Can teacher trainees use inductive arguments during their problem solving activity?

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The Hungarian curriculum for mathematics teachers' training specializes in a problem-solving Seminar aimed at teaching heuristic strategies. This fact motivated our research focusing on problem-solving competency of teacher trainees. In this study we deal with some aspects of inductive reasoning. We summarize the results of a diagnostic survey. We chose a closed problem which could be solved through inductive reasoning, and analyzed problem solving process of 94 students. Our primary interest was how students apply general phases of inductive reasoning, if they use it at all; that is, how they conclude general statements after pattern recognition, and whether they close it deductively or not.

Keywords: Problem solving, inductive reasoning, proof and proving, rational errors.

Introduction

In the midst of a long-term discussion on the role of metacognition and teaching heuristic strategies in order to enhance mathematical problem solving skills (Schoenfeld, 1985; Cai, 2010), the new Hungarian curriculum for mathematics teachers' training[1] explicitly specializes such a course. We think that there is no final rule concerning this polemics; moreover there is no comprehensive research focused on this student group in Hungary in this respect. One of the antecedent studies, the PhD dissertation (Pintér, 2012) focused solely on primary teacher trainees. Motivated by these facts we have begun a research to map the status quo in Hungary, with the aim to give didactical consequences and finding ways of teaching heuristic strategies, general problem solving skills effectively.

According to Polya (1954), heuristic reasoning is based on induction or analogy. In this study we focus on inductive reasoning process only. Csapo (1997) supports the proposition that inductive reasoning and skills of proof develop during broad age range (Grade 1-11). We therefore assume that problem solving skills, especially proper utilization of inductive reasoning strategy develop after entering higher education, and should be subject to development.

Inductive reasoning and inductive problem-solving strategy

The word "induction" means a scientific procedure starting from experience. In inductive reasoning, one makes a series of observations and infers a new claim based on them. The mathematics education offers the possibility of learning the way of inductive reasoning beside the deductive one. Within the process of inductive reasoning Polya (1954) distinguishes stages such as observation of particular cases, formulating a conjecture (generalization), testing the conjecture with other particular cases. Haverty, Koedinger, Klahr, & Alibali (2000) identify the "function finding task" as the "representative of inductive reasoning" and use this term in a narrower sense as we use it, thinking

[1] This curriculum was introduced in 2013.

only on open problems and determine three basic inductive activity such as data gathering, pattern finding and hypothesis generation. Yeo and Yeap (2009) make the difference between inductive observation and inductive reasoning clearer. If students observe a pattern when specialising, the pattern is only a conjecture and they call it ‘inductive observation’. But if students use the underlying mathematical structure to argue that the observed pattern will always continue, this can be called ‘inductive reasoning’. Motivated by these approaches we describe inductive reasoning with five phases. (1) Observation of particular cases including looking for possible pattern as well. (2) Following the observed pattern, i.e. applying it for other cases. It often happens without formulation of a general statement. (3) Formulating a general conjecture. (4) Testing it by other particular cases. The result of the inductive reasoning is a general statement, but the mathematical problem solving process requires its deductive closure (5). The form of deductive closure could be either a rigorous proof or justification using the underlying mathematical structure (Mason, Burton, & Stacey, 2010). Moreover, Rivera (2013) uses the term of *empirical structural argument*, as a type of justification. Empirical structural argument means that one uses the steps of a logical deductive proof with concrete numbers or objects instead of variables. This process is closely related to the phenomenon of *generic example* (Stylianides, 2009) and *transformational proof-scheme* (Harel & Sowder, 2007). Thus, we look not only for the clues of the formal proof but the clues of empirical structural argument too while investigating Phase 5.

Haverty et al. (2000) argue, in accordance with other studies, that the detection of patterns is crucial to inductive reasoning. In patterning activity there is a difference between near generalization, a description of a pattern allowing one to determine the next term in a sequence, and far generalization, the construction of a general rule or a far stage in the pattern (Rivera, 2013). Solving the problem we investigated in this research requires mainly the near generalization activity; the formulation of a general rule could be useful, but is not necessary.

We identified in many cases that a mental manipulation process led to the inductive observations. The same phenomenon was detected by Simon (1996) who defined the concept of transformational reasoning, which is rather a dynamic process. Transformational reasoning visualizes the transformation of a mathematical situation and the results of that transformation. The conjecture is drawn from the result of the mental manipulation.

Besides inductive strategy, some other strategies may work for many closed mathematical problems. As Ben-Zeev (1996) pointed out, the schema-based thinking could be a useful way for organizing mathematical experiences. Using a schema – the knowledge structure for a particular class of concepts – in a proper way can predict the solution of the problem. However, schematic reasoning often lead to rational errors when applied rigidly or without understanding the context. The term rational error “refers to process where student first induces an incorrect rule and then proceeds to follow it “correctly” in a logical consistent manner” (Ben-Zeev, 1996, p. 65).

In this current study our main focus is on the way of problem solving of our teacher trainees with special interest in inductive reasoning. Thus, we have formulated the following research questions.

- Q1 Whether they use inductive arguments during the problem solving process or not?
- Q2 If not, what are the frequent types of their reasoning?
- Q3 What are the characteristics and the typical errors of their inductive reasoning process?

In order to answer these questions we constructed a problem which may be solved in different ways, among others using inductive strategy.

The problem

In Figure 1 $AC_1 = C_1B_1 = B_1C_2 = C_2B_2 = B_2C_3 = C_3B_3$.

1. If $\alpha = 15°$, find β.
2. How many isosceles triangles can be drawn following the algorithm presented in the figure?
3. For some α, we can draw exactly 9 isosceles triangles. Find α.

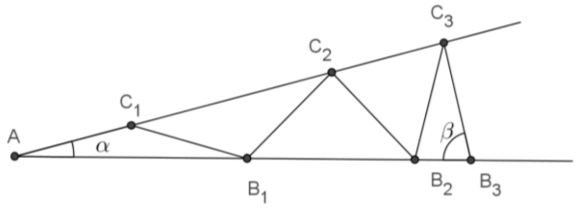

Figure 1: The initial problem

The first question requires only minimal geometrical knowledge; moreover, the completion is the only cognitive operation needed, where by completion we mean finishing arithmetic operations in this context. The second question tests whether the student could follow the algorithm given by the figure. Our hypothesis was that the third question should be a mathematical problem for our students. Since the solution is completely determined by the underlying geometrical structure, this problem is suitable for examination Phases 1-5 of inductive reasoning.

A possible strategy is based on two steps. (1) The n^{th} isosceles triangle has angles with measure $n\alpha$ on its base. (2) If we can draw the 9$^{\text{th}}$ triangle, then $9\alpha < 90°$; moreover, we cannot draw the 10^{th} triangle, thus $10\alpha \geq 90°$. It means $9° \leq \alpha < 10°$.

Our primary interest was in Step (1). If the student uses this general statement, how he or she concludes it. This “general approach”, i.e. when we use a general n in the solution instead of a concrete number of triangles, may appear in all parts of the solution; however it is not necessary for this particular problem. The reason is that only near generalization is involved here, i.e. direct methods (drawing, counting, and determining all the angles) could be effective (Rivera, 2013).

We highlight here only one more question: how students deal with the last possible triangle? We briefly refer to this question as “condition for halt”.

Dimensions and methodology of the research

In academic year 2015/16 we investigated the solution of the problem described above with involvement of 94 students, including 49 prospective primary school teachers and 45 prospective secondary and upper secondary school Mathematics teachers. Solving of the problem does not require advanced mathematical knowledge and skills. Thus, we do not distinguish between these two groups in our research. The base group consists of 83 students (S01-S83 in the transcripts). In this group we investigated students’ written elaborations. The interview with 11 other students completed the frame of this research (S84-S94). During the interviews we followed students’ activities and made sound records. Students were asked to say out loud what they are thinking of when solving the problem. We corrected numerical errors immediately; otherwise we did not put guiding questions.

Results: Students' activity during problem solving

Overview

Analysis of students' performance handling the third question represents the overall problem solving process well. We identified two classes of solutions (Figure 2). *Concrete solution class* means that student deals with 9 triangles only and sticks to the text verbatim. Because the third question is a near generalization of the previous one, this plan is acceptable. By *general solution class* we mean, that solution works for arbitrary number of triangles. Some students used more than one strategy. 18 students didn't show up any strategy, 5 of them ignored the problem, and 13 students could only compute the angle of the 5th triangle.

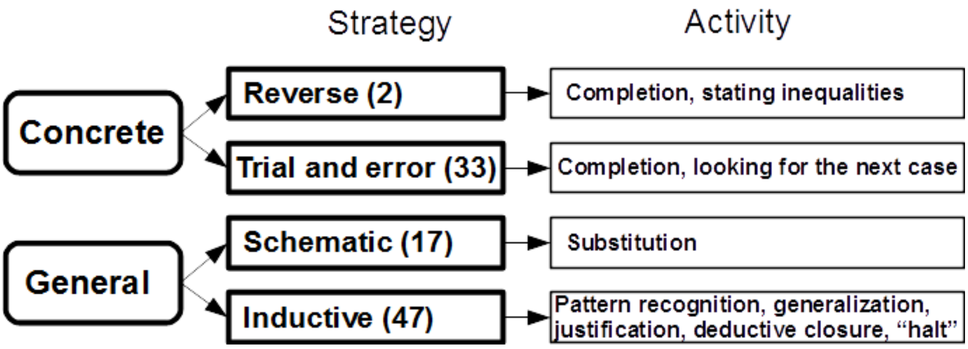

Figure 2: Strategies and activities with number of students following the particular strategy

Reverse strategy

By reverse strategy we mean here that student's starting point is the final configuration with 9 triangles. This is a successful approach, where the student investigates the figure with 9 and 10 isosceles triangles and computes all the necessary angles directly, with or without showing signs of pattern recognition. We encounter this approach in 2 interviews, but nobody completed the third question using this strategy in the base group.

Trial and error approach

Trial and error strategy is characterized by repeated, varied attempts which can be continued until success. Although this approach appeared 33 times, in most cases it played certain role in the inductive reasoning. 10 students applying this strategy did not show inductive or any other strategies; however, in 4 cases the activity was controlled with the (unproven) hypothesis that $n(\alpha)$ is a decreasing function, where n is the number of isosceles triangles. One student in the interviewed group followed this pattern. Her view demonstrates that trial and error could be a rational activity even for this problem. After reading the text, her first and immediate reaction was applying trial and error method. After two trials with angle measure 10 and 5 the interviewer interrupted her:

Interviewer: Do you think that the solution is an integer?

S86: Certainly.

Interviewer: Why?

S86: I don't know… It is a nice problem and the solution should be a 'nice' integer.

Interviewer: [He gave a hint that $\alpha \in \mathbb{R}$.] In what cases is trial and error effective?

S86: When we have small number of cases to check. [She gave up.]

The transcript points out the role of student's belief in the problem solving process (Schoenfeld, 1985). Theoretically she knows that her effort is hopeless, but her belief in 'nice' solution overwrites this knowledge. In this context the false trial and error strategy is a rational error here in the sense of Ben-Zeev (1996), because if α is an integer then we have finite number of integers to check. We detected 24 students with belief that the solution is an integer but in some cases with sign of uncertainty, e.g. "If we reject the condition that α is an integer, then we have infinite possibilities" (S37).

Schematic reasoning, false scheme

Schematic reasoning is the process of reasoning by which new information is interpreted according to a particular schema. In our problem the number of isosceles triangles is $n(\alpha) = \left\lceil \frac{90}{\alpha} \right\rceil - 1$, where α is the given angle. This function, to be more precise, some approximate idea of $n(\alpha)$ appeared in students' responses. First of all, $n(\alpha)$ is a decreasing function, and 15 students referred to this property properly or erroneously (i.e. in strict form) without proof or explanation. The following transcript demonstrates the typical usage of this observation. Previously this student settled that for $\alpha = 9°$ there are 9 triangles. "If $\alpha < 9°$, then the number of isosceles triangles is more than 9" (S10). We presume that the transformation reasoning (Simon, 1996) is behind this recognition. Some students showed explicit evidence of transformational reasoning. We encountered sentences like the following transcript 5 times in the base group. "If we decrease the angle, then we get more triangles" (S03). Students' observations are the result of the mental transformation of the angle.

In some cases it invoked the scheme of inverse proportionality or the misconception of *strictly* decreasing $n(\alpha)$ function. Two false solutions with inverse proportionality scheme appeared in our experiment and caused a rational error. Two other students referred to inverse proportionality, but later revised the idea.

Other false scheme was the direct proportionality scheme. Perhaps the following interpretation of the problem invokes it. "In case of $15°$ we have 5 triangles, how much is the angle if we have 9 triangles?" This is a common pattern in elementary word problems. Transcriptions of data demonstrated in Figure 3 strengthen this presumption. Data from the second question is not necessary to answer the third question, but students who applied the direct proportionality scheme connected data in this way. S13 misprinted the angle and used 30 instead of 15. S36 revised her outcome. S08 finds x by the 'proper' way: $5x = 9 \cdot 15$. He just began the division (the tick between digits 3 and 5 indicates this), but presumably rejects the result which he found too big and finishes the calculation "forcing" a more reasonable result. Direct proportionality appeared 5 times, but 1 student revised this solution.

Figure 3: Direct proportionality (left: S13, center: S36, right: S08)

Looking for a general solution using inductive strategy

We consider that the inductive strategy appears if a student reaches at least the first phase of the inductive reasoning process i.e. at least observes particular cases and looks for a possible pattern. Half of the students in this research (47 people from 94) used or tried to use this problem solving strategy. (Some of them used other strategies too.) Eight students stopped at the first stage because of the possible lack of near generalization ability. In the second phase (near generalization) the others determined the 9th and the 10th angle in the sequence in a way that they skipped some members and tried to transfer the "condition for halt" observed before. In our problem we didn't ask to formulate a general statement; however 11 students made it (far generalization, Phase 3). The statements were expressed either by symbols or by words, like S85 told "Thus the length of one step is equal to the opener." [The difference between the base angles in two consecutive triangles is equal to the given angle.][2]

We wondered whether the students feel the need of testing their conjecture by other particular cases or not (Phase 4). The following transcript represents this phenomenon well. After calculating $\alpha, 2\alpha$, and 3α S88 said: "I'm sure the result will be something similar. β equals probably 5α, but I will compute it." 13 solutions contained test of the near/far generalized conjecture.

Concerning Phase 5 (deductive closure) we confronted with the dilemma of "proving or not". The near generalization feature of the problem probably caused the fact that no one has felt the necessity of proving of the observed and applied conjecture. The following transcript represents a typical attitude during the interviews:

Interviewer: Why are you sure that the 9th angle equals 9α?"

S93: Because it was clearly visible, and I felt that it will work always in the same way.

The clue of empirical structural argument (Figure 4) appeared only in 5 works. Previously S88 determined the 5th angle without any skipping, after that she skipped to the 9th angle directly.

Here the recursive counting procedure confirms that the measure of the angle increases by α.

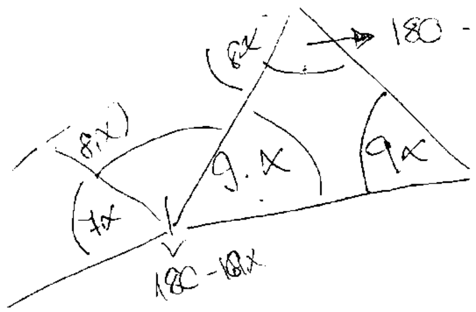

Figure 4: Empirical structural argument of S88

Four students were able to make a correct deductive closure of the inductive reasoning by mathematical induction proof after the interviewer asked them to prove their conjecture. One of them said "I can prove if you wish." (S93)

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[2] Rephrased by the authors

Typical error during inductive reasoning: spurious abstraction from irrelevant feature

Solving the first and second problem, students have some previous experience in the third problem. In 18 solutions we found that they abstracted a false rule from a previous experience, what is more, from one particular case. We highlighted only a few examples here. In the third part of the problem 2 students used the same difference (i.e. 15) for the arithmetic sequence of base angles as in the first part of the problem. In 2 cases the starting point was that the measure of base angles of the last possible triangle always equals 75°. The most frequent spurious abstraction concerns the "condition for halt" (12 students). In Figure 1 $\measuredangle AC_3B_3 = 90°$ causes the halt. Generally this condition is $\measuredangle AC_nB_n \leq 90°$ (for the smallest n), but these students kept the equality instead of inequality. The following transcript is a typical answer to the third question: "$90/10 = 9$, because in this way the tenth triangle would have two right angles" (S16).

Findings and interpretation of results

The students involved in this research dealt with the presented problem in many different ways, and we detected many different solution strategies. Thus, we conclude that the chosen problem was an appropriate instrument to answer our research question in particular and to make some conclusions in general. We have summarized our findings for research questions as follows.

Q1 50% of the students used inductive arguments during their problem solving process.

Q2 In the other cases the most frequent type of their reasoning was trial and error strategy. Other strategies appeared, namely schematic, and reverse as well. Furthermore, we found that lot of students (19% in this research) did not go beyond the computational activity; they did not have any other idea. Yeo and Yeap (2009) describe the same phenomenon for weaker students.

Q3 We found an uncertainty in inductive reasoning: students formulated conjecture from a few particular cases; moreover, they did not confirm it and 95% of students did not make any form of deductive justification. They often abstracted a false rule from a previous experience, what is more, from one particular case. Students relied on their intuitions without doubt; and this behavior calls for rigid schemes. They often mixed or changed these strategies without any result.

Possible explanations of these findings are complex. First of all, our students are not familiar with heuristic strategies, especially with strategy for determining patterns. The recognized pattern which described the relation between the angles and the number of triangles was a plausible one in their mind instead of a definite pattern in situation with well-defined mathematical structure. Moreover, the common misconception appears in the interviews that particular examples prove a general statement. With respect to the function concept we conclude that it is not deep enough, students have difficulty with step function. In many cases our students had in mind natural numbers instead of real numbers, as possible values of an angle, which suggests that their number concept is very simple and/or their belief in "nice whole number" solution is very strong.

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