Affine reductive spaces of small dimension and left A-loops

Ágota Figula

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Abstract

In this paper we determine the at least 4-dimensional affine reductive homogeneous manifolds for an at most 9-dimensional simple Lie group or an at most 6-dimensional semi-simple Lie group. Those reductive spaces among them which admit a sharply transitive differentiable section yield local almost differentiable left A-loops. Using this we classify all global almost differentiable left A-loops L having either a 6-dimensional semi-simple Lie group or the group $SL_3(\mathbb{R})$ as the group topologically generated by their left translations. Moreover, we determine all at most 5-dimensional left A-loops L with $PSU_3(\mathbb{C}, 1)$ as the group topologically generated by their left translations.

1 Introduction

The affine reductive spaces are essential objects of differential geometry (cf. [8], [19], [12]). They are homogeneous manifolds G/H such that there exists an $Ad(H)$-invariant subspace $\mathbf{m}$ of the Lie algebra $\mathbf{g}$ of G that is complementary to the subalgebra $\mathbf{h}$ in $\mathbf{g}$.

The explicite knowledge of affine reductive spaces plays an important role in many investigations (cf. [21], [4], [13]). This paper is an application to differentiable loops since the affine reductive spaces are the key for the classification of almost differentiable left A-loops L; these are loops in which any mapping $x \mapsto [(ab)^{-1}(a(bx))], a, b \in L$ is an automorphism of L. The relations between them and reductive homogeneous spaces are explicitly discussed in [10], [11] and [18].

Using the fact that the groups topologically generated by the left translations of almost differentiable left A-loops L are Lie groups (cf. [17]), we treat L as images of global differentiable sections $\sigma : G/H \to G$, where G is a connected

Lie group, H is a closed subgroup containing no non-trivial normal subgroup of G such that the subset $\sigma(G/H)$ is invariant under the conjugation with the elements of H. Since the tangent space $T_1(\sigma(G/H))$ is a complementary reductive subspace to the Lie algebra $\mathbf{h}$ of H the affine reductive spaces are crucial for the classification of almost differentiable left A-loops.

In contrast to the compact connected Lie groups in which for any connected closed subgroup there is an reductive complement (cf. [12], p. 199), for non-compact Lie groups the situation is complicated already if they have small dimension. This is documented by Section 3 and Proposition 20, where we determine all at least 4-dimensional affine reductive homogeneous spaces $(\mathbf{g}, \mathbf{h}, \mathbf{m})$, such that $\mathbf{g}$ is either an at most 9-dimensional simple Lie algebra or it is isomorphic to $\mathfrak{sl}_2(\mathbb{R}) \oplus \mathbf{g}_2$, where $\mathbf{g}_2$ is a 3-dimensional simple Lie algebra.

The exponential images $\exp \mathbf{m}$ of reductive complements $\mathbf{m}$ of the triples $(\mathbf{g}, \mathbf{h}, \mathbf{m})$ obtained in Section 3 and in Proposition 20 yield local left A-loops. In Section 4 and Proposition 21 we discuss which of these left A-loops can be extended to global ones. They are precisely those exponential images $\exp \mathbf{m}$ which form systems of representatives for the cosets $\{xH \mid x \in G\}$ in G and do not contain any element conjugate to an element of H.

Since differentiable Bruck loops have realizations on differentiable affine symmetric spaces G/H, where H is the set of fixed elements of an involutory automorphism of G and $\sigma(G/H)$ is the exponential image of the (-1)-eigenspace of the corresponding automorphism of the Lie algebra $\mathbf{g}$ of G, the class of differentiable Bruck loops form a proper subclass of almost differentiable left A-loops. An important subclass of Bruck loops are the Bruck loops of hyperbolic type which correspond to Lie groups G and involutions τ fixing elementwise a maximal compact subgroup of G (cf. [7], 64.9, 64.10). Almost differentiable left A-loops L having dimension at most 3 and semi-simple Lie groups as the groups topologically generated by their left translations are classified in [18], Section 27 and in [6]. Hence in the following main result of this paper only at most 4-dimensional almost differentiable left A-loops occur.

Theorem *Let L be a connected almost differentiable left A-loop such that dim $L \geq 4$ and the group topologically generated by the left translations of L is semi-simple.*

If dim $G = 6$ then G is isomorphic to $PSL_2(\mathbb{R}) \times G_2$, where G_2 is either $PSL_2(\mathbb{R})$ or $SO_3(\mathbb{R})$ and the loop L is either a Scheerer extension of G_2 by the hyperbolic plane loop $\mathbb{H}_2$ (cf. [18], Section 22) or the direct product $\mathbb{H}_2 \times \mathbb{H}_2$.

If the group G is simple and $7 \leq$ dim $G \leq 9$ then G is isomorphic either to $SL_3(\mathbb{R})$ or to $PSU_3(\mathbb{C}, 1)$. In the first case L is the 5-dimensional Bruck loop of hyperbolic type having the group $SO_3(\mathbb{R})$ as the stabilizer of $e \in L$ (cf. [5], *p. 12). In the case* $G \cong PSU_3(\mathbb{C}, 1)$ *every loop* L *with dim* $L < 6$ *is the complex hyperbolic plane loop* L_0 *having the group* $Spin_3 \times SO_2(\mathbb{R})/\langle(-1, -1)\rangle$ *as the stabilizer of* $e \in L_0$ *(cf. [5], p. 9).*

2 Some basic notions

A binary system $(L, \cdot)$ is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution which we denote by $y = a\backslash b$ and $x = b/a$. Let $(L_1, \cdot)$ and $(L_2, *)$ be two loops. The set $L = L_1 \times L_2 = \{(a, b) \mid a \in L_1, b \in L_2\}$ with the componentwise multiplication is again a loop, which is called the direct product of L_1 and L_2, and the loops $(L_1, \cdot)$, $(L_2, *)$ are subloops of L.

A loop is called a left A-loop if each mapping $\lambda_{x,y} = \lambda_{xy}^{-1}\lambda_x\lambda_y : L \to L$ is an automorphism of L.

Let G be the group generated by the left translations of L and let H be the stabilizer of $e \in L$ in the group G. The left translations of L form a subset of G acting on the cosets $\{xH; x \in G\}$ such that for any given cosets aH and bH there exists precisely one left translation λ_z with $\lambda_z aH = bH$.

Conversely, let G be a group, H be a subgroup containing no normal non-trivial subgroup of G and $\sigma : G/H \to G$ be a section with $\sigma(H) = 1 \in G$ such that the set $\sigma(G/H)$ of representatives for the left cosets $\{xH, x \in G\}$ acts sharply transitively on the space G/H of $\{xH, x \in G\}$ (cf. [18], p. 18). Such a section we call a sharply transitive section. Then the multiplication defined by $xH * yH = \sigma(xH)yH$ on the factor space G/H or by $x * y = \sigma(xyH)$ on $\sigma(G/H)$ yields a loop $L(\sigma)$. The group G is isomorphic to the group generated by the left translations of $L(\sigma)$.

If G is a Lie group and σ is a differentiable section satisfying the above conditions then the loop $L(\sigma)$ is almost differentiable. This loop is a left A-loop if and only if the subset $\sigma(G/H)$ is invariant under the conjugation with the elements of H. Moreover the manifold L is parallelizable since the set of the left translations is sharply transitive.

Let L_1 be a loop defined on the factor space G_1/H_1 with respect to a section $\sigma_1 : G_1/H_1 \to G_1$ the image of which is the set $M_1 \subset G_1$. Let G_2 be a group, let $\varphi : H_1 \to G_2$ be a homomorphism and $(H_1, \varphi(H_1)) = \{(x, \varphi(x)); x \in H_1\}$. A loop L is called a Scheerer extension of G_2 by L_1 if the loop L is defined on the factor space $(G_1 \times G_2)/(H_1, \varphi(H_1))$ with respect to the section $\sigma : (G_1 \times G_2)/(H_1, \varphi(H_1)) \to G_1 \times G_2$ the image of which is the set $M_1 \times G_2$.

If L is a connected almost differentiable left A-loop, then the group G topologically generated by the left translations of L within the group of autohomeomorphisms is a connected Lie group (cf. [17]; [18], Proposition 5.20.

p. 75), and we may describe L by a differentiable section.

Let L be a connected almost differentiable left A-loop. Let G be the Lie group topologically generated by the left translations of L, and let $(\mathbf{g}, [.,.])$ be the Lie algebra of G. Denote by $\mathbf{h}$ the Lie algebra of the stabilizer H of the identity $e \in L$ in G and by $\mathbf{m} = T_1\sigma(G/H)$ the tangent space at $1 \in G$ of the image of the section $\sigma : G/H \to G$ corresponding to L. Then $\mathbf{m}$ generates $\mathbf{g}$ and the homogeneous space G/H is reductive, i.e. we have $\mathbf{g} = \mathbf{m} \oplus \mathbf{h}$ and $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$. (cf. [18], Proposition 5.20. p. 75) If $[\mathbf{m}, \mathbf{m}] \subseteq \mathbf{h}$ then the factor space G/H is an affine symmetric space ([16]) and the corresponding loop L is called a Bruck loop.

In our computation we often use the following facts about the Lie algebras $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{so}_3(\mathbb{R})$.

As a real basis of $\mathfrak{sl}_2(\mathbb{R})$ we choose the following

$$(*) \qquad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(cf. [9], pp. 19-20).

With respect to this basis the Lie algebra multiplication is given by:

$$[e_1, e_2] = 2e_3, \ [e_1, e_3] = 2e_2, \ [e_3, e_2] = 2e_1.$$

1.1 An element $X = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in \mathfrak{sl}_2(\mathbb{R})$ is elliptic, parabolic or hyperbolic according whether

$$k(X) = k(X, X) = \lambda_1^2 + \lambda_2^2 - \lambda_3^2 \text{ is smaller, equal, or greater 0.}$$

The basis elements e_1, e_2 are hyperbolic, e_3 is elliptic and the elements e_2+e_3, $e_1 + e_3$ are both parabolic. All elliptic elements, all hyperbolic elements as well as all parabolic elements of $\mathfrak{sl}_2(\mathbb{R})$ are conjugate in this order to e_3, to e_1 respectively to e_2+e_3 (cf. [9], p. 23). There are 3 conjugacy classes of the one dimensional subgroups of $PSL_2(\mathbb{R})$. As representatives of these classes we can choose $\exp e_3$, $\exp e_1$, $\exp e_2 + e_3$. There is precisely one conjugacy class $\mathcal{C}$ of the two dimensional subgroups of $PSL_2(\mathbb{R})$, as a representative of $\mathcal{C}$ we choose

$$\mathcal{L}_2 = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}; \ a > 0, b \in \mathbb{R} \right\}.$$

The Lie algebra of $\mathcal{L}_2$ is generated by the elements e_1, $e_2 + e_3$.

According to [9] for the exponential function $\exp : sl_2(\mathbb{R}) \to SL_2(\mathbb{R})$ we have

$$\exp \ X = C(k(X)) \ I + S(k(X)) \ X.$$

Here is

$$C(x) = \begin{matrix} \cosh\sqrt{x} & \text{for} & 0 \le x, \\ \cos\sqrt{-x} & \text{for} & 0 > x, \end{matrix} \qquad \sqrt{|x|} \ S(x) = \begin{matrix} \sinh\sqrt{x} & \text{for} & 0 \le x, \\ \sin\sqrt{-x} & \text{for} & 0 > x. \end{matrix}$$

1.2 As a real basis of the Lie algebra $\mathfrak{so}_3(\mathbb{R}) \cong \mathfrak{su}_2(\mathbb{C})$ we can choose the basis elements $\{\mathrm{i}e_1, \mathrm{i}e_2, e_3\}$, where $\mathrm{i}^2 = -1$. Every element of $\mathfrak{so}_3(\mathbb{R})$ is conjugate to e_3.
If $X \in \mathfrak{so}_3(\mathbb{R})$ has the decomposition

$$X = \lambda_1 \mathrm{i}e_1 + \lambda_2 \mathrm{i}e_2 + \lambda_3 e_3$$

then the normalized real Cartan-Killing form $k : \mathfrak{so}_3(\mathbb{R}) \times \mathfrak{so}_3(\mathbb{R}) \to \mathbb{R}$; $k(X,Y) = \frac{1}{8}\mathrm{trace}(\mathrm{ad}X \ \mathrm{ad}Y)$ satisfies

$$k(X) = k(X,X) = -\lambda_1^2 - \lambda_2^2 - \lambda_3^2.$$

For the exponential function $\exp : \mathfrak{su}_2(\mathbb{C}) \to SU_2(\mathbb{C})$ one has

$$\exp \ X = C(k(X)) \ I + S(k(X)) \ X,$$

where $C(x) = \cosh(\sqrt{-x}\mathrm{i})$ and $S(x) = \dfrac{\sinh(\sqrt{-x}\mathrm{i})}{\sqrt{-x}\mathrm{i}}$.

Proposition 1. *There is no connected almost differentiable left A-loop L such that the group G topologically generated by its left translations is a compact quasi-simple Lie group G with dim $G \leq 9$.*

Proof. If G is a quasi-simple Lie group then it admits a continuous section if and only if G is locally isomorphic to $SO_8(\mathbb{R})$ (cf. [20], pp. 149-150). □

An important tool to exclude certain stabilizers H is the fundamental group π_1 of a connected topological space. This shows the following lemma which is proved in [5], p. 6.

Lemma 2. *Denote by G a connected Lie group and by H a connected subgroup of G. Let $\sigma : G/H \to G$ be a global section. Then $\pi_1(K) \cong \pi_1(\sigma(G/H)) \times \pi_1(K_1)$, where K respectively K_1 is a maximal compact subgroup of G respectively H.*

¿From [6] we use Lemma 2, which reads as follows.

Lemma 3. *Let L be an almost differentiable loop and denote by* **m** *the tangent space $T_1\sigma(G/H)$, where $\sigma : G/H \to G$ is the section corresponding to L. Then* **m** *does not contain any element of $Ad_g\mathbf{h}$ for some $g \in G$. Moreover, every element of G can be uniquely written as a product of an element of $\sigma(G/H)$ with an element of H.*

3 Affine reductive spaces of small dimension

In this section we determine all affine reductive homogeneous spaces $(\mathbf{g}, \mathbf{h}, \mathbf{m})$, where $\mathbf{g}$ is a simple non-compact Lie algebra of dimension at most 9 and $\mathbf{h}$ is a subalgebra of $\mathbf{g}$ such that dim $\mathbf{g}$ − dim $\mathbf{h} > 3$.

First we deal with the Lie algebra $\mathbf{g} = \mathfrak{sl}_2(\mathbb{C})$. A real basis of $\mathbf{g}$ is given by $\{e_1, e_2, e_3, \mathrm{i}e_1, \mathrm{i}e_2, \mathrm{i}e_3\}$, where $\{e_1, e_2, e_3\}$ is the basis of $\mathfrak{sl}_2(\mathbb{R})$ described by $(*)$.
Using the classification of Lie (see Theorem 15 in [15], p. 129) we obtain that every 2-dimensional Lie algebra $\mathbf{h}$ of $\mathbf{g}$ has (up to conjugation) one of the following shapes:

$$\mathbf{h}_1 = \langle e_1, e_2 + e_3 \rangle, \quad \mathbf{h}_2 = \langle \mathrm{i}(e_2 + e_3), e_2 + e_3 \rangle, \quad \mathbf{h}_3 = \langle e_3, \mathrm{i}e_3 \rangle,$$

and every 1-dimensional Lie algebra $\mathbf{h}$ of $\mathbf{g}$ is one of the following:

$$\mathbf{h}_4 = \langle e_1 \rangle, \quad \mathbf{h}_5 = \langle e_2 + e_3 \rangle, \quad \mathbf{h}_6 = \langle e_3 \rangle.$$

Proposition 4. *The Lie algebra $\mathbf{g} = \mathfrak{sl}_2(\mathbb{C})$ is reductive with respect to the following pairs $(\mathbf{h}, \mathbf{m})$, where $\mathbf{h}$ is an at most 2-dimensional subalgebra of $\mathbf{g}$ and $\mathbf{m}$ is a complementary subspace to $\mathbf{h}$ generating $\mathbf{g}$*

1) $\mathbf{h}_3 = \langle e_3, ie_3 \rangle$, $\mathbf{m} = \langle e_1, e_2, ie_1, ie_2 \rangle$,

2) $\mathbf{h}_4 = \langle e_1 \rangle$, $\mathbf{m}_a = \langle e_2, e_3, ie_1 + ae_1, ie_2, ie_3 \rangle$, *where* $a \in \mathbb{R}$,

3) $\mathbf{h}_6 = \langle e_3 \rangle$, $\mathbf{m}_b = \langle e_1, e_2, ie_1, ie_2, ie_3 + be_3 \rangle$, *where* $b \in \mathbb{R}$.

Proof. The basis elements of an arbitrary complement $\mathbf{m}_1$ to $\mathbf{h}_1$ in $\mathbf{g}$ are

$$X_1 = e_2 + a_1 e_1 + b_1(e_2 + e_3), \quad X_2 = \mathrm{i}e_1 + a_2 e_1 + b_2(e_2 + e_3),$$
$$X_3 = \mathrm{i}e_2 + a_3 e_1 + b_3(e_2 + e_3), \quad X_4 = \mathrm{i}e_3 + a_4 e_1 + b_4(e_2 + e_3),$$

where a_j, b_j, $j = 1, 2, 3, 4$ are real parameters.
An arbitrary complement $\mathbf{m}_2$ to $\mathbf{h}_2$ in $\mathbf{g}$ has as generators

$$Y_1 = e_1 + a_1(e_2 + e_3) + b_1\mathrm{i}(e_2 + e_3), \quad Y_2 = e_2 + a_2(e_2 + e_3) + b_2\mathrm{i}(e_2 + e_3),$$
$$Y_3 = \mathrm{i}e_1 + a_3(e_2 + e_3) + b_3\mathrm{i}(e_2 + e_3), \quad Y_4 = \mathrm{i}e_2 + a_4(e_2 + e_3) + b_4\mathrm{i}(e_2 + e_3),$$

where $a_j, b_j \in \mathbb{R}$, $j = 1, 2, 3, 4$.
We can choose as basis elements of an arbitrary complement $\mathbf{m}_3$ to $\mathbf{h}_3$ the following:

$$Z_1 = e_1 + a_1 e_3 + b_1 \mathrm{i}e_3, \quad Z_2 = e_2 + a_2 e_3 + b_2 \mathrm{i}e_3,$$
$$Z_3 = \mathrm{i}e_1 + a_3 e_3 + b_3 \mathrm{i}e_3, \quad Z_4 = \mathrm{i}e_2 + a_4 e_3 + b_4 \mathrm{i}e_3,$$

where $a_j, b_j \in \mathbb{R}$, $j = 1, 2, 3, 4$ are real numbers.
An arbitrary complement $\mathbf{m}_4$ to $\mathbf{h}_4$ in $\mathbf{g}$ has as basis elements

$$W_1 = e_2 + a_1 e_1, \quad W_2 = e_3 + a_2 e_1, \quad W_3 = \mathrm{i}e_1 + a_3 e_1,$$
$$W_4 = \mathrm{i}e_2 + a_4 e_1, \quad W_5 = \mathrm{i}e_3 + a_5 e_1$$

with the real parameters a_j, $j = 1, 2, 3, 4, 5$.
The generators of an arbitrary complement $\mathbf{m}_5$ to $\mathbf{h}_5$ in $\mathbf{g}$ are

$$V_1 = e_1 + a_1(e_2 + e_3), \quad V_2 = e_2 + a_2(e_2 + e_3), \quad V_3 = \mathrm{i}e_1 + a_3(e_2 + e_3),$$

$$V_4 = \mathrm{i}e_2 + a_4(e_2 + e_3), \quad V_5 = \mathrm{i}e_3 + a_5(e_2 + e_3),$$

where a_j, $j = 1, 2, 3, 4, 5$ are real parameters.
An arbitrary complement $\mathbf{m}_6$ to $\mathbf{h}_6$ in $\mathbf{g}$ has as generators

$$U_1 = e_1 + a_1 e_3, \quad U_2 = e_2 + a_2 e_3, \quad U_3 = \mathrm{i}e_1 + a_3 e_3,$$

$$U_4 = \mathrm{i}e_2 + a_4 e_3, \quad U_5 = \mathrm{i}e_3 + a_5 e_3$$

with $a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}$.
Using the relation $[\mathbf{h}_i, \mathbf{m}_i] \subseteq \mathbf{m}_i$, $i = 1, \cdots, 6$, we obtain the contradictions that $[e_2 + e_3, X_1] = 2e_1 - 2a_1(e_2 + e_3) \in \mathbf{h}_1$ and $[e_2 + e_3, Y_1] = [e_2 + e_3, V_1] = -2(e_2 + e_3) \in \mathbf{h}_2 \cap \mathbf{h}_5$ and the assertion follows. □

Now we consider the Lie algebra $\mathbf{g} = \mathfrak{sl}_3(\mathbb{R})$. It is isomorphic to the Lie algebra of matrices

$$(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + \lambda_5 e_5 + \lambda_6 e_6 + \lambda_7 e_7 + \lambda_8 e_8) \mapsto$$

$$\begin{pmatrix} -\lambda_5 - \lambda_8 & \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_5 & \lambda_6 \\ \lambda_4 & \lambda_7 & \lambda_8 \end{pmatrix}; \lambda_i \in \mathbb{R}, i = 1, \cdots, 8.$$

In this representation the Lie multiplication of $\mathbf{g}$ is given by

$$[e_1, e_2] = [e_1, e_7] = [e_2, e_6] = [e_3, e_4] = [e_3, e_6] = [e_4, e_7] = [e_5, e_8] = 0,$$

$$[e_1, e_6] = [e_2, e_5] = \frac{1}{2}[e_2, e_8] = e_2, \; [e_1, e_8] = [e_2, e_7] = \frac{1}{2}[e_1, e_5] = e_1,$$

$$[e_4, e_6] = [e_3, e_8] = \frac{1}{2}[e_3, e_5] = -e_3, \; [e_3, e_7] = [e_4, e_5] = \frac{1}{2}[e_4, e_8] = -e_4,$$

$$[e_6, e_8] = [e_5, e_6] = [e_3, e_2] = e_6, \; [e_1, e_4] = [e_5, e_7] = [e_7, e_8] = -e_7,$$

$$[e_1, e_3] = -e_5, \; [e_2, e_4] = -e_8, \; [e_6, e_7] = e_5 - e_8.$$

Now using the classification of Lie, who has determined all subalgebras of $\mathfrak{sl}_3(\mathbb{R})$ (cf. [15], pp. 288-289 and [14], p. 384) we obtain that every 4-dimensional Lie algebra $\mathbf{h}$ of $\mathbf{g}$ has (up to conjugation) one of the following forms:

$\mathbf{h}_1 = \langle e_1, e_2, e_6, e_5 + ce_8 \rangle$, $\mathbf{h}_2 = \langle e_3, e_5, e_6, e_8 \rangle$, $\mathbf{h}_3 = \langle e_1, e_2, e_6, e_8 \rangle$,
$\mathbf{h}_4 = \langle e_2, e_5, e_6, e_8 \rangle$, $\mathbf{h}_5 \cong \mathfrak{gl}_2(\mathbb{R}) = \langle e_5, e_6, e_7, e_8 \rangle$, where $c \in \mathbb{R}$.

The 3-dimensional subalgebras $\mathbf{h}$ of $\mathbf{g}$ (up to conjugation) are the following:

$\mathbf{h}_6 \cong \mathfrak{so}_3(\mathbb{R}) = \langle e_1 - e_3, e_2 - e_4, e_7 - e_6 \rangle$, $\mathbf{h}_7 \cong \mathfrak{sl}_2(\mathbb{R}) = \langle e_1 + e_3, e_2 + e_4, e_6 - e_7 \rangle$,
$\mathbf{h}_8 \cong \mathfrak{sl}_2(\mathbb{R}) = \langle e_5 - e_8, e_6, e_7 \rangle$, $\mathbf{h}_9 = \langle a(e_5 + e_8) + e_6 - e_7, e_1, e_2 \rangle$, $a \geq 0$,
$\mathbf{h}_{10} = \langle e_5 - e_8, e_2 + e_3, e_6 \rangle$, $\mathbf{h}_{11} = \langle e_3, e_6, e_8 + e_2 \rangle$, $\mathbf{h}_{12} = \langle e_2, e_6, e_5 + e_8 - e_3 \rangle$,
$\mathbf{h}_{13} = \langle e_1, e_2, e_6 \rangle$, $\mathbf{h}_{14} = \langle e_5, e_8, e_6 \rangle$, $\mathbf{h}_{15} = \langle e_2, e_5 + e_8, e_6 \rangle$, $\mathbf{h}_{16} = \langle e_3, e_6, e_8 \rangle$,
$\mathbf{h}_{17} = \langle e_2, e_6, (b-1)e_5 + be_8, \rangle$, $b \in \mathbb{R}$, $\mathbf{h}_{18} = \langle e_3, e_6, e_5 + ce_8 \rangle$, $c \in \mathbb{R}$.

The 2-dimensional subalgebras $\mathbf{h}$ of $\mathbf{g}$ are given (up to conjugation) by

$\mathbf{h}_{19} = \langle e_6, e_2 + e_3 \rangle$, $\mathbf{h}_{20} = \langle e_6, e_2 + e_8 \rangle$, $\mathbf{h}_{21} = \langle e_3, e_6 + e_5 \rangle$,
$\mathbf{h}_{22} = \langle e_3, e_5 + ae_8 \rangle$, $a \in \mathbb{R} \backslash \{0, 1\}$, $\mathbf{h}_{23} = \langle e_5, e_6 \rangle$, $\mathbf{h}_{24} = \langle e_2, e_6 \rangle$,
$\mathbf{h}_{25} = \langle e_6, e_3 \rangle$, $\mathbf{h}_{26} = \langle e_5, e_8 \rangle$, $\mathbf{h}_{27} = \langle e_6, e_5 + e_8 \rangle$, $\mathbf{h}_{28} = \langle e_6, e_8 \rangle$,
$\mathbf{h}_{29} = \langle e_5 - e_8, e_2 + e_3 \rangle$, $\mathbf{h}_{30} = \langle e_5 + e_8, e_6 - e_7 \rangle$.

Moreover, every 1-dimensional subalgebra $\mathbf{h}$ of $\mathbf{g}$ has one of the following shapes:

$\mathbf{h}_{31} = \langle e_5 + ae_8 \rangle$, $a \in \mathbb{R} \backslash \{0\}$, $\mathbf{h}_{32} = \langle e_2 + e_8 \rangle$, $\mathbf{h}_{33} = \langle e_2 + e_3 \rangle$,
$\mathbf{h}_{34} = \langle e_6 \rangle$, $\mathbf{h}_{35} = \langle e_6 - e_7 + b(e_5 + e_8) \rangle$, $b \geq 0$.

Proposition 5. *The Lie algebra* $\mathbf{g} = \mathfrak{sl}_3(\mathbb{R})$ *is reductive with respect to a* 4-*dimensional subalgebra* $\mathbf{h}$ *of* $\mathbf{g}$ *and a complementary subspace* $\mathbf{m}$ *generating* $\mathbf{g}$ *only in the case* $\mathbf{h}_5 \cong \mathfrak{gl}_2(\mathbb{R})$ *and* $\mathbf{m}_5 = \langle e_1, e_2, e_3, e_4 \rangle$.

Proof. The basis elements of an arbitrary complement $\mathbf{m}_i$ to the subalgebra $\mathbf{h}_i$ are:
For $i = 1$

$$e_3 + a_1e_1 + a_2e_2 + a_3(e_5 + ce_8) + a_4e_6, \;\; e_4 + b_1e_1 + b_2e_2 + b_3(e_5 + ce_8) + b_4e_6,$$

$$e_7 + c_1e_1 + c_2e_2 + c_3(e_5 + ce_8) + c_4e_6, \;\; e_8 + d_1e_1 + d_2e_2 + d_3(e_5 + ce_8) + d_4e_6,$$

for $i = 2$

$$e_1 + a_1e_3 + a_2e_5 + a_3e_6 + a_4e_8, \;\; e_2 + b_1e_3 + b_2e_5 + b_3e_6 + b_4e_8,$$

$$e_4 + c_1e_3 + c_2e_5 + c_3e_6 + c_4e_8, \;\; e_7 + d_1e_3 + d_2e_5 + d_3e_6 + d_4e_8,$$

for $i = 3$

$$e_3 + a_1e_1 + a_2e_2 + a_3e_6 + a_4e_8, \;\; e_4 + b_1e_1 + b_2e_2 + b_3e_6 + b_4e_8,$$

$$e_5 + c_1e_1 + c_2e_2 + c_3e_6 + c_4e_8, \;\; e_7 + d_1e_1 + d_2e_2 + d_3e_6 + d_4e_8,$$

for $i = 4$

$$e_1 + a_1e_2 + a_2e_5 + a_3e_6 + a_4e_8, \;\; e_3 + b_1e_2 + b_2e_5 + b_3e_6 + b_4e_8,$$

$$e_4 + c_1e_2 + c_2e_5 + c_3e_6 + c_4e_8, \;\; e_7 + d_1e_2 + d_2e_5 + d_3e_6 + d_4e_8,$$

for $i = 5$

$$e_1 + a_1e_5 + a_2e_6 + a_3e_7 + a_4e_8, \;\; e_2 + b_1e_5 + b_2e_6 + b_3e_7 + b_4e_8,$$

$$e_3 + c_1e_5 + c_2e_6 + c_3e_7 + c_4e_8, \;\; e_4 + d_1e_5 + d_2e_6 + d_3e_7 + d_4e_8,$$

where a_j, b_j, c_j, d_j are real numbers $j = 1, 2, 3, 4$. The assertion follows now from the relation $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$. □

Proposition 6. *The Lie algebra* $\mathbf{g} = \mathfrak{sl}_3(\mathbb{R})$ *is reductive with a* 3*-dimensional subalgebra* $\mathbf{h}$ *and a* 5*-dimensional complementary subspace* $\mathbf{m}$ *generating* $\mathbf{g}$ *in precisely one of the following cases:*

1) $\mathbf{h}_6 \cong \mathfrak{so}_3(\mathbb{R})$, $\mathbf{m}_6 = \langle e_5, e_8, e_1 + e_3, e_2 + e_4, e_7 + e_6 \rangle$,

2) $\mathbf{h}_7 = \langle e_1 + e_3, e_2 + e_4, e_6 - e_7 \rangle$, $\mathbf{m}_7 = \langle e_5, e_8, e_1 - e_3, e_2 - e_4, e_7 + e_6 \rangle$,

3) $\mathbf{h}_8 = \langle e_5 - e_8, e_6, e_7 \rangle$, $\mathbf{m}_8 = \langle e_1, e_2, e_3, e_4, e_5 + e_8 \rangle$.

Both Lie algebras $\mathbf{h}_7$ *and* $\mathbf{h}_8$ *are isomorphic to* $\mathfrak{sl}_2(\mathbb{R})$.

Proof. The generators of an arbitrary complement $\mathbf{m}_i$ to $\mathbf{h}_i$ in $\mathbf{g}$ are:
For $i = 6$

$$\begin{aligned} &e_3 + a_1(e_1 - e_3) + a_2(e_2 - e_4) + a_3(e_7 - e_6), \\ &e_4 + b_1(e_1 - e_3) + b_2(e_2 - e_4) + b_3(e_7 - e_6), \\ &e_5 + c_1(e_1 - e_3) + c_2(e_2 - e_4) + c_3(e_7 - e_6), \\ &e_6 + d_1(e_1 - e_3) + d_2(e_2 - e_4) + d_3(e_7 - e_6), \\ &e_8 + f_1(e_1 - e_3) + f_2(e_2 - e_4) + f_3(e_7 - e_6), \end{aligned}$$

for $i = 7$

$$\begin{aligned} &e_3 + a_1(e_1 + e_3) + a_2(e_2 + e_4) + a_3(e_6 - e_7), \\ &e_4 + b_1(e_1 + e_3) + b_2(e_2 + e_4) + b_3(e_6 - e_7), \\ &e_5 + c_1(e_1 + e_3) + c_2(e_2 + e_4) + c_3(e_6 - e_7), \\ &e_6 + d_1(e_1 + e_3) + d_2(e_2 + e_4) + d_3(e_6 - e_7), \\ &e_8 + f_1(e_1 + e_3) + f_2(e_2 + e_4) + f_3(e_6 - e_7), \end{aligned}$$

for $i = 8$

$$e_1 + a_1(e_5 - e_8) + a_2 e_6 + a_3 e_7, \;\; e_2 + b_1(e_5 - e_8) + b_2 e_6 + b_3 e_7,$$
$$e_3 + c_1(e_5 - e_8) + c_2 e_6 + c_3 e_7, \;\; e_4 + d_1(e_5 - e_8) + d_2 e_6 + d_3 e_7,$$
$$e_5 + f_1(e_5 - e_8) + f_2 e_6 + f_3 e_7,$$

for $i = 9$

$$\begin{aligned} &e_3 + a_1 e_1 + a_2 e_2 + a_3(e_6 - e_7 + a(e_5 + e_8)), \\ &e_4 + b_1 e_1 + b_2 e_2 + b_3(e_6 - e_7 + a(e_5 + e_8)), \\ &e_5 + c_1 e_1 + c_2 e_2 + c_3(e_6 - e_7 + a(e_5 + e_8)), \\ &e_6 + d_1 e_1 + d_2 e_2 + d_3(e_6 - e_7 + a(e_5 + e_8)), \\ &e_8 + f_1 e_1 + f_2 e_2 + f_3(e_6 - e_7 + a(e_5 + e_8)), \end{aligned}$$

for $i = 10$

$$e_1 + a_1(e_2 + e_3) + a_2(e_5 - e_8) + a_3 e_6, \;\; e_2 + b_1(e_2 + e_3) + b_2(e_5 - e_8) + b_3 e_6,$$

$$e_4 + c_1(e_2 + e_3) + c_2(e_5 - e_8) + c_3 e_6, \;\; e_5 + d_1(e_2 + e_3) + d_2(e_5 - e_8) + d_3 e_6,$$
$$e_7 + f_1(e_2 + e_3) + f_2(e_5 - e_8) + f_3 e_6,$$

for $i = 11$

$$e_1 + a_1(e_2 + e_8) + a_2 e_3 + a_3 e_6, \;\; e_2 + b_1(e_2 + e_8) + b_2 e_3 + b_3 e_6,$$
$$e_4 + c_1(e_2 + e_8) + c_2 e_3 + c_3 e_6, \;\; e_5 + d_1(e_2 + e_8) + d_2 e_3 + d_3 e_6,$$
$$e_7 + f_1(e_2 + e_8) + f_2 e_3 + f_3 e_6,$$

for $i = 12$

$$e_1 + a_1 e_2 + a_2 e_6 + a_3(e_5 + e_8 - e_3), \;\; e_3 + b_1 e_2 + b_2 e_6 + b_3(e_5 + e_8 - e_3),$$
$$e_4 + c_1 e_2 + c_2 e_6 + c_3(e_5 + e_8 - e_3), \;\; e_7 + d_1 e_2 + d_2 e_6 + d_3(e_5 + e_8 - e_3),$$
$$e_8 + f_1 e_2 + f_2 e_6 + f_3(e_5 + e_8 - e_3),$$

for $i = 13$

$$e_3 + a_1 e_1 + a_2 e_2 + a_3 e_6, \;\; e_4 + b_1 e_1 + b_2 e_2 + b_3 e_6, \;\; e_5 + c_1 e_1 + c_2 e_2 + c_3 e_6,$$
$$e_7 + d_1 e_1 + d_2 e_2 + d_3 e_6, \;\; e_8 + f_1 e_1 + f_2 e_2 + f_3 e_6,$$

for $i = 14$

$$e_1 + a_1 e_5 + a_2 e_6 + a_3 e_8, \;\; e_2 + b_1 e_5 + b_2 e_6 + b_3 e_8, \;\; e_3 + c_1 e_5 + c_2 e_6 + c_3 e_8,$$
$$e_4 + d_1 e_5 + d_2 e_6 + d_3 e_8, \;\; e_7 + f_1 e_5 + f_2 e_6 + f_3 e_8,$$

for $i = 15$

$$e_1 + a_1 e_2 + a_2(e_5 + e_8) + a_3 e_6, \;\; e_3 + b_1 e_2 + b_2(e_5 + e_8) + b_3 e_6,$$
$$e_4 + c_1 e_2 + c_2(e_5 + e_8) + c_3 e_6, \;\; e_5 + d_1 e_2 + d_2(e_5 + e_8) + d_3 e_6,$$
$$e_7 + f_1 e_2 + f_2(e_5 + e_8) + f_3 e_6,$$

for $i = 16$

$$e_1 + a_1 e_3 + a_2 e_6 + a_3 e_8, \;\; e_2 + b_1 e_3 + b_2 e_6 + b_3 e_8, \;\; e_4 + c_1 e_3 + c_2 e_6 + c_3 e_8,$$
$$e_5 + d_1 e_3 + d_2 e_6 + d_3 e_8, \;\; e_7 + f_1 e_3 + f_2 e_6 + f_3 e_8,$$

for $i = 17$ and $b \neq 0$

$$e_1 + a_1 e_2 + a_2 e_6 + a_3((b-1)e_5 + b e_8), \;\; e_3 + b_1 e_2 + b_2 e_6 + b_3((b-1)e_5 + b e_8),$$
$$e_4 + c_1 e_2 + c_2 e_6 + c_3((b-1)e_5 + b e_8), \;\; e_5 + d_1 e_2 + d_2 e_6 + d_3((b-1)e_5 + b e_8),$$
$$e_7 + f_1 e_2 + f_2 e_6 + f_3((b-1)e_5 + b e_8),$$

for $i = 17$ and $b = 0$

$$e_1 + a_1 e_2 + a_2 e_6 - a_3 e_5, \;\; e_3 + b_1 e_2 + b_2 e_6 - b_3 e_5, \;\; e_4 + c_1 e_2 + c_2 e_6 - c_3 e_5,$$
$$e_7 + d_1 e_2 + d_2 e_6 - d_3 e_5, \;\; e_8 + f_1 e_2 + f_2 e_6 - f_3 e_5,$$

for $i = 18$

$$e_1 + a_1 e_3 + a_2(e_5 + c e_8) + a_3 e_6,\;\; e_2 + b_1 e_3 + b_2(e_5 + c e_8) + b_3 e_6,$$
$$e_4 + c_1 e_3 + c_2(e_5 + c e_8) + c_3 e_6,\;\; e_7 + d_1 e_3 + d_2(e_5 + c e_8) + d_3 e_6,$$
$$e_8 + f_1 e_3 + f_2(e_5 + c e_8) + f_3 e_6,$$

where $a_j, b_j, c_j, d_j, f_j \in \mathbb{R}$, $j = 1, 2, 3$. Using the relation $[\mathbf{h_i}, \mathbf{m_i}] \subseteq \mathbf{m_i}$, $i = 6, \cdots, 18$, we obtain the assertion. □

Proposition 7. *The Lie algebra $\mathbf{g} = \mathfrak{sl}_3(\mathbb{R})$ is reductive with respect to a pair $(\mathbf{h}, \mathbf{m})$, where $\mathbf{h}$ is a 2-dimensional subalgebra of $\mathbf{g}$ and $\mathbf{m}$ is a complementary subspace to $\mathbf{h}$ generating $\mathbf{g}$ in exactly one of the following cases:*

1) $\mathbf{h}_{26} = \langle e_5, e_8 \rangle$ *and* $\mathbf{m}_{26} = \langle e_1, e_2, e_3, e_4, e_6, e_7 \rangle$.

2) $\mathbf{h}_{30} = \langle e_5 + e_8, e_6 - e_7 \rangle$ *and* $\mathbf{m}_{30} = \langle e_1, e_2, e_3, e_4, e_5 - e_8, e_6 + e_7 \rangle$.

Proof. An arbitrary complement $\mathbf{m}_i$ to the subalgebra $\mathbf{h}_i$, $i = 19, \cdots, 30$, in $\mathbf{g}$ has as generators in the case $i = 19$

$$e_1 + b_1 e_6 + c_1(e_2 + e_3),\;\; e_2 + b_2 e_6 + c_2(e_2 + e_3),\;\; e_4 + b_3 e_6 + c_3(e_2 + e_3)$$
$$e_5 + b_4 e_6 + c_4(e_2 + e_3),\;\; e_7 + b_5 e_6 + c_5(e_2 + e_3),\;\; e_8 + b_6 e_6 + c_6(e_2 + e_3),$$

in the case $i = 20$

$$e_1 + b_1 e_6 + c_1(e_2 + e_8),\;\; e_2 + b_2 e_6 + c_2(e_2 + e_8),\;\; e_3 + b_3 e_6 + c_3(e_2 + e_8),$$
$$e_4 + b_4 e_6 + c_4(e_2 + e_8),\;\; e_5 + b_5 e_6 + c_5(e_2 + e_8),\;\; e_7 + b_6 e_6 + c_6(e_2 + e_8),$$

in the case $i = 21$

$$e_1 + b_1 e_3 + c_1(e_6 + e_5),\;\; e_2 + b_2 e_3 + c_2(e_6 + e_5),\;\; e_4 + b_3 e_3 + c_3(e_6 + e_5),$$
$$e_5 + b_4 e_3 + c_4(e_6 + e_5),\;\; e_7 + b_5 e_3 + c_5(e_6 + e_5),\;\; e_8 + b_6 e_3 + c_6(e_6 + e_5),$$

in the case $i = 22$

$$e_1 + b_1 e_3 + c_1(e_5 + a e_8),\;\; e_2 + b_2 e_3 + c_2(e_5 + a e_8),\;\; e_4 + b_3 e_3 + c_3(e_5 + a e_8),$$
$$e_6 + b_4 e_3 + c_4(e_5 + a e_8),\;\; e_7 + b_5 e_3 + c_5(e_5 + a e_8),\;\; e_8 + b_6 e_3 + c_6(e_5 + a e_8),$$

in the case $i = 23$

$$e_1 + b_1 e_5 + c_1 e_6,\;\; e_2 + b_2 e_5 + c_2 e_6,\;\; e_3 + b_3 e_5 + c_3 e_6,$$
$$e_4 + b_4 e_5 + c_4 e_6,\;\; e_7 + b_5 e_5 + c_5 e_6,\;\; e_8 + b_6 e_5 + c_6 e_6,$$

in the case $i = 24$

$$e_1 + b_1 e_2 + c_1 e_6,\;\; e_3 + b_2 e_2 + c_2 e_6,\;\; e_4 + b_3 e_2 + c_3 e_6,$$
$$e_5 + b_4 e_2 + c_4 e_6,\;\; e_7 + b_5 e_2 + c_5 e_6,\;\; e_8 + b_6 e_2 + c_6 e_6,$$

in the case $i = 25$

$$e_1 + b_1 e_3 + c_1 e_6,\;\; e_2 + b_2 e_3 + c_2 e_6,\;\; e_4 + b_3 e_3 + c_3 e_6,$$
$$e_5 + b_4 e_3 + c_4 e_6,\;\; e_7 + b_5 e_3 + c_5 e_6,\;\; e_8 + b_6 e_3 + c_6 e_6,$$

in the case $i = 26$

$$e_1 + b_1 e_5 + c_1 e_8, \;\; e_2 + b_2 e_5 + c_2 e_8, \;\; e_3 + b_3 e_5 + c_3 e_8,$$
$$e_4 + b_4 e_5 + c_4 e_8, \;\; e_6 + b_5 e_5 + c_5 e_8, \;\; e_7 + b_6 e_5 + c_6 e_8,$$

in the case $i = 27$

$$e_1 + b_1 e_6 + c_1(e_5 + e_8), \;\; e_2 + b_2 e_6 + c_2(e_5 + e_8) \;\; e_3 + b_3 e_6 + c_3(e_5 + e_8),$$
$$e_4 + b_4 e_6 + c_4(e_5 + e_8), \;\; e_5 + b_5 e_6 + c_5(e_5 + e_8), \;\; e_7 + b_6 e_6 + c_6(e_5 + e_8),$$

in the case $i = 28$

$$e_1 + b_1 e_6 + c_1 e_8, \;\; e_2 + b_2 e_6 + c_2 e_8 \;\; e_3 + b_3 e_6 + c_3 e_8,$$
$$e_4 + b_4 e_6 + c_4 e_8, \;\; e_5 + b_5 e_6 + c_5 e_8, \;\; e_7 + b_6 e_6 + c_6 e_8,$$

in the case $i = 29$

$$e_1 + b_1(e_2 + e_3) + c_1(e_5 - e_8), \;\; e_2 + b_2(e_2 + e_3) + c_2(e_5 - e_8),$$
$$e_4 + b_3(e_2 + e_3) + c_3(e_5 - e_8), \;\; e_5 + b_4(e_2 + e_3) + c_4(e_5 - e_8),$$
$$e_6 + b_5(e_2 + e_3) + c_5(e_5 - e_8), \;\; e_7 + b_6(e_2 + e_3) + c_6(e_5 - e_8),$$

in the case $i = 30$

$$e_1 + b_1(e_5 + e_8) + c_1(e_6 - e_7), \;\; e_2 + b_2(e_5 + e_8) + c_2(e_6 - e_7),$$
$$e_3 + b_3(e_5 + e_8) + c_3(e_6 - e_7), \;\; e_4 + b_4(e_5 + e_8) + c_4(e_6 - e_7),$$
$$e_5 + b_5(e_5 + e_8) + c_5(e_6 - e_7), \;\; e_6 + b_6(e_5 + e_8) + c_6(e_6 - e_7),$$

where $b_j, c_j \in \mathbb{R}$, $j = 1, \cdots, 6$. The relation $[\mathbf{h_i}, \mathbf{m_i}] \subseteq \mathbf{m_i}$, $i = 19, \cdots, 30$, yields the assertion. □

Proposition 8. *The Lie algebra* $\mathbf{g} = \mathfrak{sl}_3(\mathbb{R})$ *is reductive with a* 1*-dimensional subalgebra* $\mathbf{h}$ *and a* 7*-dimensional complementary subspace* $\mathbf{m}$ *generating* $\mathbf{g}$ *in precisely one of the following cases:*

1) $\mathbf{h}_{31,1} = \langle e_5 + a e_8 \rangle$, $a \in \mathbb{R} \backslash \{0, 1, -\frac{1}{2}, -2\}$ *and*
$\mathbf{m}_b = \langle e_1, e_2, e_3, e_4, e_6, e_7, e_8 + b(e_5 + a e_8) \rangle$, $b \in \mathbb{R}$,

2) $\mathbf{h}_{31,2} = \langle e_5 - 2e_8 \rangle$ *and*
$\mathbf{m}_{b,c,d} = \langle e_6, e_7, e_1 + b(e_5 - 2e_8), e_3 + c(e_5 - 2e_8), e_2, e_4, e_8 + d(e_5 - 2e_8) \rangle$, $b, c, d \in \mathbb{R}$,

3) $\mathbf{h}_{31,3} = \langle e_5 - \frac{1}{2} e_8 \rangle$ *and*
$\mathbf{m}_{b,c,d} = \langle e_6, e_7, e_1, e_2 + b(e_5 - \frac{1}{2} e_8), e_3, e_4 + c(e_5 - \frac{1}{2} e_8), e_8 + d(e_5 - \frac{1}{2} e_8) \rangle$, $b, c, d \in \mathbb{R}$,

4) $\mathbf{h}_{31,4} = \langle e_5 + e_8 \rangle$ *and*
$\mathbf{m}_{b,c,d} = \langle e_1, e_2, e_3, e_4, e_6 + b(e_5 + e_8), e_7 + c(e_5 + e_8), e_8 + d(e_5 + e_8) \rangle$, $b, c, d \in \mathbb{R}$,

5) $\mathbf{h}_{32} = \langle e_2 + e_8 \rangle$ *and* $\mathbf{m}_d = \langle e_1, e_2, e_3, -e_8 + 2e_4, e_6, e_7, e_5 + d e_8 \rangle$, $d \in \mathbb{R}$,

6) $\mathbf{h}_{35} = \langle e_6 - e_7 + b(e_5 + e_8) \rangle$, $b \geq 0$ *and*
$\mathbf{m}_c = \langle e_1, e_2, e_3, e_4, e_6 + e_7, e_5 - e_8, e_8 - 2c e_7 + 2cb e_8 \rangle$, $c \in \mathbb{R}$.

Proof. An arbitrary complement $\mathbf{m}_i$ to the subalgebra $\mathbf{h}_i$, $i = 31, \cdots, 35$, in $\mathbf{g}$ has as generators in the case $i = 31$

$$e_1 + a_1(e_5 + ae_8),\ \ e_2 + a_2(e_5 + ae_8),\ \ e_3 + a_3(e_5 + ae_8),\ \ e_4 + a_4(e_5 + ae_8),$$
$$e_6 + a_5(e_5 + ae_8),\ \ e_7 + a_6(e_5 + ae_8),\ \ e_8 + a_7(e_5 + ae_8),$$

in the case $i = 32$

$$e_1 + a_1(e_2 + e_8),\ \ e_3 + a_2(e_2 + e_8),\ \ e_4 + a_3(e_2 + e_8),\ \ e_5 + a_4(e_2 + e_8),$$
$$e_6 + a_5(e_2 + e_8),\ \ e_7 + a_6(e_2 + e_8),\ \ e_8 + a_7(e_2 + e_8),$$

in the case $i = 33$

$$e_1 + a_1(e_2 + e_3),\ \ e_3 + a_2(e_2 + e_3),\ \ e_4 + a_3(e_2 + e_3),\ \ e_5 + a_4(e_2 + e_3),$$
$$e_6 + a_5(e_2 + e_3),\ \ e_7 + a_6(e_2 + e_3),\ \ e_8 + a_7(e_2 + e_3),$$

in the case $i = 34$

$$e_1 + a_1e_6,\ \ e_2 + a_2e_6,\ \ e_3 + a_3e_6,\ \ e_4 + a_4e_6,\ \ e_5 + a_5e_6,$$
$$e_7 + a_6e_6,\ \ e_8 + a_7e_6,$$

in the case $i = 35$

$$e_1 + a_1(e_6 - e_7 + b(e_5 + e_8)),\ \ e_2 + a_2(e_6 - e_7 + b(e_5 + e_8)),$$
$$e_3 + a_3(e_6 - e_7 + b(e_5 + e_8)),\ \ e_4 + a_4(e_6 - e_7 + b(e_5 + e_8)),$$
$$e_5 + a_5(e_6 - e_7 + b(e_5 + e_8)),\ \ e_7 + a_6(e_6 - e_7 + b(e_5 + e_8)),$$
$$e_8 + a_7(e_6 - e_7 + b(e_5 + e_8)),$$

where $a_j \in \mathbb{R}$, $j = 1, \cdots 7$. Using the relation $[\mathbf{h_i}, \mathbf{m_i}] \subseteq \mathbf{m_i}$, $i = 31, \cdots, 35$, we obtain the assertion. $\square$

Now we deal with the Lie algebra $\mathfrak{su}_3(\mathbb{C}, 1)$. It can be treated as the Lie algebra of matrices

$$(\lambda_1e_1 + \lambda_2e_2 + \lambda_3e_3 + \lambda_4e_4 + \lambda_5e_5 + \lambda_6e_6 + \lambda_7e_7 + \lambda_8e_8) \mapsto$$
$$\begin{pmatrix} -\lambda_1 i & -\lambda_2 - \lambda_3 i & \lambda_4 + \lambda_5 i \\ \lambda_2 - \lambda_3 i & \lambda_1 i + \lambda_6 i & \lambda_7 + \lambda_8 i \\ \lambda_4 - \lambda_5 i & \lambda_7 - \lambda_8 i & -\lambda_6 i \end{pmatrix}; \lambda_i \in \mathbb{R}, i = 1, \cdots, 8.$$

Then the multiplication of $\mathbf{g}$ is given by the following:

$$[e_1, e_6] = 0,\ \ [e_3, e_2] = 2e_1,\ \ [e_4, e_5] = 2(e_1 - e_6),\ \ [e_8, e_7] = 2e_6,$$
$$[e_6, e_3] = [e_7, e_4] = [e_8, e_5] = \frac{1}{2}[e_1, e_3] = e_2,$$
$$[e_2, e_6] = [e_4, e_8] = [e_7, e_5] = \frac{1}{2}[e_2, e_1] = e_3,$$

$$[e_7, e_2] = [e_3, e_8] = [e_5, e_6] = [e_1, e_5] = e_4,$$
$$[e_8, e_2] = [e_7, e_3] = [e_6, e_4] = [e_4, e_1] = e_5,$$
$$[e_2, e_4] = [e_3, e_5] = [e_8, e_1] = \frac{1}{2}[e_8, e_6] = e_7,$$
$$[e_2, e_5] = [e_4, e_3] = [e_1, e_7] = \frac{1}{2}[e_6, e_7] = e_8.$$

The normalized Cartan-Killing form $k : \mathfrak{su}_3(\mathbb{C}, 1) \times \mathfrak{su}_3(\mathbb{C}, 1) \to \mathbb{R}$ is the map $(X, Y) \mapsto \frac{1}{12}\text{trace}(\text{ad}X\text{ad}Y) = \frac{1}{2}\text{trace}(XY)$. An element $X = \lambda_i e_i \in \mathfrak{su}_3(\mathbb{C}, 1)$, $\lambda_i \in \mathbb{R}$, $i = 1, \cdots, 8$, is elliptic, parabolic or loxodromic according whether

$$k(X) = k(X, X) = -\lambda_1^2 - \lambda_2^2 - \lambda_3^2 - \lambda_6^2 + \lambda_4^2 + \lambda_5^2 + \lambda_7^2 + \lambda_8^2 - 2\lambda_1\lambda_6$$

is smaller, equal or greater 0.

Let H be a connected closed subgroup of the group $PSU_3(\mathbb{C}, 1)$. Then according to [1], Satz 1, p. 251 and [2], Section 5, p. 276, the group H is, up to conjugacy, one of the following:
(1) H is a subgroup of $Spin_3 \times SO_2(\mathbb{R})/\langle(-1, -1)\rangle$,
(2) H is a subgroup of the 5-dimensional solvable group $NG_{1,1}$ in [1], p. 253,
(3) H is the group $SL_2(\mathbb{R}) \times SO_2(\mathbb{R})/\langle(-1, -1)\rangle$,
(4) H is the group $SL_2(\mathbb{R}) \times \{1\}/\langle(-1, 1)\rangle \cong PSL_2(\mathbb{R})$,
(5) H is the connected component of the group $SO_3(\mathbb{R}, 1) \cong PSL_2(\mathbb{R})$.
The Lie algebras $\mathbf{h}_i$, $i = 1, \cdots, 5$, of H in the cases (1) till (5) are given in this order by

$$\widehat{\mathbf{h}_1} = \langle e_1, e_2, e_3, e_6\rangle,\ \widehat{\mathbf{h}_2} = \langle e_1 - \tfrac{1}{2}e_6, e_8, e_4 - e_3, e_5 + e_2, e_6 + e_7\rangle,$$
$$\widehat{\mathbf{h}_3} = \langle e_1, e_6, e_7, e_8\rangle,\ \widehat{\mathbf{h}_4} = \langle e_6, e_7, e_8\rangle,\ \widehat{\mathbf{h}_5} = \langle e_2, e_4, e_7\rangle.$$

After a straightforward calculation in $\widehat{\mathbf{h}_2}$ we obtain that the conjugacy classes of the 4-dimensional subalgebras of $\mathfrak{su}_3(\mathbb{C}, 1)$ are the following:

$$\mathbf{h}_1 = \langle e_1, e_2, e_3, e_6\rangle, \quad \mathbf{h}_2 = \langle e_4 - e_3, e_2 + e_5, e_6 + e_7, e_8\rangle,$$
$$\mathbf{h}_3 = \langle e_1 - \tfrac{1}{2}e_6 + ae_8, e_4 - e_3, e_2 + e_5, e_6 + e_7\rangle, \quad \mathbf{h}_4 = \langle e_1, e_6, e_7, e_8\rangle,$$

where $a \in \mathbb{R}$.
Computations in $\widehat{\mathbf{h}_1}$ and $\widehat{\mathbf{h}_2}$ yield that the 3-dimensional subalgebras of $\mathfrak{su}_3(\mathbb{C}, 1)$ have one of the following shapes:

$$\mathbf{h}_5 = \langle e_1, e_2, e_3\rangle, \quad \mathbf{h}_6 = \langle e_2, e_4, e_7\rangle, \quad \mathbf{h}_7 = \langle e_6, e_7, e_8\rangle,$$
$$\mathbf{h}_8 = \langle e_5 + e_2, e_6 + e_7, e_8\rangle,\ \mathbf{h}_9 = \langle e_4 - e_3 + be_8, e_5 + e_2, e_6 + e_7\rangle,$$
$$\mathbf{h}_{10} = \langle e_4 - e_3 + b(e_5 + e_2), e_6 + e_7, e_8 + c(e_5 + e_2)\rangle,$$

$\mathbf{h}_{11} = \langle e_1 - \frac{1}{2}e_6 + \frac{3}{2}c(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2), e_8 + b(e_4 - e_3) + c(e_5 + e_2),$
$e_6 + e_7\rangle$, where $b, c \in \mathbb{R}$.

Similarly we obtain that every 2-dimensional subalgebra of $\mathfrak{su}_3(\mathbb{C},1)$ has one of the following forms:

$$\mathbf{h}_{12} = \langle e_1, e_6\rangle, \quad \mathbf{h}_{13} = \langle e_4 - e_3, e_6 + e_7\rangle,$$

$$\mathbf{h}_{14} = \langle e_5 + e_2 + b(e_4 - e_3), e_6 + e_7\rangle, \;\; \mathbf{h}_{15} = \langle e_4 - e_3, e_8 + b(e_6 + e_7)\rangle,$$

$$\mathbf{h}_{16} = \langle e_5 + e_2 + b(e_4 - e_3), e_8 + c(e_6 + e_7)\rangle,$$

$$\mathbf{h}_{17} = \langle e_6 + e_7, e_8 + b(e_4 - e_3) + c(e_5 + e_2)\rangle,$$

$$\mathbf{h}_{18} = \langle e_6 + e_7 + a(e_4 - e_3) + b(e_5 + e_2), e_8 + c(e_4 - e_3) + d(e_5 + e_2)\rangle,$$

$$\mathbf{h}_{19} = \langle e_1 - \tfrac{1}{2}e_6 + ae_8 + b(e_4 - e_3) + c(e_5 + e_2), e_6 + e_7\rangle,$$

$$\mathbf{h}_{20} = \langle e_1 - \tfrac{1}{2}e_6 + \tfrac{3}{2}a(e_4 - e_3) - \tfrac{3}{2}b(e_5 + e_2) - \tfrac{3(a^2+b^2)}{2}(e_6 + e_7), \\ e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7)\rangle,$$

where $a, b, c, d \in \mathbb{R}$ and in the Lie algebra $\mathbf{h}_{18}$ one has $bc - ad = \frac{1}{2}$.

Moreover, every 1-dimensional subalgebra $\mathbf{h}$ of $\mathbf{g}$ is given by

$$\mathbf{h}_{21} = \langle e_1 + ae_6\rangle, \quad \mathbf{h}_{22} = \langle e_6\rangle, \quad \mathbf{h}_{23} = \langle e_8\rangle,$$

$$\mathbf{h}_{24} = \langle e_6 + e_7 + ce_8\rangle, \quad \mathbf{h}_{25} = \langle e_5 + e_2 + b(e_6 + e_7) + ce_8\rangle,$$

$$\mathbf{h}_{26} = \langle e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8\rangle,$$

$$\mathbf{h}_{27} = \langle e_1 - \tfrac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8\rangle,$$

where a, b, c, d are real numbers.

Proposition 9. *The Lie algebra $\mathfrak{su}_3(\mathbb{C},1)$ is reductive with a 4-dimensional subalgebra $\mathbf{h}$ and a complementary subspace $\mathbf{m}$ generating $\mathbf{g}$ if and only if the following holds:*

1) $\mathbf{h}_1 \cong \mathfrak{so}_3(\mathbb{R}) \oplus \mathfrak{so}_2(\mathbb{R}) = \langle e_1, e_2, e_3, e_6\rangle$ *and* $\mathbf{m}_1 = \langle e_4, e_5, e_7, e_8\rangle$,

2) $\mathbf{h}_4 \cong \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{so}_2(\mathbb{R}) = \langle e_1, e_6, e_7, e_8\rangle$ *and* $\mathbf{m}_4 = \langle e_2, e_3, e_4, e_5\rangle$.

Proof. For the basis elements of an arbitrary complement $\mathbf{m}$ to $\mathbf{h}_1$ in $\mathbf{g}$ we have

$$e_4 + a_1e_1 + b_1e_2 + c_1e_3 + d_1e_6, \quad e_5 + a_2e_1 + b_2e_2 + c_2e_3 + d_2e_6,$$

$$e_7 + a_3e_1 + b_3e_2 + c_3e_3 + d_3e_6, \quad e_8 + a_4e_1 + b_4e_2 + c_4e_3 + d_4e_6$$

with the real numbers a_i, b_i, c_i, d_i, $i = 1, 2, 3, 4$.
An arbitrary complement $\mathbf{m}$ to $\mathbf{h}_2$ in $\mathbf{g}$ has as generators

$$e_1 + a_1(e_4 - e_3) + b_1(e_5 + e_2) + c_1(e_6 + e_7) + d_1e_8,$$

$$e_2 + a_2(e_4 - e_3) + b_2(e_5 + e_2) + c_2(e_6 + e_7) + d_2e_8,$$

$$e_3 + a_3(e_4 - e_3) + b_3(e_5 + e_2) + c_3(e_6 + e_7) + d_3e_8,$$

$$e_6 + a_4(e_4 - e_3) + b_4(e_5 + e_2) + c_4(e_6 + e_7) + d_4e_8,$$

where a_i, b_i, c_i, d_i, $i = 1, 2, 3, 4$ are real parameters.
The basis elements of an arbitrary complement $\mathbf{m}$ to $\mathbf{h}_3$ in $\mathbf{g}$ are

$$e_3 + a_1(e_1 - \tfrac{1}{2}e_6 + ae_8) + b_1(e_4 - e_3) + c_1(e_2 + e_5) + d_1(e_6 + e_7),$$
$$e_5 + a_2(e_1 - \tfrac{1}{2}e_6 + ae_8) + b_2(e_4 - e_3) + c_2(e_2 + e_5) + d_2(e_6 + e_7),$$
$$e_7 + a_3(e_1 - \tfrac{1}{2}e_6 + ae_8) + b_3(e_4 - e_3) + c_3(e_2 + e_5) + d_3(e_6 + e_7),$$
$$e_8 + a_4(e_1 - \tfrac{1}{2}e_6 + ae_8) + b_4(e_4 - e_3) + c_4(e_2 + e_5) + d_4(e_6 + e_7),$$

where $a_i, b_i, c_i, d_i \in \mathbb{R}$, $i = 1, 2, 3, 4$.
As the generators of an arbitrary complement $\mathbf{m}$ to $\mathbf{h}_4$ in $\mathbf{g}$ we can choose the following:

$$e_2 + a_1e_1 + b_1e_6 + c_1e_7 + d_1e_8, \quad e_3 + a_2e_1 + b_2e_6 + c_2e_7 + d_2e_8,$$
$$e_4 + a_3e_1 + b_3e_6 + c_3e_7 + d_3e_8, \quad e_5 + a_4e_1 + b_4e_6 + c_4e_6 + d_4e_8,$$

where a_i, b_i, c_i, d_i, $i = 1, 2, 3, 4$ are real numbers.
Now the assertion follows from the relation $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$. □

Proposition 10. *The Lie algebra* $\mathbf{g} = \mathfrak{su}_3(\mathbb{C}, 1)$ *is reductive with respect to precisely one of the following pairs* $(\mathbf{h}, \mathbf{m})$*, where* $\mathbf{h}$ *is a* 3*-dimensional subalgebra of* $\mathbf{g}$ *and* $\mathbf{m}$ *is a complementary subspace to* $\mathbf{h}$ *generating* $\mathbf{g}$*:*

1) $\mathbf{h}_6 \cong \mathfrak{sl}_2(\mathbb{R}) = \langle e_2, e_4, e_7 \rangle$ *and* $\mathbf{m}_6 = \langle e_1, e_3, e_5, e_6, e_8 \rangle$*,*

2) $\mathbf{h}_7 \cong \mathfrak{sl}_2(\mathbb{R}) = \langle e_6, e_7, e_8 \rangle$ *and* $\mathbf{m}_7 = \langle e_1 - \frac{1}{2}e_6, e_2, e_3, e_4, e_5 \rangle$*.*

Proof. An arbitrary complement $\mathbf{m}_i$ to the subalgebra $\mathbf{h}_i$, $i = 5, \cdots, 11$, in $\mathbf{g}$ has as generators in the case $i = 5$

$$e_4 + a_1e_1 + b_1e_2 + c_1e_3, \quad e_5 + a_2e_1 + b_2e_2 + c_2e_3, \quad e_6 + a_3e_1 + b_3e_2 + c_3e_3,$$
$$e_7 + a_4e_1 + b_4e_2 + c_4e_3, \quad e_8 + a_5e_1 + b_5e_2 + c_5e_3,$$

in the case $i = 6$

$$e_1 + a_1e_2 + b_1e_4 + c_1e_7, \quad e_3 + a_2e_2 + b_2e_4 + c_2e_7, \quad e_5 + a_3e_2 + b_3e_4 + c_3e_7,$$
$$e_6 + a_4e_2 + b_4e_4 + c_4e_7, \quad e_8 + a_5e_2 + b_5e_4 + c_5e_7,$$

in the case $i = 7$

$$e_1 + a_1e_6 + b_1e_7 + c_1e_8, \quad e_2 + a_2e_6 + b_2e_7 + c_2e_8, \quad e_3 + a_3e_6 + b_3e_7 + c_3e_8,$$
$$e_4 + a_4e_6 + b_4e_7 + c_4e_8, \quad e_5 + a_5e_6 + b_5e_7 + c_5e_8,$$

in the case $i = 8$

$$e_1 + a_1(e_2 + e_5) + b_1(e_6 + e_7) + c_1e_8, \quad e_2 + a_2(e_2 + e_5) + b_2(e_6 + e_7) + c_2e_8,$$
$$e_3 + a_3(e_2 + e_5) + b_3(e_6 + e_7) + c_3e_8, \quad e_4 + a_4(e_2 + e_5) + b_4(e_6 + e_7) + c_4e_8,$$
$$e_6 + a_5(e_2 + e_5) + b_5(e_6 + e_7) + c_5e_8,$$

in the case $i = 9$

$$e_1 + a_1(e_2 + e_5) + b_1(e_6 + e_7) + c_1(e_4 - e_3 + be_8),$$
$$e_2 + a_2(e_2 + e_5) + b_2(e_6 + e_7) + c_2(e_4 - e_3 + be_8),$$
$$e_3 + a_3(e_2 + e_5) + b_3(e_6 + e_7) + c_3(e_4 - e_3 + be_8),$$
$$e_6 + a_4(e_2 + e_5) + b_4(e_6 + e_7) + c_4(e_4 - e_3 + be_8),$$
$$e_8 + a_5(e_2 + e_5) + b_5(e_6 + e_7) + c_5(e_4 - e_3 + be_8),$$

in the case $i = 10$

$$e_1 + a_1(e_4 - e_3 + b(e_2 + e_5)) + b_1(e_6 + e_7) + c_1(e_8 + c(e_2 + e_5)),$$
$$e_2 + a_2(e_4 - e_3 + b(e_2 + e_5)) + b_2(e_6 + e_7) + c_2(e_8 + c(e_2 + e_5)),$$
$$e_3 + a_3(e_4 - e_3 + b(e_2 + e_5)) + b_3(e_6 + e_7) + c_3(e_8 + c(e_2 + e_5)),$$
$$e_5 + a_4(e_4 - e_3 + b(e_2 + e_5)) + b_4(e_6 + e_7) + c_4(e_8 + c(e_2 + e_5)),$$
$$e_6 + a_5(e_4 - e_3 + b(e_2 + e_5)) + b_5(e_6 + e_7) + c_5(e_8 + c(e_2 + e_5)),$$

and in the case $i = 11$

$$e_2 + a_1(e_1 - \tfrac{1}{2}e_6 + \tfrac{3}{2}c(e_4 - e_3) - \tfrac{3}{2}b(e_5 + e_2)) + \\ b_1(e_8 + b(e_4 - e_3) + c(e_5 + e_2)) + c_1(e_6 + e_7),$$
$$e_3 + a_2(e_1 - \tfrac{1}{2}e_6 + \tfrac{3}{2}c(e_4 - e_3) - \tfrac{3}{2}b(e_5 + e_2)) + \\ b_2(e_8 + b(e_4 - e_3) + c(e_5 + e_2)) + c_2(e_6 + e_7),$$
$$e_4 + a_3(e_1 - \tfrac{1}{2}e_6 + \tfrac{3}{2}c(e_4 - e_3) - \tfrac{3}{2}b(e_5 + e_2)) + \\ b_3(e_8 + b(e_4 - e_3) + c(e_5 + e_2)) + c_3(e_6 + e_7),$$
$$e_5 + a_4(e_1 - \tfrac{1}{2}e_6 + \tfrac{3}{2}c(e_4 - e_3) - \tfrac{3}{2}b(e_5 + e_2)) + \\ b_4(e_8 + b(e_4 - e_3) + c(e_5 + e_2)) + c_4(e_6 + e_7),$$
$$e_7 + a_5(e_1 - \tfrac{1}{2}e_6 + \tfrac{3}{2}c(e_4 - e_3) - \tfrac{3}{2}b(e_5 + e_2)) + \\ b_5(e_8 + b(e_4 - e_3) + c(e_5 + e_2)) + c_5(e_6 + e_7),$$

where $a_j, b_j, c_j \in \mathbb{R}$, $j = 1, \cdots, 5$. The relation $[\mathbf{h_i}, \mathbf{m_i}] \subseteq \mathbf{m_i}$, $i = 5, \cdots, 11$, yields the assertion. $\square$

Proposition 11. *The Lie algebra* $\mathbf{g} = \mathfrak{su}_3(\mathbb{C}, 1)$ *is reductive with respect to the following pairs* $(\mathbf{h}, \mathbf{m})$*, where* $\mathbf{h}$ *is a 2-dimensional subalgebra of* $\mathbf{g}$ *and* $\mathbf{m}$ *is a complementary subspace to* $\mathbf{h}$ *generating* $\mathbf{g}$*, if and only if one of the following holds:*

1) $\mathbf{h}_{12} = \langle e_1, e_6 \rangle$ *and* $\mathbf{m}_{12} = \langle e_2, e_3, e_4, e_5, e_7, e_8 \rangle$,

2) $\mathbf{h}_{20} = \langle e_1 - \frac{1}{2}e_6 + \frac{3}{2}a(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2) - \frac{3(a^2+b^2)}{2}(e_6 + e_7),$
$e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7)\rangle$

and

$\mathbf{m}_{20} = \langle e_6 + e_7, e_2 + e_5, e_4 - e_3, e_4 - be_8 + 2ae_1 - ae_6, e_2 + ae_8 + 2be_1 - be_6, e_6 + ce_8 + be_5 - ae_4 \rangle$, $a, b, c \in \mathbb{R}$.

Proof. An arbitrary complement $\mathbf{m}_i$ to the subalgebra $\mathbf{h}_i$, $i = 12, \cdots, 20$, in $\mathbf{g}$ has as generators in the case $i = 12$

$$e_2 + a_1 e_1 + b_1 e_6, \quad e_3 + a_2 e_1 + b_2 e_6, \quad e_4 + a_3 e_1 + b_3 e_6,$$

$$e_5 + a_4 e_1 + b_4 e_6, \quad e_7 + a_5 e_1 + b_5 e_6, \quad e_8 + a_6 e_1 + b_6 e_6,$$

in the case $i = 13$

$$e_1 + a_1(e_4 - e_3) + b_1(e_6 + e_7), \quad e_2 + a_2(e_4 - e_3) + b_2(e_6 + e_7),$$

$$e_3 + a_3(e_4 - e_3) + b_3(e_6 + e_7), \quad e_5 + a_4(e_4 - e_3) + b_4(e_6 + e_7),$$

$$e_6 + a_5(e_4 - e_3) + b_5(e_6 + e_7), \quad e_8 + a_6(e_4 - e_3) + b_6(e_6 + e_7),$$

in the case $i = 14$

$$e_1 + a_1(e_2 + e_5 + b(e_4 - e_3)) + b_1(e_6 + e_7),$$

$$e_2 + a_2(e_2 + e_5 + b(e_4 - e_3)) + b_2(e_6 + e_7),$$

$$e_3 + a_3(e_2 + e_5 + b(e_4 - e_3)) + b_3(e_6 + e_7),$$

$$e_4 + a_4(e_2 + e_5 + b(e_4 - e_3)) + b_4(e_6 + e_7),$$

$$e_6 + a_5(e_2 + e_5 + b(e_4 - e_3)) + b_5(e_6 + e_7),$$

$$e_8 + a_6(e_2 + e_5 + b(e_4 - e_3)) + b_6(e_6 + e_7),$$

in the case $i = 15$

$$e_1 + a_1(e_4 - e_3) + b_1(e_8 + b(e_6 + e_7)),$$

$$e_2 + a_2(e_4 - e_3) + b_2(e_8 + b(e_6 + e_7)),$$

$$e_3 + a_3(e_4 - e_3) + b_3(e_8 + b(e_6 + e_7)),$$

$$e_5 + a_4(e_4 - e_3) + b_4(e_8 + b(e_6 + e_7)),$$

$$e_6 + a_5(e_4 - e_3) + b_5(e_8 + b(e_6 + e_7)),$$

$$e_7 + a_6(e_4 - e_3) + b_6(e_8 + b(e_6 + e_7)),$$

in the case $i = 16$

$$e_1 + a_1(e_5 + e_2 + b(e_4 - e_3)) + b_1(e_8 + c(e_6 + e_7)),$$

$$e_2 + a_2(e_5 + e_2 + b(e_4 - e_3)) + b_2(e_8 + c(e_6 + e_7)),$$

$$e_3 + a_3(e_5 + e_2 + b(e_4 - e_3)) + b_3(e_8 + c(e_6 + e_7)),$$

$$e_4 + a_4(e_5 + e_2 + b(e_4 - e_3)) + b_4(e_8 + c(e_6 + e_7)),$$

$$e_6 + a_5(e_5 + e_2 + b(e_4 - e_3)) + b_5(e_8 + b(e_6 + e_7)),$$

$$e_7 + a_6(e_5 + e_2 + b(e_4 - e_3)) + b_6(e_8 + b(e_6 + e_7)),$$

in the case $i = 17$

$$e_1 + a_1(e_6 + e_7) + b_1(e_8 + b(e_4 - e_3) + c(e_5 + e_2)),$$

$$e_2 + a_2(e_6 + e_7) + b_2(e_8 + b(e_4 - e_3) + c(e_5 + e_2)),$$
$$e_3 + a_3(e_6 + e_7) + b_3(e_8 + b(e_4 - e_3) + c(e_5 + e_2)),$$
$$e_4 + a_4(e_6 + e_7) + b_4(e_8 + b(e_4 - e_3) + c(e_5 + e_2)),$$
$$e_5 + a_5(e_6 + e_7) + b_5(e_8 + b(e_4 - e_3) + c(e_5 + e_2)),$$
$$e_6 + a_6(e_6 + e_7) + b_6(e_8 + b(e_4 - e_3) + c(e_5 + e_2)),$$

in the case $i = 18$

$$e_1 + a_1(e_6 + e_7 + a(e_4 - e_3) + b(e_5 + e_2)) + b_1(e_8 + c(e_4 - e_3) + d(e_5 + e_2)),$$
$$e_2 + a_2(e_6 + e_7 + a(e_4 - e_3) + b(e_5 + e_2)) + b_2(e_8 + c(e_4 - e_3) + d(e_5 + e_2)),$$
$$e_3 + a_3(e_6 + e_7 + a(e_4 - e_3) + b(e_5 + e_2)) + b_3(e_8 + c(e_4 - e_3) + d(e_5 + e_2)),$$
$$e_4 + a_4(e_6 + e_7 + a(e_4 - e_3) + b(e_5 + e_2)) + b_4(e_8 + c(e_4 - e_3) + d(e_5 + e_2)),$$
$$e_5 + a_5(e_6 + e_7 + a(e_4 - e_3) + b(e_5 + e_2)) + b_5(e_8 + c(e_4 - e_3) + d(e_5 + e_2)),$$
$$e_6 + a_6(e_6 + e_7 + a(e_4 - e_3) + b(e_5 + e_2)) + b_6(e_8 + c(e_4 - e_3) + d(e_5 + e_2)),$$

in the case $i = 19$

$$e_2 + a_1(e_1 - \tfrac{1}{2}e_6 + ae_8 + b(e_4 - e_3) + c(e_5 + e_2)) + b_1(e_6 + e_7),$$
$$e_3 + a_2(e_1 - \tfrac{1}{2}e_6 + ae_8 + b(e_4 - e_3) + c(e_5 + e_2)) + b_2(e_6 + e_7),$$
$$e_4 + a_3(e_1 - \tfrac{1}{2}e_6 + ae_8 + b(e_4 - e_3) + c(e_5 + e_2)) + b_3(e_6 + e_7),$$
$$e_5 + a_4(e_1 - \tfrac{1}{2}e_6 + ae_8 + b(e_4 - e_3) + c(e_5 + e_2)) + b_4(e_6 + e_7),$$
$$e_7 + a_5(e_1 - \tfrac{1}{2}e_6 + ae_8 + b(e_4 - e_3) + c(e_5 + e_2)) + b_5(e_6 + e_7),$$
$$e_8 + a_6(e_1 - \tfrac{1}{2}e_6 + ae_8 + b(e_4 - e_3) + c(e_5 + e_2)) + b_6(e_6 + e_7),$$

and in the case $i = 20$

$$e_2 + a_1(e_1 - \tfrac{1}{2}e_6 + \tfrac{3}{2}a(e_4 - e_3) - \tfrac{3}{2}b(e_5 + e_2) - \tfrac{3(a^2+b^2)}{2}(e_6 + e_7))+$$
$$+b_1(e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7)),$$

$$e_3 + a_2(e_1 - \tfrac{1}{2}e_6 + \tfrac{3}{2}a(e_4 - e_3) - \tfrac{3}{2}b(e_5 + e_2) - \tfrac{3(a^2+b^2)}{2}(e_6 + e_7))+$$
$$+b_2(e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7)),$$

$$e_4 + a_3(e_1 - \tfrac{1}{2}e_6 + \tfrac{3}{2}a(e_4 - e_3) - \tfrac{3}{2}b(e_5 + e_2) - \tfrac{3(a^2+b^2)}{2}(e_6 + e_7))+$$
$$+b_3(e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7)),$$

$$e_5 + a_4(e_1 - \tfrac{1}{2}e_6 + \tfrac{3}{2}a(e_4 - e_3) - \tfrac{3}{2}b(e_5 + e_2) - \tfrac{3(a^2+b^2)}{2}(e_6 + e_7))+$$
$$+b_4(e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7)),$$

$$e_6 + a_5(e_1 - \tfrac{1}{2}e_6 + \tfrac{3}{2}a(e_4 - e_3) - \tfrac{3}{2}b(e_5 + e_2) - \tfrac{3(a^2+b^2)}{2}(e_6 + e_7))+$$
$$+b_5(e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7)),$$

$$e_7 + a_6(e_1 - \tfrac{1}{2}e_6 + \tfrac{3}{2}a(e_4 - e_3) - \tfrac{3}{2}b(e_5 + e_2) - \tfrac{3(a^2+b^2)}{2}(e_6 + e_7))+$$
$$+b_6(e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7)),$$

where $a_j, b_j \in \mathbb{R}$, $j = 1, \cdots 6$. Using the relation $[\mathbf{h_i}, \mathbf{m_i}] \subseteq \mathbf{m_i}$, $i = 12, \cdots, 20$, we obtain the assertion. $\square$

Proposition 12. *The Lie algebra* $\mathbf{g} = \mathfrak{su}_3(\mathbb{C}, 1)$ *is reductive with a 1-dimensional subalgebra* $\mathbf{h}$ *and a 7-dimensional complementary subspace* $\mathbf{m}$ *generating* $\mathbf{g}$ *in precisely one of the following cases:*

1) $\mathbf{h} = \langle e_1 - 2e_6 \rangle$, $\mathbf{m}_{b,c,d} = \langle e_2 + b(e_1 - 2e_6), e_3 + c(e_1 - 2e_6), e_6 + d(e_1 - 2e_6), e_4, e_5, e_7, e_8 \rangle$, *when* $b, c, d \in \mathbb{R}$,

2) $\mathbf{h} = \langle e_1 + e_6 \rangle$ *and* $\mathbf{m}_{b,c,d} = \langle e_2, e_3, e_7, e_8, e_4 + d(e_1 + e_6), e_5 + b(e_1 + e_6), e_6 + c(e_1 + e_6) \rangle$ *with* $b, c, d \in \mathbb{R}$,

3) $\mathbf{h} = \langle e_1 - \frac{1}{2}e_6 \rangle$, $\mathbf{m}_{b,c,d} = \langle e_2, e_3, e_4, e_5, e_6 + b(e_1 - \frac{1}{2}e_6), e_7 + c(e_1 - \frac{1}{2}e_6), e_8 + d(e_1 - \frac{1}{2}e_6) \rangle$ *and* $b, c, d \in \mathbb{R}$,

4) $\mathbf{h}_a = \langle e_1 + ae_6 \rangle$ *and* $\mathbf{m}_b = \langle e_2, e_3, e_4, e_5, e_6 + b(e_1 + ae_6), e_7, e_8 \rangle$, *where* $a \in \mathbb{R} \backslash \{-\frac{1}{2}, -2, 1\}$, $b, c, d \in \mathbb{R}$,

5) $\mathbf{h} = \langle e_6 \rangle$ *and* $\mathbf{m}_a = \langle e_1 + ae_6, e_2, e_3, e_4, e_5, e_7, e_8 \rangle$, $a \in \mathbb{R}$,

6) $\mathbf{h} = \langle e_8 \rangle$ *and* $\mathbf{m}_a = \langle e_1 + ae_8, e_2, e_3, e_4, e_5, e_6, e_7 \rangle$, $a \in \mathbb{R}$,

7) $\mathbf{h} = \langle e_6 + e_7 + ce_8 \rangle$ *and* $\mathbf{m}_b = \langle e_1 + bce_8, e_2, e_3, e_4, e_5, e_6 + e_7, e_7 - \frac{1}{c}e_8 \rangle$ *with* $c \in \mathbb{R} \backslash \{0\}$, $b \in \mathbb{R}$,

8) $\mathbf{h}_{b,c} = \langle e_5 + e_2 + b(e_6 + e_7) + ce_8 \rangle$ *and* $\mathbf{m}_d = \langle e_1 - \frac{c^3d - cd - b}{2c}e_8, e_2 + \frac{1}{c}e_8, e_3 + cde_8, e_7 - \frac{b+cd}{c}e_8, e_4 - e_3, e_2 + e_5, e_6 + e_7 \rangle$, *where* $b, c, d \in \mathbb{R}, c \neq 0$,

9) $\mathbf{h}_{a,b,c} = \langle e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8 \rangle$ *and* $\mathbf{m}_d = \langle e_2 - dce_8, e_3 - \frac{1+dc^2a+a^2}{c}e_8, e_6 - \frac{a^3+a-bc+dc^2+dc^2a^2}{c^2}e_8, e_5 + e_2, e_6 + e_7, e_4 - e_3, e_1 + \frac{bc+c^2a-a-a^3+c^4d-c^2d-c^2a^2d}{2c^2}e_8 \rangle$ *with* $a, b, c, d \in \mathbb{R}, c \neq 0$,

10) $\mathbf{h}_{a,b,c,d} = \langle e_1 - \frac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8 \rangle$ *and* $\mathbf{m}_f = \langle e_6 + e_7, e_4 - e_3, e_5 + e_2, e_3 + f(e_1 - \frac{1}{2}e_6 + ce_8), e_2 - \frac{2c}{3}e_4 - \frac{4a}{3}e_1 - \frac{2a}{3}e_7 + \frac{2d}{3}e_8, e_7 - \frac{b}{c}e_8 + \frac{a}{c}e_4 + \frac{d}{c}e_2, e_8 - \frac{8ac-4fc^2+24fd^2-9f+12d}{2(8dc-3a+4ac^2)}(e_1 - \frac{1}{2}e_6 + ce_8) \rangle$,

where $a, b, c, d, f \in \mathbb{R}$, $c \neq 0$, $8dc - 3a + 4ac^2 \neq 0$.

Proof. An arbitrary complement $\mathbf{m}_i$ to the subalgebra $\mathbf{h}_i$, $i = 21, \cdots, 27$, in $\mathbf{g}$ has as generators in the case $i = 21$

$$e_2 + a_1(e_1 + ae_6),\ \ e_3 + a_2(e_1 + ae_6),\ \ e_4 + a_3(e_1 + ae_6),\ e_5 + a_4(e_1 + ae_6),$$
$$e_6 + a_5(e_1 + ae_6),\ \ e_7 + a_6(e_1 + ae_6),\ \ \ e_8 + a_7(e_1 + ae_6),$$

in the case $i = 22$

$$e_1 + a_1e_6,\ \ e_2 + a_2e_6,\ \ e_3 + a_3e_6,\ e_4 + a_4e_6,$$
$$e_5 + a_5e_6,\ \ e_7 + a_6e_6,\ \ \ e_8 + a_7e_6,$$

in the case $i = 23$

$$e_1 + a_1e_8,\ \ e_2 + a_2e_8,\ \ e_3 + a_3e_8,\ e_4 + a_4e_8,$$
$$e_5 + a_5e_8,\ \ e_6 + a_6e_8,\ \ \ e_7 + a_7e_8,$$

in the case $i = 24$

$$e_1 + a_1(e_6 + e_7 + ce_8), \;\; e_2 + a_2(e_6 + e_7 + ce_8),$$
$$e_3 + a_3(e_6 + e_7 + ce_8), \; e_4 + a_4(e_6 + e_7 + ce_8),$$
$$e_5 + a_5(e_6 + e_7 + ce_8), \;\; e_7 + a_6(e_6 + e_7 + ce_8),$$
$$e_8 + a_7(e_6 + e_7 + ce_8),$$

in the case $i = 25$

$$e_1 + a_1(e_5 + e_2 + b(e_6 + e_7) + ce_8), \;\; e_2 + a_2(e_5 + e_2 + b(e_6 + e_7) + ce_8),$$
$$e_3 + a_3(e_5 + e_2 + b(e_6 + e_7) + ce_8), \;\; e_4 + a_4(e_5 + e_2 + b(e_6 + e_7) + ce_8),$$
$$e_6 + a_5(e_5 + e_2 + b(e_6 + e_7) + ce_8), \;\; e_7 + a_6(e_5 + e_2 + b(e_6 + e_7) + ce_8),$$
$$e_8 + a_7(e_5 + e_2 + b(e_6 + e_7) + ce_8),$$

in the case $i = 26$

$$e_1 + a_1(e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_2 + a_2(e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_3 + a_3(e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_5 + a_4(e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_6 + a_5(e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_7 + a_6(e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_8 + a_6(e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$

in the case $i = 27$

$$e_2 + a_1(e_1 - \tfrac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_3 + a_2(e_1 - \tfrac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_4 + a_3(e_1 - \tfrac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_5 + a_4(e_1 - \tfrac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_6 + a_5(e_1 - \tfrac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_7 + a_6(e_1 - \tfrac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_8 + a_7(e_1 - \tfrac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$

where a_j, $j = 1, \cdots, 7$, are real parameters. The relation $[\mathbf{h_i}, \mathbf{m_i}] \subseteq \mathbf{m_i}$, $i = 21, \cdots, 27$, yields the assertion. $\square$

4 Left A-loops as sections in simple Lie groups

The connected almost differentiable left A-loops L with dim $L \le 2$ are classified in [18], Section 27 and Theorem 18.14. Furthermore, all 3-dimensional left A-loops which are differentiable sections in a non-solvable Lie group are determined in [6]. In this section we deal with the at least 4-dimensional almost differentiable left A-loops having an at most 9-dimensional simple Lie group G as the group topologically generated by their left translations. According to Lemma 1 the group G is not compact.

Proposition 13. *There exists no at least* 4*-dimensional differentiable left A-loop having a group locally isomorphic to* $PSL_2(\mathbb{C})$ *as the group topologically generated by its left translations.*

Proof. Since the tangent space T_eL for an almost differentiable left A-loop L is reductive only the pairs $(\mathbf{h}, \mathbf{m})$ in Proposition 4 can occur as the tangent objects (T_1H, T_eL), where H is the stabilizer of the identity e of L. A maximal compact subalgebra of the Lie algebra $\mathbf{h}_3$ as well as of $\mathbf{h}_6$ is isomorphic to $so_2(\mathbb{R})$. Hence the Lie group corresponding to $\mathbf{h}_3$ as well as to $\mathbf{h}_6$ cannot be the stabilizer of $e \in L$ (cf. Lemma 2). Moreover, the hyperbolic elements $e_1 \in \mathbf{h}_4$ and $e_2 \in \mathbf{m}_a$ are conjugate (see **1.1**). This contradiction to Lemma 3 yields the assertion. □

Proposition 14. *Let* G *be locally isomorphic to* $SL_3(\mathbb{R})$*. Every connected almost differentiable left A-loop having* G *as the group topologically generated by its left translations is isomorphic to the* 5*-dimensional Bruck loop* L_0 *of hyperbolic type having the group* $SO_3(\mathbb{R})$ *as the stabilizer of* $e \in L_0$*.*

Proof. Since the tangent space T_eL for an almost differentiable left A-loop L is reductive we have to investigate the pairs $(\mathbf{h}, \mathbf{m})$ listed in Propositions 5, 6, 7 and 8. According to Lemma 2 the Lie groups belonging to the Lie algebras $\mathbf{h}_5$, $\mathbf{h}_7$, $\mathbf{h}_8$, $\mathbf{h}_{30}$ and $\mathbf{h}_{35}$ for $b = 0$ cannot be stabilizers of $e \in L$. The element $-e_5 + e_8 \in \mathbf{h}_{26}$ is conjugate to $\frac{1}{2}e_1 + 2e_3 \in \mathbf{m}_{26}$ under $g = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -\frac{1}{2} & 0 \\ 1 & \frac{1}{2} & 0 \end{pmatrix}$, the element $e_2 + e_8 \in \mathbf{h}_{32}$ is conjugate to $e_1 + 2e_7 - e_8 + 2e_4 \in \mathbf{m}_d$ under $g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \\ 2 & 2 & 0 \end{pmatrix}$ and $e_6 - e_7 + b(e_5 + e_8) \in \mathbf{h}_{35}$, $b > 0$, is conjugate to $(b^2+1)e_1 - e_3 + 2b(e_5 - e_8) \in \mathbf{m}_c$ under $g = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -b & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Moreover, the element $e_8 + \frac{1}{a}e_5 \in \mathbf{h}_{31,1}$ is conjugate to $\frac{-a^2+a+1}{a^2}e_1 + e_2 + e_3 + e_4 - e_6 - e_7 \in \mathbf{m}_b$ under $g = \begin{pmatrix} 1 & -\frac{1}{a} & -1 \\ 1 & \frac{a+1}{a} & -1 \\ 0 & \frac{a}{2+a} & \frac{a}{2+a} \end{pmatrix}$.

In the case 2) of Proposition 8 we choose $k \in \mathbb{R}\backslash\{0\}$ in such a way that $l := k^2c + k + b \neq 0$. Then the element $l(e_5 - 2e_8) \in \mathbf{h}_{31,2}$ is conjugate to $e_1 + b(e_5 - 2e_8) + 3l(e_2 - ke_6) + k(e_3 + c(e_5 - 2e_8)) + \frac{3k^2c+k+3b}{3l}(ke_4 - e_7) \in \mathbf{m}_{b,c,d}$ under $g = \begin{pmatrix} 0 & -\frac{3k^2c+k+3b}{3kl} & 1 \\ k & 1 & 0 \\ \frac{-k}{3l} & \frac{1}{3l} & 1 \end{pmatrix}$.
In the case 3) of Proposition 8 we take $k \in \mathbb{R}$ such that $n := k^2b - 2k + c \neq 0$. Then the element $n(e_5 - \frac{1}{2}e_8) \in \mathbf{h}_{31,3}$ is conjugate to $-ke_1 + k^2(e_2 + b(e_5 - \frac{1}{2}e_8)) + \frac{3k^2b-2k+3c}{2}(e_3 - ke_6) + e_4 + c(e_5 - \frac{1}{2}e_8) + e_7 \in \mathbf{m}_{b,c,d}$ under $g = \begin{pmatrix} 0 & \frac{2}{3n} & \frac{-3k^2b+2b-3c}{3n} \\ 1 & 1 & -k \\ 1 & 0 & k \end{pmatrix}$.
In the case 4) of Proposition 8 we take $k \in \mathbb{R}$ such that $m := k^2b + k + c \neq 0$. Then the element $m(e_5 + e_8) \in \mathbf{h}_{31,4}$ is conjugate to $(3c + 3k^2b + k)(ke_2 - e_1) + e_4 - ke_3 + e_7 + c(e_5 + e_8) + k^2(e_6 + b(e_5 + e_8)) \in \mathbf{m}_{b,c,d}$ under $g = \begin{pmatrix} 1 & 1 & -k \\ -\frac{1}{3m} & 0 & \frac{-3c-k-3k^2b}{3m} \\ 0 & 1 & k \end{pmatrix}$. These facts contradict Lemma 3.
In the remaining case one has $[\mathbf{m}_6, \mathbf{m}_6] = \mathbf{h}_6$ and the loop L with $T_eL = \mathbf{m}_6$ is a Bruck loop. The assertion follows now from the proof of the Theorem 13 in [5], p. 12. □

Since the exponential image of the Lie algebra $\mathbf{g} = \mathfrak{su}_3(\mathbb{C}, 1)$ is much more complicated than the exponential image of $\mathbf{g} = \mathfrak{sl}_3(\mathbb{R})$ we treat the almost differentiable left A-loops having $PSU_3(\mathbb{C}, 1)$ as the group topologically generated by the left translations under the assumption that their dimension is at most 5.

Proposition 15. *Let G be locally isomorphic to $PSU_3(\mathbb{C}, 1)$. Every at most 5-dimensional connected almost differentiable left A-loop having G as the group topologically generated by the left translations is isomorphic to the complex hyperbolic plane loop L_0 having the group $Spin_3 \times SO_2(\mathbb{R})/\langle(-1, -1)\rangle$ as the stabilizer of $e \in L_0$.*

Proof. Since the tangent space T_eL for an almost differentiable left A-loop L is reductive we have to deal only with the pairs $(\mathbf{h}, \mathbf{m})$ described in the Propositions 9, 10. The complex hyperbolic plane loop L_0 is realized on the exponential image of the subspace $\mathbf{m}_1$ (cf. [5], p. 8). The Lie group corresponding to $\mathbf{h}_4$ cannot be the stabilizer of a 4-dimensional topological loop L (see Lemma 2). According to **1.2** the element $e_2 \in \mathbf{h}_6$ is conjugate to $e_1 \in \mathbf{m}_6$, which is a contradiction to Lemma 3. Two loxodromic elements of $\mathfrak{su}_3(\mathbb{C}, 1)$ are conjugate in $SU_3(\mathbb{C}, 1)$ if and only if they have the same eigenvalues (cf. Prop. 3.2.3 (d) in [3], p. 65) and therefore they are conjugate in $SL_3(\mathbb{C})$. Since the elements $e_7 \in \mathbf{h}_7$ and $e_4 \in \mathbf{m}_7$ are loxodromic and $Ad_g(e_7) = e_4$ with $g = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \sqrt{2} \\ \sqrt{2} & 0 & 1 \end{pmatrix} \in SL_3(\mathbb{C})$ we have also a contradiction to Lemma 3. □

At the end of this section we show that several reductive spaces $(\mathbf{g}, \mathbf{h}, \mathbf{m})$, where $\mathbf{g} = \mathfrak{su}_3(\mathbb{C}, 1)$ and dim $\mathbf{h} \leq 2$ can not correspond to an almost differentiable left A-loop.

Proposition 16. *There is no almost differentiable left A-loop corresponding to one of the following triples:* $(\mathbf{g}, \mathbf{h}_{12}, \mathbf{m}_{12})$ *in Proposition 11 and* $(\mathbf{g}, \mathbf{h}, \mathbf{m}_a)$ *in the case 6) as well as* $(\mathbf{g}, \mathbf{h}, \mathbf{m}_b)$ *in the case 7) of Proposition 12.*

Proof. Since the elements $e_1 \in \mathbf{h}_{12}$ and $e_2 \in \mathbf{m}_{12}$ are elliptic in a subalgebra isomorphic to $\mathfrak{so}_3(\mathbb{R})$ of $\mathbf{g}$ (see **1.2**) they are conjugate under $Ad\, PSU_3(\mathbb{C}, 1)$. Since he element $e_8 \in \mathbf{h}$ in the case 6 as well as $e_6 + e_7 + ce_8 \in \mathbf{h}$, $c \neq 0$, in the case 7 of Proposition 12 is hyperbolic in a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ of $\mathbf{g}$ (see **1.1**), we have that e_8 and $e_7 \in \mathbf{m}_a$ respectively $e_6 + e_7 + ce_8$ and $-\frac{1}{\sqrt{2+2c^2}}(e_7 - \frac{1}{c}e_8) \in \mathbf{m}_b$ are conjugate under $Ad\, PSU_3(\mathbb{C}, 1)$. This contradicts Lemma 3. □

5 Reductive loops corresponding to semi-simple Lie groups of dimension 6

Let $G = G_1 \times G_2$ be the group topologically generated by the left translations of a connected almost differentiable left A-loop L, such that G_i, $i = 1, 2$, is a 3-dimensional quasi-simple Lie group. In contrast to the nonexistence of 3-dimensional almost differentiable left A-loops belonging to G (cf. Propositions 5 and 8 in [6]) we will show that there are such loops L with $G = G_1 \times G_2$ as the group topologically generated by the left translations if dim $L \geq 4$.

The following fact is well known from linear algebra:

Lemma 17. *Let* $\mathbf{g} = \mathbf{g}_1 \oplus \mathbf{g}_2$, *where* $\mathbf{g}_i$, $i = 1, 2$ *are simple Lie algebras of dimension* 3. *For any subspace* $\mathbf{m}$ *with dimension* 4 *respectively* 5 *the intersections* $\mathbf{m} \cap \mathbf{g}_1$ *and* $\mathbf{m} \cap \mathbf{g}_2$ *have dimension at least* 1 *respectively at least* 2.

The fact that the coset space G/H is parallelizable is reflected in the following lemma.

Lemma 18. *Let G be isomorphic to the Lie group $G_1 \times G_2$, such that $G_2 \cong SO_3(\mathbb{R})$ and for the subgroup H of G one has $H = H_1 \times H_2$ with $1 \neq H_2 \leq G_2$. Then G cannot be the group topologically generated by the left translations of a topological loop.*

For the proof see Lemma 2 in [5], p. 5.

First let G be locally isomorphic to $SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$. Since the at most 2-dimensional connected subgroups of G are tori and $\dim L \geq 4$ Lemma 2 gives

Proposition 19. *There is no left A-loop as differentiable section in a group locally isomorphic to $SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$.*

Now let G be locally isomorphic to $PSL_2(\mathbb{R}) \times G_2$, where G_2 is either the group $SO_3(\mathbb{R})$ or $PSL_2(\mathbb{R})$. Using the real basis of $\mathfrak{sl}_2(\mathbb{R})$ respectively of $\mathfrak{so}_3(\mathbb{R})$ introduced in **1.1** respectively in **1.2** we can choose $(e_1, 0)$, $(e_2, 0)$, $(e_3, 0)$, $(0, \varepsilon e_1)$, $(0, \varepsilon e_2)$, $(0, e_3)$ as a real basis of the Lie algebra $\mathbf{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathbf{g}_2$, where $\varepsilon = \mathrm{i}$ with $\mathrm{i}^2 = -1$ for $\mathbf{g}_2 = \mathfrak{so}_3(\mathbb{R})$ and $\varepsilon = 1$ for $\mathbf{g}_2 = \mathfrak{sl}_2(\mathbb{R})$.
Denote by H a subgroup of G. First we assume that H is decomposable into a direct product. If H has dimension 2 then with Lemma 18 we obtain that H is (up to interchanging the components) either $\mathcal{L}_2 \times \{1\}$ or $K_1 \times K_2$, where K_i, $i = 1, 2$ are 1-dimensional subgroups of $PSL_2(\mathbb{R})$. Now according to **1.1** the Lie algebra $\mathbf{h}$ of H has one of the following forms:

$$\mathbf{h}_1 = \langle (e_3, 0), (0, e_3) \rangle, \quad \mathbf{h}_2 = \langle (e_3, 0), (0, e_2 + e_3) \rangle, \quad \mathbf{h}_3 = \langle (e_3, 0), (0, e_1) \rangle,$$
$$\mathbf{h}_4 = \langle (e_1, 0), (0, e_1) \rangle, \quad \mathbf{h}_5 = \langle (e_1, 0), (0, e_2 + e_3) \rangle,$$
$$\mathbf{h}_6 = \langle (e_2 + e_3, 0), (0, e_2 + e_3) \rangle, \quad \mathbf{h}_7 = \langle (e_1, 0), (e_2 + e_3, 0) \rangle.$$

The Lie algebras $\mathbf{h}_1$ till $\mathbf{h}_7$ are subalgebras of $\mathbf{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$ but $\mathbf{h}_7$ is also a subalgebra of $\mathbf{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{so}_3(\mathbb{R})$.
If $\dim H = 1$ then H has the shape $K_1 \times \{1\}$ with a 1-dimensional subgroup K_1 of $PSL_2(\mathbb{R})$. Then according to **1.1** the Lie algebra $\mathbf{h}$ of H has (up to interchanging the components) one of the following forms:

$$\mathbf{h}_8 = \langle (e_3, 0) \rangle, \quad \mathbf{h}_9 = \langle (e_1, 0) \rangle, \quad \mathbf{h}_{10} = \langle (e_2 + e_3, 0) \rangle.$$

These algebras are subalgebras of $\mathbf{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$ as well as $\mathbf{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{so}_3(\mathbb{R})$.
Now we suppose that H is not a direct product of two subgroups. In the case $\dim H = 2$ one has $H = \{(x, \varphi(x)) |\ x \in \mathcal{L}_2\}$, where $\varphi \neq 1$ is a homomorphism of $\mathcal{L}_2$ into $PSL_2(\mathbb{R})$. If φ is injective then the Lie algebra of H is a subalgebra of $\mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$ and has the shape

$$\mathbf{h}_{11} = \langle (e_1, e_1), (e_2 + e_3, e_2 + e_3) \rangle.$$

If φ has 1-dimensional kernel then the Lie algebra of H is given by

$$\mathbf{h}_{12} = \langle (e_1, k), (e_2 + e_3, 0) \rangle,$$

where k denotes either the element e_1 or $e_2 + e_3$ of $\mathfrak{sl}_2(\mathbb{R})$ or e_3 of $\mathfrak{sl}_2(\mathbb{R}) \cap \mathfrak{so}_3(\mathbb{R})$ (see **1.1** and **1.2**).

In the case $\dim H = 1$ one has $H = \{(k_1, \varphi(k_1)) |\ k_1 \in K_1\}$, where K_1 is a 1-dimensional subgroup of $PSL_2(\mathbb{R})$ and $\varphi \neq 1$ is a homomorphism of K_1 into $PSL_2(\mathbb{R})$ or $SO_3(\mathbb{R})$. Then the Lie algebra $\mathbf{h}$ of H has (up to interchanging the components) one of the following forms:

$$\mathbf{h}_{13} = \langle (e_1, e_1) \rangle, \quad \mathbf{h}_{14} = \langle (e_1, e_2 + e_3) \rangle, \quad \mathbf{h}_{15} = \langle (e_2 + e_3, e_2 + e_3) \rangle,$$
$$\mathbf{h}_{16} = \langle (e_1, e_3) \rangle, \quad \mathbf{h}_{17} = \langle (e_2 + e_3, e_3) \rangle, \quad \mathbf{h}_{18} = \langle (e_3, e_3) \rangle.$$

The Lie algebra $\mathbf{h}_{13}$ till $\mathbf{h}_{18}$ are subalgebras of $\mathbf{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$ but $\mathbf{h}_{16}, \mathbf{h}_{17}, \mathbf{h}_{18}$ are also subalgebras of $\mathbf{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{so}_3(\mathbb{R})$.

Proposition 20. *The Lie algebra $\mathbf{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathbf{g}_2$, where $\mathbf{g}_2$ is a 3-dimensional simple Lie algebra, is reductive with an at most 2-dimensional subalgebra $\mathbf{h}$ and a complementary subspace $\mathbf{m}$ generating $\mathbf{g}$ in exactly one of the following cases:*

1) $\mathbf{h}_8 = \langle (e_3, 0) \rangle$, $\mathbf{m}_a = \langle (e_1, 0), (e_2, 0), (0, \varepsilon e_1), (0, \varepsilon e_2), (a e_3, e_3) \rangle$,

2) $\mathbf{h}_8 = \langle (e_3, 0) \rangle$, $\mathbf{m}_b = \langle (e_1, 0), (e_2, 0), (0, \varepsilon e_1), (b e_3, \varepsilon e_2), (0, e_3) \rangle$,

3) $\mathbf{h}_8 = \langle (e_3, 0) \rangle$, $\mathbf{m}_c = \langle (e_1, 0), (e_2, 0), (c e_3, \varepsilon e_1), (0, \varepsilon e_2), (0, e_3) \rangle$,

4) $\mathbf{h}_9 = \langle (e_1, 0) \rangle$, $\mathbf{m}_d = \langle (e_2, 0), (e_3, 0), (0, \varepsilon e_1), (0, \varepsilon e_2), (d e_1, e_3) \rangle$,

5) $\mathbf{h}_9 = \langle (e_1, 0) \rangle$, $\mathbf{m}_f = \langle (e_2, 0), (e_3, 0), (0, \varepsilon e_1), (f e_1, \varepsilon e_2), (0, e_3) \rangle$,

6) $\mathbf{h}_9 = \langle (e_1, 0) \rangle$, $\mathbf{m}_g = \langle (e_2, 0), (e_3, 0), (g e_1, \varepsilon e_1), (0, \varepsilon e_2), (0, e_3) \rangle$,

7) $\mathbf{h}_{16} = \langle (e_1, e_3) \rangle$, $\mathbf{m}_h = \langle (e_2, 0), (e_3, 0), (0, \varepsilon e_1), (0, \varepsilon e_2), (h e_1, (1 + h) e_3) \rangle$,

8) $\mathbf{h}_{17} = \langle (e_2 + e_3, e_3) \rangle$, $\mathbf{m}_k = \langle (e_3, k e_3), (e_1, 0), (0, \varepsilon e_1), (0, \varepsilon e_2), (e_2 + e_3, 0) \rangle$,

9) $\mathbf{h}_{18} = \langle (e_3, e_3) \rangle$, $\mathbf{m}_l = \langle (l e_3, (1 + l) e_3), (e_1, 0), (e_2, 0), (0, \varepsilon e_1), (0, \varepsilon e_2) \rangle$,

10) $\mathbf{h}_1 = \langle (e_3, 0), (0, e_3) \rangle$, $\mathbf{m}_1 = \langle (e_1, 0), (e_2, 0), (0, e_1), (0, e_2) \rangle$,

11) $\mathbf{h}_3 = \langle (e_3, 0), (0, e_1) \rangle$, $\mathbf{m}_3 = \langle (e_1, 0), (e_2, 0), (0, e_2), (0, e_3) \rangle$,

12) $\mathbf{h}_4 = \langle (e_1, 0), (0, e_1) \rangle$, $\mathbf{m}_4 = \langle (e_2, 0), (e_3, 0), (0, e_2), (0, e_3) \rangle$,

13) $\mathbf{h}_{13} = \langle (e_1, e_1) \rangle$, $\mathbf{m}_m = \langle (e_2, 0), (e_3, 0), (0, e_3), (0, e_2), (m e_1, (1 + m) e_1) \rangle$,

14) $\mathbf{h}_{14} = \langle (e_1, e_2 + e_3) \rangle$, $\mathbf{m}_n = \langle (e_2, 0), (e_3, 0), (0, e_1), (0, e_2 + e_3), (n e_1, e_2) \rangle$,

where $a, b, c, d, f, g, h, k, l, m, n \in \mathbb{R}$ and $\varepsilon = i$ for $\mathbf{g}_2 = \mathfrak{so}_3(\mathbb{R})$ whereas $\varepsilon = 1$ for $\mathbf{g}_2 = \mathfrak{sl}_2(\mathbb{R})$. The cases 1) till 10) occur for both simple 3-dimensional Lie algebras whereas the cases 10) till 14) occur only for $\mathbf{g}_2 = \mathfrak{sl}_2(\mathbb{R})$.

Proof. The basis elements of an arbitrary complement $\mathbf{m}_i$ to $\mathbf{h}_i$, $i = 1, \cdots, 18$, in $\mathbf{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathbf{g}_2$, where $\mathbf{g}_2$ is either $\mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{so}_3(\mathbb{R})$, are:
In the case $i = 1$

$$(e_1+a_1e_3, a_2e_3),\quad (e_2+b_1e_3, b_2e_3),\quad (c_1e_3, e_1+c_2e_3),\quad (d_1e_3, e_2+d_2e_3),$$

in the case $i=2$

$$(e_1+a_1e_3, a_2(e_2+e_3)),\quad (e_2+b_1e_3, b_2(e_2+e_3)),$$
$$(c_1e_3, e_1+c_2(e_2+e_3)),\quad (d_1e_3, e_3+d_2(e_2+e_3)),$$

in the case $i=3$

$$(e_1+a_1e_3, a_2e_1),\quad (e_2+b_1e_3, b_2e_1),\quad (c_1e_3, e_2+c_2e_1),\quad (d_1e_3, e_3+d_2e_1),$$

in the case $i=4$

$$(e_2+a_1e_1, a_2e_1),\quad (e_3+b_1e_1, b_2e_1),\quad (c_1e_1, e_2+c_2e_1),\quad (d_1e_1, e_3+d_2e_1),$$

in the case $i=5$

$$(e_2+a_1e_1, a_2(e_2+e_3)),\quad (e_3+b_1e_1, b_2(e_2+e_3)),$$
$$(c_1e_1, e_1+c_2(e_2+e_3)),\quad (d_1e_1, e_3+d_2(e_2+e_3)),$$

in the case $i=6$

$$(e_1+a_1(e_2+e_3), a_2(e_2+e_3)),\quad (e_3+b_1(e_2+e_3), b_2(e_2+e_3)),$$
$$(c_1(e_2+e_3), e_1+c_2(e_2+e_3)),\quad (d_1(e_2+e_3), e_3+d_2(e_2+e_3)),$$

in the case $i=7$

$$(e_3+a_1e_1+a_2(e_2+e_3), 0),\quad (b_1e_1+b_2(e_2+e_3), \varepsilon e_1),$$
$$(c_1e_1+c_2(e_2+e_3), \varepsilon e_2),\quad (d_1e_1+d_2(e_2+e_3), e_3),$$

in the case $i=8$

$$(e_1+a_1e_3, 0),\quad (e_2+a_2e_3, 0),\quad (a_3e_3, \varepsilon e_1),\quad (a_4e_3, \varepsilon e_2),\quad (a_5e_3, e_3),$$

in the case $i=9$

$$(e_2+a_1e_1, 0),\quad (e_3+a_2e_1, 0),\quad (a_3e_1, \varepsilon e_1),\quad (a_4e_1, \varepsilon e_2),\quad (a_5e_1, e_3),$$

in the case $i=10$

$$(e_2+a_1(e_2+e_3), 0),\quad (e_1+a_2(e_2+e_3), 0),\quad (a_3(e_2+e_3), \varepsilon e_1),$$
$$(a_4(e_2+e_3), \varepsilon e_2),\quad (a_5(e_2+e_3), e_3),$$

in the case $i=11$

$$(e_3+a_1e_1+a_2(e_2+e_3), a_1e_1+a_2(e_2+e_3)),$$
$$(b_1e_1+b_2(e_2+e_3), e_1+b_1e_1+b_2(e_2+e_3)),$$
$$(c_1e_1+c_2(e_2+e_3), e_2+c_1e_1+c_2(e_2+e_3)),$$
$$(d_1e_1+d_2(e_2+e_3), e_3+d_1e_1+d_2(e_2+e_3)),$$

in the case $i=12$

$$(e_3+a_1e_1+a_2(e_2+e_3), a_1k),\ \ (b_1e_1+b_2(e_2+e_3), \varepsilon e_1+b_1k),$$

$$(c_1e_1 + c_2(e_2 + e_3), \varepsilon e_2 + c_1k), \ \ (d_1e_1 + d_2(e_2 + e_3), e_3 + d_1k),$$

in the case $i = 13$

$$(e_2 + a_1e_1, a_1e_1), \quad (e_3 + a_2e_1, a_2e_1), \quad (a_3e_1, e_1 + a_3e_1),$$
$$(a_4e_1, e_2 + a_4e_1), \quad (a_5e_1, e_3 + a_5e_1),$$

in the case $i = 14$

$$(e_2 + a_1e_1, a_1(e_2 + e_3)), \quad (e_3 + a_2e_1, a_2(e_2 + e_3)), \quad (a_3e_1, e_1 + a_3(e_2 + e_3)),$$
$$(a_4e_1, e_2 + a_4(e_2 + e_3)), \quad (a_5e_1, e_3 + a_5(e_2 + e_3)),$$

in the case $i = 15$

$$(e_2 + a_1(e_2 + e_3), a_1(e_2 + e_3)), \quad (e_1 + a_2(e_2 + e_3), a_2(e_2 + e_3)),$$
$$(a_3(e_2 + e_3), e_1 + a_3(e_2 + e_3)), \quad (a_4(e_2 + e_3), e_2 + a_4(e_2 + e_3)),$$
$$(a_5(e_2 + e_3), e_3 + a_5(e_2 + e_3)),$$

in the case $i = 16$

$$(e_2 + a_1e_1, a_1e_3), \quad (e_3 + a_2e_1, a_2e_3), \quad (a_3e_1, \varepsilon e_1 + a_3e_3),$$
$$(a_4e_1, \varepsilon e_2 + a_4e_3), \quad (a_5e_1, e_3 + a_5e_3),$$

in the case $i = 17$

$$(e_2 + a_1(e_2 + e_3), a_1e_3), \quad (e_1 + a_2(e_2 + e_3), a_2e_3), \quad (a_3(e_2 + e_3), \varepsilon e_1 + a_3e_3),$$
$$(a_4(e_2 + e_3), \varepsilon e_2 + a_4e_3), \quad (a_5(e_2 + e_3), e_3 + a_5e_3),$$

in the case $i = 18$

$$(e_1 + a_1e_3, a_1e_3), \quad (e_2 + a_2e_3, a_2e_3), \quad (a_3e_3, \varepsilon e_1 + a_3e_3),$$
$$(a_4e_3, \varepsilon e_2 + a_4e_3), \quad (a_5e_3, e_3 + a_5e_3),$$

where a_i, $i = 1, 2, \cdots, 5$, b_j, $j = 1, 2$, c_k, $k = 1, 2$, d_l, $l = 1, 2$, are real parameters, $\varepsilon = \mathrm{i}$ for $\mathbf{g}_2 = \mathfrak{so}_3(\mathbb{R})$ and $\varepsilon = 1$ for $\mathbf{g}_2 = \mathfrak{sl}_2(\mathbb{R})$.
Using the relation $[\mathbf{h}_i, \mathbf{m}_i] \subseteq \mathbf{m}_i$, $i = 1, \cdots, 18$, and Lemma 17 we obtain the assertion. ◻

Proposition 21. *Let G be locally isomorphic to $PSL_2(\mathbb{R}) \times G_2$, where G_2 is either $PSL_2(\mathbb{R})$ or $SO_3(\mathbb{R})$. If G is the group topologically generated by the left translations of a connected almost differentiable left A-loop L then L is either a Scheerer extension of G_2 by $\mathbb{H}_2$ or the direct product $\mathbb{H}_2 \times \mathbb{H}_2$, where $\mathbb{H}_2$ denotes the hyperbolic plane loop. In the second case G is isomorphic to $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$.*

Proof. Since we assume that dim $L \geq 4$ we have to consider only the pairs $(\mathbf{h}, \mathbf{m})$ in Proposition 20. Now using **1.1** and **1.2** we obtain that the element $(0, e_1) \in \mathbf{h}_3 \cap \mathbf{h}_4$, the element $(e_1, 0) \in \mathbf{h}_9$, the element $(e_1, e_1) \in \mathbf{h}_{13}$ respectively the element $(e_1, e_2 + e_3) \in \mathbf{h}_{14}$ is conjugate in this order to

$(0, e_2) \in \mathbf{m}_3 \cap \mathbf{m}_4$, to $(e_2, 0) \in \mathbf{m}_d \cap \mathbf{m}_f \cap \mathbf{m}_g$, to $(e_2, e_2) \in \mathbf{m}_m$ respectively to $(e_2, e_2 + e_3) \in \mathbf{m}_n$. Hence there exists no global left A-loop L such that $T_e L$ is a reductive complement listed in the cases 4), 5), 6), 11), 12), 13), 14) (see. Lemma 3).

Now we consider the reductive complements $\mathbf{m}_a$, $\mathbf{m}_b$, $\mathbf{m}_c$ in 1) till 3) of Proposition 20. First we assume that $a \neq 0$, $b \neq 0$, $c \neq 0$. The vectors $v_{j,l} = \left(ke_3, \frac{k}{l}\varepsilon e_j\right)$, $w_{j,l} = \left(\sqrt{k^2 - 4\pi^2}e_2 + ke_3, \frac{k}{l}\varepsilon e_j\right)$, where $k > 2\pi$ is an integer, are contained in the subspace $\mathbf{m}_a$ for $j = 3$, $l = a$ and $\varepsilon = 1$, in the subspace $\mathbf{m}_b$ for $j = 2$, $l = b$, respectively in $\mathbf{m}_c$ for $j = 1$, $l = c$, where $\varepsilon = 1$ for $\mathbf{g}_2 = \mathfrak{sl}_2(\mathbb{R})$ and $\varepsilon = \mathrm{i}$ for $\mathbf{g}_2 = \mathfrak{so}_3(\mathbb{R})$. According to **1.1** and **1.2** the images of $v_{j,l}, w_{j,l}$, $j = 1, 2, 3$, under the exponential map have the following representatives in $PSL_2(\mathbb{R}) \times G_2$:

$$m_1 = \exp v_{3,a} = \left(A, \begin{pmatrix} \cos\frac{k}{a} & \sin\frac{k}{a} \\ -\sin\frac{k}{a} & \cos\frac{k}{a} \end{pmatrix}\right),$$

$$m_2 = \exp w_{3,a} = \left(I, \begin{pmatrix} \cos\frac{k}{a} & \sin\frac{k}{a} \\ -\sin\frac{k}{a} & \cos\frac{k}{a} \end{pmatrix}\right),$$

$$m_3 = \exp v_{2,b} = \left(A, \begin{pmatrix} \cosh\left(\frac{k}{b}\varepsilon\right) & \sinh\left(\frac{k}{b}\varepsilon\right) \\ -\sinh\left(\frac{k}{b}\varepsilon\right) & \cosh\left(\frac{k}{b}\varepsilon\right) \end{pmatrix}\right),$$

$$m_4 = \exp w_{2,b} = \left(\pm I, \begin{pmatrix} \cosh\left(\frac{k}{b}\varepsilon\right) & \sinh\left(\frac{k}{b}\varepsilon\right) \\ -\sinh\left(\frac{k}{b}\varepsilon\right) & \cosh\left(\frac{k}{b}\varepsilon\right) \end{pmatrix}\right),$$

$$m_5 = \exp v_{1,c} = \left(A, \begin{pmatrix} \cosh\left(\frac{k}{c}\varepsilon\right) + \sinh\left(\frac{k}{c}\varepsilon\right) & 0 \\ 0 & \cosh\left(\frac{k}{c}\varepsilon\right) - \sinh\left(\frac{k}{c}\varepsilon\right) \end{pmatrix}\right),$$

$$m_6 = \exp w_{1,c} = \left(I, \begin{pmatrix} \cosh\left(\frac{k}{c}\varepsilon\right) + \sinh\left(\frac{k}{c}\varepsilon\right) & 0 \\ 0 & \cosh\left(\frac{k}{c}\varepsilon\right) - \sinh\left(\frac{k}{c}\varepsilon\right) \end{pmatrix}\right),$$

where $A = \begin{pmatrix} \cos k & \sin k \\ -\sin k & \cos k \end{pmatrix}$, $\varepsilon = \mathrm{i}$ for $\mathbf{g}_2 = \mathfrak{so}_3(\mathbb{R})$, whereas $\varepsilon = 1$ for $\mathbf{g}_2 = \mathfrak{sl}_2(\mathbb{R})$. For the representatives

$$g_1 = \left(I, \begin{pmatrix} \cos\frac{k}{a} & \sin\frac{k}{a} \\ -\sin\frac{k}{a} & \cos\frac{k}{a} \end{pmatrix}\right),$$

$$g_2 = \left(I, \begin{pmatrix} \cosh\left(\frac{k}{b}\varepsilon\right) & \sinh\left(\frac{k}{b}\varepsilon\right) \\ -\sinh\left(\frac{k}{b}\varepsilon\right) & \cosh\left(\frac{k}{b}\varepsilon\right) \end{pmatrix}\right),$$

$$g_3 = \left(I, \begin{pmatrix} \cosh\left(\frac{k}{c}\varepsilon\right) + \sinh\left(\frac{k}{c}\varepsilon\right) & 0 \\ 0 & \cosh\left(\frac{k}{c}\varepsilon\right) - \sinh\left(\frac{k}{c}\varepsilon\right) \end{pmatrix}\right)$$

we have $g_1 = m_1 \cdot h_1 = m_2$, $g_2 = m_3 \cdot h_1 = m_4$, $g_3 = m_5 \cdot h_1 = m_6$ such that $h_1 = \left(A^{-1}, I\right)$. These facts again contradict Lemma 3.

For $a = 0$, $b = 0$, $c = 0$ the complements $\mathbf{m}_a$, $\mathbf{m}_b$, $\mathbf{m}_c$ in 1) till 3) of Proposition 20 reduce to $\mathbf{m}_0 = \langle (e_1, 0), (e_2, 0), (0, \varepsilon e_1), (0, \varepsilon e_2), (0, e_3) \rangle$. The exponential image $\exp \mathbf{m}_0$ is the direct product $M \times G_2$, such that M is the image of the section corresponding to the hyperbolic plane loop $\mathbb{H}_2$ (cf. [18], pp. 283-284) and G_2 is the group $PSL_2(\mathbb{R})$ respectively $SO_3(\mathbb{R})$ according whether $\varepsilon = 1$ or $\varepsilon = \mathrm{i}$. Since H has the shape $H_1 \times \{1\}$, where $H_1 \cong SO_2(\mathbb{R}) \le PSL_2(\mathbb{R})$ the global loop L_0 realized on $\exp \mathbf{m}_0$ is the direct product of $\mathbb{H}_2$ and G_2.

Now we treat the complements $\mathbf{m}_h$, $\mathbf{m}_k$, $\mathbf{m}_l$, $h, k, l \in \mathbb{R}$ of the cases 7) till 9) in Proposition 20. The reductive complement $\mathbf{m}_a, a \in \mathbb{R}$, $\mathbf{m}_b, b \in \mathbb{R}$, respectively $\mathbf{m}_c, c \in \mathbb{R}$ of Lemma 12 in [6], p. 404, is in this order a subspace of $\mathbf{m}_h$, $\mathbf{m}_k$, respectively $\mathbf{m}_l$. Moreover, the subalgebra $\mathbf{h}_{16}$ in the case 7) coincides with the subalgebra $\mathbf{h}$ in case 1) of Lemma 12 in [6], the subalgebra $\mathbf{h}_{17}$ in the case 8) is equal with the subalgebra $\mathbf{h}$ in case 2) of Lemma 12 in [6], and the subalgebra $\mathbf{h}_{18}$ in the case 9) coincides with the subalgebra $\mathbf{h}$ in case 3) of Lemma 12 in [6], p. 404. Hence the same computations as in the proof of Proposition 13 in [6], pp. 404-406, show that for $h \neq -1$ the complement $\mathbf{m}_h$, for $k \neq 0$ the complement $\mathbf{m}_k$ and for $l \notin \{0, -1\}$ the complement $\mathbf{m}_l$ cannot be the tangent space of a global almost differentiable left A-loop.

It remains to consider the complements $\mathbf{m}_{h=-1}$, $\mathbf{m}_{k=0}$, $\mathbf{m}_{l=0}$ and $\mathbf{m}_{l=-1}$. First let $\varepsilon = \mathrm{i}$. Then the element $(e_1, e_3) \in \mathbf{h}_{16}$ is conjugate to $(e_2, \mathrm{i}e_1) \in \mathbf{m}_{h=-1}$, the element $(e_2 + e_3, e_3) \in \mathbf{h}_{17}$ is conjugate to $(e_2 + e_3, \mathrm{i}e_1) \in \mathbf{m}_{k=0}$ and the element $(e_3, e_3) \in \mathbf{h}_{18}$ is conjugate to $(e_3, \mathrm{i}e_1) \in \mathbf{m}_{l=-1}$ (see **1.2**), which are contradictions to Lemma 3. Since the exponential image of the Lie algebra $\mathbf{h}_{18}$ has the shape $H_n = \{(x, x^n) \mid x \in SO_2(\mathbb{R}), n \in \mathbb{N}\backslash\{0\}\}$ the exponential image $M \times SO_3(\mathbb{R})$ of the complement $\mathbf{m}_{l=0}$, where M is the image of the section belonging to the hyperbolic plane loop $\mathbb{H}_2$ (cf. [18], pp. 283-284), yields Scheerer extensions of $SO_3(\mathbb{R})$ by $\mathbb{H}_2$ (cf. [18], Section 2).

Finally let $\varepsilon = 1$. The complements $\mathbf{m}_{h=-1}$, $\mathbf{m}_{k=0}$, $\mathbf{m}_{l=-1}$ and $\mathbf{m}_{l=0}$ are (up to interchanging the components) equal to the vector space

$$\mathbf{m}' = \langle (e_1, 0), (e_2, 0), (e_3, 0), (0, e_1), (0, e_2) \rangle$$

and its exponential image $\exp \mathbf{m}'$ is the direct product $PSL_2(\mathbb{R}) \times M$, where M is the image of the section corresponding to $\mathbb{H}_2$. The group $H = \{(\varphi(x), x) \mid x \in SO_2(\mathbb{R})\}$ coincides with the group H_{16} belonging to $\mathbf{h}_{16}$ respectively with H_{17} of $\mathbf{h}_{17}$ if φ is a homomorphism from $SO_2(\mathbb{R})$ onto a hyperbolic respectively a parabolic 1-parameter subgroup of $PSL_2(\mathbb{R})$. The subgroup H_{18} of $\mathbf{h}_{18}$ has the form: $H'_n = \{(x^n, x) \mid x \in SO_2(\mathbb{R}), n \in \mathbb{N}\backslash\{0\}\}$. According to [18], Section 2, any loop L realized on the factor space G/H_n, $n = 16, 17, 18$, and having $\exp \mathbf{m}'$ as the image of its section is a Scheerer extension of the Lie group $PSL_2(\mathbb{R})$ by $\mathbb{H}_2$.

All Scheerer extensions having $PSL_2(\mathbb{R}) \times SO_3(\mathbb{R})$ or $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ as the group topologically generated by their left translations satisfy the Bol identity because of $\big[[\mathbf{m}, \mathbf{m}], \mathbf{m}\big] \subset \mathbf{m}$ but they are not Bruck loops since there is no involutory automorphism $\sigma : \mathbf{g} \to \mathbf{g}$ such that $\sigma(\mathbf{m}) = -\mathbf{m}$ and $\sigma(\mathbf{h}) = \mathbf{h}$.

In the remaining case 10) in Proposition 20 the subgroup H_1 of $\mathbf{h}_1$ is the direct product $SO_2(\mathbb{R}) \times SO_2(\mathbb{R})$ and the exponential image M_1 of $\mathbf{m}_1$ is the direct product $M \times M$, where M is the image of the section belonging to $\mathbb{H}_2$. According to Proposition 1.19 in [18], p. 28, the loop L is the direct product $\mathbb{H}_2 \times \mathbb{H}_2$. □

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Author address:
Mathematisches Institut der Universität Erlangen-Nürnberg,
Bismarckstr. 1 $\frac{1}{2}$,
D-91054 Erlangen, Germany;
and Institute of Mathematics, University of Debrecen,
P.O.B. 12, H-4010 Debrecen, Hungary
figula@mi.uni-erlangen.de; figula@math.klte.hu