Akivis algebra of the tangent prolongation of a differentiable local loop

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Abstract

In a recent paper, we generalized the basic constructions of the tangent prolongation of Lie groups to $\mathcal{C}^r$-differentiable local loops. The prolongation leads to a $\mathcal{C}^{r-1}$-differentiable linear abelian extension of a tangent vector space by the local loop. The purpose of this article is to investigate the tangent algebra of the tangent prolongation of a local loop. The study of tangent algebras of loops was initiated in 1976 by M. A. Akivis. The abstract version is now called Akivis algebra, defined by a skew-symmetric bilinear and a trilinear operation connected only by the Akivis identity, which generalizes the Jacobi identity. Using the power series expansion of the multiplication, we prove that the tangent Akivis algebras of linear abelian extensions are semidirect sums of Akivis algebras. Applying to the tangent prolongation of a $\mathcal{C}^r$-differentiable local loop, $r \geq 4$, we obtain the operations of the tangent Akivis algebra, which are similar in structure to the associative case.

1 Introduction

First, we remember the main constructions and results of the theory of tangent prolongation of a Lie group G with Lie algebra $\mathfrak{g}$. Denote $\lambda_x : G \to G$ and $\rho_x : G \to G$ the left, respectively, right multiplication of G with identity element $e \in G$. The tangent bundle $\mathrm{T}(G)$ can be identified with the product $G \times \mathrm{T}_e(G)$ by the map $(x, \xi) \mapsto (x, \mathrm{d}_x\lambda_x^{-1}\xi)$ for $\xi \in \mathrm{T}_e(G)$. The manifold $G \times \mathrm{T}_e(G)$ has a natural Lie group structure, called *tangent prolongation of* G, determined by the multiplication

$$(x, X) \cdot (y, Y) = \big(xy, \mathrm{d}_{xy}\lambda_{xy}^{-1}\frac{d}{dt}\big|_{t=0}(x \exp tX \cdot y \exp tY)\big) = \big(xy, \mathrm{Ad}_y^{-1}X + Y\big), \qquad (1)$$

where $x, y \in G$, $X, Y \in \mathrm{T}_e(G)$ and $\mathrm{Ad}_g = \mathrm{d}_e(\lambda_g\rho_g^{-1}) : \mathrm{T}_e(G) \to \mathrm{T}_e(G)$, $g \in G$, is the adjoint action of G on $\mathrm{T}_e(G)$. This means that the tangent prolongation is a semidirect product

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$G \ltimes \mathrm{T}_e(G)$ determined by the adjoint representation. The Lie algebra of $G \ltimes \mathrm{T}_e(G)$ is the semidirect sum $\mathfrak{g} \oplus_\alpha \mathfrak{a}$ of $\mathfrak{g}$ with the abelian Lie algebra $\mathfrak{a}$ on $\mathrm{T}_e(G)$, which is determined by the homomorphism $\mathfrak{g} \to \mathrm{End}(\mathfrak{a})$ given by $\alpha : \xi \mapsto \theta^{-1} \cdot \mathrm{ad}_\xi \cdot \theta$, where $\theta : \mathfrak{a} \to \mathfrak{g}$ is the identity map of underlying vector space. The Lie bracket of $\mathfrak{g} \oplus_\alpha \mathfrak{a}$ is given by

$$[(\xi, X), (\eta, Y)] = ([\xi, \eta], \alpha_\xi Y - \alpha_\eta X) = ([\xi, \eta], \theta^{-1}([\xi, \theta(Y)] + [\theta(X), \eta])), \qquad (2)$$

cf. e.g. [16], §V.1. and [15], §3.15.

The purpose of this article is to generalize the basic constructions of tangent prolongation of Lie groups to differentiable non-associative systems and to examine the resulting structures. There are two main differences between Lie theory of groups and loops. As long as a differentiable group multiplication is necessarily analytic, loop multiplications can satisfy various differentiability conditions. Moreover, the tangent Lie algebra of a Lie group determines the multiplication up to local isomorphism, but differentiable local loops corresponding to the same Akivis algebra are not necessarily locally isomorphic. These claims justify the interest in studying the construction of tangent Akivis algebras in the context of local loops.

The development of Lie's theory of differentiable local loops and their tangent algebras was initiated by M. A. Akivis (see [1], [2], [3]). The tangent space of a local loop at the identity element has a natural algebra structure with a skew-symmetric bilinear commutator and a trilinear associator operation, connected only by a particular relationship known as the Akivis identity that generalizes the Jacobi identity. The abstract version of this algebra was later called the Akivis algebra, a theory of which has been developed by many authors in the last few decades (see e.g. [7], [8], [11], [12], [13]).

The tangent prolongation of a $\mathcal{C}^r$-differentiable local loop L can be obtained by generalization of the multiplication (1) to non-associative system, which is studied in our paper [6]. It became clear that the tangent prolongation of a $\mathcal{C}^r$-differentiable loop L is a $\mathcal{C}^{r-1}$-differentiable loop which is a linear abelian extension (cf. [5]) of the tangent space $\mathrm{T}_e(L)$ at the identity element e by L given by suitable $\mathcal{C}^{r-1}$-differentiable loop cocycle (P, Q). This loop extension has the same classical weak inverse and weak associative properties as the initial loop. We will now determine Akivis algebras of linear abelian extensions, in particular of the tangent prolongation of $\mathcal{C}^r$-differentiable local loops with $r \geq 4$.

In §2 we introduce the basic concepts and methods of our research, in particular the tools of Taylor expansions of differentiable local loop operations. In §3, after defining abstract Akivis algebras and tangent Akivis algebras of differentiable local loops, we define and examine the class of semidirect sums of Akivis algebras, which are called linear semidirect sums. These extensions of Akivis algebras are determined by one bilinear and three trilinear maps from the first Akivis algebra to the endomorphism algebra of the second Akivis algebra, which fulfill the so-called generalized Akivis identity. §4 is devoted to the computation of the operations of the tangent Akivis algebra of linear abelian extensions of L. Using power series expansion of the cocycle maps P, Q we prove that this tangent Akivis algebra is a special semidirect sum of Akivis algebras called a linear semidirect sum. In §5 we apply the results on tangent Akivis algebras of linear abelian extensions to the case of tangent prolongation of differentiable loops. We obtain a remarkable form of the commutator and the associator of the tangent Akivis algebra and prove that both operations of the tangent Akivis algebra are given by analogous formulas to the expression (2) of the commutator of the Lie algebra of the tangent prolongation of a Lie group.

2 Preliminaries

A *loop* is a set L with three binary operations $\cdot, \backslash, / : L \times L \to L$ in which the identities

$$(x/y) \cdot y = x,\ y\backslash(y \cdot x) = x,\ (x \cdot y)/y = x,\ y \cdot (y\backslash x) = x, \quad x, y \in L \tag{3}$$

are fulfilled and there is an identity element $e \in L$ satisfying

$$e \cdot x = x \cdot e = x/e = e\backslash x = x \quad \text{for all} \quad x \in L \tag{4}$$

Left translations $\lambda_x : L \to L$, $\lambda_x y = x \cdot y$, and right translations $\rho_x : L \to L$, $\rho_x y = y \cdot x$, of the *multiplication operation* $x \cdot y$ are bijective maps and the *left and right division operations* of L satisfy $x\backslash y = \lambda_x^{-1} y$, respectively $x/y = \rho_y^{-1} x$.

$\mathcal{C}^r$-differentiable local loops

For a differentiable map $\varphi : M \to N$ between differentiable manifolds M and N we denote by $\mathrm{d}_x\varphi : \mathrm{T}_x(M) \to \mathrm{T}_{\varphi(x)}(N)$ the linear differential map between the tangent spaces at a point $x \in M$. Let V^n be a real vector space of dimension n and F a k-variable differentiable map defined in a neighbourhood of $(0, \dots, 0) \in V^n \times \dots \times V^n$, then $F'_i(u)$, $i = 1, \dots, k$, will denote the linear differential map of F at the point $(0, \dots, 0)$ with respect to the i-th vector variable, applied to the vector $u \in V^n$. Similarly, $F''_{ij}(u, v)$ denotes the bilinear second, respectively, $F'''_{ijk}(u, v, w)$ the trilinear third differential map at $(0, \dots, 0)$ with respect to the i-th and j-th, respectively, the i-th, j-th and k-th vector variables, applied to the adequate number of vectors $u, v \in V^n$ or $u, v, w \in V^n$.
An n-dimensional $\mathcal{C}^r$-differentiable manifold L equipped with a $\mathcal{C}^r$-differentiable partial operation $(x, y) \mapsto x \cdot y$ (called partial multiplication) that is defined in an open domain $(e, e) \in U \subset L \times L$ and satisfies $e \cdot x = x \cdot e = x$ for all $x \in L$ with a fixed $e \in L$ is called $\mathcal{C}^r$*-differentiable local H-space* with identity element $e \in L$.
If two more $\mathcal{C}^r$-differentiable partial operations $\backslash, / : U \to L$ (called left and right partial divisions) are defined in a $\mathcal{C}^r$-differentiable local H-space L, and $\cdot, \backslash, / : U \to L$ satify (3) and (4), if the terms connected by equal sign have meaning, then L is a $\mathcal{C}^r$*-differentiable local loop* with identity element $e \in L$.
Let L be a $\mathcal{C}^r$-differentiable local H-space ($r \geq 4$) covered by a coordinate neighbourhood. We identify L with its coordinate chart in the euclidean vector space $(V^n, \langle ., . \rangle)$ and the identity element $e \in L$ with the zero element $0 \in V^n$. The coordinate function of the local multiplication has the Taylor expansion

$$x \cdot y = x + y + q(x, y) + r(x, x, y) + s(x, y, y) + M(x, y), \tag{5}$$

in a neighbourhood of $(0, 0) \in V^n \times V^n$ with an error term $M(x, y)$ satisfying

$$\lim_{x,y \to 0} \frac{M(x, y)}{(|x| + |y|)^3} = 0.$$

The bilinear and trilinear monomials in (5) are expressed by

$$q = (x \cdot y)''_{xy}(0, 0), \quad r = \frac{1}{2}(x \cdot y)'''_{xxy}(0, 0), \quad s = \frac{1}{2}(x \cdot y)'''_{xyy}(0, 0), \tag{6}$$

on the vector space V^n, (e.g. Corollary 4.4. in [10]), hence r and s are symmetric in the first, respectively, in the last two variables.

Remark 2.1. We notice that $q(x,y)$ is skew-symmetric in canonical coordinate systems (cf. [9] and [4]), having the same differentiability property as the local multiplication. This property of the bilinear form q can also be provided by a locally invertible coordinate change $\phi(x) = x - \frac{1}{2}q(x,x)$ in a neighbourhood of $0 \in V^n$. Indeed, denoting the multiplication by $\phi(x) \star \phi(y)$ with respect to the coordinates $\phi(x) \in V^n$ we have

$$\phi(x) \star \phi(y) = \phi(x \cdot y) = x - \frac{1}{2}q(x,x) + y - \frac{1}{2}q(y,y) + q(x,y) - \frac{1}{2}q(x,y) - \frac{1}{2}q(y,x) + \cdots .$$

The inverse of the map $\phi(x) = x - \frac{1}{2}q(x,x)$ is of the form $\phi^{-1}(x) = x + \frac{1}{2}q(x,x) + o(2)$, hence with $\tilde{x} = \phi(x)$ and $\tilde{y} = \phi(y)$ we get the expansion

$$\tilde{x} \star \tilde{y} = \tilde{x} + \tilde{y} + \frac{1}{2}\left(q(\tilde{x},\tilde{y}) - q(\tilde{y},\tilde{x})\right) + o(2).$$

In the following we will assume that the bilinear map $q : V^n \times V^n \to V^n$ in (5) is skew-symmetric.

According to the implicit mapping theorem the partial left and right division operations are implicitly determined by the equation $x \cdot y - z = 0$ in a neighbourhood of $(0,0,0)$ in $V^n \times V^n \times V^n \to V^n$ and have the same differentiability properties as the multiplication $(x,y) \mapsto x \cdot y$, since the tangent maps $(x \cdot y - z)'_y(0,0,0)$ and $(x \cdot y - z)'_x(0,0,0)$ are invertible (see e.g. Theorem 5.9. in [10]). It follows

Proposition 2.2. Any $\mathcal{C}^r$-differentiable local H-space L is a $\mathcal{C}^r$-differentiable local loop on a neighbourhood of the identity element $e \in L$ (cf. [4] (1.3) Proposition).

An immediate computation shows that the Taylor expansions of the coordinate functions of the left and right divisions are of the form

$$\begin{aligned} y/x =& y - x - q(y-x,x) + q(q(y-x,x),x) - r(y-x,y-x,x) - s(y-x,x,x) + o(3), \\ x\backslash y =& y - x - q(x,y-x) + q(x,q(x,y-x)) - r(x,x,y-x) - s(x,y-x,y-x) + o(3), \end{aligned} \tag{7}$$

where $o(3)$ is an error term up to order 3.

Definition 2.1. Let L be a $\mathcal{C}^r$-differentiable local loop and $\alpha(t)$, $\beta(t)$, $\gamma(t)$ differentiable curves in L with initial data

$$\alpha(0) = \beta(0) = \gamma(0) = e,\ \alpha'(0) = X,\ \beta'(0) = Y,\ \gamma'(0) = Z, \quad X,Y,Z \in \mathrm{T}_e(L).$$

The bilinear *tangent commutator* $(X,Y) \mapsto [X,Y]$ of the local loop L on the tangent space $\mathrm{T}_e(L)$ is defined by

$$[X,Y] = \frac{1}{2}\frac{d^2t}{d\,t^2}\Big|_{t=0}(\alpha(t)\cdot\beta(t))/(\beta(t)\cdot\alpha(t)) = \frac{1}{2}\frac{d^2t}{d\,t^2}\Big|_{t=0}(\beta(t)\cdot\alpha(t))\backslash(\alpha(t)\cdot\beta(t)). \tag{8}$$

The trilinear *tangent associator* $(X,Y,Z) \mapsto \langle X,Y,Z\rangle$ of L on the tangent space $\mathrm{T}_e(L)$ is defined by

$$\begin{aligned} \langle X,Y,Z\rangle =& \frac{1}{6}\frac{d^3t}{d\,t^3}\Big|_{t=0}\left((\alpha(t)\cdot\beta(t))\cdot\gamma(t)\right)/\left(\alpha(t)\cdot(\beta(t)\cdot\gamma(t))\right) = \\ =& \frac{1}{6}\frac{d^3t}{d\,t^3}\Big|_{t=0}\left(\alpha(t)\cdot(\beta(t)\cdot\gamma(t))\right)\backslash\left((\alpha(t)\cdot\beta(t))\cdot\gamma(t)\right). \end{aligned} \tag{9}$$

3 Akivis algebras and their semidirect sum

Definition 3.1. An *Akivis algebra* $\mathcal{A}^n = (V^n, [.,.], \langle .,.,. \rangle)$ is a vector space V^n equipped with a skew-symmetric bilinear and a trilinear operation:

$$(X,Y) \mapsto [X,Y], \quad (X,Y,Z) \mapsto \langle X,Y,Z\rangle,$$

fulfilling the so-called *Akivis identity*:

$$\begin{aligned}\langle X,Y,Z\rangle - \langle Y,X,Z\rangle + \langle Y,Z,X\rangle - \langle Z,Y,X\rangle + \langle Z,X,Y\rangle - \langle X,Z,Y\rangle = \\ =[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y].\end{aligned} \tag{10}$$

An Akivis algebra $\mathcal{A}^n$ is *abelian* if $[X,Y]=0$ and $\langle X,Y,Z\rangle = 0$ for any $X,Y,Z \in V^n$.
A *derivation* of $\mathcal{A}^n$ is a map $\partial \in \mathrm{End}(V^n)$ satisfying the identity:

$$\partial[X,Y] = [\partial X, Y] + [X, \partial Y], \qquad X,Y \in V^n.$$

Let L be a local loop identified with a neighbourhood W^n of 0 in the vector space V^n, the identity element of L with $0 \in V^n$ and the tangent space T_eL with V^n. The bilinear and trilinear maps $q : \mathrm{T}_eL \times \mathrm{T}_eL \to \mathrm{T}_eL$ and $r,s : \mathrm{T}_eL \times \mathrm{T}_eL \times \mathrm{T}_eL \to \mathrm{T}_eL$ are well defined by (6), using this identification. According to 2.1.Lemma and 2.2.Lemma in [7] (or IX.6.6. Theorem in [8]) the commutator (8) and the associator (9) of the local multiplication (5) are expressed by the first non-vanishing term of the Taylor series of

$$(x\cdot y)/(y\cdot x) \text{ or } (y\cdot x)\backslash(x\cdot y) \text{ and } \big((x\cdot y)\cdot z\big)/\big(x\cdot(y\cdot z)\big) \text{ or } \big(x\cdot(y\cdot z)\big)\backslash\big((x\cdot y)\cdot z\big),$$

respectively. It follows for any $X,Y,Z \in \mathrm{T}_e(L)$

$$\begin{aligned}[X,Y] &= 2q(X,Y),\\ \langle X,Y,Z\rangle &= q(q(X,Y),Z) - q(X,q(Y,Z)) + 2r(X,Y,Z) - 2s(X,Y,Z).\end{aligned} \tag{11}$$

Moreover, these operations $(X,Y) \mapsto [X,Y]$ and $(X,Y,Z) \mapsto \langle X,Y,Z\rangle$ determine an Akivis algebra on the tangent space $\mathrm{T}_e(L)$.

Definition 3.2. The *tangent Akivis algebra* $\mathcal{A}(L)$ of a $\mathcal{C}^r$-differentiable local loop L is the tangent space T_eL equipped with the tangent commutator (8) and tangent associator (9) operations expressed by (11).

Let $\mathcal{A}^n = (V^n, [.,.]_A, \langle .,.,.\rangle_A)$ and $\mathcal{B}^k = (U^k, [.,.]_B, \langle .,.,.\rangle_B)$ be Akivis algebras defined on the vector spaces V^n and U^k, respectively. Let $\alpha : V^n \to \mathrm{End}(U^k)$ be a linear map and $\lambda, \mu, \nu : V^n \times V^n \to \mathrm{End}(U^k)$ bilinear maps. We define the operations

$$\begin{aligned}[(\xi,X),(\eta,Y)] &= ([\xi,\eta]_A, [X,Y]_B + \alpha_\xi Y - \alpha_\eta X),\\ \langle(\xi,X),(\eta,Y),(\zeta,Z)\rangle &= \big(\langle\xi,\eta,\zeta\rangle_A, \langle X,Y,Z\rangle_B + \lambda_{(\eta,\zeta)}X + \mu_{(\zeta,\xi)}Y + \nu_{(\xi,\eta)}Z\big)\end{aligned} \tag{12}$$

$(\xi,X),(\eta,Y),(\zeta,Z) \in V^n \oplus U^k$ on the direct sum $V^n \oplus U^k$.

Theorem 3.1. The operations (12) on $V^n \oplus U^k$ determined by the linear, respectively, bilinear maps

$$\alpha : V^n \to \mathrm{End}(U^k), \quad \lambda, \mu, \nu : V^n \times V^n \to \mathrm{End}(U^k)$$

satisfy the Akivis identity if and only if

(i) the maps $\alpha_\xi \in \mathrm{End}(U^k)$, $\xi \in V^n$, are derivations of $\mathcal{B}^k$,

(ii) the *general Akivis identity*

$$\lambda_{(\xi,\eta)} - \lambda_{(\eta,\xi)} + \mu_{(\xi,\eta)} - \mu_{(\eta,\xi)} + \nu_{(\xi,\eta)} - \nu_{(\eta,\xi)} = \alpha_{[\xi,\eta]_A} - \alpha_\xi \alpha_\eta + \alpha_\eta \alpha_\xi \tag{13}$$

is satisfied for any $\xi, \eta \in V^n$.

Proof. The left and right hand sides of the Akivis identity for the operations $[(\xi, X), (\eta, Y)]$ and $\langle(\xi, X), (\eta, Y), (\zeta, Z)\rangle$ defined by (12) and (10) give the expressions

$$\begin{aligned}
(\langle\xi,\eta,\zeta\rangle_A + \langle\eta,\zeta,\xi\rangle_A + \langle\zeta,\xi,\eta\rangle_A - \langle\eta,\xi,\zeta\rangle_A - \langle\zeta,\eta,\xi\rangle_A - \langle\xi,\zeta,\eta\rangle_A,\\
\langle X,Y,Z\rangle_B + \langle Y,Z,X\rangle_B + \langle Z,X,Y\rangle_B - \langle Y,X,Z\rangle_B - \langle Z,Y,X\rangle_B - \langle X,Z,Y\rangle_B +\\
+ (\lambda_{(\eta,\zeta)} + \mu_{(\eta,\zeta)} + \nu_{(\eta,\zeta)} - \lambda_{(\zeta,\eta)} - \mu_{(\zeta,\eta)} - \nu_{(\zeta,\eta)})X +\\
+ (\lambda_{(\zeta,\xi)} + \mu_{(\zeta,\xi)} + \nu_{(\zeta,\xi)} - \lambda_{(\xi,\zeta)} - \mu_{(\xi,\zeta)} - \nu_{(\xi,\zeta)})Y +\\
+ (\lambda_{(\xi,\eta)} + \mu_{(\xi,\eta)} + \nu_{(\xi,\eta)} - \lambda_{(\eta,\xi)} - \mu_{(\eta,\xi)} - \nu_{(\eta,\xi)})Z),
\end{aligned}$$

and

$$\begin{aligned}
&([[\xi,\eta]_A,\zeta]_A + [[\zeta,\xi]_A,\eta]_A + [[\eta,\zeta]_A,\xi]_A,\\
&[[X,Y]_B,Z]_B + [[Z,X]_B,Y]_B + [[Y,Z]_B,X]_B + [\alpha_\eta Z,X]_B + [Z,\alpha_\eta X]_B - \alpha_\eta[Z,X]_B +\\
&+ [\alpha_\xi Y,Z]_B + [Y,\alpha_\xi Z]_B - \alpha_\xi[Y,Z]_B + [\alpha_\zeta X,Y]_B + [X,\alpha_\zeta Y]_B - \alpha_\zeta[X,Y]_B +\\
&+ (\alpha_{[\eta,\zeta]_A} - \alpha_\eta\alpha_\zeta + \alpha_\zeta\alpha_\eta)X + (\alpha_{[\zeta,\xi]_A} - \alpha_\zeta\alpha_\xi + \alpha_\xi\alpha_\zeta)Y + (\alpha_{[\xi,\eta]_A} - \alpha_\xi\alpha_\eta + \alpha_\eta\alpha_\xi)Z.
\end{aligned}$$

The operations (12) of Akivis algebras $\mathcal{A}^n$ and $\mathcal{B}^k$ satisfy the Akivis identity precisely if

$$\begin{aligned}
[\alpha_\eta Z,X]_B + [Z,\alpha_\eta X]_B - \alpha_\eta[Z,X]_B + (\alpha_{[\zeta,\xi]_A} - \alpha_\zeta\alpha_\xi + \alpha_\xi\alpha_\zeta)Y +\\
+[\alpha_\xi Y,Z]_B + [Y,\alpha_\xi Z]_B - \alpha_\xi[Y,Z]_B + (\alpha_{[\eta,\zeta]_A} - \alpha_\eta\alpha_\zeta + \alpha_\zeta\alpha_\eta)X +\\
+[\alpha_\zeta X,Y]_B + [X,\alpha_\zeta Y]_B - \alpha_\zeta[X,Y]_B + (\alpha_{[\xi,\eta]_A} - \alpha_\xi\alpha_\eta + \alpha_\eta\alpha_\xi)Z =\\
=(\lambda_{(\eta,\zeta)} - \lambda_{(\zeta,\eta)} + \mu_{(\eta,\zeta)} - \mu_{(\zeta,\eta)} + \nu_{(\eta,\zeta)} - \nu_{(\zeta,\eta)})X +\\
+(\mu_{(\zeta,\xi)} - \mu_{(\xi,\zeta)} + \nu_{(\zeta,\xi)} - \nu_{(\xi,\zeta)} + \lambda_{(\zeta,\xi)} - \lambda_{(\xi,\zeta)})Y +\\
+(\nu_{(\xi,\eta)} - \nu_{(\eta,\xi)} + \lambda_{(\xi,\eta)} - \lambda_{(\eta,\xi)} + \mu_{(\xi,\eta)} - \mu_{(\eta,\xi)})Z
\end{aligned} \tag{14}$$

for all ξ, η, $\zeta \in V^n$ and $X, Y, Z \in U^k$. Putting $\eta = \zeta = 0$ we get

$$[\alpha_\xi Y, Z]_B + [Y, \alpha_\xi Z]_B - \alpha_\xi[Y, Z]_B = 0,$$

for all $\xi \in V^n$, which means that $\alpha_\xi \in \mathrm{End}(U^k)$ is a derivation of the Akivis algebra $\mathcal{B}^k$, giving the condition (i). Hence putting $X = Y = 0$ into (14) we obtain that the identity (14) is equivalent to the condition (ii). Hence the assertion is proved. □

Definition 3.3. If the linear $\alpha : V^n \to \mathrm{End}(U^k)$ and bilinear $\lambda, \mu, \nu : V^n \times V^n \to \mathrm{End}(U^k)$ maps satisfy the conditions (i) and (ii) of Theorem 3.1, then the operations (12) on $V^n \oplus U^k$ define an Akivis algebra $\mathcal{A}^n \rtimes \mathcal{B}^k$, called the *linear semidirect sum* of Akivis algebras $\mathcal{A}^n$ and $\mathcal{B}^k$, determined by $\alpha : V^n \to \mathrm{End}(U^k)$ and $\lambda, \mu, \nu : V^n \times V^n \to \mathrm{End}(U^k)$.

Note that the linear semidirect sum $\mathcal{A}^n \rtimes \mathcal{B}^k$ defined above belongs to a special class of semidirect sums of Akivis algebras, which are characterized by the condition that the values of the bilinear and trilinear operations (12) determined by the mappings $\alpha, \lambda, \mu, \nu$ are linear maps on U^k, i.e. contained in $\mathrm{End}(U^k) \cong U^{k^*} \otimes U^k$. More generally, the trilinear operation could also contain components that are linear maps of the form $V^n \to U^{k^*} \otimes U^{k^*} \otimes U^k$.

Remark 3.2. Let $\mathcal{A}^n = (V^n, [.,.], \langle .,.,. \rangle)$ be an Akivis algebra. In the case $\alpha_X Y = [X, Y]$, $\lambda_{(X,Y)} Z = \langle Z, X, Y \rangle$, $\mu_{(X,Y)} Z = \langle Y, Z, X \rangle$, $\nu_{(X,Y)} Z = \langle X, Y, Z \rangle$ the general Akivis identity (13) applied to $U^k = V^n$

$$\lambda_{(X,Y)} - \lambda_{(Y,X)} + \mu_{(X,Y)} - \mu_{(Y,X)} + \nu_{(X,Y)} - \nu_{(Y,X)} = \alpha_{[X,Y]} - \alpha_X \alpha_Y + \alpha_Y \alpha_X. \tag{15}$$

is reduced to the classical Akivis identity (10).

We now formulate a special construction of linear semidirect sum of Akivis algebras, which will be useful in the investigation of tangent Akivis algebras of tangent prolongations.

Proposition 3.3. Let $\mathcal{A}^n = (V^n, [.,.]_A, \langle .,.,. \rangle_A)$, $\mathcal{A}^{*n} = (V^n, [.,.]_A, \langle .,.,. \rangle^*)$ be Akivis algebras, $(V^n)^+$ the abelian Akivis algebra on the vector space V^n and $\theta : \{0\} \oplus V^n \to V^n \oplus \{0\}$ a bijective linear map. The maps

$$\alpha_\xi Z = \theta^{-1}[\xi, \theta Z]_A,\ \lambda_{(\xi,\eta)} Z = \theta^{-1}\langle \theta Z, \xi, \eta \rangle^*,\ \mu_{(\xi,\eta)} Z = \theta^{-1}\langle \eta, \theta Z, \xi \rangle^*,\ \nu_{(\xi,\eta)} Z = \theta^{-1}\langle \xi, \eta, \theta Z \rangle^*$$

determine a linear semidirect sum $\mathcal{A}^n \rtimes (V^n)^+$ of Akivis algebras.

Proof. The condition (i) of Theorem 3.1 is satisfied since $(V^n)^+$ is abelian. The identity (13) can be obtained by conjugation with the map θ of the Akivis identity $\mathcal{A}^{*n}$. For example:

$$\alpha_\xi \alpha_\eta Z = \theta^{-1}[\xi, \theta \cdot \theta^{-1}[\eta, \theta Z]_A]_A = \theta^{-1}[\xi, [\eta, \theta Z]_A]_A, \quad \lambda_{(\xi,\eta)} Z = \theta^{-1}\langle \theta Z, \xi, \eta \rangle^*.$$

Likewise, we can get the same conjugation relationship for the other terms of the Akivis identity. □

4 Akivis algebra of linear abelian extensions

Let $L = (L, \cdot, /, \backslash)$ be a $\mathcal{C}^r$-differentiable local loop with $r \geq 4$, U^k a vector space and $\mathrm{GL}(U^k)$ the general linear group of U^k. For any given pair $P, Q : L \times L \to \mathrm{GL}(U^k)$ of $\mathcal{C}^q$-differentiable maps $(3 \leq q \leq r)$ satisfying $P(a, e) = \mathrm{Id} = Q(e, b)$ we construct a $\mathcal{C}^q$-differentiable local loop with identity $(e, 0)$ on the product manifold $L \times U^k$, (this construction is investigated in [5]).

Definition 4.1. A $\mathcal{C}^q$-differentiable *loop cocycle* on the product manifold $L \times U^k$ is a pair of $\mathcal{C}^q$-differentiable maps:

$$P, Q : L \times L \to \mathrm{GL}(U^k) \quad \text{with} \quad P(x, e) = \mathrm{Id} = Q(e, y) \quad \text{for all} \quad x, y \in L. \tag{16}$$

The *linear abelian extension* $\mathcal{F}(P,Q)$ is the $\mathcal{C}^q$-differentiable local loop on $L\times U^k$ defined by the $\mathcal{C}^q$-differentiable operations

$$\begin{aligned}(x,X)\cdot(y,Y)&=(xy,P(x,y)X+Q(x,y)Y),\\ (y,Y)/(x,X)&=\big(y/x,P(y/x,x)^{-1}(Y-Q(y/x,x)X)\big),\\ (x,X)\backslash(y,Y)&=\big(x\backslash y,Q(x,x\backslash y)^{-1}(Y-P(x,x\backslash y)X)\big)\end{aligned}$$

and identity element $(e,0)$.

Assume that L is covered by a coordinate neighbourhood and hence L can be identified with a coordinate chart $W^n\subset V^n$ containing $0\in V^n$ such that $e\in L$ corresponds to $0\in W^n$. We investigate the linear abelian extension $\mathcal{F}(P,Q)$ determined by the cocycle (16). The power series expansion of the maps $P,Q:W^n\times W^n\to \mathrm{GL}(U^k)$ in a neighbourhood of $(0,0)$ has the form

$$\begin{aligned}P(x,y)&=\mathrm{Id}+P_2'(y)+\ P_{12}''(x,y)+\frac{1}{2}P_{22}''(y,y)+\ o(2),\\ Q(x,y)&=\mathrm{Id}+Q_1'(x)+\frac{1}{2}Q_{11}''(x,x)+Q_{12}''(x,y)+o(2),\end{aligned}\tag{17}$$

since $P(x,0)=\mathrm{Id}=Q(0,y)$, $x,y\in W^n$, where $P_{22}''(x,y)$ and $Q_{11}''(x,y)$ are symmetric bilinear forms. Consequently we have the expansion at $(0,0)$ with respect to the variables $(x,X),(y,Y)\in W^n\times U^k$:

$$\begin{aligned}P(x,y)X+Q(x,y)Y=X+Y+P_2'(y)X+Q_1'(x)Y+\\ +P_{12}''(x,y)X+\frac{1}{2}P_{22}''(y,y)X+\frac{1}{2}Q_{11}''(x,x)Y+Q_{12}''(x,y)Y+o(3).\end{aligned}\tag{18}$$

The trilinear maps

$$\begin{aligned}((x,X),(y,Y),(z,Z))&\mapsto P_{12}''(y,z)X+P_{12}''(x,z)Y+Q_{11}''(x,y)Z,\\ ((x,X),(y,Y),(z,Z))&\mapsto P_{22}''(y,z)X+Q_{12}''(x,z)Y+Q_{12}''(x,y)Z\end{aligned}$$

are symmetric in the first, respectively, last two variables. Introducing the notations

$$\begin{aligned}\mathcal{Q}\left((x,X),(y,Y)\right)&=\big(q(x,y),P_2'(y)X+Q_1'(x)Y\big),\\ \mathcal{R}\left((x,X),(y,Y),(z,Z)\right)&=\left(r(x,y,z),\frac{1}{2}\left(P_{12}''(y,z)X+P_{12}''(x,z)Y+Q_{11}''(x,y)Z\right)\right),\\ \mathcal{S}\left((x,X),(y,Y),(z,Z)\right)&=\left(s(x,y,z),\frac{1}{2}\left(P_{22}''(y,z)X+Q_{12}''(x,z)Y+Q_{12}''(x,y)Z\right)\right),\end{aligned}\tag{19}$$

we obtain the expansion

$$\begin{aligned}(x,X)\cdot(y,Y)=(x,X)+(y,Y)+\mathcal{Q}\left((x,X),(y,Y)\right)+\mathcal{R}\left((x,X),(x,X),(y,Y)\right)+\\ +\mathcal{S}\left((x,X),(y,Y),(y,Y)\right)+o(3),\quad (x,X),(y,Y)\in W^n\times U^k\end{aligned}\tag{20}$$

of the multiplication of $\mathcal{F}(P,Q)$. Now we can compute the commutator:

$$\begin{aligned}[(x,X),(y,Y)]=\mathcal{Q}\left((x,X),(y,Y)\right)-\mathcal{Q}\left((y,Y),(x,X)\right)=\\ =([x,y],\left(P_2'(y)-Q_1'(y)\right)X+\left(Q_1'(x)-P_2'(x)\right)Y).\end{aligned}\tag{21}$$

and the associator:

$$\begin{aligned}\langle(x,X),(y,Y),&(z,Z)\rangle=\mathcal{Q}(\mathcal{Q}((x,X),(y,Y)),(z,Z))-\mathcal{Q}((x,X),\mathcal{Q}((y,Y),(z,Z)))+\\&+2\mathcal{R}((x,X),(y,Y),(z,Z))-2\mathcal{S}((x,X),(y,Y),(z,Z))=\\&=\big(q(q(x,y),z),P_2'(z)\left(P_2'(y)X+Q_1'(x)Y\right)+Q_1'(q(x,y))Z\big)-\\&-\big(q(x,q(y,z)),P_2'(q(y,z))X+Q_1'(x)\left(P_2'(z)Y+Q_1'(y)Z\right)\big)+\\&+\big(2r(x,y,z),P_{12}''(y,z)X+P_{12}''(x,z)Y+Q_{11}''(x,y)Z\big)-\\&-\big(2s(x,y,z),P_{22}''(y,z)X+Q_{12}''(x,z)Y+Q_{12}''(x,y)Z\big)=\\&=\big(\langle x,y,z\rangle,\left(P_2'(z)P_2'(y)-P_2'(q(y,z))+P_{12}''(y,z)-P_{22}''(y,z)\right)X+\\&+\left(P_2'(z)Q_1'(x)-Q_1'(x)P_2'(z)+P_{12}''(x,z)-Q_{12}''(x,z)\right)Y+\\&+\left(Q_1'(q(x,y))-Q_1'(x)Q_1'(y)+Q_{11}''(x,y)-Q_{12}''(x,y)\right)Z\big)\end{aligned} \tag{22}$$

of the tangent Akivis algebra of the linear abelian extension $\mathcal{F}(P,Q)$. Substituting arbitrary tangent vectors into the variables x,y,z in the equations (21) and (22), we obtain

Theorem 4.1. The tangent Akivis algebra $\mathcal{A}(\mathcal{F}(P,Q))$ of the linear abelian extension $\mathcal{F}(P,Q)$ is a linear semidirect sum $\mathcal{A}(L)\rtimes(U^k)^+$ of the tangent Akivis algebra $\mathcal{A}(L)$ of L and the abelian Akivis algebra $(U^k)^+$ on the vector space U^k. The linear, respectively, bilinear maps

$$\alpha:\mathrm{T}_e(L)\to\mathrm{End}(U^k),\quad \lambda,\mu,\nu:\mathrm{T}_e(L)\times\mathrm{T}_e(L)\to\mathrm{End}(U^k)$$

that determine the linear semidirect sum $\mathcal{A}(L)\rtimes(U^k)^+$ are expressed by

$$\begin{aligned}\alpha_\xi&=Q_1'(\xi)-P_2'(\xi),\\ \lambda_{(\xi,\eta)}&=P_2'(\eta)P_2'(\xi)-P_2'(q(\xi,\eta))+P_{12}''(\xi,\eta)-P_{22}''(\xi,\eta),\\ \mu_{(\xi,\eta)}&=P_2'(\xi)Q_1'(\eta)-Q_1'(\eta)P_2'(\xi)+P_{12}''(\eta,\xi)-Q_{12}''(\eta,\xi),\\ \nu_{(\xi,\eta)}&=Q_1'(q(\xi,\eta))-Q_1'(\xi)Q_1'(\eta)+Q_{11}''(\xi,\eta)-Q_{12}''(\xi,\eta)\end{aligned}$$

for any $\xi,\eta\in\mathrm{T}_e(L)$.

5 Tangent Akivis algebra of the tangent prolongation

Definition 5.1. Let L be a $\mathcal{C}^r$-differentiable local loop and $\alpha(t)$, $\beta(t)$ differentiable curves in L with initial data $\alpha(0)=\beta(0)=e$, $\alpha'(0)=X$, $\beta'(0)=Y$, where $X,Y\in\mathrm{T}_e(L)$. The *tangent prolongation* $\mathscr{T}(L\times\mathrm{T}_e(L))$ of L is the manifold $L\times\mathrm{T}_e(L)$ equipped with the multiplication

$$(x,X)\cdot(y,Y)=\left(xy,\mathrm{d}_{xy}\lambda_{xy}^{-1}\frac{d}{dt}\Big|_{t=0}(x\alpha(t)\cdot y\beta(t))\right)=(xy,\mathrm{d}_e\lambda_{xy}^{-1}\rho_y\lambda_xX+\mathrm{d}_e\lambda_{xy}^{-1}\lambda_x\lambda_yY),$$

for all $(x,X),(y,Y)\in L\times\mathrm{T}_e(L)$.

We can see immediately, (cf. [6], Lemma 4.1.):

Lemma 5.1. The tangent prolongation $\mathscr{T}(L \times \mathrm{T}_e(L))$ of a $\mathcal{C}^r$-differentiable local loop L is a $\mathcal{C}^{r-1}$-differentiable linear abelian extension $\mathcal{F}(P, Q)$ of L determined by the $\mathcal{C}^{r-1}$-differentiable cocycle

$$P(x,y) := \mathrm{d}_e(\lambda_{xy}^{-1}\rho_y\lambda_x), \quad Q(x,y) := \mathrm{d}_e(\lambda_{xy}^{-1}\lambda_x\lambda_y). \tag{23}$$

Lemma 5.2. The monomial terms of the power series expansion (18) of

$$P(x,y)X + Q(x,y)Y = \mathrm{d}_e(\lambda_{xy}^{-1}\rho_y\lambda_x)X + \mathrm{d}_e(\lambda_{xy}^{-1}\lambda_x\lambda_y)Y$$

are expressed by

$$\begin{aligned}
&P'_2(y)X = 2q(X,y),\\
&P''_{12}(x,y)X = q(q(x,X),y) - q(q(x,y),X) - 2q(x,q(X,y)) +\\
&+ 2r(x,X,y) - 2r(x,y,X),\\
&\frac{1}{2}P''_{22}(x,y)X = -q(x,q(X,y)) - q(y,q(X,x)) - r(x,y,X) + s(X,x,y),
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
&Q'_1(x) = Q''_{11}(x,y) = 0,\\
&Q''_{12}(x,y)Y = q(x,q(y,Y)) - q(q(x,y),Y) + 2s(x,y,Y) - 2r(x,y,Y).
\end{aligned} \tag{25}$$

Proof. Let us denote

$$\Sigma = x \cdot y, \quad \Gamma(z) = \rho_y\lambda_x z, \quad \Delta(z) = \lambda_x\lambda_y z.$$

The map $P(x,y) = \mathrm{d}_e(\lambda_{xy}^{-1}\rho_y\lambda_x) = \mathrm{d}_e(\Sigma\backslash\Gamma)$ is the linear part of $\Sigma\backslash\Gamma(z)$ with respect to the variable z. We have the expansions

$$\begin{aligned}
\Gamma(z) = (x\cdot z)\cdot y = x + y + z + q(x,z) + q(x+z,y) + q(q(x,z),y) +\\
+ r(x,x,z) + r(x+z,x+z,y) + s(x,z,z) + s(x+z,y,y) + o(3)
\end{aligned}$$

and

$$\begin{aligned}
\Gamma(z) - \Sigma = z + q(x,z) + q(z,y) + q(q(x,z),y) + r(x,x,z) +\\
+ 2r(x,z,y) + r(z,z,y) + s(x,z,z) + s(z,y,y) + o(3).
\end{aligned}$$

Using

$$x\backslash y = y - x - q(x,y-x) + q(x,q(x,y-x)) - r(x,x,y-x) - s(x,y-x,y-x) + o(3)$$

we obtain

$$\begin{aligned}
&\Sigma\backslash\Gamma(z) = \Gamma(z) - \Sigma - q(\Sigma,\Gamma(z)-\Sigma) + q(\Sigma,q(\Sigma,\Gamma(z)-\Sigma)) -\\
&- r(\Sigma,\Sigma,\Gamma(z)-\Sigma) - s(\Sigma,\Gamma(z)-\Sigma,\Gamma(z)-\Sigma) + \cdots =\\
&= z + 2q(z,y) + q(q(x,z),y) - q(q(x,y),z) -\\
&- q(x+y,q(z,y)-q(y,z)) + 2r(x,z,y) - 2r(x,y,z) +\\
&+ r(z,z,y) - r(y,y,z) + s(x,z,z) + s(z,y,y) - s(x+y,z,z) + o(3).
\end{aligned}$$

The linear part of $\Sigma\backslash\Gamma$ with respect to the variable z gives

$$\begin{aligned} P(x,y)Z = Z + 2q(Z,y) + q(q(x,Z),y) - q(q(x,y),Z) - 2q(x+y,q(Z,y)) + \\ + 2r(x,Z,y) - 2r(x,y,Z) - r(y,y,Z) + s(Z,y,y) + o(3). \end{aligned} \tag{26}$$

Similarly, $Q(x,y) = \mathrm{d}_e(\lambda_{xy}^{-1}\lambda_x\lambda_y) = \mathrm{d}_e(\Sigma\backslash\Delta)$ is the linear part of $\lambda_{xy}^{-1}\lambda_x\lambda_y(z) = \Sigma\backslash\Delta(z)$ with respect to the variable z. We have

$$\begin{aligned} &\Delta(z) - \Sigma = z + q(x+y,z) + q(x,q(y,z)) + r(x,x,z) + r(y,y,z) + \\ &+ s(y,z,z) + 2s(x,y,z) + s(x,z,z) + o(3), \end{aligned}$$

and

$$\begin{aligned} &\Sigma\backslash\Delta(z) = \Delta(z) - \Sigma - q(\Sigma,\Delta(z)-\Sigma) + q(\Sigma,q(\Sigma,\Delta(z)-\Sigma)) - \\ &- r(\Sigma,\Sigma,\Delta(z)-\Sigma) - s(\Sigma,\Delta(z)-\Sigma,\Delta(z)-\Sigma) + \cdots = \\ &= z + q(x,q(y,z)) - q(q(x,y),z) - 2r(x,y,z) + 2s(x,y,z) + o(3). \end{aligned}$$

It follows

$$Q(x,y)Z = Z + q(x,q(y,Z)) - q(q(x,y),Z) + 2s(x,y,Z) - 2r(x,y,Z) + o(3). \tag{27}$$

According to (17), (26) and (27) we have

$$\begin{aligned} &P_2'(y)X + \ P_{12}''(x,y)X + \frac{1}{2}P_{22}''(y,y)X = \\ &= 2q(X,y) + q(q(x,X),y) - q(q(x,y),X) - 2q(x+y,q(X,y)) + \\ &+ 2r(x,X,y) - 2r(x,y,X) - r(y,y,X) + s(X,y,y), \end{aligned} \tag{28}$$

and

$$\begin{aligned} &Q_1'(x)Y + \frac{1}{2}Q_{11}''(x,x)Y + Q_{12}''(x,y)Y = \\ &= q(x,q(y,Y)) - q(q(x,y),Y) + 2s(x,y,Y) - 2r(x,y,Y). \end{aligned} \tag{29}$$

The assertion of lemma follows from the formulas (28) and (29). □

Proposition 5.3. The commutator and the associator of the tangent Akivis algebra of the tangent prolongation $\mathscr{T}(L\times \mathrm{T}_e(L))$ are expressed by

$$[(x,X),(y,Y)] = ([x,y],[X,y]+[x,Y])$$

and

$$\langle(x,X),(y,Y),(z,Z)\rangle = \big(\langle x,y,z\rangle, \langle X,y,z\rangle + \langle x,Y,z\rangle + \langle x,y,Z\rangle\big)$$

in a distinguished coordinate chart $W^n\times V^n$.

Proof. We apply the results of Theorem 4.1 to linear abelian prolongation $\mathcal{F}(P,Q)$ of L determined by the cocycle (23). We compute the commutator

$$\begin{aligned}[(x,X),(y,Y)] = ([x,y], \big(P_2'(y) - Q_1'(y)\big)\, X + \big(Q_1'(x) - P_2'(x)\big)\, Y) = \\ = ([x,y], 2(q(X,y) - q(Y,x)) = ([x,y], [X,y] + [x,Y]),\end{aligned}$$

and the associator

$$\begin{aligned}&\langle (x,X),(y,Y),(z,Z)\rangle = \\ &=\big(\langle x,y,z\rangle, (P_2'(z)P_2'(y) - P_2'(q(y,z)) + P_{12}''(y,z) - P_{22}''(y,z))X+ \\ &+(P_2'(z)Q_1'(x) - Q_1'(x)P_2'(z) + P_{12}''(x,z) - Q_{12}''(x,z))Y+ \\ &+(Q_1'(q(x,y)) - Q_1'(x)Q_1'(y) + Q_{11}''(x,y) - Q_{12}''(x,y))Z\big) = \\ &=\big(\langle x,y,z\rangle, q(q(X,y),z) - q(X,q(y,z)) + 2r(X,y,z) - 2s(X,y,z)+ \\ &+q(q(x,Y),z) - q(x,q(Y,z)) + 2r(x,Y,z) - 2s(x,Y,z)- \\ &-q(x,q(y,Z)) + q(q(x,y),Z) - 2s(x,y,Z) + 2r(x,y,Z)\big) = \\ &=\big(\langle x,y,z\rangle, \langle X,y,z\rangle + \langle x,Y,z\rangle + \langle x,y,Z\rangle\big).\end{aligned}$$

Hence the assertion is proved. □

In Proposition (5.3) we identified the local loop L with the coordinate chart $W^n \subset V^n$, the tangent space $\mathrm{T}_e(L)$ with the vector space V^n, the tangent prolongation $\mathscr{T}(L \times \mathrm{T}_e(L))$ with $W^n \times V^n$, and computed the commutator and associator in the tangent space $\mathrm{T}_{(0,0)}(W^n \times V^n)$. Now we will find a coordinate-free expression for the operations of the tangent Akivis algebra of tangent prolongation $\mathscr{T}(L \times \mathrm{T}_e(L))$. Using the fact that the tangent spaces of a vector space are canonically isomorphic to the vector space we get a canonical linear isomorphism of the tangent space $\mathrm{T}_{(e,0)}\,(L \times \mathrm{T}_e(L))$ to the direct sum $\mathrm{T}_e(L) \oplus \mathrm{T}_e(L)$. The bilinear, respectively, trilinear forms q, r, s, the commutator and the associator of L are defined on the subspace $\mathrm{T}_e(L) \oplus \{0\} \cong \mathrm{T}_e(L)$. Let

$$\theta : \{0\} \oplus \mathrm{T}_e(L) \to \mathrm{T}_e(L) \oplus \{0\}, \quad \theta : (0,X) \mapsto (X,0)$$

be the canonical linear isomorphism induced by the identity map of $\mathrm{T}_e(L)$. In the expressions in Proposition (5.3) of the operations of the tangent Akivis algebra of $\mathscr{T}(L \times \mathrm{T}_e(L))$ we replace $x,y,z \in W^n$ with $\xi,\eta,\zeta \in \mathrm{T}_e(L) \oplus \{0\}$. Using the map $\theta : (0,X) \mapsto (X,0)$ we can express the commutator

$$[(\xi,X),(\eta,Y)] = ([\xi,\eta], 2\theta^{-1}(q(\theta(X),\eta) + q(\xi,\theta(Y))) = ([\xi,\eta], \theta^{-1}([\theta(X),\eta]) + \theta^{-1}([\xi,\theta(Y)])),$$

and the associator

$$\begin{aligned}&\langle (\xi,X),(\eta,Y),(\zeta,Z)\rangle = \\ &= \big(\langle \xi,\eta,\zeta\rangle, \theta^{-1}\big(q(q(\theta(X),\eta),\zeta) - q(\theta(X),q(\eta,\zeta)) + 2r(\theta(X),\eta,\zeta) - 2s(\theta(X),\eta,\zeta)+ \\ &+ q(q(\xi,\theta(Y)),\zeta) - q(\xi,q(\theta(Y),\zeta)) + 2r(\xi,\theta(Y),\zeta) - 2s(\xi,\theta(Y),\zeta)- \\ &- q(\xi,q(\eta,\theta(Z))) + q(q(\xi,\eta),\theta(Z)) - 2s(\xi,\eta,\theta(Z)) + 2r(\xi,\eta,\theta(Z))\big)\big) = \\ &= \big(\langle \xi,\eta,\zeta\rangle, \theta^{-1}(\langle \theta(X),\eta,\zeta\rangle) + \theta^{-1}(\langle \xi,\theta(Y),\zeta\rangle) + \theta^{-1}(\langle \xi,\eta,\theta(Z)\rangle)\big).\end{aligned}$$

Hence we obtain:

Theorem 5.4. The tangent Akivis algebra $\mathcal{A}(\mathscr{T}(L \times \mathrm{T}_e(L)))$ of the tangent prolongation $\mathscr{T}(L \times \mathrm{T}_e(L))$ of a $\mathcal{C}^r$-differentiable local loop L is a linear semidirect sum

$$\mathcal{A}(\mathscr{T}(L \times \mathrm{T}_e(L))) \cong \mathcal{A}(L) \rtimes \mathrm{T}_e(L)^+$$

of the tangent Akivis algebra $\mathcal{A}(L)$ and the abelian Akivis algebra $\mathrm{T}_e(L)^+$ on the tangent space $\mathrm{T}_e(L)$. The commutator and the associator of $\mathcal{A}(\mathscr{T}(L \times \mathrm{T}_e(L)))$ are expressed by

$$[(\xi, X), (\eta, Y)] = ([\xi, \eta], \theta^{-1}([\theta(X), \eta]) + \theta^{-1}([\xi, \theta(Y)])),$$
$$\langle(\xi, X), (\eta, Y), (\zeta, Z)\rangle = \big(\langle\xi, \eta, \zeta\rangle, \theta^{-1}(\langle\theta(X), \eta, \zeta\rangle) + \theta^{-1}(\langle\xi, \theta(Y), \zeta\rangle) + \theta^{-1}(\langle\xi, \eta, \theta(Z)\rangle)\big)$$

for any $(\xi, X), (\eta, Y), (\zeta, Z) \in \mathrm{T}_e(L) \oplus \mathrm{T}_e(L)$.

The expressions obtained for the commutator and associator of the tangent Akivis algebra of the tangent prolongation show that this semidirect sum of Akivis algebras is constructed as described in Proposition (3.3) in the case if $\mathcal{A}^n = \mathcal{A}^{*n}$ and $\theta : \{0\} \oplus \mathrm{T}_e(L) \to \mathrm{T}_e(L) \oplus \{0\}$ is induced by the identity map of $\mathrm{T}_e(L)$.

Corollary 5.5. The tangent Akivis algebra $\mathcal{A}(\mathscr{T}(L \times \mathrm{T}_e(L)))$ is the linear semidirect sum $\mathcal{A}(L) \rtimes \mathrm{T}_e(L)^+$ determined by the maps

$$\alpha_\xi Z = \theta^{-1}[\xi, \theta Z],\ \lambda_{(\xi,\eta)} Z = \theta^{-1}\langle\theta Z, \xi, \eta\rangle,\ \mu_{(\xi,\eta)} Z = \theta^{-1}\langle\eta, \theta Z, \xi\rangle,\ \nu_{(\xi,\eta)} Z = \theta^{-1}\langle\xi, \eta, \theta Z\rangle$$

where $\theta : (0, X) \mapsto (X, 0)$.

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