ON A GENERALIZATION OF A PROBLEM OF ERDŐS AND GRAHAM

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Dedicated to Professor Lajos Tamássy on his 90th birthday

Abstract. In this paper we provide bounds for the size of the solutions of the Diophantine equation $\frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} = y^2$, where $a, b \in \mathbb{Z}, a \neq b$ are parameters. We also determine all integral solutions for $a, b \in \{-4, -3, -2, -1, 4, 5, 6, 7\}$.

1. introduction

Let us define

$$f(x, k, d) = x(x + d) \cdots (x + (k - 1)d).$$

Erdős [7] and independently Rigge [19] proved that $f(x, k, 1)$ is never a perfect square. A celebrated result of Erdős and Selfridge [8] states that $f(x, k, 1)$ is never a perfect power of an integer, provided $x \geq 1$ and $k \geq 2$. That is, they completely solved the Diophantine equation

$$f(x, k, d) = y^l \tag{1}$$

with $d = 1$. The literature of this type of Diophantine equations is very rich. First consider some results related to $l = 2$. Euler proved (see [5] pp. 440 and 635) that a product of four terms in arithmetic progression is never a square solving (1) with $k = 4, l = 2$. Obláth [18] obtained a similar statement for $k = 5$. Saradha and Shorey [23] proved that (1) has no solutions with $k \geq 4$, provided that d is a power of a prime number. Laishram and Shorey [16] extended this result to the case where either $d \leq 10^{10}$, or d has at most six prime divisors. Bennett, Bruin, Győry and Hajdu [2] solved (1) with $6 \leq k \leq 11$ and $l = 2$. Hirata-Kohno, Laishram, Shorey and Tijdeman [15] completely solved (1) with $3 \leq k < 110$.

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Now assume for this paragraph that $l \geq 3$. Many authors have considered the more general equation

$$f(x,k,d) = by^l, \tag{2}$$

where $b > 0$ and the greatest prime factor of b does not exceed k. Saradha [22] proved that (2) has no solution with $k \geq 4$. Győry [11] studied the cases $k = 2, 3$, he determined all solutions. Győry, Hajdu and Saradha [12] proved that the product of four or five consecutive terms of an arithmetical progression of integers cannot be a perfect power, provided that the initial term is coprime to the difference. Hajdu, Tengely and Tijdeman [13] proved that the product of k coprime integers in arithmetic progression cannot be a cube when $2 < k < 39$. Győry, Hajdu and Pintér proved that for any positive integers x, d and k with $\gcd(x,d) = 1$ and $3 < k < 35$, the product $x(x+d)\cdots(x+(k-1)d)$ cannot be a perfect power.

Erdős and Graham [6] asked if the Diophantine equation

$$\prod_{i=1}^{r} f(x_i, k_i, 1) = y^2$$

has, for fixed $r \geq 1$ and $\{k_1, k_2, \ldots, k_r\}$ with $k_i \geq 4$ for $i = 1, 2, \ldots, r$, at most finitely many solutions in positive integers $(x_1, x_2, \ldots, x_r, y)$ with $x_i + k_i \leq x_{i+1}$ for $1 \leq i \leq r-1$. Skałba [25] provided a bound for the smallest solution and estimated the number of solutions below a given bound. Ulas [27] answered the above question of Erdős and Graham in the negative when either $r = k_i = 4$, or $r \geq 6$ and $k_i = 4$. Bauer and Bennett [1] extended this result to the cases $r = 3$ and $r = 5$. Bennett and Van Luijk [3] constructed an infinite family of $r \geq 5$ non-overlapping blocks of five consecutive integers such that their product is always a perfect square. Luca and Walsh [17] studied the case $(r, k_i) = (2, 4)$.

In this paper we study the Diophantine equation

$$\frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} = y^2, \tag{3}$$

where $a, b \in \mathbb{Z}, a \neq b$ are parameters. We provide bounds for the size of solutions and an algorithm to determine all solutions $(x, y) \in \mathbb{Z}^2$. The method of proof is based on Runge's method [10, 14, 20, 21, 24, 26, 28]. In 2008, Sankaranarayanan and Saradha established improved upper bounds for the size of the solutions of the Diophantine equations $F(x) = y^m$ and $F(x) = G(y)$, for which Runge's method can be applied. They generalized the method to obtain bounds for the solutions of equations of the form $P(x)/Q(x) = y^m$. Based on this latter result we provide bounds for the solutions of equation (3).

Theorem 1. *(I) If* $(x, y) \in \mathbb{Z}^2$ *is a solution of* (3) *with* $a \equiv b \pmod 2$, *then*

$$|x| \leq \max\{|A_2|, |A_1|^{1/2}, |A_0|^{1/3}, |B_2|, |B_1|^{1/2}, |B_0|^{1/3}, |\frac{1}{4}(a+b-6)^2 ab|\},$$

where

$$\begin{aligned}
A_2 &= \frac{3}{4}a^2 + \frac{1}{2}ab + \frac{3}{4}b^2 - 2a - 2b + 7\\
A_1 &= -\frac{1}{4}a^3 + \frac{1}{4}a^2b + \frac{1}{4}ab^2 + 2a^2 - \frac{1}{4}b^3 + 2b^2 - 4a - 4b + 6\\
A_0 &= -\frac{1}{4}(a+b-4)^2 ab\\
B_2 &= \frac{3}{4}a^2 + \frac{1}{2}ab + \frac{3}{4}b^2 - 4a - 4b - 5\\
B_1 &= -\frac{1}{4}a^3 + \frac{1}{4}a^2b + \frac{1}{4}ab^2 + 4a^2 - \frac{1}{4}b^3 + 4b^2 - 16a - 16b + 6\\
B_0 &= -\frac{1}{4}(a+b-8)^2 ab.
\end{aligned}$$

(II) If $(x, y) \in \mathbb{Z}^2$ *is a solution of* (3) *with* $a \not\equiv b \pmod 2$, *then*

$$|x| \leq 2\max\{|C_2|, |C_1|^{1/2}, |C_0|^{1/3}, |D_2|, |D_1|^{1/2}, |D_0|^{1/3}\},$$

where

$$\begin{aligned}
C_2 &= \frac{3}{4}a^2 + \frac{1}{2}ab + \frac{3}{4}b^2 - \frac{7}{2}a - \frac{7}{2}b - \frac{5}{4}\\
C_1 &= -\frac{1}{4}a^3 + \frac{1}{4}a^2b + \frac{1}{4}ab^2 + \frac{7}{2}a^2 - \frac{1}{4}b^3 + \frac{7}{2}b^2 - \frac{49}{4}a - \frac{49}{4}b + 6\\
C_0 &= -\frac{1}{4}(a+b-7)^2 ab\\
D_2 &= \frac{3}{4}a^2 + \frac{1}{2}ab + \frac{3}{4}b^2 - \frac{5}{2}a - \frac{5}{2}b + \frac{19}{4}\\
D_1 &= -\frac{1}{4}a^3 + \frac{1}{4}a^2b + \frac{1}{4}ab^2 + \frac{5}{2}a^2 - \frac{1}{4}b^3 + \frac{5}{2}b^2 - \frac{25}{4}a - \frac{25}{4}b + 6\\
D_0 &= -\frac{1}{4}(a+b-5)^2 ab.
\end{aligned}$$

We apply the above theorem to determine all integral solutions of (3) with $a, b \in \{-4, -3, -2, -1, 4, 5, 6, 7\}, a \neq b$.

Corollary 1. *All solutions* $(x,y) \in \mathbb{Z}^2, y \neq 0$ *of* (3) *with* $a,b \in \{-4,-3,-2,-1,4,5,6,7\}, a \neq b$ *are as follows*

$$\begin{aligned} a=-4, b=-3, &\quad (x,y) \in \{(-6,2),(1,2)\} \\ a=-4, b=5, &\quad (x,y) \in \{(-6,6)\} \\ a=-2, b=7, &\quad (x,y) \in \{(3,6)\} \\ a=6, b=7, &\quad (x,y) \in \{(-4,2),(3,2)\}. \end{aligned}$$

2. PROOF OF THE RESULTS

In the proof we will use the following result of Fujiwara [9].

Lemma 1. *Given* $p(z)=\sum_{i=0}^n a_i z^i, a_n \neq 0$. *Then*

$$\max\{|\zeta| : p(\zeta)=0\} \leq 2\max\left\{\left|\frac{a_{n-1}}{a_n}\right|, \left|\frac{a_{n-2}}{a_n}\right|^{1/2}, \ldots, \left|\frac{a_0}{a_n}\right|^{1/n}\right\}.$$

Proof of Theorem. The polynomial part of the Puiseux expansion of

$$\left(\frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)}\right)^{1/2}$$

is $x+3-\frac{a+b}{2}$. (I) First we deal with the case $a \equiv b \pmod 2$ that is, when $\frac{a+b}{2}$ is an integer. We have that

$$x(x+1)(x+2)(x+3)-(x+a)(x+b)\left(x+2-\frac{a+b}{2}\right)^2 =$$
$$2x^3+A_2x^2+A_1x+A_0 =: f_A(x)$$

and

$$x(x+1)(x+2)(x+3)-(x+a)(x+b)\left(x+4-\frac{a+b}{2}\right)^2 =$$
$$-2x^3+B_2x^2+B_1x+B_0 =: f_B(x).$$

If follows from Lemma 1 that $f_A(x) \neq 0$ if

$$|x| > \max\{|A_2|, |A_1|^{1/2}, |A_0|^{1/3}\} =: r_A.$$

Similarly, one has that $f_B(x) \neq 0$ if

$$|x| > \max\{|B_2|, |B_1|^{1/2}, |B_0|^{1/3}\} =: r_B.$$

Therefore $f_A(x)f_B(x) < 0$, if $|x| > \max\{r_A, r_B\}$. We obtain that either

$$\left(x+4-\frac{a+b}{2}\right)^2 < \frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} < \left(x+2-\frac{a+b}{2}\right)^2$$

or

$$\left(x+2-\frac{a+b}{2}\right)^2 < \frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} < \left(x+4-\frac{a+b}{2}\right)^2.$$

Since $\frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} = y^2$, we get that $y^2 = \left(x+3-\frac{a+b}{2}\right)^2$ in both cases. Thus x is a root of a quadratic polynomial $x(x+1)(x+2)(x+3)-(x+a)(x+b)\left(x+3-\frac{a+b}{2}\right)^2$. The constant term of this quadratic polynomial is $-\frac{1}{4}(a+b-6)^2ab$, hence

$$|x| \le |\frac{1}{4}(a+b-6)^2ab|.$$

(II) Now we consider the case $a \not\equiv b \pmod 2$. We have that

$$x(x+1)(x+2)(x+3)-(x+a)(x+b)\left(x+3-\frac{a+b-1}{2}\right)^2 =$$
$$-x^3+C_2x^2+C_1x+C_0 =: f_C(x)$$

and

$$x(x+1)(x+2)(x+3)-(x+a)(x+b)\left(x+3-\frac{a+b+1}{2}\right)^2 =$$
$$x^3+D_2x^2+D_1x+D_0 =: f_D(x).$$

Lemma 1 implies that $f_C(x) \neq 0$ if

$$|x| > 2\max\{|C_2|, |C_1|^{1/2}, |C_0|^{1/3}\} =: r_C$$

and $f_D(x) \neq 0$ if

$$|x| > 2\max\{|D_2|, |D_1|^{1/2}, |D_0|^{1/3}\} =: r_D.$$

It is clear that $f_C(x)f_D(x) < 0$, if $|x| > \max\{r_C, r_D\}$. One gets that either

$$\left(x+3-\frac{a+b-1}{2}\right)^2 < \frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} < \left(x+3-\frac{a+b+1}{2}\right)^2$$

or

$$\left(x+3-\frac{a+b+1}{2}\right)^2 < \frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} < \left(x+3-\frac{a+b-1}{2}\right)^2.$$

In both cases we get a contradiction, since $\frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} = y^2$ and there cannot be a square between consecutive squares. Thus $|x| \le \max\{r_C, r_D\}$. □

We wrote a Magma [4] code to solve equation (3). If $a \equiv b \pmod 2$, then we used the bound

$$|x| \leq \max\{|A_2|, |A_1|^{1/2}, |A_0|^{1/3}, |B_2|, |B_1|^{1/2}, |B_0|^{1/3}\}$$

and we determined the roots of the quadratic equation $x(x+1)(x+2)(x+3) - (x+a)(x+b)\left(x+3-\frac{a+b}{2}\right)^2$. Some details of the computations are given in the following table. We only indicate those cases where there is a solution with $y \neq 0$.

a	b	bound for $\vert x\vert$
-4	-3	96
-4	5	46
-2	7	50
6	7	114

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