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On the direction independence of two remarkable Finsler tensors

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Finsler manifolds some of whose characteristic tensors are direction independent provide stimulation for current research. In this paper we show that the direction independence of the Landsberg and the stretch tensor implies the vanishing of these tensors.

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1. Basic constructions

Throughout this paper, M will be an n-dimensional smooth manifold. $C^\infty(M)$ denotes the ring of real-valued smooth functions on M. T_pM is the tangent space to M at $p \in M$, $TM := \bigcup_{p\in M} T_pM$ is the tangent bundle of M, $\tau : TM \to M$ is the natural projection. $\overset{\circ}{T}M$ denotes the open subset of the nonzero tangent vectors to M, $\overset{\circ}{\tau} := \tau \upharpoonright \overset{\circ}{T}M$. $\mathfrak{X}(M)$ is the $C^\infty(M)$-module of (smooth) vector fields on M. Capitals $X, Y, \dots$ will denote vector fields on M, while, usually, Greek letters $\xi, \eta, \zeta, \dots$ will stand for vector fields on TM. i_ξ is the substitution operator induced by $\xi \in \mathfrak{X}(TM)$, d denotes the operator of the exterior derivative.

All of our considerations will be purely of local character, so we may assume without loss of generality that our base manifold M admits a global coordinate system $\left(u^i\right)_{i=1}^n$; this assumption simplifies a little the notation. Then

$$\left(x^i, y^i\right)_{i=1}^n; \quad x^i := u^i \circ \tau,\ y^i := du^i$$

is a coordinate system for TM. These coordinate systems yield the bases

$$\left(\frac{\partial}{\partial u^i}\right)_{i=1}^n \text{ and } \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right)_{i=1}^n =: \left(\partial_i, \dot{\partial}_i\right)_{i=1}^n$$

of $\mathfrak{X}(M)$ and $\mathfrak{X}(TM)$, respectively.

Let $T_s^r M := \bigcup_{p\in M} T_s^r(T_pM)$ be the bundle of type $\binom{r}{s}$ tensors over M, and let $\tau_s^r : T_s^r M \longrightarrow M$ be the natural projection. Following Z. I. Szabó,[9] by a type $\binom{r}{s}$ *Finsler tensor field* over M we mean a smooth map

$$\widetilde{A} : \overset{\circ}{T}M \longrightarrow T_s^r M \text{ such that } \tau_s^r \circ \widetilde{A} = \overset{\circ}{\tau}.$$

These tensor fields form a $C^\infty(\overset{\circ}{T}M)$-module, which will be denoted by $\mathscr{T}_s^r\left(\overset{\circ}{\tau}\right)$. In particular, $\mathfrak{X}\left(\overset{\circ}{\tau}\right) := \mathscr{T}_0^1\left(\overset{\circ}{\tau}\right)$ is the module of *Finsler vector fields*, and $\mathfrak{X}^*\left(\overset{\circ}{\tau}\right)$ is its dual. In what follows, Finsler tensor fields will simply be mentioned as *tensors*, or, for obvious reasons, *tensors along the projection* $\overset{\circ}{\tau}$. Evidently, the construction also works on the whole TM, and leads to the $C^\infty(TM)$-modules $\mathscr{T}_s^r(\tau)$. Via restrictions, $\mathscr{T}_s^r(\tau)$ may be interpreted as a submodule of $\mathscr{T}_s^r\left(\overset{\circ}{\tau}\right)$; we shall use this harmless inclusion in what follows.

If X is a vector field on M, then $\widehat{X} := X \circ \tau$ is a Finsler vector field, called a *basic vector field* along τ. In particular, $\left(\frac{\widehat{\partial}}{\partial u^i}\right)_{i=1}^n$ is a base for the module $\mathfrak{X}(\tau)$. Besides the class of basic vector fields, a distinguished role is played by the *canonical Finsler vector field* $\delta := 1_{TM}$, called classically *support element*. In coordinates, $\delta = y^i\left(\frac{\widehat{\partial}}{\partial u^i}\right)$ (with sum convention in force).

We have a canonical $C^\infty(TM)$-linear injection $\mathbf{i} : \mathfrak{X}(\tau) \longrightarrow \mathfrak{X}(TM)$ and a surjection $\mathbf{j} : \mathfrak{X}(TM) \longrightarrow \mathfrak{X}(\tau)$ such that

$$\mathbf{i}\left(\frac{\widehat{\partial}}{\partial u^i}\right) = \frac{\partial}{\partial y^i};\ \mathbf{j}\left(\frac{\partial}{\partial x^i}\right) = \frac{\widehat{\partial}}{\partial u^i},\ \mathbf{j}\left(\frac{\partial}{\partial y^i}\right) = 0;\ i \in \{1, \dots, n\}.$$

(For an intrinsic construction of $\mathbf{i}$ and $\mathbf{j}$, see e.g. Ref. 10. $\mathfrak{X}^v(TM) := \mathbf{i}\left(\mathfrak{X}(\tau)\right)$ is the module of *vertical vector fields* on TM, $X^v := \mathbf{i}\left(\widehat{X}\right)$ is the *vertical lift* of $X \in \mathfrak{X}(M)$. $C := \mathbf{i}(\delta)$ is called the *Liouville vector field*. In coordinates, $C = y^i\frac{\partial}{\partial y^i}$. $\mathbf{J} := \mathbf{i} \circ \mathbf{j}$ is said to be the *vertical endomorphism* of $\mathfrak{X}(TM)$. It follows at once that

$$Im(\mathbf{J}) = Ker(\mathbf{J}) = \mathfrak{X}^v(TM), \quad \mathbf{J}^2 = 0.$$

We define the $d_{\mathbf{J}}$*-differential* of a smooth function f on TM as the *one-form* $d_{\mathbf{J}} f := df \circ \mathbf{J}$ *on TM.* In coordinates, $d_{\mathbf{J}} f = \frac{\partial f}{\partial y^i} dx^i$.

The formalism can go on. Let $\widetilde{X}$ be a Finsler vector field over M. We define a tensor derivation $\nabla^v_{\widetilde{X}}$ on the algebra of Finsler tensor fields by the following requirements:

(i) *On functions,* $\nabla^v_{\widetilde{X}} f := \left(\mathbf{i}\widetilde{X}\right) f \; ; \; f \in C^\infty(TM)$.

(ii) *On Finsler vector fields,* $\nabla^v_{\widetilde{X}} \widetilde{Y} := \mathbf{j}\left[\mathbf{i}\widetilde{X}, \eta\right]$, *where* $\eta \in \mathfrak{X}(TM)$ *is such that* $\mathbf{j}(\eta) = \widetilde{Y}$.

(iii) *If* $\widetilde{A} \in \mathcal{T}^r_s(\tau)$, then $\nabla^v_{\widetilde{X}} \widetilde{A}$ *is given by the* product rule.

$\nabla^v_{\widetilde{X}}$ is called the *(canonical) v-covariant derivative* with respect to $\widetilde{X}$. In coordinates: if $\widetilde{X} = \xi^i \frac{\widehat{\partial}}{\partial u^i}$, $\widetilde{Y} = \eta^i \frac{\widehat{\partial}}{\partial u^i}$, then

$$\nabla^v_{\widetilde{X}} f = \xi^i \frac{\partial f}{\partial y^i}, \; \nabla^v_{\widetilde{X}} \widetilde{Y} = \xi^i \frac{\partial \eta^j}{\partial y^i} \frac{\partial}{\partial y^j}.$$

We see that $\nabla^v_{\widetilde{X}} \widetilde{Y}$ is well-defined: it does not depend on the choice of the vector field η. We have, in particular, $\nabla^v_{\widehat{X}} \widehat{Y} = 0$ for any vector fields X,Y on M.

As a final step, we define the *vertical differential* of a type $\binom{r}{s}$ Finsler tensor field $\widetilde{A}$ as the $\binom{r}{s+1}$ tensor $\nabla^v \widetilde{A}$ which 'collects all the v-covariant derivatives' of $\widetilde{A}$. For simplicity, if $r = s = 1$, then

$$\nabla^v \widetilde{A}\left(\widetilde{\alpha}, \widetilde{Y}, \widetilde{X}\right) := \left(\nabla^v_{\widetilde{X}} \widetilde{A}\right)\left(\widetilde{\alpha}, \widetilde{Y}\right) \overset{(iii)}{=}$$
$$\left(\mathbf{i}\widetilde{X}\right) \widetilde{A}\left(\widetilde{\alpha}, \widetilde{Y}\right) - \widetilde{A}\left(\nabla^v_{\widetilde{X}} \widetilde{\alpha}, \widetilde{Y}\right) - \widetilde{A}\left(\widetilde{\alpha}, \nabla^v_{\widetilde{X}} \widetilde{Y}\right)$$

for all $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\tau)$ and $\widetilde{\alpha} \in \mathfrak{X}^*(\tau)$. More generally, if the components of an $\binom{r}{s}$ tensor $\widetilde{A}$ are

$$\widetilde{A}^{i_1 \dots i_r}_{i_1 \dots j_s} := \widetilde{A}\left(\widehat{du^{i_1}}, \dots, \widehat{du^{i_r}}, \widehat{\frac{\partial}{\partial u^{j_1}}}, \dots, \widehat{\frac{\partial}{\partial u^{j_s}}}\right) \left(\widehat{du^i} := du^i \circ \tau^0_1\right),$$

then the components of $\nabla^v \widetilde{A}$ are $\dot{\partial}_j \widetilde{A}^{i_1 \dots i_r}_{i_1 \dots j_s}$; these functions will be denoted by $\widetilde{A}^{i_1 \dots i_r}_{i_1 \dots j_s \cdot j}$. We recognize that in components 'vertical differentiation reduces to partial differentiation with respect to the fibre coordinates'. Notice that $\nabla^v f$ and $d_{\mathbf{J}} f$ are related by

$$d_{\mathbf{J}} f = \nabla^v f \circ \mathbf{j} \; , \; f \in C^\infty(TM) \; .$$

2. Finsler functions. The h-Berwald derivative

By a *Finsler function* we mean a continuous function $F : TM \longrightarrow [0, \infty[$, satisfying the three conditions:

(i) *F is smooth* on $\overset{\circ}{T}M$.
(ii) *F is positive-homogeneous of degree 1, i.e., $F(\lambda v) = \lambda F(v)$ for all $\lambda \in \mathbb{R}_+^*$ and $v \in TM$.*
(iii) *The* metric tensor $g := \frac{1}{2}\nabla^v \nabla^v F^2$ *is pointwise non-degenerate on* $\overset{\circ}{T}M$.

A manifold endowed with a Finsler function is said to be a *Finsler manifold.* Quite surprisingly, under these conditions the metric tensor g is positive definite, see Ref. 6. The components $g_{ij} := g\left(\frac{\widehat{\partial}}{\partial u^i}, \frac{\widehat{\partial}}{\partial u^j}\right)$ of g are just the functions $\frac{1}{2}\dot{\partial}_i \dot{\partial}_j F^2$.

In the remainder of the paper, (M,F) will be a Finsler manifold.

We show that the Finsler function F and the metric tensor g are related by

$$g(\delta, \delta) = F^2. \tag{1}$$

Indeed, by the homogeneity of F, we have $CF^2 = 2F^2$, and it can easily be checked that $\nabla^v \delta = 1_{\mathfrak{X}(\tau)}$. Thus

$$\begin{aligned} g(\delta, \delta) &= \frac{1}{2}\nabla^v \nabla^v F^2 (\delta, \delta) = \frac{1}{2}\nabla^v_\delta \left(\nabla^v F^2\right)(\delta) = \frac{1}{2}\left(C\left(CF^2\right) - \nabla^v F^2(\delta)\right) \\ &= \frac{1}{2}\left(4F^2 - 2F^2\right) = F^2. \end{aligned}$$

In the Finslerian case the canonical vector field δ has a dual 1-form δ^* along $\overset{\circ}{\tau}$ given by

$$\delta^*\left(\widetilde{X}\right) := g\left(\widetilde{X}, \widetilde{Y}\right) \;;\; \widetilde{X} \in \mathfrak{X}^*\left(\overset{\circ}{\tau}\right). \tag{2}$$

Then $\delta^*(\delta) = F^2$ by (1). The components of δ^* can be obtained from the components of δ by index lowering:

$$y_i := \delta^*\left(\frac{\widehat{\partial}}{\partial u^i}\right) = g\left(\frac{\widehat{\partial}}{\partial u^i}, y^j \frac{\widehat{\partial}}{\partial u^j}\right) = g_{ij} y^j \;,\; i \in \{1, \ldots, n\}\,.$$

It follows immediately that

$$y_i y^i = F^2. \tag{3}$$

By the *Cartan tensor* of (M, F) we mean the type $\binom{0}{3}$ Finsler tensor $\mathscr{C}_\flat := \frac{1}{2}\nabla^v g$. Its components are

$$C_{ijk} := \mathscr{C}_\flat\left(\frac{\widehat{\partial}}{\partial u^i}, \frac{\widehat{\partial}}{\partial u^j}, \frac{\widehat{\partial}}{\partial u^k}\right) = \frac{1}{2}\dot{\partial}_k g_{ij} = \frac{1}{4}\dot{\partial}_k\dot{\partial}_j\dot{\partial}_i F^2,$$

thus $\mathscr{C}_\flat$ is totally symmetric. Raising an index, we get the *vector-valued Cartan tensor* $\mathscr{C}$, metrically equivalent to $\mathscr{C}_\flat$. More pedantically, $\mathscr{C}$ is defined by

$$g\left(\mathscr{C}\left(\widetilde{X}, \widetilde{Y}\right), \widetilde{Z}\right) = \mathscr{C}_\flat\left(\widetilde{X}, \widetilde{Y}, \widetilde{Z}\right);\ \widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \mathfrak{X}\left(\overset{\circ}{\tau}\right),$$

so its components are

$$g^{ir}C_{jkr} =: C^i_{jk}\ ;\ \left(g^{ij}\right) := \left(g_{ij}\right)^{-1}.$$

It is a fundamental fact, that F determines a unique spray $S : TM \longrightarrow TTM$ via the Euler-Lagrange equation

$$i_S dd_{\mathbf{J}} F^2 = -dF^2.$$

S is called the *canonical spray* of the Finsler manifold. In coordinates, $S = y^i\partial_i - 2G^i\dot{\partial}_i$, where

$$G^i = \frac{1}{4}g^{ij}\left(\frac{\partial^2 F^2}{\partial x^r \partial y^j}y^r - \frac{\partial F^2}{\partial x^j}\right)\ ,\ i \in \{1, \dots, n\}.$$

The *spray coefficients* G^i are of class C^1 on TM, smooth on $\overset{\circ}{T}M$ and are positively homogeneous of degree 2. The canonical spray determines the *canonical Ehresmann connection* $\mathscr{H} : \mathfrak{X}\left(\overset{\circ}{\tau}\right) \longrightarrow \mathfrak{X}(TM)$ of (M, F) by *Crampin's construction*[4]

$$\widehat{X} \in \mathfrak{X}\left(\overset{\circ}{\tau}\right) \longmapsto X^h := \mathscr{H}\left(\widehat{X}\right) := \tfrac{1}{2}\left(X^c + [X^v, S]\right),\ X \in \mathfrak{X}(M)$$

(X^c denotes the complete lift of X). X^h is called the *horizontal lift* of X. The horizontal lifts of the coordinate vector fields $\frac{\partial}{\partial u^j}$ take the form

$$\left(\frac{\partial}{\partial u^j}\right)^h = \frac{\partial}{\partial x^j} - \frac{\partial G^i}{\partial y^j}\frac{\partial}{\partial y^i},\ j \in \{1, \dots, n\};$$

the functions $G^i_j := \dot{\partial}_j G^i$ are said to be the *Christoffel symbols* of $\mathscr{H}$.

The *horizontal* and the *vertical projector* associated to $\mathscr{H}$ are $\mathbf{h} := \mathscr{H} \circ \mathbf{j}$ and $\mathbf{v} = 1_{\mathfrak{X}\left(\overset{\circ}{T}M\right)} - \mathbf{h}$, respectively. Following Berwald's terminology,[3] we call the type $\binom{1}{2}$ Finsler tensor $\mathbf{R}$ defined by

$$\mathbf{iR}\left(\widehat{X},\widehat{Y}\right):=-\mathbf{v}\left[X^h,X^v\right]\ ;\ X,Y\in\mathfrak{X}(M)$$

the *fundamental affine curvature* of the Finsler manifold. To be in harmony with Matsumoto's conventions,[8] we define the components R^i_{jk} of $\mathbf{R}$ by $R^i_{jk}\frac{\widehat{\partial}}{\partial u^i}=\mathbf{R}\left(\frac{\widehat{\partial}}{\partial u^k},\frac{\widehat{\partial}}{\partial u^j}\right)$. If $G^i_{jk}:=\dot{\partial}_k G^i_j$, then

$$R^i_{jk}=\left(\frac{\partial}{\partial u^k}\right)^h G^i_j-\left(\frac{\partial}{\partial u^j}\right)^h G^i_k=\partial_k G^i_j-\partial_j G^i_k+G^r_j G^i_{rk}-G^r_k G^i_{rj}.\quad (4)$$

In the spirit of Berwald's above mentioned paper, by the *affine curvature tensor* of (M,F) we mean the type $\binom{1}{3}$ tensor $\mathbf{H}:=\nabla^v\mathbf{R}$. The components of $\mathbf{H}$ are determined by $H^i_{jkl}\frac{\widehat{\partial}}{\partial u^i}:=\mathbf{H}\left(\frac{\widehat{\partial}}{\partial u^l},\frac{\widehat{\partial}}{\partial u^k}\right)\frac{\widehat{\partial}}{\partial u^j}$. Obviously,

$$H^i_{jkl}=\dot{\partial}_j R^i_{kl}=R^i_{kl\cdot j},\quad (5)$$

We define a further important tensor, the *Berwald curvature* $\mathbf{B}$, by

$$\mathbf{iB}\left(\widehat{X},\widehat{Y}\right)\widehat{Z}:=\left[\left[X^v,Y^h\right],Z^v\right]\ ;\ X,Y,Z\in\mathfrak{X}(M).$$

Its components are $G^i_{jkl}:=\dot{\partial}_l G^i_{jk}$.

Following the above scheme, we construct a further tensor derivation on the algebra of Finsler tensors, depending on the canonical Ehresmann connection. Let $\widetilde{X}\in\mathfrak{X}\left(\overset{\circ}{\tau}\right)$. Define the operator $\nabla^h_{\widetilde{X}}$

(i) *on functions by* $\nabla^h_{\widetilde{X}}f:=(\mathscr{H}\widetilde{X})f\ ,\ f\in C^\infty(\overset{\circ}{T}M)$;

(ii) *on Finsler vector fields by* $\mathbf{i}\nabla^h_{\widetilde{X}}\widetilde{Y}:=\mathbf{v}\left[\mathscr{H}\widetilde{X},\mathbf{i}\widetilde{Y}\right]$;

(iii) *on type* $\binom{r}{s}$ *tensors by the* product rule.

$\nabla^h_{\widetilde{X}}$ is said to be the *h-Berwald derivative* with respect to $\widetilde{X}$. Its Christoffel symbols are just the functions G^i_{jk}, i.e., we have

$$\nabla^h_{\frac{\widehat{\partial}}{\partial u^k}}\frac{\widehat{\partial}}{\partial u^j}=:G^i_{jk}\frac{\widehat{\partial}}{\partial u^i}.$$

After this the *h-Berwald differential* ∇^h can be defined in the same way as the vertical differential ∇^v (formally: replace the canonical injection $\mathbf{i}$ by the surjection $\mathcal{H}$). As for the index gymnastics, if the components of $\widetilde{A}$ are $\widetilde{A}^{i_1\dots i_r}_{j_1\dots j_s}$, then the components of $\nabla^h\widetilde{A}$ will be denoted by $\widetilde{A}^{i_1\dots i_r}_{j_1\dots j_s;j}$. These functions are much more complicated than the components of $\nabla^v\widetilde{A}$. As an illustration, we calculate the components of the h-Berwald differential of the metric tensor g:

$$\begin{aligned} g_{ij;k} :=& \nabla^h g\left(\frac{\widehat{\partial}}{\partial u^i}, \frac{\widehat{\partial}}{\partial u^j}, \frac{\widehat{\partial}}{\partial u^k}\right) := \left(\nabla^h_{\frac{\widehat{\partial}}{\partial u^k}} g\right)\left(\frac{\widehat{\partial}}{\partial u^i}, \frac{\widehat{\partial}}{\partial u^j}\right) \\ =& \left(\frac{\widehat{\partial}}{\partial u^k}\right)^h g_{ij} - g\left(\nabla^h_{\frac{\widehat{\partial}}{\partial u^k}} \frac{\widehat{\partial}}{\partial u^i} \frac{\widehat{\partial}}{\partial u^j}\right) - g\left(\frac{\widehat{\partial}}{\partial u^i}, \nabla^h_{\frac{\widehat{\partial}}{\partial u^k}} \frac{\widehat{\partial}}{\partial u^j}\right) \\ =& \left(\frac{\partial}{\partial u^k}\right)^h g_{ij} - G^r_{ik} g_{rj} - G^r_{jk} g_{ir}. \end{aligned}$$

3. Landsberg tensor depending only on the position

By the *Landsberg tensor* of a Finsler manifold (M, F) we mean the type $\binom{0}{3}$ tensor

$$\mathbf{P} := -\tfrac{1}{2}\nabla^h g$$

along $\overset{\circ}{\tau}$. Its components are

$$P_{ijk} = -\tfrac{1}{2} g_{ij;k} \ ,$$

where the functions $g_{ij;k}$ have just been calculated. The Landsberg tensor and the Cartan tensor $\mathscr{C}_\flat$ are related by

$$\mathbf{P} = \nabla^h_\delta \mathscr{C}_\flat. \tag{6}$$

In components,

$$P_{ijk} = C_{ijk;l} y^l, \tag{7}$$

which may easily be shown. A coordinate-free proof of (6) needs a little more effort, see Ref. 10, section 3.11.

Now we are in a position to show that the property $\nabla^v \mathbf{P} = 0$ implies a drastic consequence.

Proposition 3.1. *If the Landsberg tensor of a Finsler manifold depends only on the position, then it vanishes identically.*

Proof. Keeping the notation introduced above, suppose that $\dot{\partial}_l P_{ijk} = P_{ijk \cdot l} = 0$. Then differentiation of relation (7) with respect to $\dot{\partial}_l$ leads to

$$C_{ijk;l} + C_{ijk;r \cdot l} y^r = 0. \tag{8}$$

Now we use the Ricci identity (Ref. 8, 2.5.5) for the h-Berwald derivative and the vertical derivative. Then we obtain

$$C_{ijk;r\cdot l} - C_{ijk\cdot l;r} = -C_{sjk}G^s_{ilr} - C_{isk}G^s_{jlr} - C_{ijs}G^s_{klr} \tag{9}$$

(recall that $G^i_{jkl} = \dot{\partial}_l G^i_{jk}$ are the components of the Berwald tensor). Since the functions G^i_{jk} are positively homogeneous of degree 0, we have $G^i_{jkl}y^l = 0$. Thus, transvection of (9) with y^r leads to

$$C_{ijk;r\cdot l}y^r = C_{ijk\cdot l;r}y^r.$$

Hence (8) takes the form

$$C_{ijk;l} + C_{ijk\cdot l;r}y^r = 0. \tag{10}$$

Interchanging indices k and l, we obtain

$$C_{ijl;k} + C_{ijl\cdot k;r}y^r = 0. \tag{11}$$

Since $C_{ijk\cdot l} = C_{ijl\cdot k}$, if we subtract (11) from (10) we find that

$$C_{ijk;l} - C_{ijl;k} = 0. \tag{12}$$

But transvection of (12) with y^l yields

$$C_{ijk;l}y^l = 0, \tag{13}$$

since $C_{ijl}y^l = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^l}y^l = 0$ by the 0^+-homogenity of the functions g_{ij}, and by the commutation of contractions and covariant derivatives.
Relations (13) and (7) imply our assertion $\mathbf{P} = 0$. □

4. Stretch tensor depending only on the position

Inspired by a manuscript of L. Kozma,[5] we define the *stretch tensor* Σ of a Finsler manifold (M, F) by

$$\frac{1}{2}\Sigma\left(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{U}\right) := \nabla^h\mathbf{P}\left(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{U}\right) - \nabla^h\mathbf{P}\left(\widetilde{X}, \widetilde{Y}, \widetilde{U}, \widetilde{Z}\right), \tag{14}$$

where $\mathbf{P}$ is the Landsberg tensor discussed above, and $\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{U}$ are arbitrary Finsler vector fields. Since the components of $\nabla^h\mathbf{P}$ are

$$P_{ijk;l} = \left(\frac{\widehat{\partial}}{\partial u^l}\right)^h P_{ijk} - G^r_{il}P_{rjk} - G^r_{jl}P_{irk} - G^r_{kl}P_{ijr},$$

it follows that the components of Σ are

$$\Sigma_{ijkl} = 2\left(P_{ijk;l} - P_{ijl;k}\right). \tag{15}$$

This is just the formula obtained by M. Matsumoto for the stretch tensor in Ref. 7. Notice that the stretch tensor was discovered by L. Berwald.[1] He also found an important relation between the affine curvature and the stretch tensor, which may be formulated as follows:

$$\Sigma\left(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}\right) = -\delta^{*}\left(\nabla^{v}\mathbf{H}\left(\widetilde{Z},\widetilde{U},\widetilde{X},\widetilde{Y}\right)\right), \widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U} \in \mathfrak{X}\left(\overset{\circ}{\tau}\right). \quad (16)$$

In terms of tensor components, (16) leads to

$$\Sigma_{ijkl} = -y_r \dot{\partial}_j H^r_{ikl}, \quad (17)$$

this is just formula (14) of Berwald's paper.[2] From (5) and (17) it follows that we also have

$$\Sigma_{ijkl} = -y_r \dot{\partial}_j \dot{\partial}_i R^r_{kl}. \quad (18)$$

Relations (16) and (18) imply that the stretch tensor vanishes, if the fundamental affine curvature, or, equivalently, the affine curvature tensor of the Finsler manifold depends only on the position. Now we shall show that this conclusion is also true, if Σ itself has this property.

Proposition 4.1. *If the stretch tensor of a Finsler manifold depends only on the position, then it vanishes identically.*

Proof. We use the same tactics as in the previous proof. By our condition,

$$\Sigma_{ijkl\cdot m} = \dot{\partial}_m \Sigma_{ijkl} = 0 \ ,$$

so from (15) we get for the Landsberg tensor

$$P_{ijk;l\cdot m} - P_{ijl;k\cdot m} = 0. \quad (19)$$

Using the Ricci identity for $P_{ijk;l\cdot m}$ we get

$$P_{ijk;l\cdot m} = P_{ijk\cdot m;l} - P_{rjk}G^r_{iml} - P_{irk}G^r_{jml} - P_{ijr}G^r_{kml}. \quad (20)$$

Transvection of (20) with y^i leads to

$$P_{ijk;l\cdot m}y^i = P_{ijk\cdot m;l}y^i \quad (21)$$

because of $P_{ijk}y^i \overset{(7)}{=} C_{ijk;l}y^i y^l = 0$. In the same way we obtain

$$P_{ijl;k\cdot m}y^i = P_{ijl\cdot m;k}y^i. \quad (22)$$

Relations (19), (21) and (22) imply that

$$P_{ijk\cdot m;l}y^i = P_{ijl\cdot m;k}y^i. \quad (23)$$

On the other hand, from the identity $P_{rjk}y^r = 0$ we obtain by repeated covariant differentiation

$$0 = (P_{rjk}y^r)_{\cdot i;l} = P_{ijk;l} + P_{rjk\cdot i;l}y^r \ .$$

Interchanging indices k and l, we get

$$P_{ijl;k} + P_{rjl\cdot i;k}y^r = 0 \ .$$

(23) and the last two relations imply that $P_{ijk;l} - P_{ijl;k} = 0$. Hence, by (15), $\Sigma_{ijkl} = 0$. □

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References

1. L. Berwald, Über Parallelübertragung in Räumen mit allgemeiner Massbestimmung, *Jber. Deutsch Math.-Verein* **34** (1926) 213–220.
2. L. Berwald, Parallelübertragung in allgemeinen Räumen, *Atti. Congr. Intern. Mat. Bologna* **4** (1928) 263–270.
3. L. Berwald, Ueber Finslersche und Cartansche Geometrie IV, *Ann. Math.* **48** (1947) 755–781.
4. M. Crampin, On horizontal distribution on the tangent bundle of a differentiable manifold, *J. London Math. Soc. (2)* **3** (1971) 178–182.
5. L. Kozma, Unpublished manuscript.
6. R.L. Lovas, A note on Finsler-Minkowski norms, *Houston J. Math.* **33** (2007) 701–707.
7. M. Matsumoto, On the stretch curvature of a Finsler space and certain open problems, *J. Nat. Acad. Math. India* **11** (1997) 22–32.
8. M. Matsumoto, Finsler Geometry in the 20th-Century, *In: Handbook of Finsler Geometry* ((P. Antonelli, Ed.) Kluwer Academic Publishers, Dordrecht, 2003).
9. Z.I. Szabó, Über Zusammenhänge vom Finsler-Typ, *Publ. Math. Debrecen* **27** (1980) 77–88.
10. J. Szilasi, A Setting for Spray and Finsler Geometry, *In: Handbook of Finsler Geometry* ((P. Antonelli, Ed.) Kluwer Academic Publishers, Dordrecht, 2003).