t}$. By using the induction hypothesis as in the first case, there is a tuple $\underline{t}'$ in the weak component of $\underline{t}$ both of whose branches are combs. By the pigeonhole principle, there is a branch of $\underline{t}'$ with at least 3 leaves. We can use the above claim for the lowest three vertices of this branch, and replace the lowest vertex of this branch by a leaf on a new branch. By using the induction hypothesis once again, we have found an n-comb in the same component, whose top leaf is one of the lowest leaves on a beach of $\underline{t}'$. By repeating the same procedure for the n-comb, we can find another n-comb in the same component with a different top leaf. As the lower $n-1$ leaves can be arbitrarily permuted according to the induction hypothesis, the weak component is invariant under at least two copies of S_{n-1} in S_n. As they generate S_n, we obtain that any weak component contains an n-comb, and its vertices can be arbitrarily permuted to have a new tuple in the same component. Thus all n-tuples are in the same weak component.

■

Lemma 3.7. *Let $\mathcal{M}$ be a self-embedding monoid such that* $\mathrm{Emb}(\mathbb{S}_2; R) \subsetneq \mathcal{M} \subseteq {\mathbb{S}_2}^{\mathbb{S}_2}$. *Then* $\mathrm{Emb}(\mathbb{S}_2; P) \subseteq \mathcal{M}$.

Proof. According to Lemma 3.6, some $h \in \mathcal{M}$ violates R on $R_0 = C$. By composing h with an automorphism in $\mathrm{Aut}(\mathbb{S}_2; \leq, C)$ if necessary, we may assume that $\{h(a), h(b), h(c)\} = \{a, b, c\}$. Let L be a maximal antichain in $(\mathbb{S}_2, \leq)$ that contains $\{a, b, c\}$, and let $f \in \mathrm{Emb}(\mathbb{S}_2; R)$ be a flat function with image contained in L. Then $f' := f \circ h \in \mathcal{M}$ violates C on L, and then $\mathrm{Aut}(L; C) \cup \{f'\}$ locally generates $\mathrm{Aut}(L; Q)$ according to Theorem 2.2. Even though $\mathrm{Aut}(L; C)$ is not contained in $\mathrm{Emb}(\mathbb{S}_2; R)$, any function in $\mathrm{Aut}(L; C)$ can be interpolated on any finite subset of L by a function in $\mathrm{Emb}(\mathbb{S}_2; R)$. Hence, $\mathrm{Emb}(\mathbb{S}_2; R) \cup \{f'\}$ locally generates $\mathrm{Aut}(L; Q)$, and since $\mathrm{Emb}(\mathbb{S}_2; R) \subseteq \mathcal{M}$, we have that $\mathcal{M}$ locally generates $\mathrm{Aut}(L; Q)$. According to [4], the structure $(L; Q)$ is homogeneous, thus every partial isomorphism of $(L; Q)$ extends to a function generated by $\mathcal{M}$. Then the conditions of Proposition 3.2 apply with $\Delta = (\mathbb{S}_2; \leq, C)$, $\mathcal{M} = \mathcal{M}$, $\Gamma = (\mathbb{S}_2; P)$, $D' = L$, $\Delta' = (L; Q)$, $g = f$, yielding $\mathrm{Emb}(\mathbb{S}_2; P) \subseteq \mathcal{M}$. ■

Lemma 3.8. *Let $\mathcal{M}$ be a self-embedding monoid such that* $\mathrm{Emb}(\mathbb{S}_2; P) \subsetneq \mathcal{M} \subseteq {\mathbb{S}_2}^{\mathbb{S}_2}$. *Then* $\mathcal{M} = \mathrm{Emb}(\mathbb{S}_2; =)$.

Proof. We can repeat the proof of Lemma 3.6 to show that if $\mathcal{M}$ preserves P on antichains, then $\mathcal{M} = \mathrm{Emb}(\mathbb{S}_2; =)$. Indeed, whenever we used the assumption of the lemma, it was applied to the leaves of a tree, to show that we may assume that some function is identical on that set. Instead, we can now use a rerooting to get back the original C-structure on the leaves if P is preserved. Thus an $h \in \mathcal{M}$ violates P on some antichain $\{a, b, c, d\}$. Let L be a maximal antichain in $(\mathbb{S}_2, \leq)$ that contains $\{a, b, c, d\}$, and let $f \in \mathrm{Emb}(\mathbb{S}_2; R)$ be a flat function with image contained in L. By composing h with an automorphism in $\mathrm{Aut}(\mathbb{S}_2; \leq, C)$ if necessary, we may assume that $\{h(a), h(b), h(c), h(d)\} \subseteq L$. Then $f' := f \circ h \in \mathcal{M}$ violates Q on L, and then $\mathrm{Aut}(L; Q) \cup \{f'\}$ locally generates $\mathrm{Aut}(L; =) = \mathrm{Sym}(L)$ according to Theorem 2.2. Even though $\mathrm{Aut}(L; Q)$ is not contained in $\mathrm{Emb}(\mathbb{S}_2; R)$, any function in $\mathrm{Aut}(L; Q)$ can be interpolated on any finite subset of L by a function in $\mathrm{Emb}(\mathbb{S}_2; P)$. Hence, $\mathrm{Emb}(\mathbb{S}_2; P) \cup \{f'\}$ locally generates $\mathrm{Sym}(L)$, and since $\mathrm{Emb}(\mathbb{S}_2; P) \subseteq \mathcal{M}$, we have that $\mathcal{M}$ locally generates $\mathrm{Sym}(L)$. Thus every partial isomorphism of $(L; =)$ extends to a function generated by $\mathcal{M}$. Then the conditions of Proposition 3.2 apply with $\Delta = (\mathbb{S}_2; \leq, C)$, $\mathcal{M} = \mathcal{M}$, $\Gamma = (\mathbb{S}_2; =)$, $D' = L$, $\Delta' = (L; =)$, $g = f$, yielding $\mathrm{Emb}(\mathbb{S}_2; =) \subseteq \mathcal{M}$. ■

Proof of Theorem 2.7. By Lemma 3.4 we only need to find the self-embedding monoids containing $\mathrm{Emb}(\mathbb{S}_2, R)$. According to Lemmas 3.7 and 3.8, those are exactly $\mathrm{Emb}(\mathbb{S}_2, P)$ and $\mathrm{Emb}(\mathbb{S}_2, =)$. ■

Note that in this section we have classified the self-embedding supermonoids of $\mathrm{Emb}(\mathbb{S}_2, R)$, and to this end, we did not need anything else from the paper [3] but the four simple assertions cited in the previous section. That covers the existential supermonoids of $\mathrm{Emb}(\mathbb{S}_2, \leq, C)$ that contain a flat function, which was one of the major difficulty in [3]. In particular, the arguments in the present paper not only generalize the results in [3], but by properly combining the two papers, a significant part of the arguments in [3] becomes obsolete, yielding simpler proofs to the main results of that paper. The proofs rely more on the characterization result Theorem 2.2 from [4], which is about the universal C-relation, a much simpler structure to work with than $(\mathbb{S}_2; \leq, C)$.

In the next section, we generalize some of these ideas, and show that a self-embedding that collapses types (just like the flat self-embedding $f \in \mathrm{Emb}(\mathbb{S}_2, R)$ does for $(\mathbb{S}_2; \leq, C)$) always gives rise to such a simpler structure.

4. Type-collapsing self-embeddings

A function $f : \Delta \to \Gamma$ is *canonical* if it maps tuples of the same type in Δ to tuples of the same type in Γ. If Δ is homogeneous in a finite relational language with maximum arity m, then this is witnessed on finitely many types: then $f : \Delta \to \Delta$ is canonical iff tuples of the same type with arity at most m are mapped to tuples of the same type. For example, a flat self-embedding $f \in \mathrm{Emb}(\mathbb{S}_2, R)$ as a function $f : (\mathbb{S}_2, \leq, C) \to (\mathbb{S}_2, \leq, C)$ is canonical. We call a canonical injection *type-collapsing* if every weakly connected component of the digraph it induces on the injective types is a star. The above f is such a function: the 3-types are mapped to the three permuted variants of the C-relation, making the digraph on the 3-types the disjoint union of three stars, the 2-types are all mapped to incomparable pairs, and the unique 1-type is mapped to itself.

Theorem 4.1. *Let Δ be a homogeneous structure in a finite relational language of maximum arity m. Let $f : \Delta \to \Delta$ be a type-collapsing injection.*

(1) *There exists a homogeneous structure Γ in a finite relational language of maximum arity m, whose relations are precisely the injective types of Δ that occur as the type of the f-image of an injective tuple, and* $\mathrm{Age}(\Gamma)$ *consists of the finite structures in* $\mathrm{Age}(\Delta)$ *all of whose types are among these f-images.*
(2) *There is an isomorphic copy of Γ in Δ containing the image of a canonical f' with the same behavior as f.*
(3) *If Δ has an ω-categorical ordered Ramsey expansion Δ^*, then Γ also has such an expansion Γ^*.*
(4) *If Δ is ordered Ramsey, then so is Γ.*

Proof. Let Σ be the image of f. Then $\mathrm{Age}(\Sigma)$ consists of the injective types of Δ that occur as the type of the f-image of an injective tuple. In particular, this class of substructures is the age of a countable structure, and thus it is countable up to isomorphism, hereditary, and has the joint embedding property. To show that it is the age of a homogeneous structure, we need to verify the amalgamation property for $\mathrm{Age}(\Sigma)$, as well.

Let $A, B_1, B_2 \in \mathrm{Age}(\Sigma)$ be given together with embeddings $f_1 : A \hookrightarrow B_1, f_2 : A \hookrightarrow B_2$. Then there is a $C \in \mathrm{Age}(\Delta)$ and embeddings $g_1 : B_1 \hookrightarrow C, g_2 : B_2 \hookrightarrow C$ such that $g_1 \circ f_1 = g_2 \circ f_2$, since Δ is homogeneous. Let C' be a copy of C in Δ; i.e., there is an embedding $h : C \hookrightarrow \Delta$ whose image is C'. Let C'' be the f-image of C'. Put $g_1' = f \circ h \circ g_1$ and $g_2' = f \circ h \circ g_2$. Then $g_1' : B_1 \hookrightarrow C''$, $g_2' : B_2 \hookrightarrow C''$ are embeddings with $g_1' \circ f_1 = g_2' \circ f_2$. As

C'' is contained in the image of f, we have $C'' \in \mathrm{Age}(\Sigma)$, proving the amalgamation property for $\mathrm{Age}(\Sigma)$. Then $\mathrm{Age}(\Sigma)$ is the age of a homogeneous structure Γ (see [16]), and Item (1) is shown.

As Δ is homogeneous and $\mathrm{Age}(\Gamma) \subseteq \mathrm{Age}(\Delta)$, there are isomorphic copies of Γ in Δ. Let us identify Γ with one of these isomorphic copies. Similarly, as $\mathrm{Age}(\Sigma) \subseteq \mathrm{Age}(\Gamma)$ and Γ is homogeneous, there exist a copy Γ' of Γ (not necessarily contained in Δ) with $\Sigma \leq \Gamma'$ and an isomorphism $\iota : \Gamma' \to \Gamma$. Let $f' := \iota \circ f$; then for any tuple $\underline{t}$ the type of the f-image of $\underline{t}$ coincides with that of the f'-image of $\underline{t}$. Hence, f' is a canonical injection with the same behavior as f, and the image of f' is contained in Γ, finishing the proof of Item (2).

For the third item, we may assume that f' is also canonical as a function $\Delta^* \to \Delta^*$ by using the ordered Ramsey property of Δ^*; see Theorem 5 in [9]. Let $A \leq B \in \mathrm{Age}(\Gamma)$ and $k \in \mathbb{N}$. Let $t(A)$ be the number of different types in Δ^* whose f'-image in Γ is isomorphic to A: this is a finite number that only depends on A. Let $C \in \mathrm{Age}(\Delta^*)$ be the structure that witnesses the Ramsey property with k colors for all these possible pre-images of A and B: that is, however we color all structures in C whose f' image in Γ is isomorphic to A by k colors, there is a B' in C whose f' image in Γ is isomorphic to B such that all these different isomorphism types are monochromatic in B'. Let C' be the f'-image of C in Γ. Any coloring of all copies of A in C' induces a coloring of all preimages of A in C. Thus the f'-image of the above B' is at most $t(A)$-colored. This shows that every $A \in \mathrm{Age}(\Gamma)$ has a finite Ramsey degree at most $t(A)$. By [26, 23] Γ has an ω-categorical ordered Ramsey expansion Γ^*.

Finally for Item (4), assume that Δ is ordered Ramsey. Then types coincide with labelled isomorphism types. Let $A \leq B$ be finite structures in $\mathrm{Age}(\Gamma)$, and let $\chi : \binom{\Gamma}{A} \to \{0,1\}$ be a 2-coloring of the copies of A in Γ. This defines the 2-coloring $\chi' : \binom{\Delta}{A} \to \{0,1\}$ by $\chi'(X) := \chi(f'(X))$, since the image of f' is contained in Γ and the isomorphism type of A is preserved by f'. According to the Ramsey property of Δ, there is a copy B_0 of B in $\binom{\Delta}{B}$ that is monochromatic with respect to χ'. As B is in the image of the type-collapsing function f', we have that $f' \restriction_{B_0}$ is type-preserving. Hence, $f'(B_0) \in \binom{\Gamma}{B}$, and f' induces a bijection between $\binom{B_0}{A}$ and $\binom{f'(B_0)}{A}$, i.e., all copies of A in $f'(B_0)$ are of the form $f'(X)$ for some $X \in \binom{B_0}{A}$. The fact that χ' is constant on $\binom{B_0}{A}$ then translates to χ being constant on $\binom{f'(B_0)}{A}$. By a standard compactness argument, this conclusion verifies the Ramsey property of Γ. As every type of Γ is inherited from Δ, all of whose substructures are rigid, the same holds for Γ, making it an ordered Ramsey structure according to [18]. ■

According to Item (4) of Theorem 4.1, type-collapsing embeddings define a preorder (a.k.a. quasiorder) on the class of all homogeneous ordered Ramsey structures in a finite relational language: $\Gamma \preceq \Delta$ if there is a type-collapsing injection $f : \Delta \to \Gamma$. The induced partial order is obtained by factoring out with the equivalence relation $\Delta \sim \Gamma$ defined by $\Gamma \preceq \Delta$ and $\Delta \preceq \Gamma$. This equivalence relation is coarser than first-order interdefinability.

Proposition 4.2. *The partial order induced by type-collapsing injections has a least element, namely the class of structures first-order interdefinable with* $(\mathbb{Q}, <)$. *Furthermore, the order ideal generated by any element is finite.*

Proof. Let Δ be a homogeneous ordered Ramsey structure in a finite relational language with maximum arity m. For all $1 \leq n \leq m$, let us color each n-element subset A of Δ by the isomorphism type of the n tuple in which the elements of A are indexed in increasing order with respect to a fixed total order in the language of Δ. Then according to the original

Ramsey theorem [22] (about sets), for every $N \in \mathbb{N}$ there is an N-element set in Δ such that for all $1 \leq n \leq m$ the n-element subsets are monochromatic. By using Kőnig's lemma, there is an infinite subset $S \subseteq \Delta$ with the same property.

Let $f : \Delta \to S$ be a bijection. Then f is a type-collapsing injection, and according to Theorem 4.1, there is a homogeneous ordered Ramsey structure $\Gamma \subseteq \Delta$ containing the image of f, whose age consists of the unique n-element structure for all n that was selected by the monochromatic coloring above. We show that Γ is first-order interdefinable with $(\mathbb{Q}, <)$. Assume that there is an endpoint in Γ, say a biggest element x. Pick another element $y < x$. Then there is an automorphism of Γ mapping y to x. If z is the image of x under this automorphism, then $x < z$, a contradiction. Assume that the order is not dense, that is, there are elements $a < b$ in Γ with no element in between. Let $c < d < e$ be a three element chain in Γ. As there is only one 2-element substructure in the age of the homogeneous Γ, there is an automorphism mapping c to a and e to b. Then the image of d is in between a and b, a contradiction. Hence, the order structure of Γ is isomorphic to $(\mathbb{Q}, <)$. Both structures have the property that there is a unique n-element structure in their age, providing a first-order definition of n-ary relations of one in those of the other.

The second assertion follows trivially from Theorem 4.1: there are only finitely many ways to collapse m-types. ■

Theorem 4.1 and Proposition 4.2 provide us with the opportunity to prove partial results to Thomas' conjecture, and in general about reducts of homogeneous ordered Ramsey structures by an inductive argument. For example, one technical difficulty in proving general results are the existence of type-collapsing canonical function, as it was illustrated by the semilinear order: the existence of the reduct $(\mathbb{S}_2, R)$ together with the canonical flat self embedding preserving R caused a major challenge in the classification. We show that in some sense, we can disregard such reducts completely when we are interested in the Thomas conjecture for ordered Ramsey structures. According to the Bodirsky-Pinsker conjecture, we do not lose anything by restricting our attention to such structures: to this day, there is no known example to a homogeneous structure in a finite relational language that does not have an ordered Ramsey expansion with the same properties. We say that a function $f : \Delta \to \Delta$ has a full image if the age of the image of f coincides with the age of Δ.

Corollary 4.3. *Let Δ be a minimal counterexample to the Thomas conjecture for ordered Ramsey structures in terms of the size of the order ideal with respect to type-collapsing injections. Then* $\mathrm{Aut}(\Delta)$ *has infinitely many closed supergroups in* $\mathrm{Sym}(\Delta)$ *that only generate functions with a full image.*

Proof. If the automorphism group of a reduct Σ generates a function whose image is not full, then the canonization of that function must be a type-collapsing injection f. According to Theorem 4.1, there is a homogeneous ordered Ramsey substructure Θ of Δ containing the image of f with the same age as that of the image of f. Then Θ has a strictly smaller order ideal than Δ, as Θ is strictly below Δ in the partial order. By the minimality of Δ, Θ has finitely many reducts up to first-order interdefinability, that is, $\mathrm{Aut}(\Theta)$ has finitely many closed supergroups in $\mathrm{Sym}(\Theta)$. Let Γ be the reduct of Δ whose relations are the full f pre-images of the definable relations in Θ. Then the conditions of Proposition 3.2 apply to $\Delta = \Delta$, $\Gamma = \Gamma$, $f = g$, $\Delta' = \Theta$. Moreover, f is generated by automorphisms, hence locally invertible in the sense that the restriction of its inverse to any finite set can be interpolated by an automorphism of Σ. Hence, we can avoid the zig-zag argument, and conclude that reducts of Γ correspond to some reducts of Θ. Similarly, each reduct of Σ corresponds to a reduct of Γ. Thus Σ has finitely many reducts up to first-order interdefinability. ■

As we mentioned in the introduction, it is worth studying self-embedding supermonoids of the self-embedding monoid of a structure Δ satisfying the conditions of the Thomas conjecture: there is no known counterexample to the existential variant of that conjecture, and the Ramsey-type methods combined with the techniques presented in this paper work very effectively to self-embedding monoids.

Question 4.4. *Does the zig-zag argument presented in the proof of Lemma 3.6 apply to every reduct Γ of a homogeneous Ramsey structure Δ in a finite relational language that is the image of a type-collapsing injection f (cf. Theorem 4.1)? That is, given self-embedding monoid* $\mathrm{Emb}(\Gamma) \subseteq \mathcal{M}$ *that preserves the pre-image of the basic relations in Γ, is it true that* $\mathrm{Emb}(\Gamma) = \mathcal{M}$*?*

Conjecture 4.5. *Every homogeneous Ramsey structure Δ in a finite relational language has finitely many reducts up to existential interdefinability.*

Proposition 4.6. *Provided a positive answer to Question 4.4, Conjecture 4.5 is equivalent to its first-order variant.*

Proof. According to [5], the self-embedding supermonoids of $\mathrm{Emb}(\mathbb{Q},<)$ are exactly the monoids obtained as the monoid closure of each of the five closed supergroups of $\mathrm{Aut}(\mathbb{Q},<)$. Hence, the assertion holds for the structure $(\mathbb{Q},<)$, which is the least element of the partial order induced by type-collapsing injections according to Proposition 4.2. Let Δ be a minimal counterexample in terms of the size of the order ideal with respect to type-collapsing injections. That is, Δ has finitely many reducts up to first-order interdefinability and infinitely many up to existential interdefinability. There are only finitely many self-embedding supermonoids of $\mathrm{Emb}(\Delta)$ that contain at least one of the essentially finitely many type-collapsing injections f. Indeed, using the positive answer to Question 4.4, these monoids correspond to the self-embedding supermonoids of $\mathrm{Emb}(\Gamma)$ (cf. Theorem 4.1). As Γ is strictly below Δ with respect to the partial order induced by type-collapsing injections, using the assumption on the minimality of Δ and Proposition 4.2, there are finitely many self-embedding supermonoids of $\mathrm{Emb}(\Gamma)$.

Thus all but finitely many self-embedding supermonoids $\mathcal{M}$ of $\mathrm{Emb}(\Delta)$ consist of functions with a full image. In particular, the induced digraph relation on types is symmetrical, i.e., if $\mathcal{M}$ maps a tuple with type a to a tuple with type b, then it also maps a tuple with type b to a tuple with type a. It follows from Theorem 3.4.12 of [2] that such a monoid $\mathcal{M}$ is the monoid closure of its invertible elements, i.e., the monoid closure of one of the finitely many closed supergroups of $\mathrm{Aut}(\Delta)$, a contradiction. ■

We remark that in a personal communication, Michael Pinsker has shown the author a construction of a continuum of closed supermonoids of $\mathrm{Emb}(\mathbb{Q},<,0,1)$ generated by bijections (none of which is a self-embedding monoid). This construction and the fact that the endomorphism monoid of the simplest structure $(\mathbb{N},=)$ has a continuum of closed supermonoids indicate that there is little room for improvement in Conjecture 4.5.

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