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DIFFERENTIAL EQUATIONS AND THEIR LIE SYMMETRY GROUPS

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Abstract. Our aim is to show how one can find those differential equations which have a given Lie group as their symmetry group. The method of Lie is discussed. Furthermore we present examples of ordinary differential equations and first order systems of ordinary differential equations illustrating the effective application of this method.

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1. Introduction

Symmetry analysis [5, 6, 22, 23] is a useful tool for finding smooth solutions of differential equations. With help of symmetries, integration or lowering of the order of differential equations could be effected successfully. Several examples come from Physics (see [17, 24]), as well as from Biology (see [2, 18, 19]). Also systems of equations are studied using this framework. The Fitzhugh–Nagumo model [4, 15] and the model for the population of Easter Island [19] can be described as a first order system of two equations. In contrast to this, many physical systems (see. [8, 20, 21]) are based on second order systems. One advantage for investigating systems of order one is that any system of differential equations is equivalent to a first order system. Their symmetry groups have infinite dimension. On the contrary, the symmetry groups of higher order systems have finite dimension.

Following the works of Lie we consider small dimensional Lie groups such that their canonical form is given by [14, Sections 3, 4, 5, pp. 28–78], [14, Section 19, pp. 360–392] (see also [7], p. 341). Lie classified the ordinary differential equations such that the groups of their symmetries are these given groups (cf. [10, Sections X, XI, XIV, XVI]). In Section 3 we explain his method in detail. We demonstrate this method in Section 4 to find the ordinary differential equations which have some given Lie groups as their symmetry groups. We use the given symmetries to find the solutions of these equations. In Section 5 we deal with first order systems of ordinary differential equations and give necessary conditions for them admitting a given Lie group as a subgroup of their symmetries. We illustrate these conditions on examples. We restrict us mostly to semi-simple Lie groups. To obtain our examples we use the REDUCE program [16].

We note two remarkable facts of our study. In Section 4.2 we take the four different representations of the Lie algebra $\mathbf{sl}_2(\mathbb{R})$ in the (x, y)-plane and illustrate that the invariant differential equations depend strongly on the representation of the tangential Lie algebra of their Lie symmetry group.

In Section 4.3 it is proved that there is an ordinary differential equation of order two which is invariant under the action of the simple Lie group $SO_3(\mathbb{R})$ without any 2-dimensional solvable subgroup. Since to completely solve a second order equation it needs a two dimensional solvable Lie subgroup of its symmetry group (cf. [22], [6]) this differential equation cannot be solved. The considered real Lie algebras $\mathbf{g}$ together with their representation and the invariant differential equations are summarized (see [3]).

(I) Example 4.4: 3-dimensional simple Lie algebra $\mathbf{sl_2}(\mathbb{R})$ with generators

$$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \qquad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_3 = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}. \tag{1}$$

The differential equations

$$y^{(2)} + 2\frac{(y')^2 + y' + c\,(y')^{3/2}}{x-y} = 0 \quad c \in \mathbb{R}, \quad y' = 0 \tag{2}$$

of order ≤ 2 allow this symmetry group.

(II) Example 4.5: 3-dimensional simple Lie algebra $\mathbf{sl_2}(\mathbb{R})$ with generators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \tag{3}$$

The differential equations of order ≤ 2 allowing this group of symmetries are

$$y^{(2)} = \frac{a}{y^3} \qquad a \in \mathbb{R}. \tag{4}$$

(III) Example 4.6: 3-dimensional simple Lie algebra $\mathbf{sl_2}(\mathbb{R})$ with generators

$$X_1 = \frac{\partial}{\partial y}, \qquad X_2 = y\frac{\partial}{\partial y}, \qquad X_3 = y^2\frac{\partial}{\partial y}. \tag{5}$$

The differential equations of order ≤ 3 allowing this group of symmetries are

$$y^{(3)} = \frac{3\left(y^{(2)}\right)^2}{2y'} + y'f(x), \tag{6}$$

where f is an arbitrary real function.

(IV) Example 4.8: 3-dimensional simple Lie algebra $\mathbf{sl_2}(\mathbb{R})$ with generators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad X_3 = (x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}. \tag{7}$$

The differential equations of order ≤ 2 allowing this group of symmetries are

$$y^{(2)} = -\frac{1 + (y')^2}{y} + d\frac{\left(1 + (y')^2\right)^{3/2}}{y} \qquad d \in \mathbb{R}. \tag{8}$$

(V) Example 4.9: 3-dimensional simple Lie algebra $\mathbf{so_3}(\mathbb{R}) \cong \mathbf{su_2}(\mathbb{C})$ with generators

$$X_1 = \left(1 + x^2\right)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \qquad X_2 = xy\frac{\partial}{\partial x} + \left(1 + y^2\right)\frac{\partial}{\partial y}, \tag{9}$$

$$X_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

The differential equations of order ≤ 2 permitting this group of symmetries are

$$y^{(2)} = c\left(\frac{1 + y^2 - 2xyy' + \left(1 + x^2\right)(y')^2}{1 + x^2 + y^2}\right)^{3/2} \qquad c \in \mathbb{R}. \tag{10}$$

(VI) Example 4.11: The Lie algebra $\mathbf{g}_\alpha$, $\alpha \geq 0$, is 3-dimensional solvable with generators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \alpha\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{11}$$

There is no differential equation of order ≤ 1 allowing this group of symmetries.

(VII) Example 4.12: The Lie algebra $\mathbf{g}_\beta$, $0 < |\beta| \leq 1$, is 3-dimensional solvable with generators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x} + \beta y\frac{\partial}{\partial y}. \tag{12}$$

The differential equation of order ≤ 1 allowing this group of symmetries is

$$y' = 0.$$

(VIII) Example 4.13: solvable Lie algebra of dimension 4 with generators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad X_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{13}$$

The differential equation of order ≤ 2 allowing this group of symmetries is

$$y^{(2)} = 0.$$

(IX) Example 4.14: solvable Lie algebra of dimension 4 with generators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x}, \qquad X_4 = y\frac{\partial}{\partial y}. \tag{14}$$

The differential equations of order ≤ 2 allowing this group of symmetries are

$$y' = 0, \qquad\qquad y^{(2)} = 0.$$

(X) Example 4.15: $\mathbf{sl_2}(\mathbb{R}) \times \mathbb{R}$ with generators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x}, \qquad X_4 = x^2\frac{\partial}{\partial x}. \tag{15}$$

The invariant differential equation of order ≤ 2 under the action of this group of symmetries is

$$y' = 0.$$

(XI) Example 4.16: $\mathbf{gl_2}(\mathbb{R})$ with generators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x}, \quad X_3 = y\frac{\partial}{\partial y}, \quad X_4 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \tag{16}$$

The differential equation of order ≤ 2 allowing this group of symmetries is

$$y^{(2)} = 0.$$

(XII) Example 4.17: $\mathbf{sl_2}(\mathbb{R})\times L_2$, where L_2 is the non-abelian 2-dimensional Lie algebra, with generators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x}, \quad X_4 = y\frac{\partial}{\partial y}, \quad X_5 = x^2\frac{\partial}{\partial x}. \tag{17}$$

The differential equations of order ≤ 3 permitting this group of symmetries are

$$y' = 0, \qquad\qquad 2y'y^{(3)} = 3\left(y^{(2)}\right)^2. \tag{18}$$

(XIII) Example 4.18: $\mathbf{sl_2}(\mathbb{R}) \ltimes \mathbb{R}^2$ with generators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad X_4 = y\frac{\partial}{\partial x}, \quad X_5 = x\frac{\partial}{\partial y}. \tag{19}$$

The only differential equation of order ≤ 3 allowing this group of symmetries is

$$y^{(2)} = 0.$$

(XIV) Example 4.1: the simple Lie algebra $\mathbf{sl_2}(\mathbb{C})$ with generators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad X_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \tag{20}$$

$$X_5 = \left(x^2 - y^2\right)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}, \qquad X_6 = 2xy\frac{\partial}{\partial x} + \left(y^2 - x^2\right)\frac{\partial}{\partial y}.$$

The only differential equation of order ≤ 4 allowing this group of symmetries is

$$y^{(3)}\left(1 + (y')^2\right) = 3y'\left(y^{(2)}\right)^2. \tag{21}$$

(XV) Example 4.2: the semi-simple Lie algebra $\mathbf{sl_2}(\mathbb{R}) \times \mathbf{sl_2}(\mathbb{R})$ with generators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = y\frac{\partial}{\partial y}, \tag{22}$$

$$X_4 = x\frac{\partial}{\partial x}, \qquad X_5 = y^2\frac{\partial}{\partial y}, \qquad X_6 = x^2\frac{\partial}{\partial x}.$$

The differential equations of order ≤ 4 allowing this group of symmetries are $y' = 0$ and

$$2y'y^{(3)} = 3\left(y^{(2)}\right)^2. \tag{23}$$

(XVI) Example 4.19: the Lie algebra $\mathbf{gl_2}(\mathbb{R}) \ltimes \mathbb{R}^2$ with generators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial x}, \tag{24}$$

$$X_4 = y\frac{\partial}{\partial x}, \qquad X_5 = x\frac{\partial}{\partial y}, \qquad X_6 = y\frac{\partial}{\partial y}.$$

The differential equations of order ≤ 4 allowing this group of symmetries are

$$y^{(2)} = 0, \qquad 3y^{(4)}y^{(2)} = 5\left(y^{(3)}\right)^2. \tag{25}$$

(XVII) Example 4.3: the simple Lie algebra $\mathbf{sl_3}(\mathbb{R})$ with generators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial y}, \qquad X_4 = y\frac{\partial}{\partial y}, \tag{26}$$

$$X_5 = x\frac{\partial}{\partial x}, \quad X_6 = y\frac{\partial}{\partial y}, \quad X_7 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \quad X_8 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}.$$

The differential equations of order ≤ 6 which are invariant under the action of this group of symmetries are

$$y^{(2)} = 0, \qquad 9\left(y^{(2)}\right)^2 y^{(5)} = 45 y^{(2)} y^{(3)} y^{(4)} - 40\left(y^{(3)}\right)^3. \tag{27}$$

Now we collect the considered real Lie algebras $\mathbf{g}$ together with their representation in the space $\mathbb{R}^3$ and the invariant first order systems of differential equations (see [3]).

(i) Example 5.2: the simple Lie algebra $\mathbf{so_3}(\mathbb{R}) \cong \mathbf{su_2}(\mathbb{C})$ with generators

$$X_1 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \qquad X_2 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \qquad X_3 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}. \tag{28}$$

The time-dependent system allowing the Lie algebra $\mathbf{so_3}(\mathbb{R})$ as the Lie algebras of the group of symmetries is

$$\begin{aligned} y' &= \frac{y}{x}, \\ z' &= \frac{z}{x}. \end{aligned} \tag{29}$$

The time-independent invariant system is trivial:

$$\begin{aligned} x' &= 0, \\ y' &= 0, \\ z' &= 0. \end{aligned} \tag{30}$$

(ii) Examples 5.4 and 5.5: the semi-simple Lie algebra $\mathbf{sl_2}(\mathbb{R}) \times \mathbf{sl_2}(\mathbb{R})$ with generators

$$X_1 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \qquad X_2 = y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \tag{31}$$

$$X_3 = (xy - z)\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z}, \quad X_4 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z},$$

$$X_5 = x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}, \qquad X_6 = x^2\frac{\partial}{\partial x} + (xy - z)\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}.$$

The time-independent invariant system is trivial (30), but the time-dependent explicit invariant system of order one is missing.

(iii) Example 5.3: the simple Lie algebra $\mathbf{sl_3}(\mathbb{R})$ with generators

$$X_1 = z\frac{\partial}{\partial x}, \qquad X_2 = z\frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial y}, \qquad X_4 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \tag{32}$$

$$X_5 = y\frac{\partial}{\partial x}, \qquad X_6 = x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}, \qquad X_7 = x\frac{\partial}{\partial z}, \qquad X_8 = y\frac{\partial}{\partial z}.$$

The time-dependent invariant system is (29). The time-independent system allowing the Lie algebra $\mathbf{sl_3}(\mathbb{R})$ as the Lie algebras of the group of symmetries is trivial (30).

2. Preliminaries

A map which continuously transforms a solution of a differential equation into another solution is called a symmetry of the equation. Hence every symmetry can be given by a vector field and its name is the infinitesimal generator of the symmetry. The jet space is the space of the variables $x, y, y', \dots, y^{(m)}$ for ordinary differential equations of order m. The hull of the differential equation $f\left(x, y, y', \dots, y^{(m)}\right) = 0$ is an $(m+1)$-dimensional surface in the jet space defined by f. A continuously differentiable function $\varphi(x)$ is called a smooth solution if the curve $y = \varphi(x)$ with $y' = \frac{\partial\varphi(x)}{\partial x}, \dots, y^{(m)} = \frac{\partial^m\varphi(x)}{\partial x^m}$ is part of the hull, that is $f\left(x, \varphi(x), \dots, \frac{\partial^m\varphi(x)}{\partial x^m}\right) = 0$ identically holds for all x. The group of symmetries of a differential equation consists of transformations of the (x, y)-plane whose prolongation to the derivatives $y', \dots, y^{(m)}$ leaves the hull of the equation invariant.

3. Method of Lie

In [10, Section X, pp. 243–248] S. Lie developed a method to find those ordinary differential equations which allow a given Lie group G as their symmetry group. Here we present it. The Lie algebra $\mathbf{g}$ of the r-dimensional real Lie group G is determined by the vector fields

$$X_i = \phi_i(x, y)\frac{\partial}{\partial x} + \eta_i(x, y)\frac{\partial}{\partial y}, \qquad i = 1, 2, \dots, r. \tag{33}$$

Applying the total derivative of η_i as well as of ϕ_i with respect to the variable x we define recursively $\eta_i^{(k)}$, $i = 1, 2, \dots, r$, $k = 1, 2, \dots, m$:

$$\eta_i^{(k)} = \frac{d\eta_i^{(k-1)}}{dx} - y^{(k)}\frac{d\phi_i}{dx}, \text{ that is}$$

$$\eta_i^{(1)}(x, y, y') = \frac{\partial\eta_i}{\partial x} + \frac{\partial\eta_i}{\partial y}y' - \frac{\partial\phi_i}{\partial x}y' - \frac{\partial\phi_i}{\partial y}(y')^2, \tag{34}$$

$$\begin{aligned}\eta_i^{(2)}(x, y, y', y^{(2)}) = {} & \frac{\partial\eta_i^{(1)}}{\partial x} + \frac{\partial\eta_i^{(1)}}{\partial y}y' + \frac{\partial\eta_i^{(1)}}{\partial y'}y^{(2)} \\ & - \frac{\partial\phi_i}{\partial x}y^{(2)} - \frac{\partial\phi_i}{\partial y}y'y^{(2)},\end{aligned} \tag{35}$$

$$\begin{aligned}\eta_i^{(3)}(x, y, y', y^{(2)}, y^{(3)}) = {} & \frac{\partial\eta_i^{(2)}}{\partial x} + \frac{\partial\eta_i^{(2)}}{\partial y}y' + \frac{\partial + \eta_i^{(2)}}{\partial y'}y^{(2)} + \frac{\partial\eta_i^{(2)}}{\partial y^{(2)}}y^{(3)} \\ & - \frac{\partial\phi_i}{\partial x}y^{(3)} - \frac{\partial\phi_i}{\partial y}y'y^{(3)},\end{aligned} \tag{36}$$

$$\eta_i^{(4)}(x,y,y',y^{(2)},y^{(3)},y^{(4)}) = \frac{\partial \eta_i^{(3)}}{\partial x} + \frac{\partial \eta_i^{(3)}}{\partial y} y' + \frac{\partial \eta_i^{(3)}}{\partial y'} y^{(2)} + \frac{\partial \eta_i^{(3)}}{\partial y^{(2)}} y^{(3)} + \frac{\partial \eta_i^{(3)}}{\partial y^{(3)}} y^{(4)} - \frac{\partial \phi_i}{\partial x} y^{(4)} - \frac{\partial \phi_i}{\partial y} y' y^{(4)}, \text{ etc.} \tag{37}$$

According to [10, Section X, p. 245] the m^{th} prolonged vector fields $X_i^{(m)}$, $i = 1, 2, \ldots, r$ are computed by

$$X_i^{(m)} = \phi_i(x,y)\frac{\partial}{\partial x} + \eta_i(x,y)\frac{\partial}{\partial y} + \eta_i^{(1)}(x,y,y')\frac{\partial}{\partial y'} + \ldots + \eta_i^{(m)}\left(x,y,\ldots,y^{(m)}\right)\frac{\partial}{\partial y^{(m)}}.$$

It can be established that these vector fields depend on the variables x, y, y', $\ldots$, $y^{(m)}$. The Lie algebra generated by them is isomorphic to $\mathbf{g}$ (see [10, Section X, p. 245] or [22, Theorem 2.39, p. 117]). A differential equation $f\left(x,y,y',\ldots,y^{(m)}\right) = 0$ allows a Lie symmetry group with Lie algebra $\mathbf{g}$ precisely if the following system of partial differential equations holds whenever $f\left(x,y,y',\ldots,y^{(m)}\right) = 0$ is satisfied:

$$\begin{array}{rcl}
\phi_1 \frac{\partial f}{\partial x} + \eta_1 \frac{\partial f}{\partial y} + \eta_1^{(1)} \frac{\partial f}{\partial y'} + \cdots + \eta_1^{(m)} \frac{\partial f}{\partial y^{(m)}} & = & 0, \\
\phi_2 \frac{\partial f}{\partial x} + \eta_2 \frac{\partial f}{\partial y} + \eta_2^{(1)} \frac{\partial f}{\partial y'} + \cdots + \eta_2^{(m)} \frac{\partial f}{\partial y^{(m)}} & = & 0, \\
& \vdots & \\
\phi_i \frac{\partial f}{\partial x} + \eta_i \frac{\partial f}{\partial y} + \eta_i^{(1)} \frac{\partial f}{\partial y'} + \cdots + \eta_i^{(m)} \frac{\partial f}{\partial y^{(m)}} & = & 0, \\
& \vdots & \\
\phi_r \frac{\partial f}{\partial x} + \eta_r \frac{\partial f}{\partial y} + \eta_r^{(1)} \frac{\partial f}{\partial y'} + \cdots + \eta_r^{(m)} \frac{\partial f}{\partial y^{(m)}} & = & 0.
\end{array} \tag{38}$$

This system (38) is equivalent to the following system of 'linear equations':

$$\left(\begin{array}{ccccc} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial y'} & \cdots & \frac{\partial f}{\partial y^{(m)}} \end{array} \right) \cdot M = \left(\begin{array}{ccc} 0 & \ldots & 0 \end{array} \right), \tag{39}$$

where $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial y'}$, $\ldots$, $\frac{\partial f}{\partial y^{(m)}}$ are the variables of the linear system and its $(m+2) \times r$ coefficient matrix M is defined by

$$M = \left(\begin{array}{ccccc}
\phi_1 & \phi_2 & \phi_3 & \cdots & \phi_r \\
\eta_1 & \eta_2 & \eta_3 & \cdots & \eta_r \\
\eta_1^{(1)} & \eta_2^{(1)} & \eta_3^{(1)} & \ldots & \eta_r^{(1)} \\
\vdots & & & \ddots & \vdots \\
\eta_1^{(m)} & \eta_2^{(m)} & \eta_3^{(m)} & \ldots & \eta_r^{(m)}
\end{array} \right). \tag{40}$$

The necessary and sufficient condition to find non-trivial solution of the system (38) is

$$\text{rank} M < m + 2.$$

Since rank$M \le r$ is always true, if $r < m+2$, then the rank condition is evidently fulfilled. In this case one solves (38) in $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \dots, \frac{\partial f}{\partial y(m)}$, and checks if any solution belongs to a nontrivial differential equation f.

Now we discuss the case $r \ge m+2$.

First, assume $m+2 = r$. Then M is an $(m+2)\times(m+2)$-matrix. It can be established that there exists a non-trivial solution f of the linear system (39) of equations precisely if rank$M < m+2$. Since the determinant

$$D = \begin{vmatrix} \phi_1 & \phi_2 & \phi_3 & \dots & \phi_r \\ \eta_1 & \eta_2 & \eta_3 & \dots & \eta_r \\ \eta_1^{(1)} & \eta_2^{(1)} & \eta_3^{(1)} & \dots & \eta_r^{(1)} \\ \vdots & & & \ddots & \vdots \\ \eta_1^{(r-2)} & \eta_2^{(r-2)} & \eta_3^{(r-2)} & \dots & \eta_r^{(r-2)} \end{vmatrix} \tag{41}$$

of the coefficient matrix M of (39) is a polynomial of the variables x, y, $y^{(i)}$, $i = 1, 2, \dots, r-2$, the rank condition says that D has to be 0. In the rare situation that D is identically 0, one solves (38) in $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \dots, \frac{\partial f}{\partial y(m)}$, and also checks if any solution belongs to a nontrivial differential equation f.

However, if D is not identically 0, then the polynomial D of the variables x, y, $y^{(i)}$, $i = 1, 2, \dots, r-2$, has to be 0 and the factors of D provide the only possibilities for a nontrivial differential equation f.

The reversed statement is also valid (see [13, p. 475]) and it is proved in [11, Abh. I, No. 24, pp. 36–37].

Now, let us suppose $m+2 < r$. In this case the coefficient matrix M of the linear system (39) of equations obtaining from (38) is an $(m+2)\times r$-matrix. The necessary condition to get a non-trivial solution of the linear system (39) is that rank$M < m+2$. Therefore the determinant of every $(m+2)\times(m+2)$ submatrix of M has to be 0. Since these subdeterminants are polynomials of the variables x, y, $y^{(i)}$, $i = 1, 2, \dots, m$, too, their common factors give the only possibilities for nontrivial differential equations f leaving invariant under the action of the group G. Summarizing our discussion we obtain.

Theorem 3.1. *To obtain the differential equations* $f\left(x, y, y', \dots, y^{(m)}\right) = 0$ *of order* m*, which allow a symmetry group with a given* r*-dimensional real Lie algebra* $\mathbf{g} = \langle X_i = \phi_i(x,y)\frac{\partial}{\partial x} + \eta_i(x,y)\frac{\partial}{\partial y}, i = 1, 2, \dots, r\rangle$ *such that* $m \le r-2$*, one has to create the matrix* M *defined by (40) and determine the greatest common divisor of all its* $(m+2)\times(m+2)$ *subdeterminants. If this polynomial fails to be identically 0, then its factors are the only possibilities for the sought differential equations.*

4. Examples of differential equations allowing a given Lie symmetry group

In this section using Lie's method we find the ordinary differential equations admitting a symmetry Lie group such that its Lie algebra is listed in Section 1.

4.1. **Examples where $m \leq r-2$.** This subsection is devoted to obtain differential equations of order m, where $m \leq r-2$ and r is the dimension of the Lie algebra of the given Lie group. Applying the known symmetries we solve some of these differential equations.

Example 4.1. The Lie algebra $\mathbf{g} = \mathbf{sl_2}(\mathbb{C})$ has as generators the vector fields given by (20). Therefore one has

$$(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) = (1, 0, -y, x, x^2 - y^2, 2xy), \tag{42}$$

$$(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6) = (0, 1, x, y, 2xy, y^2 - x^2). \tag{43}$$

Using (34), (35), (36), (37) we obtain $\eta_i^{(1)}$, $\eta_i^{(2)}$, $\eta_i^{(3)}$, $\eta_i^{(4)}$, $i = 1, 2, \ldots, 6$:

(44)
$$(\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}, \eta_4^{(1)}, \eta_5^{(1)}, \eta_6^{(1)}) = (0, 0, 1 + (y')^2, 0, 2y(1 + (y')^2), -2x(1 + (y')^2)),$$

(45)
$$\begin{aligned}(\eta_1^{(2)}, \eta_2^{(2)}, \eta_3^{(2)}, \eta_4^{(2)}, \eta_5^{(2)}, \eta_6^{(2)}) &= (0, 0, 3y'y^{(2)}, -y^{(2)}, 2y' - 2xy^{(2)} + 2(y')^3 + 6yy'y^{(2)}, \\ &\quad - 2 - 2(y')^2 - 2yy^{(2)} - 6xy'y^{(2)}),\end{aligned}$$

(46)
$$\begin{aligned}(\eta_1^{(3)}, \eta_2^{(3)}, \eta_3^{(3)}, \eta_4^{(3)}, \eta_5^{(3)}, \eta_6^{(3)}) &= (0, 0, 3(y^{(2)})^2 + 4y'y^{(3)}, -2y^{(3)}, \\ &\quad 12(y')^2y^{(2)} + 6y(y^{(2)})^2 - 4xy^{(3)} + 8yy'y^{(3)}, \\ &\quad - 12y'y^{(2)} - 6x(y^{(2)})^2 - 8xy'y^{(3)} - 4yy^{(3)}),\end{aligned}$$

(47)
$$\begin{aligned}(\eta_1^{(4)}, \eta_2^{(4)}, \eta_3^{(4)}, \eta_4^{(4)}, \eta_5^{(4)}, \eta_6^{(4)}) &= (0, 0, 10y^{(2)}y^{(3)} + 5y'y^{(4)}, -3y^{(4)}, \\ &\quad 30y'(y^{(2)})^2 + 20(y')^2y^{(3)} + 20yy^{(2)}y^{(3)} - 4y^{(3)} - 6xy^{(4)} + 10yy'y^{(4)}, \\ &\quad - 18(y^{(2)})^2 - 24y'y^{(3)} - 20xy^{(2)}y^{(3)} - 10xy'y^{(4)} - 6yy^{(4)}).\end{aligned}$$

Taking into account (42)-(47) we obtain for the determinant (41)

$$D = 16\left(y^{(3)} + y^{(3)}\left(y'\right)^2 - 3y'\left(y^{(2)}\right)^2\right)^2\left(1+\left(y'\right)^2\right).$$

As $1+\left(y'\right)^2 > 0$ the ordinary differential equation of order ≤ 4 having the Lie algebra $\mathbf{g} = \mathbf{sl_2}(\mathbb{C})$ as the Lie algebra of its symmetry group is given by (21) (cf. Theorem 3.1). Putting $z := y'$ into equation (21) we obtain

$$z^{(2)}\left(1+z^2\right) = 3z\left(z'\right)^2.$$

As $y^{(3)} = 0$ is not an invariant differential equation belonging to the Lie algebra $\mathbf{sl_2}(\mathbb{C})$ one gets $z' \neq 0$. Hence we have

$$\begin{gathered}
\frac{z^{(2)}}{z'} = \frac{3z}{1+z^2}z' \Longleftrightarrow (\ln z')' = \left(\frac{3}{2}\ln(1+z^2)\right)' \Longleftrightarrow \\
\left(\ln\frac{z'}{(1+z^2)^{\frac{3}{2}}}\right)' = 0 \Longleftrightarrow \frac{z'}{(1+z^2)^{\frac{3}{2}}} = e^c,\ c\in\mathbb{R} \text{ is a constant} \Longleftrightarrow \\
\int\frac{dz}{(1+z^2)^{\frac{3}{2}}} = \int l dx,\ l := e^c \text{ is a constant} \Longleftrightarrow \\
\frac{z}{\sqrt{1+z^2}} = lx+k,\ l,k\in\mathbb{R} \text{ are constants} \Longleftrightarrow \\
\frac{1}{z^2} = \frac{1}{(lx+k)^2} - 1 = \frac{1-(lx+k)^2}{(lx+k)^2} \Longleftrightarrow \\
y'(x) = \pm\sqrt{\frac{(lx+k)^2}{1-(lx+k)^2}} \Longleftrightarrow \\
y(x) = \pm\int\sqrt{\frac{(lx+k)^2}{1-(lx+k)^2}}dx = \pm\frac{1}{l}\sqrt{1-(lx+k)^2} + p,
\end{gathered}$$

with the real constants $l, k, p \in \mathbb{R}$.

Example 4.2. The generators of Lie algebra $\mathbf{g} = \mathbf{sl_2}(\mathbb{R}) \oplus \mathbf{sl_2}(\mathbb{R})$ are given by (22). Hence one has

$$\begin{gathered}
(\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6) = (1,0,0,x,0,x^2), \\
(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5,\eta_6) = (0,1,y,0,y^2,0).
\end{gathered}$$

Using (34), (35), (36), (37) the functions $\eta_i^{(1)}$, $\eta_i^{(2)}$, $\eta_i^{(3)}$, $\eta_i^{(4)}$, $i = 1, 2, \dots, 6$, are the following:

$$\begin{aligned}
&(\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}, \eta_4^{(1)}, \eta_5^{(1)}, \eta_6^{(1)}) \\
&\quad = (0, 0, y', -y', 2yy', -2xy'), \\
&(\eta_1^{(2)}, \eta_2^{(2)}, \eta_3^{(2)}, \eta_4^{(2)}, \eta_5^{(2)}, \eta_6^{(2)}) \\
&\quad = (0, 0, y^{(2)}, -2y^{(2)}, 2(yy^{(2)} + (y')^2), -2(y' + 2xy^{(2)})) \\
&(\eta_1^{(3)}, \eta_2^{(3)}, \eta_3^{(3)}, \eta_4^{(3)}, \eta_5^{(3)}, \eta_6^{(3)}) \\
&\quad = (0, 0, y^{(3)}, -3y^{(3)}, 2(yy^{(3)} + 3y'y^{(2)}), -6(xy^{(3)} + y^{(2)})), \\
&(\eta_1^{(4)}, \eta_2^{(4)}, \eta_3^{(4)}, \eta_4^{(4)}, \eta_5^{(4)}, \eta_6^{(4)}) \\
&\quad = (0, 0, y^{(4)}, -4y^{(4)}, 2(yy^{(4)} + 4y'y^{(3)} + 3(y^{(2)})^2), -4(2xy^{(4)} + 3y^{(3)})).
\end{aligned}$$

Hence the determinant (41) is $D = -4\left(2y'y^{(3)} - 3\left(y^{(2)}\right)^2\right)^2 y'$. It follows from Theorem 3.1 that the ordinary differential equations of order ≤ 4 admitting the symmetry group belongimg to the Lie algebra $\mathbf{sl_2}(\mathbb{R}) \oplus \mathbf{sl_2}(\mathbb{R})$ are $y' = 0$ and the equation given by (23).
To find the solutions of (23) we put $z := y'$. Hence we obtain

$$\begin{gathered}
2zz^{(2)} = 3(z')^2 \iff \frac{z^{(2)}}{z'} = \frac{3}{2}\frac{z'}{z} \iff \\
\left(\ln z' - \frac{3}{2}(\ln z)\right)' = 0 \iff \ln\left(\frac{z'}{z^{\frac{3}{2}}}\right) = c,\ c \in \mathbb{R} \text{ is a constant} \iff \\
\frac{z'}{z^{\frac{3}{2}}} = e^c = k,\ k \in \mathbb{R} \text{ is a constant} \iff \\
-2z^{-\frac{1}{2}} = kx + l,\ k, l \in \mathbb{R} \text{ are constants} \iff y' = \left(\frac{-2}{kx + l}\right)^2 \iff \\
y(x) = \int \frac{4}{(kx + l)^2} dx = -\frac{4}{k(kx + l)} + p, \text{ with the real constants } k, l, p.
\end{gathered}$$

Example 4.3. The basis elements of the Lie algebra $\mathbf{g} = \mathbf{sl_3}(\mathbb{R})$ are given by (26). Therefore one has

$$\begin{aligned}
(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8) &= (1, 0, 0, 0, x, y, x^2, xy), \\
(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \eta_8) &= (0, 1, x, y, 0, 0, xy, y^2).
\end{aligned}$$

Using the formulas (34), (35), (36), (37) we get

$$(\eta_1^{(1)},\eta_2^{(1)},\eta_3^{(1)},\eta_4^{(1)},\eta_5^{(1)},\eta_6^{(1)},\eta_7^{(1)},\eta_8^{(1)})$$
$$=(0,0,1,y',-y',-(y')^2,y-xy',yy'-x(y')^2),$$
$$(\eta_1^{(2)},\eta_2^{(2)},\eta_3^{(2)},\eta_4^{(2)},\eta_5^{(2)},\eta_6^{(2)},\eta_7^{(2)},\eta_8^{(2)})$$
$$=(0,0,0,y^{(2)},-2y^{(2)},-3y'y^{(2)},-3xy^{(2)},-3xy'y^{(2)}),$$
$$(\eta_1^{(3)},\eta_2^{(3)},\eta_3^{(3)},\eta_4^{(3)},\eta_5^{(3)},\eta_6^{(3)},\eta_7^{(3)},\eta_8^{(3)})$$
$$=(0,0,0,y^{(3)},-3y^{(3)},-3(y^{(2)})^2-4y'y^{(3)},-5xy^{(3)}-3y^{(2)},$$
$$-yy^{(3)}-3y'y^{(2)}-3x(y^{(2)})^2-4xy'y^{(3)}),$$
$$(\eta_1^{(4)},\eta_2^{(4)},\eta_3^{(4)},\eta_4^{(4)},\eta_5^{(4)},\eta_6^{(4)},\eta_7^{(4)},\eta_8^{(4)})$$
$$=(0,0,0,y^{(4)},-4y^{(4)},-5(2y^{(2)}y^{(3)}+y'y^{(4)}),-8(y^{(3)})-7xy^{(4)},$$
$$-6(y^{(2)})^2-8yy^{(3)}-10xy^{(2)}y^{(3)}-5xy'y^{(4)}-2yy^{(4)}).$$

Moreover for $\eta_i^{(5)}$ and $\eta_i^{(6)}$ we obtain

$$(\eta_1^{(5)},\eta_2^{(5)},\eta_3^{(5)},\eta_4^{(5)},\eta_5^{(5)},\eta_6^{(5)},\eta_7^{(5)},\eta_8^{(5)})$$
$$=(0,0,0,y^{(5)},-5y^{(5)},-15y^{(2)}y^{(4)}-10(y^{(3)})^2-6y'y^{(5)},-15y^{(4)}-9xy^{(5)},$$
$$-30y^{(2)}y^{(3)}-15y'y^{(4)}-15xy^{(2)}y^{(4)}-10x(y^{(3)})^2-6xy'y^{(5)}-3yy^{(5)}),$$
$$(\eta_1^{(6)},\eta_2^{(6)},\eta_3^{(6)},\eta_4^{(6)},\eta_5^{(6)},\eta_6^{(6)},\eta_7^{(6)},\eta_8^{(6)})$$
$$=(0,0,0,y^{(6)},-6y^{(6)},-7(3y^{(2)}y^{(5)}+5y^{(3)}y^{(4)}+y'y^{(6)}),$$
$$-24y^{(5)}-11xy^{(6)},-60y^{(2)}y^{(4)}-40(yy^{(3)})^2-24y'y^{(5)}-$$
$$-21xy^{(2)}y^{(5)}-35xy^{(3)}y^{(4)}-7xy'y^{(6)}-4yy^{(6)}).$$

We get for the determinant (41)

$$D=-2\left(9\left(y^{(2)}\right)^2y^{(5)}-45y^{(2)}y^{(3)}y^{(4)}+40\left(y^{(3)}\right)^3\right)^2y^{(2)}.$$

By Theorem 3.1 the ordinary differential equations of order ≤ 6 admitting the Lie algebra $\mathbf{sl_3}(\mathbb{R})$ as the Lie algebra of their symmetry group are determined by (27).

4.2. **Examples for $\mathbf{sl_2}(\mathbb{R})$.** There are three representations of the Lie algebra $\mathbf{g}=\mathbf{sl_2}(\mathbb{R})$ in [12, p. 501]. These belong to the imprimitive actions of the corresponding Lie groups on the plane (cf. [7, p. 341]). We treat these cases in Examples 4.4, 4.5, 4.6. There exists one representation of $\mathbf{sl_2}(\mathbb{R})$ describing the primitive action of the corresponding Lie group on the plane (cf. [7, p. 341], see also [14, (16), p. 374]). This case is considered in Example 4.8.

In this subsection we determine differential equations of order m such that $m \leq r-2$ as well as $m = r-1$, where r is the dimension of the Lie algebra of the given Lie group. Utilizing the known symmetries we find the solutions of some differential equations.

In Section 4.2 we illustrate that the differential equation allowing a Lie algebra as infinitesimal generators of symmetries strongly depends on the representation of the Lie algebra. For Examples 4.4, 4.5 and 4.8 there is one or a family of second order ordinary differential equations allowing the particular Lie algebra. In contrast to this, for Example 4.6 there does not exist any second order ordinary differential equation allowing the particular Lie algebra.

Example 4.4. Firstly we deal with the representation of the Lie algebra $\mathbf{g}_1 = \mathbf{sl_2}(\mathbb{R})$ generated by the vector fields (1). Hence one gets

$$(\phi_1, \phi_2, \phi_3) = (1, x, x^2), (\eta_1, \eta_2, \eta_3) = (1, y, y^2). \tag{48}$$

Using the formulas (34), (35) we have

$$(\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}) = (0, 0, 2(y-x)y'), \tag{49}$$

$$(\eta_1^{(2)}, \eta_2^{(2)}, \eta_3^{(2)}) = (0, -y^{(2)}, -2(y' - (y')^2 + 2xy^{(2)} - yy^{(2)})). \tag{50}$$

As the determinant (41) is $D = 2y'(y-x)^2$, we can conclude from Theorem 3.1 that the invariant first order differential equation is $y' = 0$.

To obtain the ordinary differential equations of order 2 admitting the basis elements of the Lie algebra $\mathbf{g_1}$ as infinitesimal generators of their symmetries, we suppose an explicit form:

$$f(x, y, y', y^{(2)}) = y^{(2)} - g(x, y, y') = 0,$$

and for the case $m = 2$, $r = 3$ we solve the system (38) of partial differential equations. Substituting (48), (49), (50) into (38) we receive the following system of partial differential equations:

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} = 0, \tag{51}$$

$$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} + g = 0, \tag{52}$$

$$-x^2\frac{\partial g}{\partial x} - y^2\frac{\partial g}{\partial y} - 2(y-x)y'\frac{\partial g}{\partial y'} + 2(y')^2 - 2y' + 2(y-2x)g = 0. \tag{53}$$

Taking into consideration (51) one has $g = g(x-y, y')$. Introducing the new variable $u = x - y$ we get $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial u}$, $\frac{\partial g}{\partial y} = -\frac{\partial g}{\partial u}$. Substituting these into equation (52) we obtain $g + u\frac{\partial g}{\partial u} = 0$. Therefore for g one has

$$g = \frac{h(y')}{u}. \tag{54}$$

Furthermore one gets

$$\frac{\partial g}{\partial y'} = \frac{h'}{u}, \ \frac{\partial g}{\partial x} = -\frac{h}{u^2}, \ \frac{\partial g}{\partial y} = \frac{h}{u^2}. \tag{55}$$

After putting (54) and (55) into (53) one has

$$2\left(y'\right)^2 - 2y' - 3h\left(y'\right) + 2y'h'\left(y'\right) = 0. \tag{56}$$

Substituting $z := y' \neq 0$ as a new variable into (56) we obtain

$$2h'(z) - 3\frac{h(z)}{z} + 2z - 2 = 0. \tag{57}$$

Solving (57) we receive $h(z) = -2\left(z^2 + z + cz^{3/2}\right)$ with a real constant c. Hence the second order differential equations which leave invariant under the action of the infinitesimal generators of the Lie algebra $\mathbf{g}_1$ are given by (2). (See also Table 8 in [6, p. 151]).

To solve the ordinary differential equation (2) of order 2 we take a two-dimensional solvable subalgebra $\langle X_1, X_2\rangle$ with the Lie bracket $[X_1, X_2] = X_1$ of $\mathbf{g}_1$ (see Section 2.1.2 in [6]). The differential equation (2) has the form

$$\frac{dy'}{dx} = -2\frac{\left(y'\right)^2 + y' + c\left(y'\right)^{3/2}}{x-y} =: \omega(x, y, y'). \tag{58}$$

We denote by Y the vector field

$$Y = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + \omega(x, y, y')\frac{\partial}{\partial y'}$$

such that the partial derivatives $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial y'}$ have the coefficients $\frac{dx}{dx} = 1$, $\frac{dy}{dx} = y'$, $\frac{dy'}{dx} = \omega(x, y, y')$. The vector field Y transforms the equation (58) into the linear partial differential equation

$$Y(f) = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \omega(x, y, y')\frac{\partial f}{\partial y'} = 0 \tag{59}$$

of the variables x, y, y'. The first prolonged vector fields

$$X_1^{(1)} = X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \ \ X_2^{(1)} = X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

leave invariant the equation (59). Therefore the integration of the differential equation (2) is reduced to the integration of the equation (59) (see [12, Kapitel 20, 4, pp. 457–464]). As the first prolonged vector fields do not depend on the term $\frac{\partial f}{\partial y'}$, we can calculate a first integral of (59) as follows (see [12, Kapitel 20, 2, pp. 443–444]). The coefficient matrix for the linear system of equations

$$0 = \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} + \omega(x, y, y') \frac{\partial f}{\partial y'}$$
$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$
$$0 = x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}$$

of variables $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y'}$ has determinant

$$D_1 = \begin{vmatrix} 1 & y' & \omega \\ 1 & 1 & 0 \\ x & y & 0 \end{vmatrix} = \omega(y - x) \neq 0.$$

Taking into account the determinant

$$D_2 = \begin{vmatrix} dx & dy & dy' \\ 1 & y' & \omega \\ 1 & 1 & 0 \end{vmatrix},$$

for a first integral of (59) one has

$$\int \frac{D_2}{D_1} = \int \frac{\omega(dy - dx) + (1 - y')dy'}{\omega(y - x)} =$$
$$\int \frac{dx - dy}{x - y} + \frac{(1 - y')dy'}{2((y')^2 + y' + c(y')^{\frac{3}{2}})} = \int \frac{du}{u} + \int \frac{(1 - y')dy'}{2((y')^2 + y' + c(y')^{\frac{3}{2}})} =$$
$$\ln(x - y) + \frac{1}{2} \ln y' - \ln(1 + c\sqrt{y'} + y'),$$

where $u = x - y$ is an invariant of X_1 (cf. [12, p. 533]). By exponentiating this, one gets a first integral

$$\varphi(x, y) = \frac{(x - y)\sqrt{y'}}{1 + c\sqrt{y'} + y'}.$$

By integrating it we have

$$\varphi = \frac{(x - y)\sqrt{y'}}{1 + c\sqrt{y'} + y'} = \text{constant} = \frac{1}{b}.$$

From this we express y' and solve the obtained equation. As a result, we have

$$\frac{1}{b(x - y)} = \frac{\sqrt{y'}}{1 + c\sqrt{y'} + y'} \Leftrightarrow$$
$$1 + (c - b(x - y))\sqrt{y'} + y' = 0 \Leftrightarrow$$
$$y' = (v + \sqrt{v^2 - 1})^2 = v^2 + 2v\sqrt{v^2 - 1} + v^2 - 1, \text{ with } v := \frac{b(x - y) - c}{2}.$$

Since $\frac{dv}{dx} = v' = \frac{b}{2}(1-y')$ we obtain $\frac{v'}{b} = \frac{1}{2}(1-y') = 1 - v^2 - v\sqrt{v^2-1}$. The solution v of this separable differential equation is

$$\int \frac{dv}{1-v^2-v\sqrt{v^2-1}} = \int b dx \Longleftrightarrow$$

$$\frac{1}{v+\sqrt{v^2-1}} = bx + a \Leftrightarrow$$

$$2v = bx + a + \frac{1}{bx+a}.$$

Hence the solutions $y(x)$ of (2) are

$$by(x) = -\frac{1}{bx+a} - a - c,$$

with the real constants a, b, c.

Example 4.5. Secondly, the infinitesimal generators of the Lie algebra $\mathbf{g}_2 = \mathbf{sl_2}(\mathbb{R})$ are defined by (3). Therefore for ϕ_i, η_i, $i = 1, 2, 3$, we have

$$(\phi_1, \phi_2, \phi_3) = (1, 2x, x^2), \ (\eta_1, \eta_2, \eta_3) = (0, y, xy). \tag{60}$$

Computing the formulas (34), (35) we get

$$(\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}) = (0, -y', y - xy'), \tag{61}$$

$$(\eta_1^{(2)}, \eta_2^{(2)}, \eta_3^{(2)}) = (0, -3y^{(2)}, -3xy^{(2)}). \tag{62}$$

For the determinant (41) we have $D = y^2$. Hence we can conclude from Theorem 3.1 that there does not exist any first order invariant differential equation under the action of the infinitesimal symmetries of the Lie algebra $\mathbf{g}_2$.

To receive the differential equations having the form $y^{(2)} - g(x, y, y') = 0$ and admitting the Lie algebra $\mathbf{g}_2$ as the Lie algebra of their infinitesimal symmetries, for the case $m = 2$, $r = 3$, $f(x, y, y', y^{(2)}) = y^{(2)} - g(x, y, y') = 0$ we have to determine the solution of (38). Applying (60), (61), (62) the system (38) of partial differential equations is equivalent to

$$\frac{\partial g}{\partial x} = 0 \tag{63}$$

$$-3g - y\frac{\partial g}{\partial y} - 2x\frac{\partial g}{\partial x} + y'\frac{\partial g}{\partial y'} = 0 \tag{64}$$

$$3xg + (y - xy')\frac{\partial g}{\partial y'} + x^2\frac{\partial g}{\partial x} + xy\frac{\partial g}{\partial y} = 0. \tag{65}$$

It can be concluded that the differential equation $y^{(2)} = 0$, i.e. $g(x, y, y') = 0$, fulfills equations (63), (64), (65). We may suppose that $g(x, y, y') \neq 0$. Using

(63) we come to the conclusion that the function g is independent of the variable x, that is $g(x,y,y') = g(y,y')$. Hence equation (64) changes for

$$-3 + y'\frac{\partial \ln g}{\partial y'} - y\frac{\partial \ln g}{\partial y} = 0. \tag{66}$$

Equivalently we arrive at the following ordinary differential equation (characteristic equation):

$$\frac{dy'}{y'} = \frac{dy}{-y} = \frac{d\ln g}{3} = 0. \tag{67}$$

Equation (67) provides the first integrals $yy' = c_1$ and $\frac{g}{y'^3} = c_2$. Therefore we get $g = (y')^3 f(yy')$. Using $z := yy'$ as a new variable for the function $f(yy') = f(z)$ we obtain $\frac{\partial f}{\partial y} = y'\frac{df}{dz}$ and $\frac{\partial f}{\partial y'} = y\frac{df}{dz}$. The application of these changes equation (66) for $3f(z) + zf'(z) = 0$. This last differential equation has the solution $f(z) = az^{-3}$ with a real constant a. Hence we obtain $g = ay^{-3}$. As a result, the second order differential equations (4) allow the Lie algebra $\mathbf{g}_2$ defined by (3) as the Lie algebra of their infinitesimal symmetries. (See also Table 8 in [6], p. 151.)

After the multiplication of both sides of (4) by $2y'$ the solutions of these differential equations can be received as follows

$$2y'y^{(2)} - \frac{2ay'}{y^3} = 0 \Longleftrightarrow (y')^2 + \frac{a}{y^2} = \text{const.} = b \Longleftrightarrow$$
$$y' = \frac{\sqrt{by^2 - a}}{y} \Longleftrightarrow \int \frac{ydy}{\sqrt{by^2-a}} = \int 1 \cdot dx \Longleftrightarrow$$

$$\frac{1}{b}\sqrt{by^2 - a} = x + c, \text{ where c is a constant} \Longleftrightarrow$$
$$by^2 = b^2(x+c)^2 + a, \; a, b, c \in \mathbb{R}.$$

Example 4.6. Thirdly, the infinitesimal generators of the Lie algebra $\mathbf{g}_3 = \mathbf{sl_2}(\mathbb{R})$ are defined by (5). Therefore for ϕ_i, η_i, $i = 1, 2, 3$, we have

$$(\phi_1, \phi_2, \phi_3) = (0, 0, 0), \; (\eta_1, \eta_2, \eta_3) = (1, y, y^2). \tag{68}$$

Computing the formulas (34), (35), (36) we get

$$(\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}) = (0, y', 2yy'), \tag{69}$$
$$(\eta_1^{(2)}, \eta_2^{(2)}, \eta_3^{(2)}) = (0, y^{(2)}, 2((y')^2 + yy^{(2)})), \tag{70}$$
$$(\eta_1^{(3)}, \eta_2^{(3)}, \eta_3^{(3)}) = (0, y^{(3)}, 2(3y'y^{(2)} + yy^{(3)})). \tag{71}$$

Since in this case the determinant D in (41) is identically 0, therefore we have to find the solution the following system of partial differential equations for f:

$$\begin{aligned} \frac{\partial f}{\partial y} &= 0, \\ y\frac{\partial f}{\partial y} + y'\frac{\partial f}{\partial y'} &= 0, \\ y^2\frac{\partial f}{\partial y} + 2yy'\frac{\partial f}{\partial y'} &= 0. \end{aligned}$$

The first equation yields that f is independent of the variable y, i.e. $f = f(x, y')$. Using the second equation we receive $y'\frac{\partial f}{\partial y'} = 0$, i.e. the function f has the form $f = g(x)y'$, where g is an arbitrary real function. The third equation is the same. Hence we arrive at $g(x)y' = 0$. Therefore the invariant differential equations under the action of the infinitesimal symmetries of the Lie algebra $\mathbf{g}_3$ are trivial, that is $y' = 0$.

Now we prove that there does not exist any second order differential equation admitting Lie algebra $\mathbf{g}_3$ as the Lie algebra of their infinitesimal symmetries. Let us assume that there is a differential equation $f\left(x, y, y', y^{(2)}\right) = 0$ which leaves invariant under the action of the infinitesimal generators of the Lie algebra $\mathbf{g}_3$. Applying (38), (68), (69), (70) the function f will fulfill the following system of partial differential equations:

$$\frac{\partial f}{\partial y} = 0 \tag{72}$$

$$y\frac{\partial f}{\partial y} + y'\frac{\partial f}{\partial y'} + y^{(2)}\frac{\partial f}{\partial y^{(2)}} = 0 \tag{73}$$

$$y^2\frac{\partial f}{\partial y} + 2yy'\frac{\partial f}{\partial y'} + \left(2\left(y'\right)^2 + 2yy^{(2)}\right)\frac{\partial f}{\partial y^{(2)}} = 0. \tag{74}$$

Taking into account (72) the function f does not depend on the variable y, i.e. $f = f\left(x, y', y^{(2)}\right)$. Using this to equations (73), (74) we obtain

$$y'\frac{\partial f}{\partial y'} + y^{(2)}\frac{\partial f}{\partial y^{(2)}} = 0 \tag{75}$$

$$yy'\frac{\partial f}{\partial y'} + \left(\left(y'\right)^2 + yy^{(2)}\right)\frac{\partial f}{\partial y^{(2)}} = 0. \tag{76}$$

We multiply equation (75) by $-y$ and add the obtained equation to (76). After performing this we obtain

$$\left(y'\right)^2\frac{\partial f}{\partial y^{(2)}} = 0.$$

Therefore the function f is independent of $y^{(2)}$, that is $f = f(x, y')$, which contradicts the assumption that f is a second order differential equation.

Now we seek the differential equations having the form $y^{(3)}-g(x,y,y',y^{(2)}) = 0$ and admitting the Lie algebra $\mathbf{g}_3$ as the Lie algebra of their infinitesimal symmetries. To receive them we need to find the solutions of the system (38) of partial differential equations for the case $f(x,y,y',y^{(2)},y^{(3)}) = y^{(3)} - g(x,y,y',y^{(2)}) = 0$. Utilizing (68), (69), (70) the system (38) can be reduced to

$$\frac{\partial g}{\partial y} = 0 \tag{77}$$

$$-y\frac{\partial g}{\partial y} - y'\frac{\partial g}{\partial y'} - y^{(2)}\frac{\partial g}{\partial y^{(2)}} + g = 0 \tag{78}$$

$$-y^2\frac{\partial g}{\partial y} - 2yy'\frac{\partial g}{\partial y'} - \left(2\left(y'\right)^2 + 2yy^{(2)}\right)\frac{\partial g}{\partial y^{(2)}} + 6y'y^{(2)} + 2yg = 0. \tag{79}$$

We can conclude from (77) that $g = g(x,y',y^{(2)})$. Applying this, equation (78) is equivalent to

$$y'\frac{\partial g}{\partial y'} + y^{(2)}\frac{\partial g}{\partial y^{(2)}} = g \tag{80}$$

and furthermore equation (79) changes for

$$\frac{3y^{(2)}}{y'} = \frac{\partial g}{\partial y^{(2)}}. \tag{81}$$

It follows from equation (81) that $g = \frac{3\left(y^{(2)}\right)^2}{2y'} + h\left(x,y'\right)$. Replacing this form of g into (80) we get the partial differential equation $y'\frac{\partial h(x,y')}{\partial y'} = h(x,y')$, which provides that $h(x,y') = y'f(x)$. Hence the third-order invariant differential equation under the action of the infinitesimal generators of the Lie algebra $\mathbf{g}_3$ is given by (6) for arbitrary real function $f(x)$. (See also Table 8 in [6, p. 151]).

To solve the differential equation (6) we introduce the new variable $z(x) := y'(x)$. After replacing this into (6) we receive

$$z^{(2)} - \frac{3(z')^2}{2z} - zf(x) = 0 \Longleftrightarrow \tag{82}$$

$$\frac{d}{dx}\left(\frac{z'}{z}\right) = f(x) + \frac{1}{2}\left(\frac{z'}{z}\right)^2. \tag{83}$$

Substituting $l(x) := \frac{z'}{z}$ the equation (83) is reduced to the Ricatti differential equation

$$l' = \frac{1}{2}l^2 + f(x) \tag{84}$$

(see [9, Section 4.9, p. 21]). Let us denote $v := \frac{1}{2}l$. It satisfies the Ricatti differential equation $v' = v^2 + \frac{1}{2}f(x)$. Putting $v = -\frac{u'}{u}$ the function u fulfills the linear differential equation

$$0 = u^{(2)} + \frac{1}{2}uf(x). \tag{85}$$

Denoted by $\tilde{u}$ the solutions of (85) the solutions $\tilde{l}$ of (84) can be expressed in the form $\tilde{l} = -2\frac{\tilde{u}'}{\tilde{u}}$. Applying the solution $\tilde{l}$ of (84) one gets for the solution $\tilde{z}$ of (82) this form $\tilde{z} = e^{\int \tilde{l}dx}$. Therefore for the solution $\tilde{y}$ of (6) we obtain

$$\tilde{y} = \int e^{\int \tilde{l}dx}dx.$$

Remark 4.7. The fluid draining equation

$$w^{(3)} = w^{-2}$$

is discussed in [18]. It is equivalent to the Riccati equation $l' = \frac{1}{2}l^2 + x$, which is the same as (6) with the function $f\colon x \mapsto x$. In [18] Nucci used the representation of $\mathbf{sl_2}(\mathbb{R})$ given in Example 4.8 to obtain the solution of the fluid draining equation. Finally, we deal in this subsection with this representation.

Example 4.8. Now we treat the Lie algebra $\mathbf{g}_4 = \mathbf{sl_2}(\mathbb{R})$ having as infinitesimal generators defined in (7). Hence for ϕ_i, η_i, $i = 1, 2, 3$, we have

$$(\phi_1, \phi_2, \phi_3) = (1, x, x^2 - y^2), (\eta_1, \eta_2, \eta_3) = (0, y, 2xy). \tag{86}$$

Computing the formulas (34), (35), (36) we get

$$(\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}) = (0, 0, 2(1 + (y')^2)y), \tag{87}$$

$$(\eta_1^{(2)}, \eta_2^{(2)}, \eta_3^{(2)}) = (0, -y^{(2)}, 2(y' + (y')^3 + 3yy'y^{(2)} - xy^{(2)})). \tag{88}$$

As the determinant (41) is $D = 2(1+(y')^2)y^2$, we can conclude from Theorem 3.1 that there is no differential equation of order one which is invariant under the action of the infinitesimal generators of the Lie algebra $\mathbf{g}_4$.

Now we seek the differential equations which have the explicit form $y^{(2)} - g(x, y, y') = 0$ and which admit the Lie algebra $\mathbf{g}_4$ as the Lie algebra of their infinitesimal symmetries. For this purpose we have to find the solutions of the following system of partial differential equations:

$$\frac{\partial g}{\partial x} = 0 \tag{89}$$

$$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} + g = 0 \tag{90}$$

$$\begin{aligned} -\left(x^2 - y^2\right)\frac{\partial g}{\partial x} - 2xy\frac{\partial g}{\partial y} - 2y\left(1 + \left(y'\right)^2\right)\frac{\partial g}{\partial y'} \\ +2y' + 2\left(y'\right)^3 + 6yy'g - 2xg = 0 \end{aligned} \tag{91}$$

which is created applying equations (38) for the case $m = 2$, $r = 3$, $f(x, y, y', y^{(2)}) = y^{(2)} - g(x, y, y') = 0$ and using (86), (87), (88). From (89) we can conclude that the function g is independent of the variable x, that is $g(x, y, y') = g(y, y')$. Hence equation (90) changes for

$$-y\frac{\partial g}{\partial y} = g.$$

Therefore we may suppose that the form of the function g is $g = \frac{h(y')}{y}$. Substituting this form into (91) and taking into consideration that g does not depend on x, after simplification for the function $h(y')$ we get the following linear differential equation

$$y'\left(1 + \left(y'\right)^2\right) + 3y'h\left(y'\right) = \left(1 + \left(y'\right)^2\right)h'\left(y'\right). \tag{92}$$

To solve the separable differential equation $\frac{3y'}{1+(y')^2} = \frac{h'(y')}{h(y')}$ we obtain $h(y') = d\left(1 + (y')^2\right)^{3/2}$ with a real constant d. Making use of this the solution of (92) is

$$h\left(y'\right) = -\left(1 + \left(y'\right)^2\right) + d\left(1 + \left(y'\right)^2\right)^{3/2},$$

with a real constant d. Hence the invariant second order differential equations under the action of the infinitesimal symmetries of the Lie algebra $\mathbf{g}_4$ are given by (8).

Similarly to Example 4.4, we apply the symmetries belonging to the 2-dimensional subalgebra $\langle X_1, X_2\rangle$ with the Lie bracket $[X_1, X_2] = X_1$ to solve the ordinary differential equation (8) of order 2. The form of the differential equation (8) can be expressed in the following way

$$\frac{dy'}{dx} = -\frac{1}{y}\left(1 + \left(y'\right)^2\right)\left(1 - d\left(1 + \left(y'\right)^2\right)^{1/2}\right) =: \omega(y, y'). \tag{93}$$

We denote by Y the vector field

$$Y = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + \omega(y, y')\frac{\partial}{\partial y'}$$

such that the partial derivatives $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial y'}$ have the coefficients $\frac{dx}{dx} = 1$, $\frac{dy}{dx} = y'$, $\frac{dy'}{dx} = \omega(y, y')$. Using Y the equation (8) becomes equivalent to the linear partial differential equation

$$Y(f) = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \omega(y, y')\frac{\partial f}{\partial y'} = 0 \tag{94}$$

of the variables x, y, y'. The first prolonged vector fields

$$X_1^{(1)} = X_1 = \frac{\partial}{\partial x}, \quad X_2^{(1)} = X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

leaves invariant the equation (94). Therefore the integration of the differential equation (8) is reduced to the integration of the equation (94) (cf. [12, Kapitel 20, 4, pp. 457–464]). As the first prolonged vector fields do not depend on term $\frac{\partial f}{\partial y'}$, we can calculate a first integral of (8) as follows (see [12, Kapitel 20, 2, pp. 443–444]). The coefficient matrix of the linear system of equations

$$\begin{aligned}
0 &= \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \omega(y, y')\frac{\partial f}{\partial y'} \\
0 &= \frac{\partial f}{\partial x} \\
0 &= x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}
\end{aligned}$$

of variables $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial y'}$ has determinant

$$D_1 = \begin{vmatrix} 1 & y' & \omega \\ 1 & 0 & 0 \\ x & y & 0 \end{vmatrix} = y\omega \neq 0.$$

Taking into consideration the determinant

$$D_2 = \begin{vmatrix} dx & dy & dy' \\ 1 & y' & \omega \\ 1 & 0 & 0 \end{vmatrix} = \omega dy - y'dy'$$

for a first integral of (94) we obtain

$$\int \frac{D_2}{D_1} = \int \frac{\omega dy - y' dy'}{\omega y} = \int \frac{dy}{y} + \frac{y' dy'}{\left(1+(y')^2\right)\left(1-d\sqrt{1+(y')^2}\right)}$$
$$= \int \frac{dy}{y} + \int \frac{y'}{\sqrt{1+(y')^2}\left(\sqrt{1+(y')^2}-d\right)} dy'$$
$$= \ln y + \ln\left(\sqrt{1+(y')^2}-d\right).$$

By exponentiating this, one gets a first integral

$$\varphi = \left(\sqrt{1+(y')^2}-d\right) y.$$

By integrating it we have

$$\varphi = \left(\sqrt{1+(y')^2}-d\right) y = \text{constant} = c.$$

From this we express y' and we receive

$$y' = \sqrt{\left(\frac{c}{y}+d\right)^2 - 1},$$

with the constants $c, d \in \mathbb{R}$. Hence the solutions $y(x)$ of (8) are

$$\frac{\sqrt{c^2+2cdy+(d^2-1)\,y^2}}{d^2-1} - \frac{cd\ln\left(\frac{(d^2-1)y+cd}{\sqrt{d^2-1}} + \sqrt{c^2+2cdy+(d^2-1)\,y^2}\right)}{(d^2-1)^{\frac{3}{2}}}$$
$$= x + a,$$

with the constants $a, c, d \in \mathbb{R}$.

4.3. **Differential equations for $\mathbf{so_3}(\mathbb{R})$.**

Example 4.9. The generators of the Lie algebra $\mathbf{g} = \mathbf{so_3}(\mathbb{R}) \cong \mathbf{su_2}(\mathbb{C})$ are given by (9). There does not exist any 2-dimensional subalgebra of $\mathbf{so_3}(\mathbb{R})$. Using the vector fields (9) we obtain

(95) $$(\phi_1, \phi_2, \phi_3) = (1+x^2, xy, y), \; (\eta_1, \eta_2, \eta_3) = (xy, 1+y^2, -x).$$

Making use of formulas (34), (35) to (95) we get

(96) $$(\eta_1^{(1)}, \eta_2^{(1)}, \eta_3^{(1)}) = (y - xy', yy' - x(y')^2, -1-(y')^2),$$

(97) $$(\eta_1^{(2)}, \eta_2^{(2)}, \eta_3^{(2)}) = (-3xy^{(2)}, -3xy'y^{(2)}, -3y'y^{(2)}).$$

The ordinary differential equations of order one allowing as infinitesimal generators of their symmetries the basis elements of the Lie algebra $\mathbf{g} = \mathbf{so_3}(\mathbb{R})$ are missing because the determinant $D = -(1+x^2+y^2)(x^2(y')^2 - 2xyy' + 1 + y^2)$ is different from 0, since $x^2(y')^2 - 2xyy' + 1 + y^2 = x^2\left(\left(y' - \frac{y}{x}\right)^2 + \frac{1}{x^2}\right) > 0$.

Now we seek the ordinary differential equations which have the explicit form $y^{(2)} - g(x, y, y') = 0$ and which admit the Lie algebra $\mathbf{g} = \mathbf{so_3}(\mathbb{R})$ as the Lie algebra of their infinitesimal symmetries. To obtain them we need to find the solutions of the system (38) of partial differential equations when $m = 2$, $r = 3$, $f(x, y, y', y^{(2)}) = y^{(2)} - g(x, y, y') = 0$. Substituting (95), (96), (97) into (38) we receive the following system of partial differential equations:

$$\left(1 + x^2\right)\frac{\partial g}{\partial x} + xy\frac{\partial g}{\partial y} + \left(y - xy'\right)\frac{\partial g}{\partial y'} + 3xg = 0 \tag{98}$$

$$xy\frac{\partial g}{\partial x} + \left(1 + y^2\right)\frac{\partial g}{\partial y} + \left(yy' - x\left(y'\right)^2\right)\frac{\partial g}{\partial y'} + 3xy'g = 0 \tag{99}$$

$$-y\frac{\partial g}{\partial x} + x\frac{\partial g}{\partial y} + \left(1 + \left(y'\right)^2\right)\frac{\partial g}{\partial y'} - 3y'g = 0. \tag{100}$$

If $g = 0$, then the partial differential equations (98), (99), (100) hold. Therefore we can conclude that the differential equation $y^{(2)} = 0$ admits the symmetry group belonging to the Lie algebra $\mathbf{g} = \mathbf{so_3}(\mathbb{R})$. We may suppose that $g \neq 0$. The multiplication of (98) by y' and the subtraction of (99) from the new equation give

$$\left(\left(1 + x^2\right)y' - xy\right)\frac{\partial g}{\partial x} + \left(xyy' - \left(1 + y^2\right)\right)\frac{\partial g}{\partial y} = 0. \tag{101}$$

The multiplication of (100) by x and the addition of the new equation to (99) yield

$$(1 + x^2 + y^2)\frac{\partial g}{\partial y} + (x + yy')\frac{\partial g}{\partial y'} = 0. \tag{102}$$

The substitution of the expressions

$$\frac{\partial g}{\partial y} = -\frac{(x + yy')}{(1 + x^2 + y^2)}\frac{\partial g}{\partial y'},$$
$$\frac{\partial g}{\partial x} = \frac{\left(\left(1 + y^2\right) - xyy'\right)}{\left(\left(1 + x^2\right)y' - xy\right)}\frac{\partial g}{\partial y}$$

into (100) results in

$$\left(1+y^2-2xyy'+\left(1+x^2\right)\left(y'\right)^2\right)\frac{\partial g}{\partial y'}=3\left(\left(1+x^2\right)y'-xy\right)g \Longleftrightarrow$$

$$\frac{1}{g}\frac{\partial g}{\partial y'}=\frac{3}{2}\frac{\partial \ln\left(1+y^2-2xyy'+\left(1+x^2\right)\left(y'\right)^2\right)}{\partial y'} \Longleftrightarrow$$

$$\frac{\partial \ln\left(\frac{g}{\left(1+y^2-2xyy'+(1+x^2)(y')^2\right)^{3/2}}\right)}{\partial y'}=0 \Longleftrightarrow$$

$$g=K(x,y)\left(1+y^2-2xyy'+\left(1+x^2\right)\left(y'\right)^2\right)^{3/2}. \tag{103}$$

Putting (103) into (102) and simplifying the obtained equation we receive

$$\frac{1}{K(x,y)}\frac{\partial K(x,y)}{\partial y}=-\frac{3y}{(1+x^2+y^2)} \Longleftrightarrow$$

$$\frac{\partial \ln\left(K(x,y)\left(1+x^2+y^2\right)^{3/2}\right)}{\partial y}=0 \Longleftrightarrow$$

$$K(x,y)=\frac{U(x)}{(1+x^2+y^2)^{3/2}}, \text{ or equivalently}$$

$$g=U(x)\left(\frac{1+y^2-2xyy'+\left(1+x^2\right)\left(y'\right)^2}{1+x^2+y^2}\right)^{3/2}. \tag{104}$$

Inserting (104) into (101) and making some computations we have

$$U'(x)\frac{1+y^2-2xyy'+(1+x^2)(y')^2}{1+x^2+y^2}=0.$$

Hence we come to the conclusion that $U(x)=c$ with a real constant c. As a result, the invariant differential equations of order two under the action of the infinitesimal generators of the Lie algebra $\mathbf{g}=\mathbf{so_3}(\mathbb{R})$ are given by (10). (See also Table 8 in [6, p. 151]).

Remark 4.10. Taking into account that (10) is an ordinary differential equation of order two it is necessary to use a two dimensional solvable Lie subalgebra of the Lie algebra of its symmetry group to be capable of finding its solution (see e.g. [22], Section 2.1.2 in [6]). Since two dimensional subalgebras of $\mathbf{so_3}(\mathbb{R})$ are missing, we cannot use the infinitesimal generators of $\mathbf{so_3}(\mathbb{R})$ to solve equation (10) in full extent.

4.4. **Further examples.** In this subsection Lie's method can be used for r-dimensional Lie transformation groups which are non semi-simple and act on the plane $\mathbb{R}^2$. Here we deal with those ordinary differential equations which have order $m \leq r-2$ and which are invariant under the action of these Lie groups.

Example 4.11. The generators of the solvable Lie algebra $\mathbf{g}_\alpha$ are defined by (11). (See [7, p. 341], Table 1, Case 1). It can be established that the determinant (41) is $D = -\left(1+(y')^2\right)$ which is never 0. Hence we cannot find any ordinary differential equation of order one which admits the Lie algebra $\mathbf{g}_\alpha$ as the Lie algebra of its symmetry group.

Example 4.12. The basis elements of the Lie algebra $\mathbf{g}_\beta$ are defined by (12). (See [7, p. 341], Table 1, Case 12). Therefore the determinant (41) is $D = (\beta - 1)\, y'$. It follows from Theorem 3.1 that the unique differential equation which allows the Lie algebra $\mathbf{g}_\beta$ as the Lie algebra of its symmetry group is $y' = 0$.

Example 4.13. According to [7, p. 341], Table 1, Case 4, we consider the 4-dimensional solvable Lie algebra $\mathbf{g}$ whose generators are defined by (13). The equation $y^{(2)} = 0$ is the unique ordinary differential equation of order ≤ 2 which is invariant under the action of the symmetry group belonging to the Lie algebra $\mathbf{g}$ because of $D = -y^{(2)}((y')^2 + 1)$.

Example 4.14. According to [7, p. 341], Table 1, Case 13, we deal with the 4-dimensional solvable Lie algebra $\mathbf{g}$ whose generators are defined by (14). Then for the determinant (41) we have $D = y' y^{(2)}$. Therefore the ordinary differential equations of order ≤ 2 which are invariant under the action of the infinitesimal generators of the Lie algebra $\mathbf{g}$ are $y^{(2)} = 0$ and $y' = 0$ (see Theorem 3.1).

Example 4.15. The basis elements of the Lie algebra $\mathbf{sl_2}(\mathbb{R}) \times \mathbb{R}$ are defined by (15). (See [7, p. 341], Table 1, Case 14). As the determinant (41) is equal to $D = 2\,(y')^2$, the equation $y' = 0$ is the unique ordinary differential equation of order ≤ 2 which is invariant under the action of the infinitesimal generators of the Lie algebra $\mathbf{sl_2}(\mathbb{R}) \times \mathbb{R}$ (see Theorem 3.1).

Example 4.16. The basis elements of the Lie algebra $\mathbf{g} = \mathbf{gl_2}(\mathbb{R})$ are the vector fields given by (16). (See [7, p. 341], Table 1, Case 19). As the determinant (41) is equal to $D = -2y^2 y^{(2)}$, we can conclude from Theorem 3.1 that the equation $y' = 0$ is the unique ordinary differential equation of order ≤ 2 which is invariant under the action of the infinitesimal generators of the Lie algebra $\mathbf{gl_2}(\mathbb{R})$.

Example 4.17. The generators of the Lie algebra $\mathbf{sl_2}(\mathbb{R}) \times L_2$, where L_2 denotes the non-abelian 2-dimensional Lie algebra, are given by (17). (See [7, p. 341], Table 1, Case 15). Since the determinant (41) is equal to $D = 2y'\left(2y'y^{(3)} - 3\left(y^{(2)}\right)^2\right)$, it can be established from Theorem 3.1 that the equations given by (18) are precisely the invariant ordinary differential equations of order ≤ 3 under the action of the infinitesimal generators of the Lie algebra $\mathbf{sl_2}(\mathbb{R}) \times L_2$.

Example 4.18. According to [7, p. 341], Table 1, Case 5, the basis elements of the Lie algebra $\mathbf{g} = \mathbf{sl_2}(\mathbb{R}) \ltimes \mathbb{R}^2$ are defined by (19). Since the determinant (41) is equal to $D = 9\left(y^{(2)}\right)^3$, the equation $y^{(2)} = 0$ is the unique ordinary differential equation of order ≤ 3 which is invariant under the action of the infinitesimal generators of the Lie algebra $\mathbf{sl_2}(\mathbb{R}) \ltimes \mathbb{R}^2$.

Example 4.19. According to [7, p. 341], Table 1, Case 6, the generators of the Lie algebra $\mathbf{g} = \mathbf{gl_2}(\mathbb{R}) \ltimes \mathbb{R}^2$ are determined by (24). As for the determinant (41) we have $D = -2\left(3y^{(4)}y^{(2)} - 5\left(y^{(3)}\right)^2\right)\left(y^{(2)}\right)^2$, the equations in (25) are precisely the invariant ordinary differential equations of order ≤ 4 under the action of the infinitesimal generators of the Lie algebra $\mathbf{gl_2}(\mathbb{R}) \ltimes \mathbb{R}^2$ (see Theorem 3.1).

5. First order systems of ordinary differential equations that admit a given Lie group as their symmetry group

In this chapter, we propose a method for receiving systems of first order differential equations that allow a specific Lie group as their symmetry group based on Lie's initial concept in Section 3. Assume that the real Lie group G has dimension r. We begin by considering the situation in which the Lie algebra $\mathbf{g}$ of G is the direct sum of infinitesimal generators of trivial and time-preserving symmetries with non-trivial direct factors.

5.1. **Time-dependent symmetries.** Introducing the notation $y' = \frac{dy}{dx}$ and $z' = \frac{dz}{dx}$ we investigate the following first order system of ordinary differential equations, which is time-dependent:

$$\begin{aligned} f_1(x, y, z, y', z') &= 0, \\ f_2(x, y, z, y', z') &= 0. \end{aligned} \tag{105}$$

The following vector fields in $\mathbb{R}^3$ serve as basis elements for the Lie algebra $\mathbf{g}$ of G:

$$X_i(x, y, z) = \phi_i(x, y, z)\frac{\partial}{\partial x} + \eta_i(x, y, z)\frac{\partial}{\partial y} + \alpha_i(x, y, z)\frac{\partial}{\partial z}, \ i = 1, 2, \ldots, r.$$

With respect to the variable x the first prolonged vector field of $X_i(x, y, z)$, $i = 1, 2, \ldots, r$, can be written into the form:

$$X_i^{(1)}(x, y, z, y', z') = X_i + \eta_i^{(1)}(x, y, z, y', z')\frac{\partial}{\partial y'} + \alpha_i^{(1)}(x, y, z, y', z')\frac{\partial}{\partial z'},$$

where

$$\eta_i^{(1)} = \frac{\partial \eta_i}{\partial x} + \frac{\partial \eta_i}{\partial y} y' + \frac{\partial \eta_i}{\partial z} z' - y' \left(\frac{\partial \phi_i}{\partial x} + \frac{\partial \phi_i}{\partial y} y' + \frac{\partial \phi_i}{\partial z} z' \right), \tag{106}$$

$$\alpha_i^{(1)} = \frac{\partial \alpha_i}{\partial x} + \frac{\partial \alpha_i}{\partial y} y' + \frac{\partial \alpha_i}{\partial z} z' - z' \left(\frac{\partial \phi_i}{\partial x} + \frac{\partial \phi_i}{\partial y} y' + \frac{\partial \phi_i}{\partial z} z' \right). \tag{107}$$

The necessary and sufficient condition for the time-dependent system (105) in order that it could admit the given group G as its symmetries is that the functions f_j, $j = 1, 2$ need to satisfy the following system of partial differential equations

$$\begin{array}{rcl}
\phi_1 \frac{\partial f_j}{\partial x} + \eta_1 \frac{\partial f_j}{\partial y} + \alpha_1 \frac{\partial f_j}{\partial z} + \eta_1^{(1)} \frac{\partial f_j}{\partial y'} + \alpha_1^{(1)} \frac{\partial f_j}{\partial z'} & = & 0, \\
\phi_2 \frac{\partial f_j}{\partial x} + \eta_2 \frac{\partial f_j}{\partial y} + \alpha_2 \frac{\partial f_j}{\partial z} + \eta_2^{(1)} \frac{\partial f_j}{\partial y'} + \alpha_2^{(1)} \frac{\partial f_j}{\partial z'} & = & 0, \\
\vdots & & \\
\phi_i \frac{\partial f_j}{\partial x} + \eta_i \frac{\partial f_j}{\partial y} + \alpha_i \frac{\partial f_j}{\partial z} + \eta_i^{(1)} \frac{\partial f_j}{\partial y'} + \alpha_i^{(1)} \frac{\partial f_j}{\partial z'} & = & 0, \\
\vdots & & \\
\phi_r \frac{\partial f_j}{\partial x} + \eta_r \frac{\partial f_j}{\partial y} + \alpha_r \frac{\partial f_j}{\partial z} + \eta_r^{(1)} \frac{\partial f_j}{\partial y'} + \alpha_r^{(1)} \frac{\partial f_j}{\partial z'} & = & 0.
\end{array} \tag{108}$$

We denote by M the $5 \times r$-matrix

$$M = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \dots & \phi_r \\ \eta_1 & \eta_2 & \eta_3 & \dots & \eta_r \\ \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_r \\ \eta_1^{(1)} & \eta_2^{(1)} & \eta_3^{(1)} & \dots & \eta_r^{(1)} \\ \alpha_1^{(1)} & \alpha_2^{(1)} & \alpha_3^{(1)} & \dots & \alpha_r^{(1)} \end{pmatrix}.$$

The system of partial differential equations provided by (108) can therefore be considered as the system of 'linear equations' in the variables $\frac{\partial f_j}{\partial x}$, $\frac{\partial f_j}{\partial y}$, $\frac{\partial f_j}{\partial z}$, $\frac{\partial f_j}{\partial y'}$, $\frac{\partial f_j}{\partial z'}$:

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial y'} & \frac{\partial f_1}{\partial z'} \end{pmatrix} \cdot M = \begin{pmatrix} 0 & \dots & 0 \end{pmatrix},$$
$$\begin{pmatrix} \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial y'} & \frac{\partial f_2}{\partial z'} \end{pmatrix} \cdot M = \begin{pmatrix} 0 & \dots & 0 \end{pmatrix}.$$

As a result, the rank of the matrix M must be at most 5 in order to get non-trivial solutions f_j, $j = 1, 2$, of the system of equations provided by (108). If $\mathrm{rank} M = 4$, however, the received solutions for the vectors $\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial y'} & \frac{\partial f_1}{\partial z'} \end{pmatrix}$ and $\begin{pmatrix} \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial y'} & \frac{\partial f_2}{\partial z'} \end{pmatrix}$ are linearly dependent, i.e., the obtained system of differential equations contains only one equation instead of two equations for the two dependent variables y and z. Hence $\mathrm{rank} M < 4$ is a more usable criterion.

The condition $\mathrm{rank} M \leq r$ is always true. Arising from this if $r < 4$, then the requirement for the rank is obviously fulfilled. In this case one solves

(108) in $\frac{\partial f_j}{\partial x}, \frac{\partial f_j}{\partial y}, \frac{\partial f_j}{\partial z}, \frac{\partial f_j}{\partial y'}, \frac{\partial f_j}{\partial z'}$, and checks if any solution yields a nontrivial system of differential equations f_1, f_2. Now we assume that $r \geq 4$. In this case each 4×4-subdeterminant of M has to be zero to satisfy the rank criterion.

If we suppose that the functions f_1, f_2 have the reduced explicit form

$$\begin{aligned} f_1(x,y,z,y',z') &= y' - g_1(x,y,z) = 0, \\ f_2(x,y,z,y',z') &= z' - g_2(x,y,z) = 0, \end{aligned} \tag{109}$$

then f_1 is independent of z' and f_2 does not depend on y'. Hence the coefficient matrix of the linear system of equations received from (108) for the function f_1, respectively f_2 is

$$M_1 = \begin{pmatrix} \phi_1 & \dots & \phi_r \\ \eta_1 & \dots & \eta_r \\ \alpha_1 & \dots & \alpha_r \\ \eta_1^{(1)} & \dots & \eta_r^{(1)} \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} \phi_1 & \dots & \phi_r \\ \eta_1 & \dots & \eta_r \\ \alpha_1 & \dots & \alpha_r \\ \alpha_1^{(1)} & \dots & \alpha_r^{(1)} \end{pmatrix},$$

respectively. A nontrivial system consisting of f_1, f_2 exists if the requirements $\text{rank}M_1 < 4$ and $\text{rank}M_2 < 4$ have to be simultaneously satisfied. Hence to find non-trivial solutions f_1, respectively f_2 of the system (108), each 4×4-subdeterminant of the $4 \times r$-matrix M_1, respectively M_2 needs to be zero.

Remark 5.1. We represent the basis vectors of each Lie algebra $\mathbf{g}$ in Section 5.1 in the 3-dimensional space using coordinates (x, y, z). Since every coordinate can be chosen as the time we have 3 different issues for finding the systems of ordinary differential equations which are invariant under the action of the infinitesimal generators of the Lie algebra $\mathbf{g}$. In this section we always suppose that the 'x' coordinate represents the time.

Example 5.2. The infinitesimal generators of the Lie algebra $\mathbf{g} = \mathbf{so_3}(\mathbb{R}) \cong \mathbf{su_2}(\mathbb{C})$ in the 3-dimensional space are given by (28). Therefore one has

$$\phi_i = (-y, 0, z),\ \eta_i = (x, -z, 0),\ \alpha_i = (0, y, -x). \tag{110}$$

Using the formulas (106), (107) for (110) we compute

$$\eta_i^{(1)} = (1 + (y')^2, -z', -y'z'),\ \alpha_i^{(1)} = (y'z', y', -(1 + (z')^2)).$$

Hence the matrix M can be written into the form

$$M = \begin{pmatrix} -y & 0 & z \\ x & -z & 0 \\ 0 & y & -x \\ 1 + (y')^2 & -z' & -y'z' \\ y'z' & y' & -\Big(1 + (z')^2\Big) \end{pmatrix}.$$

If we multiply the third column of M with y and the first column of M with z and we add the obtained new columns, then the matrix M is transformed to a matrix M^1. The matrix M^1 is converted to

$$M^2 = \begin{pmatrix} -y & 0 & 0 \\ x & -z & 0 \\ 0 & y & 0 \\ 1+(y')^2 & -z' & -y'z'y + z\left(1+(y')^2\right) - xz' \\ y'z' & y' & -y\left(1+(z')^2\right) + zy'z' + xy' \end{pmatrix}$$

after the multiplication of the second column of M^1 with x and the addition of this new column to the third column of M^1. Hence the function f_1 and f_2 fulfil the system (108) precisely if for f_1 and f_2 the following system of partial differential equations

$$\begin{aligned} -y\tfrac{\partial f_1}{\partial x} + x\tfrac{\partial f_1}{\partial y} + 1 + (y')^2 &= 0, \\ -z\tfrac{\partial f_1}{\partial y} + y\tfrac{\partial f_1}{\partial z} - z' &= 0, \\ -y'z'y + z\left(1+(y')^2\right) - xz' &= 0, \\ -y\tfrac{\partial f_2}{\partial x} + x\tfrac{\partial f_2}{\partial y} + y'z' &= 0, \\ -z\tfrac{\partial f_2}{\partial y} + y\tfrac{\partial f_2}{\partial z} + y' &= 0, \\ -y\left(1+(z')^2\right) + zy'z' + xy' &= 0 \end{aligned}$$

holds. We consider the third and the sixth equations

$$z + z(y')^2 = xz' + yy'z'$$

$$y + y(z')^2 = xy' + zy'z'.$$

It follows from the first equation that $z' = \frac{z(1+(y')^2)}{x+yy'}$. Substituting this into the second equation and simplifying the obtained equation we have

$$\begin{aligned} &(y')^3\left(xy^2 + xz^2\right) + (y')^2\left(2x^2y - yz^2 - y^3\right) \\ &\qquad + y'\left(x^3 + xz^2 - 2xy^2\right) - \left(yx^2 + yz^2\right) = 0. \end{aligned}$$

Since the last equation has the solution $y' = \frac{y}{x}$ and therefore $z' = \frac{z}{x}$ the unique first order system of ordinary differential equations which leaves invariant under the action of the infinitesimal generators of the Lie algebra $\mathbf{g} = \mathbf{so_3}(\mathbb{R})$ is determined by (29).

Example 5.3. Here we treat the infinitesimal generators in the (x, y, z)-space of the Lie algebra $\mathbf{g} = \mathbf{sl_3}(\mathbb{R})$. They are defined by (32). Since the group $SL_3(\mathbb{R})$ contains as a maximal compact subgroup the group $SO_3(\mathbb{R})$, we cannot predict any more systems to be invariant than those which are received in Example 5.2. From (32) we get

$$\begin{aligned} \phi_i &= (z,0,0,x,y,x,0,0), \\ \eta_i &= (0,z,x,-y,0,0,0,0), \\ \alpha_i &= (0,0,0,0,0,-z,x,y). \end{aligned} \tag{111}$$

Applying (106), (107) to (111) we calculate

$$\begin{aligned} \eta_i^{(1)} &= (-y'z',z',1,-2y',-(y')^2,-y',0,0), \\ \alpha_i^{(1)} &= (-(z')^2,0,0,-z',-y'z',-2z',1,y'). \end{aligned}$$

Hence the linear system of equations derived from the system (108) has the coefficient matrix as follows

$$M = \begin{pmatrix} z & 0 & 0 & x & y & x & 0 & 0 \\ 0 & z & x & -y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -z & x & y \\ -y'z' & z' & 1 & -2y' & -(y')^2 & -y' & 0 & 0 \\ -(z')^2 & 0 & 0 & -z' & -y'z' & -2z' & 1 & y' \end{pmatrix}.$$

The necessary condition to find non-trivial solutions f_1, f_2 of the system given by (108) is that each 5×5-subdeterminant of M has to be equal to zero. Taking into account that

$$D_{5,1} = \begin{vmatrix} 0 & 0 & x & 0 & 0 \\ z & x & -y & 0 & 0 \\ 0 & 0 & 0 & x & y \\ z' & 1 & -2y' & 0 & 0 \\ 0 & 0 & -z' & 1 & y' \end{vmatrix} = (-z + xz')(y'x - y)x$$

is equal to zero if either $z' = \frac{z}{x}$ or $y' = \frac{y}{x}$ and the function f_1 is independent of the variable z', to receive non-trivial solution f_1 of the system (108) it is required that every 4×4-subdeterminant of the matrix

$$M_1 = \begin{pmatrix} \phi_i \\ \eta_i \\ \alpha_i \\ \eta_i^{(1)} \end{pmatrix} = \begin{pmatrix} z & 0 & 0 & x & y & x & 0 & 0 \\ 0 & z & x & -y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -z & x & y \\ -y'z' & z' & 1 & -2y' & -(y')^2 & -y' & 0 & 0 \end{pmatrix}$$

has to be zero. The subdeterminants of M_1

$$D_{4,1} = \begin{vmatrix} 0 & 0 & x & 0 \\ z & x & -y & 0 \\ 0 & 0 & 0 & x \\ z' & 1 & -2y' & 0 \end{vmatrix} = (z'x - z)x^2$$

$$D_{4,2} = \begin{vmatrix} 0 & x & x & 0 \\ x & -y & 0 & 0 \\ 0 & 0 & -z & x \\ 1 & -2y' & -y' & 0 \end{vmatrix} = (y'x - y)x^2$$

are equal to zero if $y' = \frac{y}{x}$ and $z' = \frac{z}{x}$ simultaneously hold. Since the functions $f_1 = y' - \frac{y}{x}$ and $f_2 = z' - \frac{z}{x}$ satisfy the system (108) of partial differential equations we can arrive at a conclusion that the unique first order system of differential equations which admits the Lie algebra $\mathbf{g} = \mathbf{sl_3}(\mathbb{R})$ as the Lie algebra of its symmetry group is determined by (29).

Example 5.4. Here we consider the infinitesimal generators in the (x, y, z)-space of the Lie algebra $\mathbf{g} = \mathbf{sl_2}(\mathbb{R}) \oplus \mathbf{sl_2}(\mathbb{R})$ given by (31) (see [7] in [13, p. 134, 140]). We can conclude from (31) that

$$\begin{aligned} \phi_i &= (0, 0, xy - z, 1, x, x^2), \\ \eta_i &= (1, y, y^2, 0, 0, xy - z), \\ \alpha_i &= (x, z, yz, y, z, xz). \end{aligned} \tag{112}$$

Utilizing (106), (107) for (112) we compute

$$\begin{aligned} \eta_i^{(1)} &= (0, y', yy' - x(y')^2 + y'z', 0, -y', y - xy' - z'), \\ \alpha_i^{(1)} &= (1, z', zy' - xy'z' + (z')^2, y', 0, z - xz'). \end{aligned}$$

Hence for the coefficient matrix M of the linear system of equations received from (108) we get

$$M = \begin{pmatrix} 0 & 0 & xy - z & 1 & x & x^2 \\ 1 & y & y^2 & 0 & 0 & xy - z \\ x & z & yz & y & z & xz \\ 0 & y' & yy' - x(y')^2 + y'z' & 0 & -y' & y - xy' - z' \\ 1 & z' & zy' - xy'z' + (z')^2 & y' & 0 & z - xz' \end{pmatrix}.$$

Determining the 5×5-subdeterminants of M we obtain that they have

$$\left(y'\right)^2 x^2 - 2xyy' - 2xy'z' + y^2 - 2z'y + \left(z'\right)^2 + 4zy'$$

as the greatest common divisor factor. This factor is equal to zero precisely if

$$z' = xy' + y \pm 2\sqrt{y'(xy - z)}. \tag{113}$$

As the function f_1 does not depend on the variable z', to find non-trivial solution f_1 of the system (108) it is required that every 4×4-subdeterminant of the matrix

$$M_1 = \begin{pmatrix} 0 & 0 & xy - z & 1 & x & x^2 \\ 1 & y & y^2 & 0 & 0 & xy - z \\ x & z & yz & y & z & xz \\ 0 & y' & yy' - x(y')^2 + y'z' & 0 & -y' & y - xy' - z' \end{pmatrix}$$

has to be equal to zero. Using (113) we obtain

$$D_{4,1} = \begin{vmatrix} 0 & 0 & xy - z & 1 \\ 1 & y & y^2 & 0 \\ x & z & yz & y \\ 0 & y' & yy' - x(y')^2 + y'z' & 0 \end{vmatrix} = \pm 2(xy - z)y'\sqrt{y'(xy - z)}.$$

Arising from this, the determinant $D_{4,1}$ is 0 if $y' = 0$ and therefore $z' = y$. We consider the first order time-dependent system $f_1 = y' = 0$, $f_2 = z' - y = 0$ of differential equations. Since $X_6(f_2) = x^2\frac{\partial f_2}{\partial x} + (xy - z)\frac{\partial f_2}{\partial y} + xz\frac{\partial f_2}{\partial z} + (z - xy)\frac{\partial f_2}{\partial z'} = 2(z - xy) \neq 0$ this system f_1, f_2 cannot fulfill the system (108). Therefore there is no first order time-dependent system (109) of differential equations which admits the Lie algebra $\mathbf{g} = \mathbf{sl_2}(\mathbb{R}) \oplus \mathbf{sl_2}(\mathbb{R})$ defined by (31) as the Lie algebra of its symmetry group.

An analogous treatment as in Example 5.4 yields the non-existence of any first order time-dependent system (109) of differential equations which permits a symmetry Lie group whose Lie algebra is any one of the following Lie algebras:

$$\mathbf{g} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, z\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial y}, x\frac{\partial}{\partial z}, y\frac{\partial}{\partial z}, z\frac{\partial}{\partial z} \right\rangle$$

$$\mathbf{g} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, z\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial y}, x\frac{\partial}{\partial z}, y\frac{\partial}{\partial z}, z\frac{\partial}{\partial z} \right\rangle$$

$$\mathbf{g} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} \right\rangle,$$

$$\mathbf{g} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \right\rangle,$$

$$\mathbf{g} = \mathbf{sl_4}(\mathbb{R}) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial x}, z\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial y}, x\frac{\partial}{\partial z}, y\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, \right.$$
$$\left. x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}, xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z}, xz\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z} \right\rangle$$

$$\mathbf{g} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, z\frac{\partial}{\partial x}, z\frac{\partial}{\partial y}, z\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}, \right.$$
$$\left. x\frac{\partial}{\partial z}, y\frac{\partial}{\partial} \right\rangle$$

$$\mathbf{g} = \left\langle \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \frac{\partial}{\partial z}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, \right.$$
$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}, z\frac{\partial}{\partial x} - y\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right),$$
$$\left. z\frac{\partial}{\partial y} + x\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right), z\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\right\rangle$$

$$\mathbf{g} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \right.$$
$$2x\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) - (x^2+y^2+z^2)\frac{\partial}{\partial x},$$
$$2y\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) - (x^2+y^2+z^2)\frac{\partial}{\partial y},$$
$$\left. 2z\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) - (x^2+y^2+z^2)\frac{\partial}{\partial z}\right\rangle.$$

5.2. **Time-preserving symmetries.** In this subsection we investigate the case that the infinitesimal generators of the Lie algebra $\mathbf{g}$ of the given real Lie group G are time-preserving symmetries. We deal with the following first order system of ordinary differential equations:

$$\begin{aligned} f_1(t, x(t), y(t), z(t), x'(t), y'(t), z'(t)) &= 0, \\ f_2(t, x(t), y(t), z(t), x'(t), y'(t), z'(t)) &= 0, \\ f_3(t, x(t), y(t), z(t), x'(t), y'(t), z'(t)) &= 0, \end{aligned}$$

where $x' = \frac{dx}{dt}$, $y' = \frac{dy}{dt}$, $z' = \frac{dz}{dt}$. This system is time-independent. Let $\dim(\mathbf{g}) = r$. The basis elements of $\mathbf{g}$ can be written as the vector fields:

$$X_i(x(t), y(t), z(t)) = \phi_i(x(t), y(t), z(t))\frac{\partial}{\partial x} + \eta_i(x(t), y(t), z(t))\frac{\partial}{\partial y} + \alpha_i(x(t), y(t), z(t))\frac{\partial}{\partial z},$$

$i = 1, 2, \ldots, r$. The formula

$$X_i^{(1)}(x(t), y(t), z(t), x', y', z') = X_i + \phi_i^{(1)}(x(t), y(t), z(t), x', y', z')\frac{\partial}{\partial x'} + \eta_i^{(1)}(x(t), y(t), z(t), x', y', z')\frac{\partial}{\partial y'} + \alpha_i^{(1)}(x(t), y(t), z(t), x', y', z')\frac{\partial}{\partial z'},$$

with

$$\phi_i^{(1)} = \frac{\partial \phi_i}{\partial x}x' + \frac{\partial \phi_i}{\partial y}y' + \frac{\partial \phi_i}{\partial z}z',$$
$$\eta_i^{(1)} = \frac{\partial \eta_i}{\partial x}x' + \frac{\partial \eta_i}{\partial y}y' + \frac{\partial \eta_i}{\partial z}z',$$
$$\alpha_i^{(1)} = \frac{\partial \alpha_i}{\partial x}x' + \frac{\partial \alpha_i}{\partial y}y' + \frac{\partial \alpha_i}{\partial z}z',$$

defines the first prolonged vector field of $X_i(x(t), y(t), z(t))$, $i = 1, 2, \ldots, r$ with respect to the variable t. The system including f_1, f_2, f_3 admits the given group G as its symmetry group if and only if the functions f_l, $l = 1, 2, 3$, fulfil the following system of partial differential equations

$$(112)\qquad \begin{array}{rcl} \phi_1\frac{\partial f_l}{\partial x} + \eta_1\frac{\partial f_l}{\partial y} + \alpha_1\frac{\partial f_l}{\partial z} + \phi_1^{(1)}\frac{\partial f_l}{\partial x'} + \eta_1^{(1)}\frac{\partial f_l}{\partial y'} + \alpha_1^{(1)}\frac{\partial f_l}{\partial z'} & = & 0, \\ \phi_2\frac{\partial f_l}{\partial x} + \eta_2\frac{\partial f_l}{\partial y} + \alpha_2\frac{\partial f_l}{\partial z} + \phi_2^{(1)}\frac{\partial f_l}{\partial x'} + \eta_2^{(1)}\frac{\partial f_l}{\partial y'} + \alpha_2^{(1)}\frac{\partial f_l}{\partial z'} & = & 0, \\ \vdots & & \\ \phi_i\frac{\partial f_l}{\partial x} + \eta_i\frac{\partial f_l}{\partial y} + \alpha_i\frac{\partial f_l}{\partial z} + \phi_i^{(1)}\frac{\partial f_l}{\partial x'} + \eta_i^{(1)}\frac{\partial f_l}{\partial y'} + \alpha_i^{(1)}\frac{\partial f_l}{\partial z'} & = & 0, \\ \vdots & & \\ \phi_r\frac{\partial f_l}{\partial x} + \eta_r\frac{\partial f_l}{\partial y} + \alpha_r\frac{\partial f_l}{\partial z} + \phi_r^{(1)}\frac{\partial f_l}{\partial x'} + \eta_r^{(1)}\frac{\partial f_l}{\partial y'} + \alpha_r^{(1)}\frac{\partial f_l}{\partial z'} & = & 0. \end{array}$$

We denote by M the $6 \times r$-matrix

$$M = \begin{pmatrix} \phi_1 & \ldots & \phi_r \\ \eta_1 & \ldots & \eta_r \\ \alpha_1 & \ldots & \alpha_r \\ \phi_1^{(1)} & \ldots & \phi_r^{(1)} \\ \eta_1^{(1)} & \ldots & \eta_r^{(1)} \\ \alpha_1^{(1)} & \ldots & \alpha_r^{(1)} \end{pmatrix}.$$

The system (112) is equivalent to the following system of 'linear equations' of the variables $\frac{\partial f_l}{\partial x}$, $\frac{\partial f_l}{\partial y}$, $\frac{\partial f_l}{\partial z}$, $\frac{\partial f_l}{\partial x'}$, $\frac{\partial f_l}{\partial y'}$, $\frac{\partial f_l}{\partial z'}$:

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial x'} & \frac{\partial f_1}{\partial y'} & \frac{\partial f_1}{\partial z'} \end{pmatrix} \cdot M = \begin{pmatrix} 0 & \ldots & 0 \end{pmatrix},$$
$$\begin{pmatrix} \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial x'} & \frac{\partial f_2}{\partial y'} & \frac{\partial f_2}{\partial z'} \end{pmatrix} \cdot M = \begin{pmatrix} 0 & \ldots & 0 \end{pmatrix},$$
$$\begin{pmatrix} \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} & \frac{\partial f_3}{\partial x'} & \frac{\partial f_3}{\partial y'} & \frac{\partial f_3}{\partial z'} \end{pmatrix} \cdot M = \begin{pmatrix} 0 & \ldots & 0 \end{pmatrix}.$$

To receive non-trivial solutions f_l, $l = 1, 2, 3$, of the system defined by (112) it is required that for the rank of the coefficient matrix M one should have rank$M < 6$. Similarly to Section 5.1 the necessary condition for the vectors $\begin{pmatrix} \frac{\partial f_l}{\partial x} & \ldots & \frac{\partial f_l}{\partial z'} \end{pmatrix}$ to be linearly independent is that rank$M < 4$. If $r < 4$, then it is obviously holds. In this case we have to find the solution of the system

(112), and check if any solution yields a nontrivial system of differential equations f_1, f_2, f_3. Now let us assume that $r \geq 4$. In this situation the rank condition holds if and only if each 4×4-subdeterminant of the coefficient matrix M is equal to zero.
We deal with systems having the explicit form

$$\begin{aligned}
f_1(t, x(t), y(t), z(t), x') &= x' - g_1(t, x(t), y(t), z(t)) = 0, \\
f_2(t, x(t), y(t), z(t), y') &= y' - g_2(t, x(t), y(t), z(t)) = 0, \\
f_3(t, x(t), y(t), z(t), z') &= z' - g_3(t, x(t), y(t), z(t)) = 0,
\end{aligned}$$

that is the function f_1, respectively f_2, respectively f_3 is independent of the variables y', z', respectively x', z', respectively x', y'. Therefore in (112) for f_1, f_2, f_3 we have the $4 \times r$-coefficient matrices

$$M_1 = \begin{pmatrix} \phi_1 & \dots & \phi_r \\ \eta_1 & \dots & \eta_r \\ \alpha_1 & \dots & \alpha_r \\ \phi_1^{(1)} & \dots & \phi_r^{(1)} \end{pmatrix}, \qquad M_2 = \begin{pmatrix} \phi_1 & \dots & \phi_r \\ \eta_1 & \dots & \eta_r \\ \alpha_1 & \dots & \alpha_r \\ \eta_1^{(1)} & \dots & \eta_r^{(1)} \end{pmatrix},$$
$$M_3 = \begin{pmatrix} \phi_1 & \dots & \phi_r \\ \eta_1 & \dots & \eta_r \\ \alpha_1 & \dots & \alpha_r \\ \alpha_1^{(1)} & \dots & \alpha_r^{(1)} \end{pmatrix},$$

respectively. To get non-trivial solutions f_1, f_2, f_3 of the system (112) in the above explicit form it is required that for all $i = 1, 2, 3$ one should have $\text{rank} M_i < 4$, or equivalently every 4×4-subdeterminant of the matrices M_i, $i = 1, 2, 3$, should be zero. We demonstrate the above method for the Lie algebra $\mathbf{g}$ discussed in Example 5.4.

Example 5.5. The Lie algebra $\mathbf{g} = \mathbf{sl_2}(\mathbb{R}) \oplus \mathbf{sl_2}(\mathbb{R})$ has the infinitesimal generators defined by (31). We compute from (31) that

$$\begin{aligned}
\phi_i &= (0, 0, x(t)y(t) - z(t), 1, x(t), x(t)^2), \\
\eta_i &= (1, y(t), y(t)^2, 0, 0, x(t)y(t) - z(t)), \\
\alpha_i &= (x(t), z(t), y(t)z(t), y(t), z(t), x(t)z(t)).
\end{aligned}$$

Hence we calculate

$$\begin{aligned}
\phi_i^{(1)} &= (0, 0, y(t)x' + x(t)y' - z', 0, x', 2x(t)x'), \\
\eta_i^{(1)} &= (0, y', 2y(t)y', 0, 0, y(t)x' + x(t)y' - z'), \\
\alpha_i^{(1)} &= (x', z', z(t)y' + y(t)z', y', z', z(t)x' + x(t)z').
\end{aligned}$$

Therefore the coefficient matrix M of the linear system of equations is

$$M = \begin{pmatrix} 0 & 0 & xy-z & 1 & x & x^2 \\ 1 & y & y^2 & 0 & 0 & xy-z \\ x & z & yz & y & z & xz \\ 0 & 0 & yx'+xy'-z' & 0 & x' & 2xx' \\ 0 & y' & 2yy' & 0 & 0 & xy'+x'y-z' \\ x' & z' & zy'+yz' & y' & z' & zx'+xz' \end{pmatrix}.$$

The matrix M has determinant 0. To receive non-trivial solution f_1 of the system (112) in explicit form it is required that every 4×4-subdeterminant of the matrix

$$M_1 = \begin{pmatrix} 0 & 0 & xy-z & 1 & x & x^2 \\ 1 & y & y^2 & 0 & 0 & xy-z \\ x & z & yz & y & z & xz \\ 0 & 0 & yx'+xy'-z' & 0 & x' & 2xx' \end{pmatrix}$$

has to be equal to 0. Since $D_1 = \begin{vmatrix} 0 & 0 & 1 & x \\ 1 & y & 0 & 0 \\ x & z & y & z \\ 0 & 0 & 0 & x' \end{vmatrix} = (z-xy)x'$ is 0 if $x'=0$ we obtain $f_1 = x' = 0$. Applying this,

$$D_2 = \begin{vmatrix} 0 & 0 & 1 & xy-z \\ 1 & y & 0 & y^2 \\ x & z & y & yz \\ 0 & 0 & 0 & xy'-z' \end{vmatrix} = (z'-xy')(xy-z)$$

is 0 if $z' = xy'$. To find non-trivial solution f_2 of the system (112) in explicit form it is required that each 4×4-subdeterminant of the matrix

$$M_2 = \begin{pmatrix} 0 & 0 & xy-z & 1 & x & x^2 \\ 1 & y & y^2 & 0 & 0 & xy-z \\ x & z & yz & y & z & xz \\ 0 & 0 & yx'+xy'-z' & 0 & x' & 2xx' \\ 0 & y' & 2yy' & 0 & 0 & xy'+x'y-z' \end{pmatrix}$$

should be 0. As $D_3 = \begin{vmatrix} 0 & 0 & xy-z & x \\ 1 & y & y^2 & 0 \\ x & z & yz & z \\ 0 & y' & 2yy' & 0 \end{vmatrix} = (xy-z)^2 y'$ is 0 if $y' = 0$ and hence $z' = 0$, the time-independent system admitting the Lie algebra $\mathbf{g} = \mathbf{sl_2}(\mathbb{R}) \oplus \mathbf{sl_2}(\mathbb{R})$ defined by (31) as the Lie algebra of its symmetry group is trivial (cf. (30)).

Analogously to the Example 5.5 one can prove that the time-independent systems of differential equations of order one that admit one of the following

Lie algebras $\mathbf{so_3}(\mathbb{R})$, $\mathbf{sl_3}(\mathbb{R})$, $\mathbf{sl_4}(\mathbb{R})$ as the Lie algebra of its infinitesimal symmetries are trivial.

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References

[1] W. F. Ames, R. L. Anderson, V. A. Dorodnitsyn, E. V. Ferapontov, R. K. Gazizov, N. H. Ibragimov and S. R. Svirshchevskiĭ, *CRC handbook of Lie group analysis of differential equations. Vol. 1, Symmetries, exact solutions and conservation laws*, CRC Press, 1994.

[2] M. Edwards and M. C. Nucci, *Application of Lie group analysis to a core group model for sexually transmitted diseases*, J. Nonlinear Math. Phys. **13** (2006), 211–230.

[3] Á. Figula, *Ordinary differential equations allowing a given Lie group as a subgroup of their symmetries*, Lie Group Analysis of Biological Systems, Deliverable D 1.2, BIOMICS, pp. 46, 2021.

[4] R. Fitzhugh, *Impulses and Physiological States in Theoretical Models of Nerve Membrane*, Biophys J. **1** (1961), 445–466.

[5] P. E. Hydon, *Symmetry methods of differential equations*, Cambridge University Press, 2000.

[6] N. Kh. Ibragimov, *Group analysis of ordinary differential equations and the invariance principle in mathematical physics*, Russian Math. Surveys **47:4** (1992), 89-156.

[7] A. Gonzáles-López, N. Kamran and P. J. Olver, *Lie Algebras of Vector Fields in the Real Plane*, Proc. London Math. Soc. **64:2** (1992), 339-368.

[8] T. Harko, S. N. F. Lobo and M. K. Mak, *A class of exact solutions of the Liénard-type ordinary nonlinear differential equation*, Journal of Engineering Mathematics, **89** (2014), 193–205.

[9] E. Kamke, *Differentialgleichungen. Lösungsmethoden und Lösungen. Teil I: Gewöhnliche Differentialgleichungen. 6. Aufl.; Teil II: Partielle Differentialgleichungen erster Ordnung für eine gesuchte Funktion. 4. Aufl*, Akademische Verlagsgesellschaft, Geest & Portig K.-G. 1959.

[10] S. Lie, *Gesammelte Abhandlungen, Fünfter Band*, B.G. Teubner, 1924.

[11] S. Lie, *Gesammelte Abhandlungen, Sechster Band*, B.G. Teubner, 1924.

[12] S. Lie, *Vorlesungen ueber Differentialgleichungen mit bekannten infinitesimal Transformationen*, B.G. Teubner, 1891.

[13] S. Lie, *Theorie der Transformationsgruppen I*, Math. Ann., Band XVI, p. 475.

[14] S. Lie and F. Engel, *Theorie der Transformationsgruppen*, B.G. Teubner, Band III, 1893.

[15] J. Nagumo, S. Arimoto and S. Yoshizawa, *An active pulse transmission line simulating nerve axon*, Proc. IRE. **50** (1962), 2061–2070.

[16] M. C. Nucci, *Interactive REDUCE programs for calculating Lie point, non-classical, Lie-Bäcklund, and approximate symmetries of differential equations: Manual and*

floppy disk, in CRC Handbook of Lie Croup Analysis of Differential Equations, Vol. III: New Trends (Ed: N. H. Ibragimov), CRC Press, 1995.

[17] M. C. Nucci, *The complete Kepler group can be derived by Lie group analysis*, J. Math. Phys. **37** (1996), 1772–1775.

[18] M. C. Nucci, *The Role of Symmetries in Solving Differential Equations*, Mathl. Comput. Modelling, **25** (1997), 181–193.

[19] M. C. Nucci and G. Sanchini, *Symmetries, Lagrangians and Conservation Laws of an Easter Island Population Model*, Symmetry, **7** (2015), 1613–1632.

[20] B. van der Pol, *On relaxation-oscillations*, The Lond, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, **2** (1927), 978–992.

[21] B. van der Pol and J. van der Mark, *The heartbeat considered as a relaxation oscillation, and an electrical model of the heart*, The Lond, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, **6** (1928), 763–775.

[22] P. J. Olver, *Applications of Lie groups to differential equations*, Springer, 1993.

[23] J. Starrett, *Solving differential equations by symmetry groups*, Amer. Math. Monthly, **114** (2007), 778–792.

[24] A. V. Bocharov, V. N. Chetverikov, S. V. Duzhin, N. G. Khor'Kova, I. S. Krasil'shchik, A. V. Samokhin, Yu. N. Torkhov, A. M. Verbovetsky and A. M. Vinogradov, *Symmetries and conservation laws for differential equations of mathematical physics*, American Mathematical Society, 1999.

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