Proceedings of the Conference RIGA 2021
Riemannian Geometry and Applications
Bucharest, Romania

DIFFERENTIAL EQUATIONS HAVING A GIVEN LIE SYMMETRY GROUP

ÁGOTA FIGULA AND GIORGI LOMJARIA

Abstract. We present Lie's method to find the ordinary differential equations having a given Lie group as a group of their symmetries. We illustrate this method on examples of ordinary differential equations and systems of first order ordinary differential equations admitting a given Lie group G as a subgroup of their symmetries.

2020 *MSC*: 34C14, 34C20, 17B66, 17B15, 22E60, 58F07.
Keywords and Phrases: symmetries of differential equations, point symmetries, prolongation of vector fields, Lie algebra, first integral.

1. Introduction

Symmetry analysis [4, 5, 21, 22] is a framework for constructing analytic solutions for differential equations. Several examples come from Physics (see [16, 23]), as well as from Biology (see [2, 17, 18]). Finding some symmetries for a differential equation can be used to derive an appropriate change of coordinates which then helps to eliminate the independent variables or to decrease the order of the system and then transform it to an integrable form. In many cases (e.g. the Fitzhugh–Nagumo model [3, 14] or the model for the population of Easter Island [18]) the model is based on a first order system of two equations. One advantage for investigating first order systems is that any system of differential equations has an equivalent system of first order system. However, the symmetry groups of higher order systems behave very differently than first order systems. They have finite dimensional symmetry groups as opposed to the infinite dimensional Lie groups for the first order system. Many physical systems (see. [7, 19, 20]) are naturally governed by second order systems.

Here we tackle the problem from the other way around. Following the works of Lie we look for small dimensional Lie groups and see what systems of ordinary differential equations admit them as symmetries. Lie determined the groups of transformations of the (x, y)-plane and written these into canonical form (cf. [13, Sections 3, 4, 5, pp. 28–78] and [13, Section 19, pp. 360–392], see also [6], p. 341) In [9, Sections X, XI, XIV, XVI] he provided a classification of all ordinary differential equations of arbitrary order which admit these given groups as groups of their symmetries. For a given Lie algebra **g** of dimension r, represented in an n-dimensional space, one can determine every differential equation of order at most $r-2$ whose Lie group's tangential Lie algebra contains **g** as a Lie subalgebra. He described also the equations for which integration or lowering of the order could be effected by group theoretical methods. Section 3 is devoted to present his method. In Section 4 we apply this method to find the ordinary differential equations which admit some given Lie groups as groups of their symmetries. We use the given symmetries to find the solutions of these equations. Analogously to the method of Lie in Section 5 we formulate the appropriate necessary condition for a system of first order ordinary differential equations allowing a given Lie group as a subgroup of their symmetries and give examples for systems which are invariant under Lie groups of symmetries. We restrict us mostly to the cases where the tangential Lie algebra of the Lie group G is semi-simple. To obtain our examples we use the REDUCE program [15].

We note two remarkable facts of our study. The differential equations obtained by this method do not only depend on the isomorphism class of the given Lie group, but rather on the representation of its tangential Lie algebra. This will be illustrated in more detail in Section 4.2 taking the four different representations of the Lie algebra $\mathbf{sl}_2(\mathbb{R})$ in the plane $\mathbb{R}^2$. On the other hand, in Section 4.3 we obtain a second order ordinary differential equation admitting the Lie group $SO_3(\mathbb{R})$ as its symmetry group. This three-dimensional simple Lie group has no two-dimensional solvable subgroup. Therefore, the obtained second order differential equation is not solvable. To completely solve a second order equation it needs a two dimensional solvable Lie subgroup of its symmetry group (cf. [21], [5]).

We list the considered real Lie algebras **g** by increasing dimension together with their representation in the (x, y)-plane and we list the differential equations admitting a symmetry group whose tangential Lie algebra is the one we listed.

(I) Example 4.4: $\mathbf{sl_2}(\mathbb{R})$, 3-dimensional simple, its generators are

$$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \qquad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_3 = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}. \tag{1}$$

The differential equations

$$y^{(2)} + 2\frac{(y')^2 + y' + c\,(y')^{3/2}}{x-y} = 0 \qquad (c \in \mathbb{R}) \tag{2}$$

of order at most 2 admit this symmetry group.

(II) Example 4.5: $\mathbf{sl_2}(\mathbb{R})$, 3-dimensional simple, its generators are

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \tag{3}$$

The differential equations of order at most 2 admitting this symmetry group are

$$y^{(2)} - \frac{a}{y^3} = 0 \qquad (a \in \mathbb{R}). \tag{4}$$

(III) Example 4.6: $\mathbf{sl_2}(\mathbb{R})$, 3-dimensional simple, its generators are

$$X_1 = \frac{\partial}{\partial y}, \qquad X_2 = y\frac{\partial}{\partial y}, \qquad X_3 = y^2\frac{\partial}{\partial y}. \tag{5}$$

The differential equations of order at most 3 allowing this symmetry group are

$$y^{(3)} - \frac{3\left(y^{(2)}\right)^2}{2y'} - y' f(x) = 0, \tag{6}$$

for an arbitrary real function f.

(IV) Example 4.8: $\mathbf{sl_2}(\mathbb{R})$, 3-dimensional simple, its generators are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad X_3 = (x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}. \tag{7}$$

The differential equations of order at most 2 admitting this symmetry group are

$$y^{(2)} = -\frac{1 + (y')^2}{y} + d\frac{\left(1 + (y')^2\right)^{3/2}}{y} \qquad (d \in \mathbb{R}). \tag{8}$$

(V) Example 4.9: $\mathbf{so_3}(\mathbb{R}) \cong \mathbf{su_2}(\mathbb{C})$, 3-dimensional simple, its generators are

$$X_1 = \left(1 + x^2\right)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \qquad X_2 = xy\frac{\partial}{\partial x} + \left(1 + y^2\right)\frac{\partial}{\partial y}, \tag{9}$$

$$X_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

The differential equations of order at most 2 allowing this symmetry group are

$$y^{(2)} = c\left(\frac{1 + y^2 - 2xyy' + \left(1 + x^2\right)(y')^2}{1 + x^2 + y^2}\right)^{3/2} \qquad (c \in \mathbb{R}). \tag{10}$$

(VI) Example 4.11: 3-dimensional solvable Lie algebra $\mathbf{g}_\alpha$, $\alpha \geq 0$, its generators are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \alpha\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{11}$$

There is no differential equation of order at most 1 admitting this symmetry group.

(VII) Example 4.12: 3-dimensional solvable Lie algebra $\mathbf{g}_\beta$, $0 < |\beta| \leq 1$, its generators are

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x} + \beta y \frac{\partial}{\partial y}. \tag{12}$$

The differential equation of order at most 1 admitting this symmetry group is

$$y' = 0.$$

(VIII) Example 4.13: a 4-dimensional solvable Lie algebra, its generators are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad X_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{13}$$

The differential equation of order at most 2 admitting this symmetry group is

$$y^{(2)} = 0.$$

(IX) Example 4.14: 4-dimensional solvable Lie algebra, its generators are

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x}, \qquad X_4 = y\frac{\partial}{\partial y}. \tag{14}$$

The differential equations of order at most 2 allowing this symmetry group are

$$y' = 0, \qquad\qquad y^{(2)} = 0.$$

(X) Example 4.15: 4-dimensional Lie algebra $\mathbf{sl_2}(\mathbb{R}) \times \mathbb{R}$, its generators are

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x}, \qquad X_4 = x^2\frac{\partial}{\partial x}. \tag{15}$$

The differential equation of order at most 2 admitting this symmetry group is

$$y' = 0.$$

(XI) Example 4.16: 4-dimensional Lie algebra $\mathbf{gl_2}(\mathbb{R})$, its generators are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x}, \quad X_3 = y\frac{\partial}{\partial y}, \quad X_4 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \tag{16}$$

The differential equation of order at most 2 admitting this symmetry group is

$$y^{(2)} = 0.$$

(XII) Example 4.17: 5-dimensional Lie algebra $\mathbf{sl_2}(\mathbb{R}) \times L_2$, where L_2 is the 2-dimensional non-abelian Lie algebra, its generators are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x}, \quad X_4 = y\frac{\partial}{\partial y}, \quad X_5 = x^2\frac{\partial}{\partial x}. \tag{17}$$

There are two differential equations of order at most 3 admitting this symmetry group:

$$y' = 0, \qquad 2y'y^{(3)} - 3\left(y^{(2)}\right)^2 = 0. \tag{18}$$

(XIII) Example 4.18: 5-dimensional Lie algebra $\mathbf{sl_2}(\mathbb{R}) \ltimes \mathbb{R}^2$, its generators are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad X_4 = y\frac{\partial}{\partial x}, \quad X_5 = x\frac{\partial}{\partial y}. \tag{19}$$

The differential equation of order at most 3 allowing this symmetry group is

$$y^{(2)} = 0.$$

(XIV) Example 4.1: $\mathbf{sl_2}(\mathbb{C})$, 6-dimensional simple, its generators are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad X_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \tag{20}$$

$$X_5 = \left(x^2 - y^2\right)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}, \qquad X_6 = 2xy\frac{\partial}{\partial x} + \left(y^2 - x^2\right)\frac{\partial}{\partial y}.$$

The differential equation of order at most 4 admitting this symmetry group is

$$y^{(3)} + y^{(3)}\left(y'\right)^2 - 3y'\left(y^{(2)}\right)^2 = 0. \tag{21}$$

(XV) Example 4.2: $\mathbf{sl_2}(\mathbb{R}) \times \mathbf{sl_2}(\mathbb{R})$, 6-dimensional semi-simple, its generators are

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = y\frac{\partial}{\partial y}, \tag{22}$$

$$X_4 = x\frac{\partial}{\partial x}, \qquad X_5 = y^2\frac{\partial}{\partial y}, \qquad X_6 = x^2\frac{\partial}{\partial x}.$$

The differential equations of order at most 4 admitting this symmetry group are $y' = 0$ and

$$2y'y^{(3)} - 3\left(y^{(2)}\right)^2 = 0. \tag{23}$$

(XVI) Example 4.19: 6-dimensional Lie algebra $\mathbf{gl_2}(\mathbb{R}) \ltimes \mathbb{R}^2$, its generators are

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x}, \tag{24}$$

$$X_4 = y\frac{\partial}{\partial x}, \qquad X_5 = x\frac{\partial}{\partial y}, \qquad X_6 = y\frac{\partial}{\partial y}.$$

The differential equations of order at most 4 admitting this symmetry group are

$$y^{(2)} = 0, \qquad 3y^{(4)}y^{(2)} - 5\left(y^{(3)}\right)^2 = 0. \tag{25}$$

(XVII) Example 4.3: $\mathbf{sl_3}(\mathbb{R})$, 8-dimensional simple, its generators are

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial y}, \qquad X_4 = y\frac{\partial}{\partial y}, \tag{26}$$

$$X_5 = x\frac{\partial}{\partial x}, \quad X_6 = y\frac{\partial}{\partial y}, \quad X_7 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \quad X_8 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}.$$

The differential equations of order at most 6 admitting this symmetry group are

$$y^{(2)} = 0, \qquad 9\left(y^{(2)}\right)^2 y^{(5)} - 45y^{(2)}y^{(3)}y^{(4)} + 40\left(y^{(3)}\right)^3 = 0. \tag{27}$$

In the following list we collect the real Lie algebras $\mathbf{g}$, their type and infinitesimal generators in the (x, y, z)-space and the appropriate invariant systems of first order ordinary differential equations.

(i) Example 5.2: $\mathbf{so_3}(\mathbb{R}) \cong \mathbf{su_2}(\mathbb{C})$, 3-dimensional simple, its generators are

$$X_1 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \qquad X_2 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \qquad X_3 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}. \tag{28}$$

The time-dependent invariant system is

$$\begin{aligned} y' &= \frac{y}{x}, \\ z' &= \frac{z}{x}. \end{aligned} \tag{29}$$

Only the trivial time-independent system is invariant under the action of $\mathbf{so_3}(\mathbb{R})$:

$$\begin{aligned} x' &= 0, \\ y' &= 0, \\ z' &= 0. \end{aligned} \tag{30}$$

(ii) Examples 5.4 and 5.5: $\mathbf{sl_2}(\mathbb{R}) \times \mathbf{sl_2}(\mathbb{R})$, 6-dimensional semi-simple, its generators are

$$X_1 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \qquad X_2 = y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \tag{31}$$

$$X_3 = (xy - z)\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z}, \quad X_4 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z},$$
$$X_5 = x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}, \qquad X_6 = x^2\frac{\partial}{\partial x} + (xy - z)\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}.$$

There does not exist any time-dependent explicit invariant system of the first order ordinary differential equations. The invariant time-independent system is trivial (30).

(iii) Example 5.3: $\mathbf{sl_3}(\mathbb{R})$, 8-dimensional simple, its generators are

$$X_1 = z\frac{\partial}{\partial x}, \quad X_2 = z\frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial y}, \quad X_4 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \tag{32}$$

$$X_5 = y\frac{\partial}{\partial x}, \quad X_6 = x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}, \quad X_7 = x\frac{\partial}{\partial z}, \quad X_8 = y\frac{\partial}{\partial z}.$$

The time-dependent invariant system is (29). Only the trivial time-independent system (30) is invariant under the action of $\mathbf{sl_3}(\mathbb{R})$.

2. Preliminaries

Symmetries of a differential equation are transformations that move (continuously) a solution of the equation into another solution. Thus for each symmetry there exists a corresponding vector field (the infinitesimal generator of the symmetry). In the case of ordinary differential equations of order m the space of the variables $x, y, y', \ldots, y^{(m)}$ is called the jet space. The differential equation $f\left(x, y, y', \ldots, y^{(m)}\right) = 0$ defines an $(m+1)$-dimensional surface in this space which is called the hull of the differential equation. A smooth solution is a continuously differentiable function $\varphi(x)$ such that the curve $y = \varphi(x)$ with $y' = \frac{\partial \varphi(x)}{\partial x}, \ldots, y^{(m)} = \frac{\partial^m \varphi(x)}{\partial x^m}$ belongs to the hull, that is, $f\left(x, \varphi(x), \ldots, \frac{\partial^m \varphi(x)}{\partial x^m}\right) = 0$ identically holds for all x. Understanding symmetries of differential equations is useful for generating a new solution from a known solution of the differential equation and creating new methods for solving it. For example, symmetries can help lowering the order of the differential equation. In the process of integrating differential equations the crucial step is the simplification of the hull by a suitable change of variables. To this end we use the symmetry group of the differential equation which is defined as the group of transformations of the (x,y)-plane whose prolongation to the derivatives $y',\dots,y^{(m)}$ leaves the hull of the equation under consideration invariant.

3. Lie's method to find the ordinary differential equations allowing a given Lie group as a group of their symmetries

In this section we present Lie's method to obtain the ordinary differential equations which admit a given Lie group as a group G of their symmetries. This method can be found in [9, Section X, pp. 243–248]. The r-dimensional real Lie group G has the tangential Lie algebra $\mathbf{g}$, which is given by the basis elements (the infinitesimal generators)

$$X_i = \phi_i(x,y)\frac{\partial}{\partial x} + \eta_i(x,y)\frac{\partial}{\partial y}, \qquad i = 1,2,\dots,r. \tag{33}$$

We define recursively $\eta_i^{(k)}$ $(i = 1,2,\dots,r,\ k = 1,2,\dots,m)$ using the total derivative of η_i as well as of ϕ_i with respect to the variable x:

$$\eta_i^{(k)} = \frac{d\eta_i^{(k-1)}}{dx} - y^{(k)}\frac{d\phi_i}{dx}, \text{ that is}$$

$$\eta_i^{(1)}(x,y,y') = \frac{\partial\eta_i}{\partial x} + \frac{\partial\eta_i}{\partial y}y' - \frac{\partial\phi_i}{\partial x}y' - \frac{\partial\phi_i}{\partial y}(y')^2, \tag{34}$$

$$\begin{aligned}\eta_i^{(2)}(x,y,y',y^{(2)}) = {} & \frac{\partial\eta_i^{(1)}}{\partial x} + \frac{\partial\eta_i^{(1)}}{\partial y}y' + \frac{\partial\eta_i^{(1)}}{\partial y'}y^{(2)} \\ & - \frac{\partial\phi_i}{\partial x}y^{(2)} - \frac{\partial\phi_i}{\partial y}y'y^{(2)},\end{aligned} \tag{35}$$

$$\begin{aligned}\eta_i^{(3)}(x,y,y',y^{(2)},y^{(3)}) = {} & \frac{\partial\eta_i^{(2)}}{\partial x} + \frac{\partial\eta_i^{(2)}}{\partial y}y' + \frac{\partial + \eta_i^{(2)}}{\partial y'}y^{(2)} + \frac{\partial\eta_i^{(2)}}{\partial y^{(2)}}y^{(3)} \\ & - \frac{\partial\phi_i}{\partial x}y^{(3)} - \frac{\partial\phi_i}{\partial y}y'y^{(3)},\end{aligned} \tag{36}$$

$$\begin{aligned}\eta_i^{(4)}(x,y,y',y^{(2)},y^{(3)},y^{(4)}) = {} & \frac{\partial\eta_i^{(3)}}{\partial x} + \frac{\partial\eta_i^{(3)}}{\partial y}y' + \frac{\partial\eta_i^{(3)}}{\partial y'}y^{(2)} + \frac{\partial\eta_i^{(3)}}{\partial y^{(2)}}y^{(3)} \\ & + \frac{\partial\eta_i^{(3)}}{\partial y^{(3)}}y^{(4)} - \frac{\partial\phi_i}{\partial x}y^{(4)} - \frac{\partial\phi_i}{\partial y}y'y^{(4)}, \text{ etc.}\end{aligned} \tag{37}$$

The m^{th} prolonged vector fields $X_i^{(m)}$ $(i = 1, 2, \ldots, r)$ are defined as

$$X_i^{(m)} = \phi_i(x,y)\frac{\partial}{\partial x} + \eta_i(x,y)\frac{\partial}{\partial y} + \eta_i^{(1)}(x,y,y')\frac{\partial}{\partial y'} + \ldots \\ + \eta_i^{(m)}\left(x,y,\ldots,y^{(m)}\right)\frac{\partial}{\partial y^{(m)}}$$

(cf. [9, Section X, p. 245]). They depend on x, y, y', $\ldots$, $y^{(m)}$ and they generate a Lie algebra isomorphic to $\mathbf{g}$ (cf. [9, Section X, p. 245] or [21, Theorem 2.39, p. 117]). A differential equation $f\left(x,y,y',\ldots,y^{(m)}\right) = 0$ of order m admits a group of symmetries whose Lie algebra is $\mathbf{g}$ if and only if the following system of partial differential equations is satisfied whenever $f\left(x,y,y',\ldots,y^{(m)}\right) = 0$ holds:

$$\begin{array}{rcl}
\phi_1\frac{\partial f}{\partial x} + \eta_1\frac{\partial f}{\partial y} + \eta_1^{(1)}\frac{\partial f}{\partial y'} + \cdots + \eta_1^{(m)}\frac{\partial f}{\partial y^{(m)}} & = & 0, \\
\phi_2\frac{\partial f}{\partial x} + \eta_2\frac{\partial f}{\partial y} + \eta_2^{(1)}\frac{\partial f}{\partial y'} + \cdots + \eta_2^{(m)}\frac{\partial f}{\partial y^{(m)}} & = & 0, \\
& \vdots & \\
\phi_i\frac{\partial f}{\partial x} + \eta_i\frac{\partial f}{\partial y} + \eta_i^{(1)}\frac{\partial f}{\partial y'} + \cdots + \eta_i^{(m)}\frac{\partial f}{\partial y^{(m)}} & = & 0, \\
& \vdots & \\
\phi_r\frac{\partial f}{\partial x} + \eta_r\frac{\partial f}{\partial y} + \eta_r^{(1)}\frac{\partial f}{\partial y'} + \cdots + \eta_r^{(m)}\frac{\partial f}{\partial y^{(m)}} & = & 0.
\end{array} \tag{38}$$

Let

$$M = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_r \\ \eta_1 & \eta_2 & \eta_3 & \cdots & \eta_r \\ \eta_1^{(1)} & \eta_2^{(1)} & \eta_3^{(1)} & \cdots & \eta_r^{(1)} \\ \vdots & & & \ddots & \vdots \\ \eta_1^{(m)} & \eta_2^{(m)} & \eta_3^{(m)} & \cdots & \eta_r^{(m)} \end{pmatrix}. \tag{39}$$

Then the system of partial differential equations given by (38) can be treated as the following system of 'linear equations' in the variables $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial y'}$, $\ldots$, $\frac{\partial f}{\partial y^{(m)}}$:

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial y'} & \cdots & \frac{\partial f}{\partial y^{(m)}} \end{pmatrix} \cdot M = \begin{pmatrix} 0 & \ldots & 0 \end{pmatrix}. \tag{40}$$

Here, the coefficient matrix M is an $(m+2)\times r$ matrix. Thus, the system (38) has more than only trivial solution (i.e. $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \cdots = \frac{\partial f}{\partial y(m)} = 0$, which does not correspond to any differential equation f) if and only if the rank of the coefficient matrix in (40) is strictly less than $(m+2)$:

$$\text{rank}M < m + 2.$$

Now, $\text{rank}M \leq r$ always holds, hence if $r < m+2$, then the rank condition is automatically satisfied. In such a situation one needs to solve (38) in

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \ldots, \frac{\partial f}{\partial y(m)}$, and see if any solution corresponds to a nontrivial differential equation f. In the following we consider the case $r \geq m+2$, where one can arrive at the possible differential equations in a somewhat more direct manner.

First, suppose $m+2 = r$. Then the coefficient matrix of the system (40) of linear equations is an $(m+2) \times (m+2)$-matrix. The system (40) has a non-trivial solution f if and only if the rank of the coefficient matrix of (40) is $< m+2$. Hence the determinant

$$D = \begin{vmatrix} \phi_1 & \phi_2 & \phi_3 & \ldots & \phi_r \\ \eta_1 & \eta_2 & \eta_3 & \ldots & \eta_r \\ \eta_1^{(1)} & \eta_2^{(1)} & \eta_3^{(1)} & \ldots & \eta_r^{(1)} \\ \vdots & & & \ddots & \vdots \\ \eta_1^{(r-2)} & \eta_2^{(r-2)} & \eta_3^{(r-2)} & \ldots & \eta_r^{(r-2)} \end{vmatrix} \tag{41}$$

of the coefficient matrix of (40) (which is a polynomial function of x, y, $y^{(i)}$, $i = 1, 2, \ldots, r-2$) has to be 0. Now, if D is identically 0 (as a polynomial), one needs to make further considerations, like solving (38) in $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \ldots, \frac{\partial f}{\partial y(m)}$, and see if such a solution corresponds to a nontrivial differential equation f. The phenomenon $D \equiv 0$ is quite rare, though. However, when D is not identically 0, then D is a polynomial function of x, y, $y^{(i)}$ ($i = 1, 2, \ldots, r-2$) which has to be 0 if a nontrivial differential equation f exist, and hence by factoring D we obtain the only possibilities for such an f.

Conversely, the differential equations arising from the condition $D = 0$ always admit the group of symmetries corresponding to the Lie algebra $\mathbf{g} = \langle X_i, i = 1, 2, \ldots, r \rangle$ given by (33) (cf. [12, p. 475]). An analytical proof of this assertion can be found in [10, Abh. I, No. 24, pp. 36–37]. The crucial step of this proof is to show that the differential equation arising from the condition $D = 0$ satisfies the system (38) of partial differential equations whenever $D = 0$ holds.

Now, assume $m+2 < r$. Then the coefficient matrix of the system (40) of linear equations arising from (38) is an $(m+2) \times r$-matrix. To obtain a non-trivial solution of the system (40) it is necessary that $\text{rank} M < m+2$. Hence the determinants of *all* $(m+2) \times (m+2)$ submatrices of M has to be 0. Again, these subdeterminants are polynomials of the variables x, y, $y^{(i)}$, $i = 1, 2, \ldots, m$, and therefore (unless all of these subdeterminants are identically 0) their common factors provide the only possibilities for nontrivial differential equations f admitting the group G as symmetries. Summarizing our discussion we obtain.

Theorem 3.1. *Finding the differential equations* $f\left(x, y, y', \ldots, y^{(m)}\right) = 0$ *of order* m*, which admit a group of symmetries whose Lie algebra is a* *given r-dimensional real Lie algebra $\mathbf{g} = \langle X_i = \phi_i(x,y)\frac{\partial}{\partial x} + \eta_i(x,y)\frac{\partial}{\partial y}, i = 1,2,\dots,r\rangle$ such that $m \le r-2$, one has to build the matrix M defined by (39) and compute the greatest common divisor of all its $(m+2)\times(m+2)$ subdeterminants. The factors of this polynomial give the only possibilities for the sought differential equations, unless this polynomial is identically 0.*

4. Examples of differential equations admitting a given Lie group as a group of their symmetries

Applying the method described in Section 3 we give examples of ordinary differential equations which allow such a Lie group as a group of their symmetries whose tangential Lie algebra is listed in Section 1.

4.1. **Examples where $m \le r-2$.**

We apply the notations introduced in Section 3. Throughout this subsection we only determine differential equations of order m such that $m \le r-2$, where r is the number of generators of the tangential Lie algebra of the given Lie group. Applying the known symmetries we solve some of these differential equations.

Example 4.1. The Lie algebra $\mathbf{g} = \mathbf{sl_2}(\mathbb{C})$ is generated by the vector fields in (20). Therefore one has

$$(\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6) = (1,0,-y,x,x^2-y^2,2xy), \tag{42}$$

$$(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5,\eta_6) = (0,1,x,y,2xy,y^2-x^2). \tag{43}$$

Using (34), (35), (36), (37) we obtain $\eta_i^{(1)}$, $\eta_i^{(2)}$, $\eta_i^{(3)}$, $\eta_i^{(4)}$, $(i = 1,2,\dots,6)$:

$$\begin{aligned}(\eta_1^{(1)},\eta_2^{(1)},\eta_3^{(1)},\eta_4^{(1)},\eta_5^{(1)},\eta_6^{(1)}) \\ = (0,0,1+(y')^2,0,2y(1+(y')^2),-2x(1+(y')^2)),\end{aligned} \tag{44}$$

$$\begin{aligned}(\eta_1^{(2)},\eta_2^{(2)},\eta_3^{(2)},\eta_4^{(2)},\eta_5^{(2)},\eta_6^{(2)}) \\ = (0,0,3y'y^{(2)},-y^{(2)},2y'-2xy^{(2)}+2(y')^3+6yy'y^{(2)}, \\ -2-2(y')^2-2yy^{(2)}-6xy'y^{(2)}),\end{aligned} \tag{45}$$

$$\begin{aligned}(\eta_1^{(3)},\eta_2^{(3)},\eta_3^{(3)},\eta_4^{(3)},\eta_5^{(3)},\eta_6^{(3)}) \\ = (0,0,3(y^{(2)})^2+4y'y^{(3)},-2y^{(3)}, \\ 12(y')^2y^{(2)}+6y(y^{(2)})^2-4xy^{(3)}+8yy'y^{(3)}, \\ -12y'y^{(2)}-6x(y^{(2)})^2-8xy'y^{(3)}-4yy^{(3)}),\end{aligned} \tag{46}$$

$$
\begin{aligned}
(47)\quad &(\eta_1^{(4)}, \eta_2^{(4)}, \eta_3^{(4)}, \eta_4^{(4)}, \eta_5^{(4)}, \eta_6^{(4)}) \\
&= (0, 0, 10y^{(2)}y^{(3)} + 5y'y^{(4)}, -3y^{(4)}, \\
&\quad 30y'(y^{(2)})^2 + 20(y')^2y^{(3)} + 20yy^{(2)}y^{(3)} - 4y^{(3)} - 6xy^{(4)} + 10yy'y^{(4)}, \\
&\quad - 18(y^{(2)})^2 - 24y'y^{(3)} - 20xy^{(2)}y^{(3)} - 10xy'y^{(4)} - 6yy^{(4)}).
\end{aligned}
$$

Applying (42)-(47) the determinant (41) is

$$D = 16\left(1 + \left(y'\right)^2\right)\left(y^{(3)} + y^{(3)}\left(y'\right)^2 - 3y'\left(y^{(2)}\right)^2\right)^2.$$

Since $\left(y'\right)^2 + 1 > 0$ the only one ordinary differential equation of order at most 4 such that the Lie algebra $\mathbf{g} = \mathbf{sl_2}(\mathbb{C})$ is the tangential Lie algebra of a subgroup of its symmetries is given by (21) (cf. Theorem 3.1). Introducing the new variable $z := y'$ equation (21) reduces to

$$z^{(2)}\left(1 + z^2\right) - 3z\left(z'\right)^2 = 0.$$

Since $y^{(3)} = 0$ is not an invariant differential equation under the symmetry group corresponding to the Lie algebra $\mathbf{g} = \mathbf{sl_2}(\mathbb{C})$ one has $z' \neq 0$. Therefore, we obtain

$$
\begin{gathered}
\frac{z^{(2)}}{z'} = \frac{3z}{1+z^2}z' \iff (\ln z')' = \left(\frac{3}{2}\ln(1+z^2)\right)' \iff \\
\left(\ln\frac{z'}{(1+z^2)^{\frac{3}{2}}}\right)' = 0 \iff \frac{z'}{(1+z^2)^{\frac{3}{2}}} = e^c,\ c \in \mathbb{R} \text{ is a constant} \iff \\
\int \frac{dz}{(1+z^2)^{\frac{3}{2}}} = \int k dx,\ k := e^c \text{ is a constant} \iff \\
\frac{z}{\sqrt{1+z^2}} = kx + l,\ k, l \in \mathbb{R} \text{ are constants} \iff \\
\frac{1}{z^2} = \frac{1}{(kx+l)^2} - 1 = \frac{1-(kx+l)^2}{(kx+l)^2} \iff \\
y'(x) = \pm\sqrt{\frac{(kx+l)^2}{1-(kx+l)^2}} \iff \\
y(x) = \pm\int\sqrt{\frac{(kx+l)^2}{1-(kx+l)^2}}dx = \pm\frac{1}{k}\sqrt{1-(kx+l)^2} + p,
\end{gathered}
$$

where $k, l, p \in \mathbb{R}$ are constants.

Example 4.2. The Lie algebra $\mathbf{g} = \mathbf{sl_2}(\mathbb{R}) \oplus \mathbf{sl_2}(\mathbb{R})$ is generated by the vector fields in (22). Therefore we have

$$(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) = (1, 0, 0, x, 0, x^2), (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6) \quad = (0, 1, y, 0, y^2, 0).$$

Using (34), (35), (36), (37) the functions $\eta_i^{(1)}$, $\eta_i^{(2)}$, $\eta_i^{(3)}$, $\eta_i^{(4)}$, $i = 1, 2, \dots, 6$, are the following:

$$\begin{aligned}
&(\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}, \eta_4^{(1)}, \eta_5^{(1)}, \eta_6^{(1)}) \\
&\qquad = (0, 0, y', -y', 2yy', -2xy'), \\
&(\eta_1^{(2)}, \eta_2^{(2)}, \eta_3^{(2)}, \eta_4^{(2)}, \eta_5^{(2)}, \eta_6^{(2)}) \\
&\qquad = (0, 0, y^{(2)}, -2y^{(2)}, 2yy^{(2)} + 2(y')^2, -2y' - 4xy^{(2)}) \\
&(\eta_1^{(3)}, \eta_2^{(3)}, \eta_3^{(3)}, \eta_4^{(3)}, \eta_5^{(3)}, \eta_6^{(3)}) \\
&\qquad = (0, 0, y^{(3)}, -3y^{(3)}, 2yy^{(3)} + 6y'y^{(2)}, -6xy^{(3)} - 6y^{(2)}), \\
&(\eta_1^{(4)}, \eta_2^{(4)}, \eta_3^{(4)}, \eta_4^{(4)}, \eta_5^{(4)}, \eta_6^{(4)}) \\
&\qquad = (0, 0, y^{(4)}, -4y^{(4)}, 2yy^{(4)} + 8y'y^{(3)} + 6(y^{(2)})^2, -8xy^{(4)} - 12y^{(3)}).
\end{aligned}$$

Therefore the determinant (41) is $D = -4y'\left(2y'y^{(3)} - 3\left(y^{(2)}\right)^2\right)^2$. From Theorem 3.1 it follows that there are two ordinary differential equations of order at most 4 which allow the group of symmetries corresponding to the Lie algebra $\mathbf{g} = \mathbf{sl_2}(\mathbb{R}) \oplus \mathbf{sl_2}(\mathbb{R})$. These differential equations are $y' = 0$ and the equation given by (23).
Finally, to obtain the solutions of (23) we introduce the new variable $z := y'$. Then we have

$$2zz^{(2)} - 3(z')^2 = 0 \Longleftrightarrow \frac{z^{(2)}}{z'} = \frac{3}{2}\frac{z'}{z} \Longleftrightarrow$$

$$\left(\ln z' - \frac{3}{2}(\ln z)\right)' = 0 \Longleftrightarrow \ln\left(\frac{z'}{z^{\frac{3}{2}}}\right) = c,\ c \in \mathbb{R} \text{ is a constant} \Longleftrightarrow$$

$$\frac{z'}{z^{\frac{3}{2}}} = e^c = d,\ d \in \mathbb{R} \text{ is a constant} \Longleftrightarrow$$

$$-2z^{-\frac{1}{2}} = dx + e,\ d, e \in \mathbb{R} \text{ are constants} \Longleftrightarrow y' = \left(\frac{-2}{dx + e}\right)^2 \Longleftrightarrow$$

$$y(x) = \int \frac{4}{(dx + e)^2} dx = -\frac{4}{d(dx + e)} + a,\ a, d, e \in \mathbb{R} \text{ are constants.}$$

Example 4.3. The Lie algebra $\mathbf{g} = \mathbf{sl_3}(\mathbb{R})$ is generated by the vector fields given in (26). Hence we have

$$(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8) = (1, 0, 0, 0, x, y, x^2, xy),$$
$$(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \eta_8) = (0, 1, x, y, 0, 0, xy, y^2).$$

Applying the formulas (34), (35), (36), (37) we obtain

$$\begin{aligned}
&(\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}, \eta_4^{(1)}, \eta_5^{(1)}, \eta_6^{(1)}, \eta_7^{(1)}, \eta_8^{(1)}) \\
&\qquad = (0, 0, 1, y', -y', -(y')^2, y - xy', yy' - x(y')^2), \\
&(\eta_1^{(2)}, \eta_2^{(2)}, \eta_3^{(2)}, \eta_4^{(2)}, \eta_5^{(2)}, \eta_6^{(2)}, \eta_7^{(2)}, \eta_8^{(2)}) \\
&\qquad = (0, 0, 0, y^{(2)}, -2y^{(2)}, -3y'y^{(2)}, -3xy^{(2)}, -3xy'y^{(2)}), \\
&(\eta_1^{(3)}, \eta_2^{(3)}, \eta_3^{(3)}, \eta_4^{(3)}, \eta_5^{(3)}, \eta_6^{(3)}, \eta_7^{(3)}, \eta_8^{(3)}) \\
&\qquad = (0, 0, 0, y^{(3)}, -3y^{(3)}, -3(y^{(2)})^2 - 4y'y^{(3)}, -5xy^{(3)} - 3y^{(2)}, \\
&\qquad - yy^{(3)} - 3y'y^{(2)} - 3x(y^{(2)})^2 - 4xy'y^{(3)}), \\
&(\eta_1^{(4)}, \eta_2^{(4)}, \eta_3^{(4)}, \eta_4^{(4)}, \eta_5^{(4)}, \eta_6^{(4)}, \eta_7^{(4)}, \eta_8^{(4)}) \\
&\qquad = (0, 0, 0, y^{(4)}, -4y^{(4)}, -10y^{(2)}y^{(3)} - 5y'y^{(4)}, -8(y^{(3)}) - 7xy^{(4)}, \\
&\qquad - 6(y^{(2)})^2 - 8yy^{(3)} - 10xy^{(2)}y^{(3)} - 5xy'y^{(4)} - 2yy^{(4)}).
\end{aligned}$$

Furthermore computing $\eta_i^{(5)}$ and $\eta_i^{(6)}$ one gets

$$\begin{aligned}
&(\eta_1^{(5)}, \eta_2^{(5)}, \eta_3^{(5)}, \eta_4^{(5)}, \eta_5^{(5)}, \eta_6^{(5)}, \eta_7^{(5)}, \eta_8^{(5)}) \\
&\quad = (0, 0, 0, y^{(5)}, -5y^{(5)}, -15y^{(2)}y^{(4)} - 10(y^{(3)})^2 - 6y'y^{(5)}, -15y^{(4)} - 9xy^{(5)}, \\
&\qquad - 30y^{(2)}y^{(3)} - 15y'y^{(4)} - 15xy^{(2)}y^{(4)} - 10x(y^{(3)})^2 - 6xy'y^{(5)} - 3yy^{(5)}), \\
&(\eta_1^{(6)}, \eta_2^{(6)}, \eta_3^{(6)}, \eta_4^{(6)}, \eta_5^{(6)}, \eta_6^{(6)}, \eta_7^{(6)}, \eta_8^{(6)}) \\
&\quad = (0, 0, 0, y^{(6)}, -6y^{(6)}, -21y^{(2)}y^{(5)} - 35y^{(3)}y^{(4)} - 7y'y^{(6)}, \\
&\quad - 24y^{(5)} - 11xy^{(6)}, -60y^{(2)}y^{(4)} - 40(yy^{(3)})^2 - 24y'y^{(5)} - \\
&\quad - 21xy^{(2)}y^{(5)} - 35xy^{(3)}y^{(4)} - 7xy'y^{(6)} - 4yy^{(6)}).
\end{aligned}$$

Hence the determinant (41) is

$$D = -2y^{(2)} \left(9 \left(y^{(2)} \right)^2 y^{(5)} - 45y^{(2)}y^{(3)}y^{(4)} + 40 \left(y^{(3)} \right)^3 \right)^2.$$

According to Theorem 3.1 there are two ordinary differential equations of order at most 6 which are invariant under the group of symmetries belonging to the Lie algebra $\mathbf{sl_3}(\mathbb{R})$. These are given by (27).

4.2. **Examples for $\mathbf{sl_2}(\mathbb{R})$.** In [11, p. 501] three representations are given of the Lie algebra $\mathbf{g} = \mathbf{sl_2}(\mathbb{R})$. These are exactly those representations for which the action of the corresponding Lie group on the plane is imprimitive (cf. [6, p. 341]). We consider them in Examples 4.4, 4.5, 4.6. There is one representative of $\mathbf{g} = \mathbf{sl_2}(\mathbb{R})$ such that the corresponding group action is primitive on the plane (cf. [6, p. 341], see also [13, (16), p. 374]). We deal with this case in Example 4.8.

Again, we apply the notations introduced in Section 3. Throughout this subsection we not only determine differential equations of order m such that $m \leq r-2$, but also those explicit equations where $m = r-1$ (where r is the number of generators of the tangential Lie algebra of the given Lie group). We solve some of these differential equations applying the known symmetries.

Section 4.2 shows that if a Lie algebra is given, then the differential equation admitting this Lie algebra as infinitesimal generators of symmetries depends not only on the isomorphism class of this Lie algebra but also its representation, that is its action on the space it is represented in. For Examples 4.4, 4.5 and 4.8 there exist one or a family of second order ODEs admitting the particular Lie algebra. However, for Example 4.6 there are no second order ODEs admitting the particular Lie algebra.

Example 4.4. Firstly we consider the representation of the Lie algebra $\mathbf{g}_1 = \mathbf{sl_2}(\mathbb{R})$ generated by the vector fields (1). Hence one gets

$$(\phi_1, \phi_2, \phi_3) = (1, x, x^2), (\eta_1, \eta_2, \eta_3) = (1, y, y^2). \tag{48}$$

Using the formulas (34), (35) we have

$$(\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}) = (0, 0, (2y-2x)y'), \tag{49}$$

$$(\eta_1^{(2)}, \eta_2^{(2)}, \eta_3^{(2)}) = (0, -y^{(2)}, -2y' + 2(y')^2 - 4xy^{(2)} + 2yy^{(2)}). \tag{50}$$

Since the determinant (41) is $D = 2(y-x)^2 y'$, one can see that the first order differential equation $y' = 0$ is invariant under the symmetry group corresponding to the Lie algebra $\mathbf{g}_1$ given by (1) (cf. Theorem 3.1).

To find the second order ordinary differential equations allowing a group of symmetries whose tangential Lie algebra $\mathbf{g}_1$, we assume an explicit form:

$$f(x, y, y', y^{(2)}) = y^{(2)} - g(x, y, y') = 0,$$

Now, we have to solve the system of partial differential equations given by (38) for the case $m = 2$, $r = 3$. Putting (48), (49), (50) into (38) we obtain the following system of partial differential equations:

$$\frac{\partial g}{\partial x}+\frac{\partial g}{\partial y}=0, \tag{51}$$

$$x\frac{\partial g}{\partial x}+y\frac{\partial g}{\partial y}+g=0, \tag{52}$$

$$-x^2\frac{\partial g}{\partial x}-y^2\frac{\partial g}{\partial y}-(2y-2x)y'\frac{\partial g}{\partial y'}+2(y')^2-2y'+(2y-4x)g=0. \tag{53}$$

From (51) it follows that $g=g(x-y,y')$. Taking $u=x-y$ as a new variable one has $\frac{\partial g}{\partial x}=\frac{\partial g}{\partial u}$, $\frac{\partial g}{\partial y}=-\frac{\partial g}{\partial u}$. Putting these into the partial differential equation (52) it reduces to $g+u\frac{\partial g}{\partial u}=0$. Hence g has the form

$$g=\frac{h(y')}{u}. \tag{54}$$

Therefore, one obtains

$$\frac{\partial g}{\partial y'}=\frac{h'}{u},\ \frac{\partial g}{\partial x}=-\frac{h}{u^2},\ \frac{\partial g}{\partial y}=\frac{h}{u^2}. \tag{55}$$

After substituting (54) and (55) into (53) we have

$$2\left(y'\right)^2-2y'-3h\left(y'\right)+2y'h'\left(y'\right)=0. \tag{56}$$

Putting the new variable $z:=y'\neq 0$ into (56) we have the following first order linear differential equation

$$2h'(z)-3\frac{h(z)}{z}+2z-2=0. \tag{57}$$

The solution of (57) is $h(z)=-2\left(z^2+z+cz^{3/2}\right)$, where c is a real constant. Therefore the invariant differential equations of order 2 have the form (2). (See also Table 8 in [5, p. 151]).

To find the solutions of the second order ordinary differential equation (2) we need a two-dimensional solvable subalgebra of $\mathbf{g}_1$ (see Section 2.1.2 in [5]). In particular, we can use the symmetries corresponding to the subalgebra $\langle X_1,X_2\rangle$ with the Lie bracket $[X_1,X_2]=X_1$. The differential equation (2) can be written into the form

$$\frac{dy'}{dx}=-2\frac{(y')^2+y'+c\,(y')^{3/2}}{x-y}=:\omega(x,y,y'). \tag{58}$$

Introducing the vector field

$$Y=\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}+\omega(x,y,y')\frac{\partial}{\partial y'},$$

where the coefficients of the partial derivatives $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial y'}$ are $\frac{dx}{dx} = 1$, $\frac{dy}{dx} = y'$, $\frac{dy'}{dx} = \omega(x, y, y')$, the equation (58) is equivalent to the linear partial differential equation

$$Y(f) = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \omega(x, y, y')\frac{\partial f}{\partial y'} = 0 \tag{59}$$

of three variables x, y, y'. The equation (59) is invariant under the first prolonged vector fields

$$X_1^{(1)} = X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad X_2^{(1)} = X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

Hence the integration of the differential equation (2) can be reduced to the integration of the equation (59) (cf. [11, Kapitel 20, 4, pp. 457–464]). We use the method of integration given by [11, Kapitel 20, 2, pp. 443–444] for the equation (59). Since the first prolonged vector fields have no term with $\frac{\partial f}{\partial y'}$, a first integral of (59) can be obtained in the following way. The determinant of the coefficient matrix for the system of linear equations

$$\begin{aligned} 0 &= \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \omega(x, y, y')\frac{\partial f}{\partial y'} \\ 0 &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \\ 0 &= x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \end{aligned}$$

of variables $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial y'}$ is

$$D_1 = \begin{vmatrix} 1 & y' & \omega \\ 1 & 1 & 0 \\ x & y & 0 \end{vmatrix} = (y - x)\omega \neq 0.$$

Consider the determinant

$$D_2 = \begin{vmatrix} dx & dy & dy' \\ 1 & y' & \omega \\ 1 & 1 & 0 \end{vmatrix}.$$

A first integral of (59) is

$$\int \frac{D_2}{D_1} = \int \frac{\omega(dy - dx) + (1 - y')dy'}{\omega(y - x)} =$$

$$\int \frac{dx - dy}{x - y} + \frac{(1 - y')dy'}{2((y')^2 + y' + c(y')^{\frac{3}{2}})} = \int \frac{du}{u} + \int \frac{(1 - y')dy'}{2((y')^2 + y' + c(y')^{\frac{3}{2}})} =$$

$$\ln(x - y) + \frac{1}{2}\ln y' - \ln(1 + c\sqrt{y'} + y'),$$

where $u = x - y$ is an invariant of X_1 (cf. [11, p. 533]). By exponentiating this, we obtain that a first integral is

$$\varphi(x,y) = \frac{(x-y)\sqrt{y'}}{1 + c\sqrt{y'} + y'}.$$

Now we can integrate the equation

$$\varphi = \frac{(x-y)\sqrt{y'}}{1 + c\sqrt{y'} + y'} = \text{constant} = \frac{1}{b}.$$

Expressing and solving it for y' we obtain

$$\frac{1}{b(x-y)} = \frac{\sqrt{y'}}{1 + c\sqrt{y'} + y'} \Leftrightarrow$$

$$1 + (c - b(x-y))\sqrt{y'} + y' = 0 \Leftrightarrow$$

$$y' = (v + \sqrt{v^2 - 1})^2 = v^2 + 2v\sqrt{v^2-1} + v^2 - 1, \text{ with } v := \frac{b(x-y) - c}{2}.$$

As $\frac{dv}{dx} = v' = \frac{b}{2}(1 - y')$ we get $\frac{v'}{b} = \frac{1}{2}(1 - y') = 1 - v^2 - v\sqrt{v^2-1}$. Solving this first order separable differential equation for v we have

$$\int \frac{dv}{1 - v^2 - v\sqrt{v^2-1}} = \int b dx \Longleftrightarrow$$

$$\frac{1}{v + \sqrt{v^2-1}} = bx + a \Leftrightarrow$$

$$2v = bx + a + \frac{1}{bx + a}.$$

Therefore for the solutions $y(x)$ of (2) one obtains

$$by(x) = -\frac{1}{bx + a} - a - c,$$

where $a, b, c \in \mathbb{R}$ are constants.

Example 4.5. Secondly, the Lie algebra $\mathbf{g}_2 = \mathbf{sl_2}(\mathbb{R})$ is generated by the vector fields given in (3). Hence one has

$$(\phi_1, \phi_2, \phi_3) = (1, 2x, x^2), \ (\eta_1, \eta_2, \eta_3) = (0, y, xy). \tag{60}$$

Applying the formulas (34), (35) we obtain

$$(\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}) = (0, -y', y - xy'), \tag{61}$$

$$(\eta_1^{(2)}, \eta_2^{(2)}, \eta_3^{(2)}) = (0, -3y^{(2)}, -3xy^{(2)}). \tag{62}$$

As the determinant (41) is $D = y^2$, Theorem 3.1 does not yield any differential equation of order 1 which is invariant under the group of symmetries corresponding to the Lie algebra $\mathbf{g}_2$ given by (3).

To obtain the second order ordinary differential equations of the form $y^{(2)} - g(x, y, y') = 0$ allowing the Lie algebra $\mathbf{g}_2$ as the tangential Lie algebra of a subgroup of their symmetries, we have to solve (38) for the case $m = 2$, $r = 3$, $f(x, y, y', y^{(2)}) = y^{(2)} - g(x, y, y') = 0$. Using (60), (61), (62) the system (38) of partial differential equations is equivalent to

$$\frac{\partial g}{\partial x} = 0 \tag{63}$$

$$-3g - y\frac{\partial g}{\partial y} - 2x\frac{\partial g}{\partial x} + y'\frac{\partial g}{\partial y'} = 0 \tag{64}$$

$$3xg + (y - xy')\frac{\partial g}{\partial y'} + x^2\frac{\partial g}{\partial x} + xy\frac{\partial g}{\partial y} = 0. \tag{65}$$

It can be seen that the differential equation $y^{(2)} = 0$, i.e. $g(x, y, y') = 0$, satisfies equations (63), (64), (65). We may assume that $g(x, y, y') \neq 0$. From (63) it follows that the function g does not depend on the variable x, i.e $g(x, y, y') = g(y, y')$. Therefore equation (64) reduces to

$$-3 + y'\frac{\partial \ln g}{\partial y'} - y\frac{\partial \ln g}{\partial y} = 0. \tag{66}$$

This is equivalent to the following ordinary differential equation (characteristic equation):

$$\frac{dy'}{y'} = \frac{dy}{-y} = \frac{d\ln g}{3} = 0. \tag{67}$$

Equation (67) yields the first integrals $yy' = c_1$ and $\frac{g}{y'^3} = c_2$. Hence the function g has the form $g = (y')^3 f(yy')$. Introducing the new variable $z := yy'$ for the function $f(yy') = f(z)$ one gets $\frac{\partial f}{\partial y} = y'\frac{df}{dz}$ and $\frac{\partial f}{\partial y'} = y\frac{df}{dz}$. From these, equation (66) reduces to $3f(z) + zf'(z) = 0$. The solution of this last differential equation is $f(z) = az^{-3}$, where a is a constant. Therefore one has $g = ay^{-3}$ and hence the differential equations (4 of order 2 are invariant under the group of symmetries belonging to the Lie algebra $\mathbf{g}_2$ given by (3). (See also Table 8 in [5], p. 151.)

To find the solutions of these differential equations we can multiply both sides of (4) by $2y'$. Then we obtain

$$2y'y^{(2)} - \frac{2ay'}{y^3} = 0 \Longleftrightarrow (y')^2 + \frac{a}{y^2} = \text{const.} = b \Longleftrightarrow$$
$$y' = \frac{\sqrt{by^2 - a}}{y} \Longleftrightarrow \int \frac{y\,dy}{\sqrt{by^2 - a}} = \int 1 \cdot dx \Longleftrightarrow$$

$$\frac{1}{b}\sqrt{by^2-a} = x+c, \text{ where c is a constant} \Longleftrightarrow$$
$$by^2 = b^2(x+c)^2 + a, \; a,b,c \in \mathbb{R}.$$

Example 4.6. Thirdly, the Lie algebra $\mathbf{g_3} = \mathbf{sl_2}(\mathbb{R})$ is generated by the vector fields (5). Hence one has

$$(\phi_1,\phi_2,\phi_3) = (0,0,0), \; (\eta_1,\eta_2,\eta_3) = (1,y,y^2). \tag{68}$$

Using (34), (35), (36) we get

$$(\eta_1^{(1)},\eta_2^{(1)},\eta_3^{(1)}) = (0,y',2yy'), \tag{69}$$
$$(\eta_1^{(2)},\eta_2^{(2)},\eta_3^{(2)}) = (0,y^{(2)},2(y')^2+2yy^{(2)}), \tag{70}$$
$$(\eta_1^{(3)},\eta_2^{(3)},\eta_3^{(3)}) = (0,y^{(3)},6y'y^{(2)}+2yy^{(3)}). \tag{71}$$

The determinant D in (41) is identically 0, thus one needs to solve the following system of partial differential equations for f:

$$\begin{aligned} \tfrac{\partial f}{\partial y} &= 0, \\ y\tfrac{\partial f}{\partial y} + y'\tfrac{\partial f}{\partial y'} &= 0, \\ y^2\tfrac{\partial f}{\partial y} + 2yy'\tfrac{\partial f}{\partial y'} &= 0. \end{aligned}$$

From the first equation we obtain that f does not depend on y, that is $f = f(x,y')$. The third equation is redundant, and finally, from the second we have $y'\frac{\partial f}{\partial y'} = 0$, that is $f = g(x)y'$ for arbitrary real function g. Thus the differential equations look like $g(x)y' = 0$. However, this means that only the trivial differential equation $y' = 0$ is invariant under the group of symmetries corresponding to the Lie algebra $\mathbf{g_3}$.

Now we show that there are no differential equations of order two allowing as a group of their symmetries the Lie group G whose tangential Lie algebra is $\mathbf{g_3}$. If there exists a differential equation $f\left(x,y,y',y^{(2)}\right) = 0$ which admits the group G as its symmetries, then using (38), (68), (69), (70) the function f would be satisfied the following system of partial differential equations:

$$\frac{\partial f}{\partial y} = 0 \tag{72}$$

$$y\frac{\partial f}{\partial y} + y'\frac{\partial f}{\partial y'} + y^{(2)}\frac{\partial f}{\partial y^{(2)}} = 0 \tag{73}$$

$$y^2\frac{\partial f}{\partial y} + 2yy'\frac{\partial f}{\partial y'} + \left(2\left(y'\right)^2 + 2yy^{(2)}\right)\frac{\partial f}{\partial y^{(2)}} = 0. \tag{74}$$

From (72) it follows that f is independent of the variable y, that is $f = f\left(x, y', y^{(2)}\right)$. Applying this to equations (73), (74) we have

$$y' \frac{\partial f}{\partial y'} + y^{(2)} \frac{\partial f}{\partial y^{(2)}} = 0 \tag{75}$$

$$2yy' \frac{\partial f}{\partial y'} + \left(2\left(y'\right)^2 + 2yy^{(2)}\right) \frac{\partial f}{\partial y^{(2)}} = 0. \tag{76}$$

Multiplying equation (75) by $-2y$ and adding the obtained equation to (76) we get

$$2\left(y'\right)^2 \frac{\partial f}{\partial y^{(2)}} = 0.$$

Hence the function f does not depend on the variable $y^{(2)}$, i.e. $f = f(x, y')$, which is a contradiction to the assumption that f is a differential equation of order 2.

To obtain the third order ordinary differential equations of the form $y^{(3)} - g(x, y, y', y^{(2)}) = 0$ allowing the Lie algebra $\mathbf{g}_3$ as a subalgebra of the Lie algebra tangential to their symmetry group, one has to solve the system (38) of partial differential equations for the case $f(x, y, y', y^{(2)}, y^{(3)}) = y^{(3)} - g(x, y, y', y^{(2)}) = 0$. Using (68), (69), (70) the system (38) is equivalent to the following system of partial differential equations:

$$\frac{\partial g}{\partial y} = 0 \tag{77}$$

$$-y\frac{\partial g}{\partial y} - y' \frac{\partial g}{\partial y'} - y^{(2)} \frac{\partial g}{\partial y^{(2)}} + g = 0 \tag{78}$$

$$-y^2\frac{\partial g}{\partial y} - 2yy' \frac{\partial g}{\partial y'} - \left(2\left(y'\right)^2 + 2yy^{(2)}\right) \frac{\partial g}{\partial y^{(2)}} + 6y'y^{(2)} + 2yg = 0. \tag{79}$$

From (77) it follows that $g = g(x, y', y^{(2)})$. Using this, equation (78) reduces to

$$y' \frac{\partial g}{\partial y'} + y^{(2)} \frac{\partial g}{\partial y^{(2)}} = g \tag{80}$$

and therefore equation (79) reduces to

$$\frac{3y^{(2)}}{y'} = \frac{\partial g}{\partial y^{(2)}}. \tag{81}$$

Equation (81) yields that $g = \frac{3\left(y^{(2)}\right)^2}{2y'} + h\left(x, y'\right)$. Substituting this form of g into (80) we obtain the partial differential equation $y' \frac{\partial h(x,y')}{\partial y'} = h(x, y')$. It yields that $h(x, y') = y'f(x)$. Therefore the ordinary differential equation (6) of order 3 is invariant under the group of symmetries corresponding to the Lie algebra $\mathbf{g}_3$ for arbitrary real function $f(x)$. (See also Table 8 in [5, p. 151]).

The solution of the differential equation (6) leads to the solution of a Ricatti differential equation (cf. [8, Section 4.9, p. 21]). Let $z(x) := y'(x)$. Substituting this into the differential equation (6) one gets

$$z^{(2)} - \frac{3(z')^2}{2z} - zf(x) = 0 \Longleftrightarrow \tag{82}$$

$$\frac{d}{dx}\left(\frac{z'}{z}\right) = f(x) + \frac{1}{2}\left(\frac{z'}{z}\right)^2. \tag{83}$$

Putting $l(x) := \frac{z'}{z}$ the equation (83) is equivalent to the Ricatti differential equation

$$l' = \frac{1}{2}l^2 + f(x). \tag{84}$$

Let $v := \frac{1}{2}l$. For v we obtain the Ricatti differential equation $v' = v^2 + \frac{1}{2}f(x)$. Substituting $v = -\frac{u'}{u}$ the function u satisfies the second order linear differential equation

$$0 = u^{(2)} + \frac{1}{2}uf(x). \tag{85}$$

With the solutions $\tilde{u}$ of (85) we obtain that the solutions $\tilde{l}$ of (84) have the form $\tilde{l} = -2\frac{\tilde{u}'}{\tilde{u}}$. With the solution $\tilde{l}$ of (84) we obtain the solution $\tilde{z}$ of (82) in the form $\tilde{z} = e^{\int \tilde{l}dx}$, and hence the solution $\tilde{y}$ of (6) in the form

$$\tilde{y} = \int e^{\int \tilde{l}dx}dx.$$

Remark 4.7. In [17] the fluid draining equation

$$w^{(3)} = w^{-2}$$

is considered. It can be rewritten to the Riccati equation $l' = \frac{1}{2}l^2 + x$, which is exactly the same as (6) for $f\colon x \mapsto x$. In particular, Nucci in [17] found an isomorphic representation of $\mathbf{sl_2}(\mathbb{R})$ as is described in Example 4.8, and used it to solve the fluid draining equation. This is our final example in this subsection.

Example 4.8. Now we deal with the Lie algebra $\mathbf{g}_4 = \mathbf{sl_2}(\mathbb{R})$ generated by the vector fields given in (7). Hence one has

$$(\phi_1, \phi_2, \phi_3) = (1, x, x^2 - y^2), (\eta_1, \eta_2, \eta_3) = (0, y, 2xy). \tag{86}$$

Applying (34), (35), (36) we obtain

$$(\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}) = (0, 0, 2y(1 + (y')^2), \tag{87}$$

$$(\eta_1^{(2)}, \eta_2^{(2)}, \eta_3^{(2)}) = (0, -y^{(2)}, 2y' + 2(y')^3 + 6yy'y^{(2)} - 2xy^{(2)}). \tag{88}$$

Since the determinant (41) is $D = 2y^2(1 + (y')^2)$, according to Theorem 3.1 there does not exist any first order differential equation which admits a Lie group of symmetries having the Lie algebra $\mathbf{g}_4$ as its tangential Lie algebra.

To find the second order ordinary differential equations of the explicit form $y^{(2)} - g(x, y, y') = 0$ which allow a Lie group of symmetries having Lie algebra $\mathbf{g}_4$ as its Lie algebra, one has to solve the following system of partial differential equations:

$$\frac{\partial g}{\partial x} = 0 \tag{89}$$

$$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} + g = 0 \tag{90}$$

$$\begin{aligned} -\left(x^2 - y^2\right)\frac{\partial g}{\partial x} - 2xy\frac{\partial g}{\partial y} - 2y\left(1 + \left(y'\right)^2\right)\frac{\partial g}{\partial y'} \\ +2y' + 2\left(y'\right)^3 + 6yy'g - 2xg = 0 \end{aligned} \tag{91}$$

which is obtained if we apply (38) for the case $m = 2$, $r = 3$, $f(x, y, y', y^{(2)}) = y^{(2)} - g(x, y, y') = 0$, and use (86), (87), (88). It follows from (89) that the function g does not depend on the variable x, i.e $g(x, y, y') = g(y, y')$. Therefore equation (90) reduces to

$$-y\frac{\partial g}{\partial y} = g.$$

Hence we may assume that the function g has the form $g = \frac{h(y')}{y}$. Putting this form into equation (91) and using the fact that g is independent of x, after simplification we obtain the following linear differential equation for the function $h(y')$

$$y'\left(1 + \left(y'\right)^2\right) + 3y'h\left(y'\right) = \left(1 + \left(y'\right)^2\right)h'\left(y'\right). \tag{92}$$

The solution of the separable differential equation $\frac{3y'}{1+(y')^2} = \frac{h'(y')}{h(y')}$ is $h(y') = d\left(1 + (y')^2\right)^{3/2}$, where d is a real constant. Applying this we obtain the following solution for (92):

$$h\left(y'\right) = -\left(1 + \left(y'\right)^2\right) + d\left(1 + \left(y'\right)^2\right)^{3/2},$$

where d is a real constant. Therefore the ordinary differential equations (8) of order 2 are invariant under the group of symmetries corresponding to the Lie algebra $\mathbf{g}_4$.

Analogously to Example 4.4, finding the solutions of the second order ordinary differential equation (8) we can use the symmetries corresponding to the 2-dimensional subalgebra $\langle X_1, X_2\rangle$ with the Lie bracket $[X_1, X_2] = X_1$. The differential equation (8) can be written into the form

$$\frac{dy'}{dx} = -\frac{1}{y}\left(1+(y')^2\right)\left(1-d\left(1+(y')^2\right)^{1/2}\right) =: \omega(y,y'). \tag{93}$$

Introducing the vector field

$$Y = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + \omega(y,y')\frac{\partial}{\partial y'},$$

where the coefficients of the partial derivatives $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial y'}$ are $\frac{dx}{dx} = 1$, $\frac{dy}{dx} = y'$, $\frac{dy'}{dx} = \omega(y,y')$, the equation (8) is equivalent to the linear partial differential equation

$$Y(f) = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \omega(y,y')\frac{\partial f}{\partial y'} = 0 \tag{94}$$

of three variables x, y, y'. The equation (94) is invariant under the first prolonged vector fields

$$X_1^{(1)} = X_1 = \frac{\partial}{\partial x}, \quad X_2^{(1)} = X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

Hence the integration of the differential equation (8) can be reduced to the integration of the equation (94) (cf. [11, Kapitel 20, 4, pp. 457–464]) and we apply again the method of integration given by [11, Kapitel 20, 2, pp. 443–444] for the equation (94). Since the first prolonged vector fields have no term with $\frac{\partial f}{\partial y'}$, a first integral of (8) can be obtained in the following way. The determinant of the coefficient matrix for the system of linear equations

$$\begin{aligned}
0 &= \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \omega(y,y')\frac{\partial f}{\partial y'}\\
0 &= \frac{\partial f}{\partial x}\\
0 &= x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}
\end{aligned}$$

of variables $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial y'}$ is

$$D_1 = \begin{vmatrix} 1 & y' & \omega \\ 1 & 0 & 0 \\ x & y & 0 \end{vmatrix} = \omega y \neq 0.$$

Consider the determinant

$$D_2 = \begin{vmatrix} dx & dy & dy' \\ 1 & y' & \omega \\ 1 & 0 & 0 \end{vmatrix} = \omega dy - y'dy'.$$

A first integral of (94) is

$$\int \frac{D_2}{D_1} = \int \frac{\omega dy - y'dy'}{\omega y} = \int \frac{dy}{y} + \frac{y'dy'}{\left(1+(y')^2\right)\left(1-d\sqrt{1+(y')^2}\right)}$$

$$= \int \frac{dy}{y} + \int \frac{y'}{\sqrt{1+(y')^2}\left(\sqrt{1+(y')^2}-d\right)} dy'$$

$$= \ln y + \ln\left(\sqrt{1+(y')^2}-d\right).$$

Exponentiating, we obtain the first integral

$$\varphi = y\left(\sqrt{1+(y')^2}-d\right).$$

Now we can integrate the equation

$$\varphi = y\left(\sqrt{1+(y')^2}-d\right) = \text{constant} = c.$$

Expressing y' from the last relation and solving it for y' we obtain

$$y' = \sqrt{\left(\frac{c}{y}+d\right)^2 - 1},$$

where c is a real constant. Therefore for the solution $y(x)$ of (8) we have

$$\frac{\sqrt{c^2+2cdy+(d^2-1)\,y^2}}{d^2-1} - \frac{cd\ln\left(\frac{(d^2-1)y+cd}{\sqrt{d^2-1}} + \sqrt{c^2+2cdy+(d^2-1)\,y^2}\right)}{(d^2-1)^{\frac{3}{2}}} = x + a,$$

where a, c, d are real constants.

4.3. **Differential equations for $\mathbf{so_3}(\mathbb{R})$.**

Example 4.9. The Lie algebra $\mathbf{g} = \mathbf{so_3}(\mathbb{R}) \cong \mathbf{su_2}(\mathbb{C})$ which has no 2-dimensional subalgebra is generated by the vector fields (9). Hence one has

$$(95) \qquad (\phi_1, \phi_2, \phi_3) = (1+x^2, xy, y), \ (\eta_1, \eta_2, \eta_3) = (xy, 1+y^2, -x).$$

Applying formulas (34), (35) to (95) we have

$$(\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}) = (y - xy', yy' - x(y')^2, -1 - (y')^2), \tag{96}$$

$$(\eta_1^{(2)}, \eta_2^{(2)}, \eta_3^{(2)}) = (-3xy^{(2)}, -3xy'y^{(2)}, -3y'y^{(2)}). \tag{97}$$

There does not exist any first order ordinary differential equation which admits the Lie algebra $\mathbf{g} = \mathbf{so_3}(\mathbb{R})$ as the Lie algebra for a group of their symmetries, because the determinant $D = -(1 + x^2 + y^2)(x^2(y')^2 - 2xyy' + 1 + y^2)$ cannot be 0, since $x^2(y')^2 - 2xyy' + 1 + y^2 = x^2\left(\left(y' - \frac{y}{x}\right)^2 + \frac{1}{x^2}\right) > 0$.

Finding the second order ordinary differential equations of the explicit form $y^{(2)} - g(x, y, y') = 0$ which allow the group of symmetries corresponding to the Lie algebra $\mathbf{g} = \mathbf{so_3}(\mathbb{R})$ we have to solve the system (38) of partial differential equations for the case $m = 2$, $r = 3$, $f(x, y, y', y^{(2)}) = y^{(2)} - g(x, y, y') = 0$. Putting (95), (96), (97) into (38) we obtain the following system of partial differential equations:

$$\left(1 + x^2\right)\frac{\partial g}{\partial x} + xy\frac{\partial g}{\partial y} + \left(y - xy'\right)\frac{\partial g}{\partial y'} + 3xg = 0 \tag{98}$$

$$xy\frac{\partial g}{\partial x} + \left(1 + y^2\right)\frac{\partial g}{\partial y} + \left(yy' - x\left(y'\right)^2\right)\frac{\partial g}{\partial y'} + 3xy'g = 0 \tag{99}$$

$$-y\frac{\partial g}{\partial x} + x\frac{\partial g}{\partial y} + \left(1 + \left(y'\right)^2\right)\frac{\partial g}{\partial y'} - 3y'g = 0. \tag{100}$$

For $g = 0$ the partial differential equations (98), (99), (100) are satisfied. Hence the differential equation $y^{(2)} = 0$ allows the group of symmetries corresponding to the Lie algebra $\mathbf{g} = \mathbf{so_3}(\mathbb{R})$. We may assume that $g \neq 0$. Multiplying (98) by y' and subtracting (99) from it we have

$$\left(\left(1 + x^2\right)y' - xy\right)\frac{\partial g}{\partial x} + \left(xyy' - \left(1 + y^2\right)\right)\frac{\partial g}{\partial y} = 0. \tag{101}$$

Multiplying (100) by x and adding the new equation to (99) one has

$$(1 + x^2 + y^2)\frac{\partial g}{\partial y} + (x + yy')\frac{\partial g}{\partial y'} = 0. \tag{102}$$

Putting the expressions

$$\frac{\partial g}{\partial y} = -\frac{(x + yy')}{(1 + x^2 + y^2)}\frac{\partial g}{\partial y'},$$
$$\frac{\partial g}{\partial x} = \frac{\left(\left(1 + y^2\right) - xyy'\right)}{\left(\left(1 + x^2\right)y' - xy\right)}\frac{\partial g}{\partial y}$$

into (100) we obtain

$$\left(1+y^2-2xyy'+\left(1+x^2\right)\left(y'\right)^2\right)\frac{\partial g}{\partial y'}=3\left(\left(1+x^2\right)y'-xy\right)g \Longleftrightarrow$$

$$\frac{1}{g}\frac{\partial g}{\partial y'}=\frac{3}{2}\frac{\partial \ln\left(1+y^2-2xyy'+\left(1+x^2\right)\left(y'\right)^2\right)}{\partial y'} \Longleftrightarrow$$

$$\frac{\partial \ln\left(\frac{g}{\left(1+y^2-2xyy'+(1+x^2)(y')^2\right)^{3/2}}\right)}{\partial y'}=0 \Longleftrightarrow$$

$$g=K(x,y)\left(1+y^2-2xyy'+\left(1+x^2\right)\left(y'\right)^2\right)^{3/2}. \tag{103}$$

After substituting (103) into (102) and simplification we get

$$\frac{1}{K(x,y)}\frac{\partial K(x,y)}{\partial y}=-\frac{3y}{(1+x^2+y^2)} \Longleftrightarrow$$

$$\frac{\partial \ln\left(K(x,y)\left(1+x^2+y^2\right)^{3/2}\right)}{\partial y}=0 \Longleftrightarrow$$

$$K(x,y)=\frac{U(x)}{(1+x^2+y^2)^{3/2}}, \text{ or equivalently}$$

$$g=U(x)\left(\frac{1+y^2-2xyy'+\left(1+x^2\right)\left(y'\right)^2}{1+x^2+y^2}\right)^{3/2}. \tag{104}$$

Putting (104) into (101) after some calculations we obtain

$$U'(x)\frac{1+y^2-2xyy'+(1+x^2)(y')^2}{1+x^2+y^2}=0.$$

This yields that $U(x)=c$, $c\in\mathbb{R}$. Therefore the second order differential equations (10) are invariant under a symmetry group whose Lie algebra $\mathbf{g}=\mathbf{so_3}(\mathbb{R})$. (See also Table 8 in [5, p. 151]).

Remark 4.10. As (10) is a second order ordinary differential equation we need to have a two dimensional solvable Lie subalgebra to being able to solve it (see e.g. [21], Section 2.1.2 in [5]). However, $\mathbf{so_3}(\mathbb{R})$ has no two dimensional subalgebras, as all its proper subalgebras are one dimensional. That is, even though we know a three dimensional subalgebra of the tangential Lie algebra of the Lie symmetry group of the second order equation (10), we cannot apply this knowledge to completely solve it.

4.4. **Further examples.** Finally we apply the method of Lie for some non semi-simple Lie transformation groups of dimension r acting on the (x,y)-plane. Here we only give a list of the ordinary differential equations of order $m\le r-2$ which are invariant under the action of these Lie groups.

Example 4.11. The 3-dimensional Lie algebra $\mathbf{g}_\alpha$ is generated by the vector fields given in (11). (See [6, p. 341], Table 1, Case 1). Since the determinant (41) is $D = -\left(1 + (y')^2\right)$ and cannot be 0, there does not exist any first order ordinary differential equation which allows the Lie algebra $\mathbf{g}_\alpha$ as the tangential Lie algebra for a group of its symmetries.

Example 4.12. The 3-dimensional Lie algebra $\mathbf{g}_\beta$ has as basis elements given in (12). (See [6, p. 341], Table 1, Case 12). Then for the determinant (41) we have $D = y'(\beta - 1)$. Hence, only the differential equation $y' = 0$ admits the Lie algebra $\mathbf{g}_\beta$ as the Lie algebra of a group of its symmetries (cf. Theorem 3.1).

Example 4.13. According to [6, p. 341], Table 1, Case 4, the 4-dimensional Lie algebra $\mathbf{g}$ is generated by the vector fields (13). As $D = -y^{(2)}((y')^2 + 1)$, the only ordinary differential equation of order at most 2 which is invariant under the group of symmetries corresponding to the Lie algebra $\mathbf{g}$ is $y^{(2)} = 0$.

Example 4.14. According to [6, p. 341], Table 1, Case 13, the 4-dimensional Lie algebra $\mathbf{g}$ has as basis elements (14). The determinant (41) of the matrix in Theorem 3.1 is $D = y^{(2)}y'$. Therefore among the ordinary differential equations of order at most 2, the differential equations $y^{(2)} = 0$ and $y' = 0$ allow the group of symmetries corresponding to the Lie algebra $\mathbf{g}$.

Example 4.15. The basis elements of the Lie algebra $\mathbf{sl_2}(\mathbb{R}) \times \mathbb{R}$ are given by (15). (See [6, p. 341], Table 1, Case 14). For the determinant (41) one has $D = 2(y')^2$. Therefore among the at most second order ordinary differential equations the differential equation $y' = 0$ allows the group of symmetries corresponding to the Lie algebra $\mathbf{g}$ (cf. Theorem 3.1).

Example 4.16. The 4-dimensional Lie algebra $\mathbf{g} = \mathbf{gl_2}(\mathbb{R})$ has as basis elements (16). (See [6, p. 341], Table 1, Case 19). Therefore the determinant (41) is $D = -2y^2y^{(2)}$. By Theorem 3.1 among the ordinary differential equations of order at most 2 the equation $y^{(2)} = 0$ admits the group of symmetries corresponding to the Lie algebra $\mathbf{g}$.

Example 4.17. The generators of the Lie algebra $\mathbf{sl_2}(\mathbb{R}) \times L_2$, where L_2 is the 2-dimensional non-abelian Lie algebra are given by (17). (See [6, p. 341], Table 1, Case 15). Since $D = 2y'\left(2y'y^{(3)} - 3\left(y^{(2)}\right)^2\right)$, according to Theorem 3.1 among the at most third order ordinary differential equations the equations given by (18) admit the group of symmetries corresponding to the Lie algebra $\mathbf{sl_2}(\mathbb{R}) \times L_2$.

Example 4.18. According to [6, p. 341], Table 1, Case 5, the Lie algebra $\mathbf{g} = \mathbf{sl_2}(\mathbb{R}) \ltimes \mathbb{R}^2$ has as basis elements in (19). Since the determinant (41) of the matrix in Theorem 3.1 is $D = 9\left(y^{(2)}\right)^3$, among the ordinary differential equations of order at most 3 the equation $y^{(2)} = 0$ allows the group of symmetries corresponding to the Lie algebra $\mathbf{g}$.

Example 4.19. The basis elements of the Lie algebra $\mathbf{g} = \mathbf{gl_2}(\mathbb{R}) \ltimes \mathbb{R}^2$ are given by (24). (See [6, p. 341], Table 1, Case 6). Since the determinant (41) is $D = -2\left(y^{(2)}\right)^2\left(3y^{(4)}y^{(2)} - 5\left(y^{(3)}\right)^2\right)$ among the ordinary differential equations of order at most 4 the equations given by (25) admit the group of symmetries corresponding to the Lie algebra $\mathbf{g}$ (cf. Theorem 3.1).

5. Systems of first order ordinary differential equations which allow a given Lie group as a group of their symmetries

In this section we devise a method based on Lie's original idea in Section 3 to obtain systems of first order ordinary differential equations which admit a given Lie group as a subgroup of their symmetries. Let G be a given r-dimensional real Lie group. First we deal with the case that the Lie algebra $\mathbf{g}$ of G is the direct sum of infinitesimal generators of trivial symmetries and time-preserving symmetries such that both direct factors are non-trivial.

5.1. **Time-dependent symmetries.** Let us use the following notation $y' = \frac{dy}{dx}$, $z' = \frac{dz}{dx}$. Let us consider the following time-dependent system of first order ordinary differential equations in $\mathbb{R}^2$:

$$\begin{aligned} f_1(x, y, z, y', z') &= 0, \\ f_2(x, y, z, y', z') &= 0. \end{aligned} \tag{105}$$

Then the Lie algebra $\mathbf{g}$ of G has as basis elements the following vector fields in the 3-dimensional space:

$$X_i(x, y, z) = \phi_i(x, y, z)\frac{\partial}{\partial x} + \eta_i(x, y, z)\frac{\partial}{\partial y} + \alpha_i(x, y, z)\frac{\partial}{\partial z},\ i = 1, 2, \dots, r.$$

The first prolonged vector field of $X_i(x, y, z)$, $i = 1, 2, \dots, r$, with respect to x has the form

$$X_i^{(1)}(x, y, z, y', z') = X_i + \eta_i^{(1)}(x, y, z, y', z')\frac{\partial}{\partial y'} + \alpha_i^{(1)}(x, y, z, y', z')\frac{\partial}{\partial z'},$$

where

$$\eta_i^{(1)} = \frac{\partial \eta_i}{\partial x} + \frac{\partial \eta_i}{\partial y}y' + \frac{\partial \eta_i}{\partial z}z' - y'\left(\frac{\partial \phi_i}{\partial x} + \frac{\partial \phi_i}{\partial y}y' + \frac{\partial \phi_i}{\partial z}z'\right), \tag{106}$$

$$\alpha_i^{(1)} = \frac{\partial \alpha_i}{\partial x} + \frac{\partial \alpha_i}{\partial y}y' + \frac{\partial \alpha_i}{\partial z}z' - z'\left(\frac{\partial \phi_i}{\partial x} + \frac{\partial \phi_i}{\partial y}y' + \frac{\partial \phi_i}{\partial z}z'\right). \tag{107}$$

The time-dependent system (105) of first order ordinary differential equations allows the given group G of symmetries if and only if the functions f_j, $j = 1, 2$, fulfil the following system of partial differential equations

$$
\begin{array}{rcl}
\phi_1 \frac{\partial f_j}{\partial x} + \eta_1 \frac{\partial f_j}{\partial y} + \alpha_1 \frac{\partial f_j}{\partial z} + \eta_1^{(1)} \frac{\partial f_j}{\partial y'} + \alpha_1^{(1)} \frac{\partial f_j}{\partial z'} & = & 0, \\
\phi_2 \frac{\partial f_j}{\partial x} + \eta_2 \frac{\partial f_j}{\partial y} + \alpha_2 \frac{\partial f_j}{\partial z} + \eta_2^{(1)} \frac{\partial f_j}{\partial y'} + \alpha_2^{(1)} \frac{\partial f_j}{\partial z'} & = & 0, \\
\vdots & & \\
\phi_i \frac{\partial f_j}{\partial x} + \eta_i \frac{\partial f_j}{\partial y} + \alpha_i \frac{\partial f_j}{\partial z} + \eta_i^{(1)} \frac{\partial f_j}{\partial y'} + \alpha_i^{(1)} \frac{\partial f_j}{\partial z'} & = & 0, \\
\vdots & & \\
\phi_r \frac{\partial f_j}{\partial x} + \eta_r \frac{\partial f_j}{\partial y} + \alpha_r \frac{\partial f_j}{\partial z} + \eta_r^{(1)} \frac{\partial f_j}{\partial y'} + \alpha_r^{(1)} \frac{\partial f_j}{\partial z'} & = & 0.
\end{array}
\tag{108}
$$

Let

$$
M = \begin{pmatrix}
\phi_1 & \phi_2 & \phi_3 & \dots & \phi_r \\
\eta_1 & \eta_2 & \eta_3 & \dots & \eta_r \\
\alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_r \\
\eta_1^{(1)} & \eta_2^{(1)} & \eta_3^{(1)} & \dots & \eta_r^{(1)} \\
\alpha_1^{(1)} & \alpha_2^{(1)} & \alpha_3^{(1)} & \dots & \alpha_r^{(1)}
\end{pmatrix}.
$$

Then the system of partial differential equations given by (108) can be treated as the following system of 'linear equations' in the variables $\frac{\partial f_j}{\partial x}$, $\frac{\partial f_j}{\partial y}$, $\frac{\partial f_j}{\partial z}$, $\frac{\partial f_j}{\partial y'}$, $\frac{\partial f_j}{\partial z'}$:

$$
\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial y'} & \frac{\partial f_1}{\partial z'} \end{pmatrix} \cdot M = \begin{pmatrix} 0 & \dots & 0 \end{pmatrix},
$$
$$
\begin{pmatrix} \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial y'} & \frac{\partial f_2}{\partial z'} \end{pmatrix} \cdot M = \begin{pmatrix} 0 & \dots & 0 \end{pmatrix}.
$$

Here, the coefficient matrix M is an $5 \times r$ matrix. Thus, to obtain nontrivial solutions f_j $(j = 1, 2)$ of the system of equations given by (108) it is necessary that the rank of the matrix M is at most 5. However, if rank$M = 4$, then the obtained solutions for the vectors $\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial y'} & \frac{\partial f_1}{\partial z'} \end{pmatrix}$ and $\begin{pmatrix} \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial y'} & \frac{\partial f_2}{\partial z'} \end{pmatrix}$ are linearly dependent, that is the obtained system of differential equations consist of only one equation rather than two for the two dependent variables y and z. Therefore rank$M < 4$ is a more useful requirement.

Now, rank$M \leq r$ always holds, hence if $r < 4$, then the rank condition is automatically satisfied. In such a situation one needs to solve (108) in $\frac{\partial f_j}{\partial x}$, $\frac{\partial f_j}{\partial y}$, $\frac{\partial f_j}{\partial z}$, $\frac{\partial f_j}{\partial y'}$, $\frac{\partial f_j}{\partial z'}$, and see if any solution corresponds to a nontrivial system of differential equations f_1, f_2. In the following we consider the case $r \geq 4$, where the rank condition is equivalent to that every 4×4 subdeterminant of M is zero.

Now, if we reduce ourselves to systems of the form

$$f_1(x,y,z,y',z') = y' - g_1(x,y,z) = 0, \qquad (109)$$
$$f_2(x,y,z,y',z') = z' - g_2(x,y,z) = 0,$$

then the function f_1 does not depend on z' and the function f_2 is independent of y'. Thus for the function f_1, respectively f_2 the system of linear equations obtained from (108) has the coefficient matrix

$$M_1 = \begin{pmatrix} \phi_1 & \dots & \phi_r \\ \eta_1 & \dots & \eta_r \\ \alpha_1 & \dots & \alpha_r \\ \eta_1^{(1)} & \dots & \eta_r^{(1)} \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} \phi_1 & \dots & \phi_r \\ \eta_1 & \dots & \eta_r \\ \alpha_1 & \dots & \alpha_r \\ \alpha_1^{(1)} & \dots & \alpha_r^{(1)} \end{pmatrix},$$

respectively. Now, for having a nontrivial system f_1, f_2, both rank$M_1 < 4$ and rank$M_2 < 4$ has to hold. That is, to obtain non-trivial functions f_1, respectively f_2 as a solution of the system (108), it is necessary that all 4×4 subdeterminants of the $4 \times r$-matrix M_1, respectively M_2 are zero.

Remark 5.1. Let the Lie algebra $\mathbf{g}$ be represented in the n-dimensional space. In Section 5.1 every Lie algebra is represented in the 3-dimensional space with coordinates (x,y,z). If we want to determine the systems of ODEs for which $\mathbf{g}$ is a subalgebra of the tangential Lie algebra of the Lie symmetry group, then we really have n different problems at hand depending on which coordinate represents the time. In the following, we consistently assume everywhere that the time is the 'x' coordinate.

Example 5.2. The Lie algebra $\mathbf{g} = \mathbf{so_3}(\mathbb{R}) \cong \mathbf{su_2}(\mathbb{C})$ is generated by the vector fields (28) in the 3-dimensional space. Hence one gets

$$\phi_i = (-y,0,z),\ \eta_i = (x,-z,0),\ \alpha_i = (0,y,-x). \qquad (110)$$

Applying the formulas (106), (107) to (110) we obtain

$$\eta_i^{(1)} = (1+(y')^2, -z', -y'z'),\ \alpha_i^{(1)} = (y'z', y', -(1+(z')^2)).$$

Therefore the matrix M has the form

$$M = \begin{pmatrix} -y & 0 & z \\ x & -z & 0 \\ 0 & y & -x \\ 1+(y')^2 & -z' & -y'z' \\ y'z' & y' & -\left(1+(z')^2\right) \end{pmatrix}.$$

Multiplying the first column of M with z and the third column of M with y and adding the new first column to the new third column the matrix M transforms to a matrix M^1. Multiplying the second column of M^1 with x

and adding this new column to the third column the matrix M^1 changes into

$$M^2 = \begin{pmatrix} -y & 0 & 0 \\ x & -z & 0 \\ 0 & y & 0 \\ 1+(y')^2 & -z' & -y'z'y + z\left(1+(y')^2\right) - xz' \\ y'z' & y' & -y\left(1+(z')^2\right) + zy'z' + xy' \end{pmatrix}.$$

Therefore the function f_1 and f_2 fulfil the system (108) of partial differential equations precisely if f_1 and f_2 satisfy the following system of partial differential equations

$$\begin{aligned} -y\tfrac{\partial f_1}{\partial x} + x\tfrac{\partial f_1}{\partial y} + 1 + (y')^2 &= 0, \\ -z\tfrac{\partial f_1}{\partial y} + y\tfrac{\partial f_1}{\partial z} - z' &= 0, \\ -y'z'y + z\left(1+(y')^2\right) - xz' &= 0, \\ -y\tfrac{\partial f_2}{\partial x} + x\tfrac{\partial f_2}{\partial y} + y'z' &= 0, \\ -z\tfrac{\partial f_2}{\partial y} + y\tfrac{\partial f_2}{\partial z} + y' &= 0, \\ -y\left(1+(z')^2\right) + zy'z' + xy' &= 0. \end{aligned}$$

To solve the equations

$$z + z(y')^2 - xz' - yy'z' = 0$$
$$-y - y(z')^2 + xy' + zy'z' = 0$$

the first equation gives $z' = \frac{z(1+(y')^2)}{x+yy'}$. Putting this into the second equation, after simplification we obtain

$$\left(y'\right)^3\left(xy^2 + xz^2\right) + \left(y'\right)^2\left(2x^2y - yz^2 - y^3\right) \\ + y'\left(x^3 + xz^2 - 2xy^2\right) - \left(yx^2 + yz^2\right) = 0.$$

The solution of the last equation is $y' = \frac{y}{x}$ and hence $z' = \frac{z}{x}$. Therefore only the system (29) of first order ordinary differential equations allows the group of symmetries corresponding to the Lie algebra $\mathbf{g} = \mathbf{so_3}(\mathbb{R})$.

Example 5.3. The Lie algebra $\mathbf{g} = \mathbf{sl_3}(\mathbb{R})$ is generated by the vector fields (32) in the 3-dimensional space. Each maximal compact subgroup of the group $SL_3(\mathbb{R})$ is isomorphic to the group $SO_3(\mathbb{R})$, therefore we cannot expect any more systems to be invariant than those already obtained in Example 5.2. Using (32) one has

$$\begin{aligned} \phi_i &= (z, 0, 0, x, y, x, 0, 0), \\ \eta_i &= (0, z, x, -y, 0, 0, 0, 0), \\ \alpha_i &= (0, 0, 0, 0, 0, -z, x, y). \end{aligned} \tag{111}$$

Applying (106), (107) to (111) we have

$$\eta_i^{(1)} = (-y'z', z', 1, -2y', -(y')^2, -y', 0, 0),$$
$$\alpha_i^{(1)} = (-(z')^2, 0, 0, -z', -y'z', -2z', 1, y').$$

Therefore the coefficient matrix M of the system of linear equations arising from the system (108) of partial differential equations is

$$M = \begin{pmatrix} z & 0 & 0 & x & y & x & 0 & 0 \\ 0 & z & x & -y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -z & x & y \\ -y'z' & z' & 1 & -2y' & -(y')^2 & -y' & 0 & 0 \\ -(z')^2 & 0 & 0 & -z' & -y'z' & -2z' & 1 & y' \end{pmatrix}.$$

To get non-trivial solutions f_1, f_2 of the system of equations given by (108), it is necessary that all 5×5 subdeterminants of M are zero. The subdeterminant

$$D_{5,1} = \begin{vmatrix} 0 & 0 & x & 0 & 0 \\ z & x & -y & 0 & 0 \\ 0 & 0 & 0 & x & y \\ z' & 1 & -2y' & 0 & 0 \\ 0 & 0 & -z' & 1 & y' \end{vmatrix} = x(-z + xz')(y'x - y)$$

is zero if either $z' = \frac{z}{x}$ or $y' = \frac{y}{x}$. Since the function f_1 does not depend on z' to obtain non-trivial function f_1 as a solution of the system (108) it is necessary that all 4×4 subdeterminants of the matrix

$$M_1 = \begin{pmatrix} \phi_i \\ \eta_i \\ \alpha_i \\ \eta_i^{(1)} \end{pmatrix} = \begin{pmatrix} z & 0 & 0 & x & y & x & 0 & 0 \\ 0 & z & x & -y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -z & x & y \\ -y'z' & z' & 1 & -2y' & -(y')^2 & -y' & 0 & 0 \end{pmatrix}$$

are zero. Consider the following subdeterminants of M_1

$$D_{4,1} = \begin{vmatrix} 0 & 0 & x & 0 \\ z & x & -y & 0 \\ 0 & 0 & 0 & x \\ z' & 1 & -2y' & 0 \end{vmatrix} = x^2(z'x - z)$$

$$D_{4,2} = \begin{vmatrix} 0 & x & x & 0 \\ x & -y & 0 & 0 \\ 0 & 0 & -z & x \\ 1 & -2y' & -y' & 0 \end{vmatrix} = -x^2(y - y'x).$$

These are zero if $y' = \frac{y}{x}$ and $z' = \frac{z}{x}$ are both satisfied. As the functions $f_1 = y' - \frac{y}{x}$ and $f_2 = z' - \frac{z}{x}$ fulfil the system of partial differential equations given by (108), we obtain that only the system (29) of first order ordinary differential equations is invariant under the action of the symmetries belonging to the Lie algebra $\mathbf{g} = \mathbf{sl_3}(\mathbb{R})$.

Example 5.4. According to [7] in [12, p. 134, 140] the infinitesimal generators which form a 6-dimensional Lie algebra isomorphic to $\mathbf{g} = \mathbf{sl_2}(\mathbb{R}) \oplus \mathbf{sl_2}(\mathbb{R})$ such that the corresponding Lie group acts on the 3-dimensional non-euclidean space are given by (31). From (31) it follows that

$$\begin{aligned} \phi_i &= (0, 0, xy - z, 1, x, x^2), \\ \eta_i &= (1, y, y^2, 0, 0, xy - z), \\ \alpha_i &= (x, z, yz, y, z, xz). \end{aligned} \tag{112}$$

Applying (106), (107) to (112) we have

$$\begin{aligned} \eta_i^{(1)} &= (0, y', yy' - x(y')^2 + y'z', 0, -y', y - xy' - z'), \\ \alpha_i^{(1)} &= (1, z', zy' - xy'z' + (z')^2, y', 0, z - xz'). \end{aligned}$$

Therefore the coefficient matrix M of the system of linear equations derived from (108) is

$$M = \begin{pmatrix} 0 & 0 & xy - z & 1 & x & x^2 \\ 1 & y & y^2 & 0 & 0 & xy - z \\ x & z & yz & y & z & xz \\ 0 & y' & yy' - x(y')^2 + y'z' & 0 & -y' & y - xy' - z' \\ 1 & z' & zy' - xy'z' + (z')^2 & y' & 0 & z - xz' \end{pmatrix}.$$

There are six 5×5 subdeterminants of M. Their greatest common divisor factor is

$$\left(y'\right)^2 x^2 - 2xyy' - 2xy'z' + y^2 - 2z'y + \left(z'\right)^2 + 4zy'. \tag{113}$$

The factor (113) is zero if and only if

$$z' = xy' + y \pm 2\sqrt{y'(xy - z)}. \tag{114}$$

Since the function f_1 is independent of z', to obtain non-trivial function f_1 as a solution of the system (108) it is necessary that all 4×4 subdeterminants of the matrix

$$M_1 = \begin{pmatrix} 0 & 0 & xy - z & 1 & x & x^2 \\ 1 & y & y^2 & 0 & 0 & xy - z \\ x & z & yz & y & z & xz \\ 0 & y' & yy' - x(y')^2 + y'z' & 0 & -y' & y - xy' - z' \end{pmatrix}$$

are zero. Using (114) we have

$$D_{4,1} = \begin{vmatrix} 0 & 0 & xy - z & 1 \\ 1 & y & y^2 & 0 \\ x & z & yz & y \\ 0 & y' & yy' - x(y')^2 + y'z' & 0 \end{vmatrix} = \pm 2y' \sqrt{y'(xy - z)}(xy - z).$$

One has $D_{4,1} = 0$ if $y' = 0$ and hence $z' = y$. The time-dependent system $f_1 = y' = 0$, $f_2 = z' - y = 0$ of first order ordinary differential equations does not satisfy the system (108) because one has $X_6(f_2) = x^2\frac{\partial f_2}{\partial x} + (xy - z)\frac{\partial f_2}{\partial y} + xz\frac{\partial f_2}{\partial z} + (z - xy)\frac{\partial f_2}{\partial z'} = 2(z - xy) \neq 0$. Hence there does not exist any time-dependent system (109) of first order ordinary differential equations which allows the group of symmetries corresponding to the Lie algebra $\mathbf{g} = \mathbf{sl_2}(\mathbb{R}) \oplus \mathbf{sl_2}(\mathbb{R})$ given by (31).

A similar consideration as in Example 5.4 shows that there does not exist any time-dependent system (109) of first order ordinary differential equations which allows a Lie group of symmetries whose Lie algebra is any one of the following Lie algebras:

$$\mathbf{g} = \mathbf{sl_4}(\mathbb{R}) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, z\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial y}, x\frac{\partial}{\partial z}, y\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, \right.$$
$$\left. x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}, xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z}, xz\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z} \right\rangle$$
$$\mathbf{g} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, z\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial y}, x\frac{\partial}{\partial z}, y\frac{\partial}{\partial z}, z\frac{\partial}{\partial z} \right\rangle$$
$$\mathbf{g} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, z\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial y}, x\frac{\partial}{\partial z}, y\frac{\partial}{\partial z}, z\frac{\partial}{\partial z} \right\rangle$$
$$\mathbf{g} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, z\frac{\partial}{\partial x}, z\frac{\partial}{\partial y}, z\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}, \right.$$
$$\left. x\frac{\partial}{\partial z}, y\frac{\partial}{\partial} \right\rangle$$
$$\mathbf{g} = \left\langle \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \frac{\partial}{\partial z}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, \right.$$
$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}, z\frac{\partial}{\partial x} - y\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right),$$
$$\left. z\frac{\partial}{\partial y} + x\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right), z\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) \right\rangle$$
$$\mathbf{g} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} \right\rangle,$$
$$\mathbf{g} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \right\rangle,$$
$$\mathbf{g} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \right.$$
$$2x\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) - (x^2 + y^2 + z^2)\frac{\partial}{\partial x},$$

$$2y\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)-(x^2+y^2+z^2)\frac{\partial}{\partial y},$$
$$2z\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)-(x^2+y^2+z^2)\frac{\partial}{\partial z}\right\rangle.$$

5.2. **Time-preserving symmetries.** Now we study the case that the Lie algebra $\mathbf{g}$ of the given r-dimensional real Lie group G consists of infinitesimal generators which are time-preserving symmetries. Let us introduce the following notation $x'=\frac{dx}{dt}$, $y'=\frac{dy}{dt}$, $z'=\frac{dz}{dt}$ and consider the following time-independent system of first order ordinary differential equations:

$$\begin{aligned}
f_1(t,x(t),y(t),z(t),x'(t),y'(t),z'(t)) =&0,\\
f_2(t,x(t),y(t),z(t),x'(t),y'(t),z'(t)) =&0,\\
f_3(t,x(t),y(t),z(t),x'(t),y'(t),z'(t)) =&0.
\end{aligned}$$

Assume that the Lie algebra $\mathbf{g}$ of G has the following vector fields as basis elements:

$$\begin{aligned}
X_i(x(t),y(t),z(t)) = \phi_i(x(t),y(t),z(t))\frac{\partial}{\partial x}+\eta_i(x(t),y(t),z(t))\frac{\partial}{\partial y}\\
+\alpha_i(x(t),y(t),z(t))\frac{\partial}{\partial z},
\end{aligned}$$

for all $i=1,2,\ldots,r$. The first prolonged vector field of $X_i(x(t),y(t),z(t))$ $(i=1,2,\ldots,r)$ with respect to the variable t has the form

$$\begin{aligned}
X_i^{(1)}(x(t),y(t),z(t),x',y',z') = X_i+\phi_i^{(1)}(x(t),y(t),z(t),x',y',z')\frac{\partial}{\partial x'}+\\
\eta_i^{(1)}(x(t),y(t),z(t),x',y',z')\frac{\partial}{\partial y'}+\alpha_i^{(1)}(x(t),y(t),z(t),x',y',z')\frac{\partial}{\partial z'},
\end{aligned}$$

where

$$\begin{aligned}
\phi_i^{(1)} &= \frac{\partial\phi_i}{\partial x}x'+\frac{\partial\phi_i}{\partial y}y'+\frac{\partial\phi_i}{\partial z}z',\\
\eta_i^{(1)} &= \frac{\partial\eta_i}{\partial x}x'+\frac{\partial\eta_i}{\partial y}y'+\frac{\partial\eta_i}{\partial z}z',\\
\alpha_i^{(1)} &= \frac{\partial\alpha_i}{\partial x}x'+\frac{\partial\alpha_i}{\partial y}y'+\frac{\partial\alpha_i}{\partial z}z'.
\end{aligned}$$

The time-independent system of first order ordinary differential equations allows the given group G of symmetries precisely if the functions f_k, $k=$

$1, 2, 3$, fulfil the following system of partial differential equations

$$\begin{array}{rcl}
\phi_1 \frac{\partial f_k}{\partial x} + \eta_1 \frac{\partial f_k}{\partial y} + \alpha_1 \frac{\partial f_k}{\partial z} + \phi_1^{(1)} \frac{\partial f_k}{\partial x'} + \eta_1^{(1)} \frac{\partial f_k}{\partial y'} + \alpha_1^{(1)} \frac{\partial f_k}{\partial z'} & = & 0, \\
\phi_2 \frac{\partial f_k}{\partial x} + \eta_2 \frac{\partial f_k}{\partial y} + \alpha_2 \frac{\partial f_k}{\partial z} + \phi_2^{(1)} \frac{\partial f_k}{\partial x'} + \eta_2^{(1)} \frac{\partial f_k}{\partial y'} + \alpha_2^{(1)} \frac{\partial f_k}{\partial z'} & = & 0, \\
\vdots & & \\
\phi_i \frac{\partial f_k}{\partial x} + \eta_i \frac{\partial f_k}{\partial y} + \alpha_i \frac{\partial f_k}{\partial z} + \phi_i^{(1)} \frac{\partial f_k}{\partial x'} + \eta_i^{(1)} \frac{\partial f_k}{\partial y'} + \alpha_i^{(1)} \frac{\partial f_k}{\partial z'} & = & 0, \\
\vdots & & \\
\phi_r \frac{\partial f_k}{\partial x} + \eta_r \frac{\partial f_k}{\partial y} + \alpha_r \frac{\partial f_k}{\partial z} + \phi_r^{(1)} \frac{\partial f_k}{\partial x'} + \eta_r^{(1)} \frac{\partial f_k}{\partial y'} + \alpha_r^{(1)} \frac{\partial f_k}{\partial z'} & = & 0.
\end{array}$$

Let

$$M = \begin{pmatrix} \phi_1 & \dots & \phi_r \\ \eta_1 & \dots & \eta_r \\ \alpha_1 & \dots & \alpha_r \\ \phi_1^{(1)} & \dots & \phi_r^{(1)} \\ \eta_1^{(1)} & \dots & \eta_r^{(1)} \\ \alpha_1^{(1)} & \dots & \alpha_r^{(1)} \end{pmatrix}.$$

The system (5.2) of partial differential equations can be seen as a system of 'linear equations' of the variables $\frac{\partial f_k}{\partial x}, \frac{\partial f_k}{\partial y}, \frac{\partial f_k}{\partial z}, \frac{\partial f_k}{\partial x'}, \frac{\partial f_k}{\partial y'}, \frac{\partial f_k}{\partial z'}$:

$$\begin{array}{l}
\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial x'} & \frac{\partial f_1}{\partial y'} & \frac{\partial f_1}{\partial z'} \end{pmatrix} \cdot M = \begin{pmatrix} 0 & \dots & 0 \end{pmatrix}, \\
\begin{pmatrix} \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial x'} & \frac{\partial f_2}{\partial y'} & \frac{\partial f_2}{\partial z'} \end{pmatrix} \cdot M = \begin{pmatrix} 0 & \dots & 0 \end{pmatrix}, \\
\begin{pmatrix} \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} & \frac{\partial f_3}{\partial x'} & \frac{\partial f_3}{\partial y'} & \frac{\partial f_3}{\partial z'} \end{pmatrix} \cdot M = \begin{pmatrix} 0 & \dots & 0 \end{pmatrix}.
\end{array}$$

To obtain non-trivial solutions f_k, $k = 1, 2, 3$, of the system of equations given by (5.2) it is necessary that the rank of the $6 \times r$-matrix M is less than 6. However, as in Section 5.1, if one wants the vectors $\begin{pmatrix} \frac{\partial f_i}{\partial x} & \dots & \frac{\partial f_i}{\partial z'} \end{pmatrix}$ to be linearly independent, $\mathrm{rank} M < 4$ is the necessary requirement. This condition is automatically satisfied if $r < 4$, and then one can only solve the system of partial differential equations (5.2), and see if any solution corresponds to a nontrivial system of differential equations f_1, f_2, f_3. In the following we consider the case $r \geq 4$, where the rank condition is equivalent to that every 4×4 subdeterminant of M is zero.
Now, if we reduce ourselves to systems of the form

$$\begin{aligned}
f_1(t, x(t), y(t), z(t), x') &= x' - g_1(t, x(t), y(t), z(t)) = 0, \\
f_2(t, x(t), y(t), z(t), y') &= y' - g_2(t, x(t), y(t), z(t)) = 0, \\
f_3(t, x(t), y(t), z(t), z') &= z' - g_3(t, x(t), y(t), z(t)) = 0,
\end{aligned}$$

then the function f_1 does not depend on y', z', the function f_2 is independent of x', z' and the function f_3 does not depend on x', y'. Hence, for f_1, f_2, f_3

in (5.2), one needs to deal with the coefficient matrices

$$M_1 = \begin{pmatrix} \phi_1 & \dots & \phi_r \\ \eta_1 & \dots & \eta_r \\ \alpha_1 & \dots & \alpha_r \\ \phi_1^{(1)} & \dots & \phi_r^{(1)} \end{pmatrix}, \qquad M_2 = \begin{pmatrix} \phi_1 & \dots & \phi_r \\ \eta_1 & \dots & \eta_r \\ \alpha_1 & \dots & \alpha_r \\ \eta_1^{(1)} & \dots & \eta_r^{(1)} \end{pmatrix},$$

$$M_3 = \begin{pmatrix} \phi_1 & \dots & \phi_r \\ \eta_1 & \dots & \eta_r \\ \alpha_1 & \dots & \alpha_r \\ \alpha_1^{(1)} & \dots & \alpha_r^{(1)} \end{pmatrix},$$

respectively. To get non-trivial functions f_1, f_2, f_3 in explicit form as a solution of the system (5.2) it is necessary that rank$M_1 < 4$, rank$M_2 < 4$, rank$M_3 < 4$ hold. That is, all 4×4-subdeterminants of the $4 \times r$-matrices M_1, M_2, M_3 have to be zero. Now we apply the above discussed method for the Lie group G given in Example 5.4.

Example 5.5. The infinitesimal generators of the Lie algebra $\mathbf{g} = \mathbf{sl_2}(\mathbb{R}) \oplus \mathbf{sl_2}(\mathbb{R})$ of G are given by (31). From (31) it follows that

$$\begin{aligned} \phi_i &= (0, 0, x(t)y(t) - z(t), 1, x(t), x(t)^2), \\ \eta_i &= (1, y(t), y(t)^2, 0, 0, x(t)y(t) - z(t)), \\ \alpha_i &= (x(t), z(t), y(t)z(t), y(t), z(t), x(t)z(t)). \end{aligned}$$

Therefore one has

$$\begin{aligned} \phi_i^{(1)} &= (0, 0, y(t)x' + x(t)y' - z', 0, x', 2x(t)x'), \\ \eta_i^{(1)} &= (0, y', 2y(t)y', 0, 0, y(t)x' + x(t)y' - z'), \\ \alpha_i^{(1)} &= (x', z', z(t)y' + y(t)z', y', z', z(t)x' + x(t)z'). \end{aligned}$$

Hence the coefficient matrix M of the system of linear equations is

$$M = \begin{pmatrix} 0 & 0 & xy - z & 1 & x & x^2 \\ 1 & y & y^2 & 0 & 0 & xy - z \\ x & z & yz & y & z & xz \\ 0 & 0 & yx' + xy' - z' & 0 & x' & 2xx' \\ 0 & y' & 2yy' & 0 & 0 & xy' + x'y - z' \\ x' & z' & zy' + yz' & y' & z' & zx' + xz' \end{pmatrix}.$$

The determinant of M is 0. To obtain non-trivial function f_1 of explicit form as a solution of the system (5.2) of partial differential equations it is necessary that all (4×4) subdeterminants of the matrix

$$M_1 = \begin{pmatrix} 0 & 0 & xy-z & 1 & x & x^2 \\ 1 & y & y^2 & 0 & 0 & xy-z \\ x & z & yz & y & z & xz \\ 0 & 0 & yx'+xy'-z' & 0 & x' & 2xx' \end{pmatrix}$$

are 0. The subdeterminant $D_1 = \begin{vmatrix} 0 & 0 & 1 & x \\ 1 & y & 0 & 0 \\ x & z & y & z \\ 0 & 0 & 0 & x' \end{vmatrix} = x'(z-xy)$ is 0 if $x' = 0$ and hence $f_1 = x' = 0$. Using this, the subdeterminant

$$D_2 = \begin{vmatrix} 0 & 0 & 1 & xy-z \\ 1 & y & 0 & y^2 \\ x & z & y & yz \\ 0 & 0 & 0 & xy'-z' \end{vmatrix} = (xy-z)(z'-xy')$$

is 0 if $z' = xy'$. To get non-trivial function f_2 given in explicit form as a solution of the system (5.2) of partial differential equations it is necessary that all (4×4) subdeterminants of the matrix

$$M_2 = \begin{pmatrix} 0 & 0 & xy-z & 1 & x & x^2 \\ 1 & y & y^2 & 0 & 0 & xy-z \\ x & z & yz & y & z & xz \\ 0 & 0 & yx'+xy'-z' & 0 & x' & 2xx' \\ 0 & y' & 2yy' & 0 & 0 & xy'+x'y-z' \end{pmatrix}$$

are 0. The subdeterminant $D_3 = \begin{vmatrix} 0 & 0 & xy-z & x \\ 1 & y & y^2 & 0 \\ x & z & yz & z \\ 0 & y' & 2yy' & 0 \end{vmatrix} = y'(xy-z)^2$ is 0 if $y' = 0$ and therefore $z' = 0$. Hence the only time-independent system that allows the Lie group G corresponding to the Lie algebra $\mathbf{g} = \mathbf{sl_2}(\mathbb{R}) \oplus \mathbf{sl_2}(\mathbb{R})$ given by (31) as its symmetries is trivial (cf. (30)).

Similarly as in the Example 5.5, only the trivial time-independent system of first order ordinary differential equations allows a Lie group of symmetries whose tangential Lie algebra is one of the following simple Lie algebras: $\mathbf{so_3}(\mathbb{R})$, $\mathbf{sl_3}(\mathbb{R})$, $\mathbf{sl_4}(\mathbb{R})$.

Acknowledgement

This paper was supported by the Talent UD program, by the National Research, Development and Innovation Office (NKFIH) Grant No. K132951 and by the EFOP-3.6.1-16-2016-00022 project. The later project has been supported by the European Union, co-financed by the European Social Fund.

References

[1] W. F. Ames, R. L. Anderson, V. A. Dorodnitsyn, E. V. Ferapontov, R. K. Gazizov, N. H. Ibragimov and S. R. Svirshchevskiĭ, *CRC handbook of Lie group analysis of differential equations. Vol. 1, Symmetries, exact solutions and conservation laws*, CRC Press, 1994.

[2] M. Edwards and M. C. Nucci, *Application of Lie group analysis to a core group model for sexually transmitted diseases*, J. Nonlinear Math. Phys. **13** (2006), 211–230.

[3] R. Fitzhugh, *Impulses and Physiological States in Theoretical Models of Nerve Membrane*, Biophys J. **1** (1961), 445–466.

[4] P. E. Hydon, *Symmetry methods of differential equations*, Cambridge University Press, 2000.

[5] N. Kh. Ibragimov, *Group analysis of ordinary differential equations and the invariance principle in mathematical physics*, Russian Math. Surveys **47:4** (1992), 89-156.

[6] A. Gonzáles-López, N. Kamran and P. J. Olver, *Lie Algebras of Vector Fields in the Real Plane*, Proc. London Math. Soc. **64:2** (1992), 339-368.

[7] T. Harko, S. N. F. Lobo and M. K. Mak, *A class of exact solutions of the Liénard-type ordinary nonlinear differential equation*, Journal of Engineering Mathematics, **89** (2014), 193–205.

[8] E. Kamke, *Differentialgleichungen. Lösungsmethoden und Lösungen. Teil I: Gewöhnliche Differentialgleichungen. 6. Aufl.; Teil II: Partielle Differentialgleichungen erster Ordnung für eine gesuchte Funktion. 4. Aufl*, Akademische Verlagsgesellschaft, Geest & Portig K.-G. 1959.

[9] S. Lie, *Gesammelte Abhandlungen, Fünfter Band*, B.G. Teubner, 1924.

[10] S. Lie, *Gesammelte Abhandlungen, Sechster Band*, B.G. Teubner, 1924.

[11] S. Lie, *Vorlesungen ueber Differentialgleichungen mit bekannten infinitesimal Transformationen*, B.G. Teubner, 1891.

[12] S. Lie, *Theorie der Transformationsgruppen I*, Math. Ann., Band XVI, p. 475.

[13] S. Lie and F. Engel, *Theorie der Transformationsgruppen*, B.G. Teubner, Band III, 1893.

[14] J. Nagumo, S. Arimoto and S. Yoshizawa, *An active pulse transmission line simulating nerve axon*, Proc. IRE. **50** (1962), 2061–2070.

[15] M. C. Nucci, *Interactive REDUCE programs for calculating Lie point, non-classical, Lie-Bäcklund, and approximate symmetries of differential equations: Manual and floppy disk*, in CRC Handbook of Lie Croup Analysis of Differential Equations, Vol. III: New Trends (Ed: N. H. Ibragimov), CRC Press, 1995.

[16] M. C. Nucci, *The complete Kepler group can be derived by Lie group analysis*, J. Math. Phys. **37** (1996), 1772–1775.

[17] M. C. Nucci, *The Role of Symmetries in Solving Differential Equations*, Mathl. Comput. Modelling, **25** (1997), 181–193.

[18] M. C. Nucci and G. Sanchini, *Symmetries, Lagrangians and Conservation Laws of an Easter Island Population Model*, Symmetry, **7** (2015), 1613–1632.

[19] B. van der Pol, *On relaxation-oscillations*, The Lond, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, **2** (1927), 978–992.

[20] B. van der Pol and J. van der Mark, *The heartbeat considered as a relaxation oscillation, and an electrical model of the heart*, The Lond, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, **6** (1928), 763–775.

[21] P. J. Olver, *Applications of Lie groups to differential equations*, Springer, 1993.

[22] J. Starrett, *Solving differential equations by symmetry groups*, Amer. Math. Monthly, **114** (2007), 778–792.

[23] A. V. Bocharov, V. N. Chetverikov, S. V. Duzhin, N. G. Khor'Kova, I. S. Krasil'shchik, A. V. Samokhin, Yu. N. Torkhov, A. M. Verbovetsky and A. M. Vinogradov, *Symmetries and conservation laws for differential equations of mathematical physics*, American Mathematical Society, 1999.

Institute of Mathematics, University of Debrecen, P.O.Box 400,
4002 Debrecen, Hungary.
E-mails: figula@science.unideb.hu and lomjariagio@yahoo.com