# EFFECTIVE RESULTS AND METHODS FOR DIOPHANTINE EQUATIONS OVER FINITELY GENERATED DOMAINS 

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To our families

## Contents

Preface ..... v
History and summary ..... ix
1 Ineffective results for Diophantine equations over finitely gener- ated domains ..... 1
1.1 Thue equations ..... 2
1.2 Unit equations in two unknowns ..... 5
1.3 Hyper- and superelliptic equations ..... 8
1.4 Curves with finitely many integral points ..... 9
1.5 Decomposable form equations and multivariate unit equations ..... 10
1.6 Discriminant equations for polynomials and integral elements ..... 14
2 Effective results for Diophantine equations over finitely generated
domains: the statements ..... 19
2.1 Notation and preliminaries ..... 19
2.2 Unit equations in two unknowns ..... 23
2.3 Thue equations ..... 25
2.4 Hyper- and superelliptic equations, the Schinzel-Tijdeman equa- ..... 26
2.5 The Catalan equation ..... 28
2.6 Decomposable form equations ..... 29
2.7 Norm form equations ..... 34
2.8 Discriminant form equations and discriminant equations ..... 36
2.9 Open problems ..... 40
3 A brief explanation of our effective methods over finitely gener- ated domains ..... 43
3.1 Sketch of the effective specialization method ..... 44
3.2 Illustration of the application of the effective specialization method to Diophantine equations ..... 50
3.3 Sketch of the method reducing equations to unit equations ..... 52
3.3.1 Effective finiteness result for systems of unit equations ..... 53
3.3.2 Reduction of decomposable form equations to unit equations ..... 54
3.3.3 Quantitative version ..... 56
3.3.4 Reduction of discriminant equations to unit equations ..... 57
3.4 Comparison of our two effective methods ..... 60
4 Effective results over number fields ..... 61
4.1 Notation and preliminaries ..... 62
4.2 Effective estimates for linear forms in logarithms ..... 71
$4.3 \quad S$-unit equations ..... 76
4.4 Thue equations ..... 80
4.5 Hyper- and superelliptic equations, the Schinzel-Tijdeman equa- ..... 82
4.6 The Catalan equation ..... 92
4.7 Decomposable form equations ..... 101
4.8 Discriminant equations ..... 107
5 Effective results over function fields ..... 111
5.1 Notation and preliminaries ..... 111
$5.2 \quad S$-unit equations ..... 116
5.3 The Catalan equation ..... 118
5.4 Thue equations ..... 120
5.5 Hyper- and superelliptic equations ..... 123
6 Tools from effective commutative algebra ..... 131
6.1 Effective linear algebra over polynomial rings ..... 132
6.2 Finitely generated fields over $\mathbb{Q}$ ..... 137
6.3 Finitely generated integral domains over $\mathbb{Z}$ ..... 140
7 The effective specialization method ..... 147
7.1 Notation ..... 148
7.2 Construction of a more convenient ground domain $B$ ..... 149
7.3 Comparison of different degrees and heights ..... 156
7.4 Specializations ..... 161
7.5 Multiplicative independence ..... 172
8 Degree-height estimates ..... 181
8.1 Definitions. ..... 181
8.2 Estimates for factors of polynomials ..... 183
8.3 Consequences ..... 188
9 Proofs of the results from Sections $\mathbf{2} 2 \boldsymbol{2} \boldsymbol{2} \mathbf{2 . 5}$; use of specializations ..... 199
9.1 A reduction ..... 200
9.1.1 Unit equations ..... 201
9.1.2 Thue equations ..... 204
9.1.3 Hyper- and superelliptic equations ..... 205
9.2 Bounding the degrees ..... 206
9.2.1 Unit equations ..... 207
9.2.2 Thue equations ..... 208
9.2.3 Hyper- and superelliptic equations ..... 209
9.3 Bounding the heights, specializations ..... 211
9.3.1 Unit equations ..... 212
9.3.2 Thue equations ..... 215
9.3.3 Hyper- and superelliptic equations ..... 220
9.4 The Catalan equation ..... 222
10 Proofs of the results from Sections $2.6+2.8$; reduction to unit equa- tions ..... 227
10.1 Proofs of the central results on decomposable form equations ..... 227
10.2 Proofs of the results for norm form equations ..... 235
10.3 Proofs of the results for discriminant form equations and dis- criminant equations ..... 237
Bibliography ..... 243
Glossary of frequently used notation ..... 257

## Preface

This book is devoted to Diophantine equations where the solutions are taken from an integral domain of characteristic 0 that is finitely generated over $\mathbb{Z}$, that is a domain of the shape $\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]$ with quotient field of characteristic 0 , where the generators $z_{1}, \ldots, z_{r}$ may be algebraic or transcendental over $\mathbb{Q}$. For instance, the ring of integers and the rings of $S$-integers of a number field are finitely generated domains where all generators are algebraic. Our aim is to prove effective finiteness results for certain classes of Diophantine equations, i.e., results that not only show that the equations from the said classes have only finitely many solutions, but whose proofs provide methods to determine the solutions in principle.

There is an extensive literature on Diophantine equations with solutions taken from the ring of rational integers $\mathbb{Z}$, or from more general domains, containing theorems on the finiteness of the set of solutions of such equations. Most of the finiteness theorems over $\mathbb{Z}$, and more generally over rings of integers and $S$-integers of number fields are ineffective. Their proofs are mainly based on techniques from Diophantine approximation (e.g., the Thue-Siegel-Roth-Schmidt theory) often combined with algebra and arithmetic geometry. These techniques yield the finiteness of the number of solutions, but do not enable one to determine the solutions. Lang (1960) and others used certain specialization arguments to extend several ineffective finiteness results to the even more general case when the solutions are taken from an arbitrary integral domain of characteristic 0 that is finitely generated over $\mathbb{Z}$.

Since the 1960 's, a great number of ineffective finiteness theorems over number fields were made effective and new theorems were obtained in effective form by means of A. Baker's effective theory of logarithmic forms. These results give effective upper bounds for the solutions, and thereby make it possible, at least in principle, to find all the solutions of the equations under consideration. Analogous theorems were established by Mason (1984) and others over function fields of characteristic 0 as well, which provide effective
upper bounds for the heights of the solutions, but do not imply the finiteness of the number of solutions.

Győry (1983,1984b) initiated to extend effective Diophantine results over number fields to the finitely generated case, and proved effective finiteness theorems over certain restricted classes of finitely generated integral domains over $\mathbb{Z}$ of zero characteristic. He developed an effective specialization method, reducing the initial equations to the number field and function field cases, and using the corresponding effective results over number fields and function fields, he derived effective bounds for the solutions of the initial equations.

In the paper Evertse and Győry (2013), Győry's specialization method was extended to the case of arbitrary finitely generated domains of characteristic 0 over $\mathbb{Z}$. The crucial new tool in this extension was work of Aschenbrenner (2004) on effective commutative algebra. Evertse's and Győry's general specialization method may be viewed as a 'machine,' which takes as input an effective Diophantine finiteness result concerning $S$-integral solutions over number fields together with an effective analogue over function fields, and produces as output a corresponding effective result over finitely generated domains. This general specialization method lead to effective finiteness results for various classes of Diophantine equations over arbitrary domains of characteristic 0 that are finitely generated over $\mathbb{Z}$ : Evertse and Győry (2013,2014,2015), Bérczes, Evertse and Győry (2014), Bérczes (2015a,b), Koymans $(2016,2017)$ established general effective finiteness theorems over finitely generated domains of characteristic 0 for several classical equations, including unit equations in two unknowns, Thue equations, hyper- and superelliptic equations and the Catalan equation. An important feature of these results is their quantitative nature, i.e., they give upper bounds for the sizes (suitable measures) of the solutions in terms of defining parameters for the domain from which the solutions are taken and for the Diophantine equation under consideration.

Our book provides the first comprehensive treatment of effective results and methods for Diophantine equations over finitely generated domains. Similarly to the above mentioned literature, most of the results in our book are proved in quantitative form, giving effective bounds for the sizes of the solutions. Apart from the results mentioned above, our book contains new material, concerning decomposable form equations over finitely generated domains. Here we have adapted the method of Győry $(1973,1980 b)$ and Győry and Papp (1978) to reduce the decomposable form equations under consideration to systems of unit equations in two unknowns. Here again, we give
effective upper bounds for the sizes of the solutions, and for this purpose we had to work out new effective procedures. As a special case, we get back the results on discriminant equations from Evertse and Győry (2017a,b).

We believe that the results in this book do not exhaust the possibilities of our techniques. Hopefully, they will inspire further investigations to obtain new effective results for other classes of Diophantine equations over finitely generated domains.

This book is aimed at anybody (graduate student and expert) with basic knowledge of algebra (groups, commutative rings, fields, Galois theory) and elementary algebraic number theory. No further specialized knowledge on commutative algebra or algebraic geometry is presupposed.

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## History and summary

First we give a brief historical overview on the equations treated in our book, and then outline the contents of the book.

We start with ineffective results. Thue (1909) developed an ingenious method for approximation of algebraic numbers by rationals. As an application he proved that if $F \in \mathbb{Z}[X, Y]$ is a binary form (i.e., a homogeneous polynomial) of degree at least 3 which is irreducible over $\mathbb{Q}$ and $\delta$ is a nonzero integer, then the equation

$$
\begin{equation*}
F(x, y)=\delta \text { in } x, y \in \mathbb{Z} \tag{1}
\end{equation*}
$$

(nowadays called a Thue equation) has only finitely many solutions. Thue's approximation result was later considerably improved and generalized by many people including Siegel, Mahler, Dyson, Gel'fond, Roth, Schmidt, and Schlickewei.

Thue's finiteness theorem concerning equation (1) has many generalizations. Siegel (1921) generalized it for the number field case when the ground ring, i.e., the ring from which the solutions are taken, is the ring of integers $\mathcal{O}_{K}$ of a number field $K$. Mahler (1933) extended Thue's theorem to the case of ground rings of the form $\mathbb{Z}\left[\left(p_{1} \cdots p_{s}\right)^{-1}\right]$, where $p_{1}, \ldots, p_{s}$ are primes, while Parry (1950) gave a common generalization of the results of Siegel and Mahler to the case where the ground ring is the ring of $S$-integers of a number field.

Siegel's theorem has the following important consequence, which was not stated explicitly by Siegel, but was implicitly proved by him. Denote by $\mathcal{O}_{K}^{*}$ the group of units of $\mathcal{O}_{K}$, and let $\alpha, \beta$ be non-zero elements of the number field $K$. Using the fact that $\mathcal{O}_{K}^{*}$ is finitely generated, it is easy to deduce from Siegel's theorem that the equation

$$
\begin{equation*}
\alpha x+\beta y=1 \tag{2}
\end{equation*}
$$

in $x, y \in \mathcal{O}_{K}^{*}$ has only finitely many solutions. Similarly, it follows from the results of Mahler and Parry that equation (2) has finitely many solutions even in $S$-units of $K$, these are elements of $K$ composed of prime ideals from a finite, possibly empty set $S$ of prime ideals of $\mathcal{O}_{K}$. Nowadays equation (2) is called a unit equation (when $S$ is empty) resp. $S$-unit equation otherwise, or more precisely unit equation and $S$-unit equation in two unknowns.

Further important equations are

$$
\begin{equation*}
f(x)=\delta y^{m} \text { in } x, y \in \mathbb{Z}, \tag{3}
\end{equation*}
$$

where $f \in \mathbb{Z}[X]$ is a polynomial of degree $n$ and $\delta \in \mathbb{Z} \backslash\{0\}$. Equation (3) is called elliptic if $n=3, m=2$, more generally hyperelliptic if $n \geq 3$, $m=2$, and superelliptic if $n \geq 2, m \geq 3$. If $m$ or $n$ is at least 3 and $f$ has no multiple zero, equation (3) has only finitely many solutions. This was proved in the elliptic case by Mordell (1922a,b,1923), in the hyperelliptic case by Siegel (1926), and in the superelliptic case by Siegel (1929). LeVeque (1964) considered (3) in the more general case when $f$ may have multiple zeros, and gave a finiteness criterion for (3) over the ring of integers of a number field.

A celebrated theorem of Siegel (1929) states that if $F(X, Y)$ is a polynomial with coefficients in a number field $K$, which is irreducible over $\bar{K}$ and the affine curve $F(x, y)=0$ is of genus $\geq 1$, then this curve has only finitely many points with integral coordinates in $K$. This theorem implies the above-mentioned finiteness results on Thue equations, unit equations and hyperelliptic/superelliptic equations over number fields.

Lang (1960) generalized Siegel's theorem to what we call the finitely generated case, when the solutions are taken from an arbitrary integral domain of characteristic 0 which is finitely generated as a $\mathbb{Z}$-algebra, that is a domain of the shape

$$
\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]=\left\{f\left(z_{1}, \ldots, z_{r}\right): f \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]\right\}
$$

where $z_{1}, \ldots, z_{r}$ may be algebraic or transcendental over $\mathbb{Q}$. Recall that both the ring of integers of a number field $K$, and the rings of $S$-integers of $K$, are of this shape, with $z_{1}, \ldots, z_{r}$ all algebraic. In his proof, Lang used a specialization argument, reducing the theorem to the case of number fields and function fields of one variable, and then applied Siegel's theorem (1929) and its function field analogue from Lang (1960). As a consequence, Lang extended the earlier finiteness results concerning Thue equations, unit equations and hyperelliptic/superelliptic equations to the finitely generated case.

Multivariate generalizations of Thue equations that have attracted much attention are the decomposable form equations

$$
\begin{equation*}
F\left(x_{1}, \ldots, x_{m}\right)=\delta \text { in } x_{1}, \ldots, x_{m} \in \mathbb{Z} \tag{4}
\end{equation*}
$$

where $\delta \in \mathbb{Z} \backslash\{0\}$ and $F\left(X_{1}, \ldots, X_{m}\right)$ is a decomposable form of degree $n>m$ in $m \geq 2$ variables with coefficients in $\mathbb{Z}$ i.e., a homogeneous polynomial which factorizes into linear forms with coefficients in the algebraic closure $\overline{\mathbb{Q}}$. Further important types of decomposable form equations are norm form equations, discriminant form equations and index form equations which are of basic importance in algebraic number theory. $\operatorname{Schmidt}(1971,1972)$ developed a multidimensional generalization of Roth's theorem on the approximation of algebraic numbers, eventually leading to his famous Subspace Theorem, and from the latter he deduced a finiteness criterion for norm form equations. Evertse and Győry (1988b) proved a general finiteness criterion for decomposable form equations of the form (4). Their proof depends on the following finiteness result on multivariate unit equations of the form

$$
\begin{equation*}
\alpha_{1} x_{1}+\cdots+\alpha_{m} x_{m}=1 \text { in } x_{1}, \ldots, x_{m} \in \mathcal{O}_{K}^{*} \tag{5}
\end{equation*}
$$

where $K$ is a number field and $\alpha_{1}, \ldots, \alpha_{m}$ are non-zero elements of $K$. A solution of (5) is called degenerate if there is a vanishing subsum on the left hand side of (5). In this case (5) has infinitely many solutions if $\mathcal{O}_{K}^{*}$ is infinite. As a generalization of Siegel's theorem on equation (2), van der Poorten and Schlickewei (1982) and Evertse (1984) proved independently of each other that equation (5) has only finitely many non-degenerate solutions. This theorem was extended by Evertse and Győry (1988a) and van der Poorten and Schlickewei (1991) to the finitely generated case, when $K$ is a finitely generated extension of $\mathbb{Q}$ and $\mathcal{O}_{K}^{*}$ is replaced by a finitely generated multiplicative subgroup of $K^{*}$. As a consequence, the above-mentioned general finiteness criterion for (4) was proved in Evertse and Győry (1988b) in a more general form, over finitely generated domains of characteristic 0 .

In the 1960's, Baker developed an effective method in transcendence theory, providing non-trivial effective lower bounds for linear forms in logarithms of algebraic numbers. This furnished a very powerful tool to prove effective finiteness results for Diophantine equations over $\mathbb{Z}$ and more generally over number fields, that enabled one to determine, at least in principle, all solutions of the equations under consideration. Using his method, Baker (1968b,c,1969) derived explicit upper bounds among others for the
solutions of Thue equations and hyperelliptic/superelliptic equations. Győry $(1974,1979)$ used Baker's theory of logarithmic forms to obtain explicit upper bounds for the solutions of unit equations and $S$-unit equations in two unknowns. With the help of his bounds Györy proved effective finiteness theorems for discriminant equations for polynomials

$$
\begin{equation*}
D(f)=\delta \text { in monic polynomials } f \in \mathbb{Z}[X] \tag{6}
\end{equation*}
$$

and for elements

$$
\begin{equation*}
D(\alpha)=\delta \text { in algebraic integers } \alpha \tag{7}
\end{equation*}
$$

Here $D()$ denotes the discriminant of a polynomial $f$ resp. of an algebraic integer $\alpha$, and $\delta$ is a non-zero integer. Two monic polynomials $f, f^{\prime} \in \mathbb{Z}[X]$ are called strongly $\mathbb{Z}$-equivalent if $f^{\prime}(X)=f(X+a)$ for some $a \in \mathbb{Z}$. Similarly, two algebraic integers $\alpha, \alpha^{\prime}$ are said to be strongly $\mathbb{Z}$-equivalent if $\alpha^{\prime}-\alpha \in \mathbb{Z}$. Clearly, strongly $\mathbb{Z}$-equivalent monic polynomials resp. algebraic integers have the same discriminant.

Győry (1973) proved that there are only finitely many pairwise $\mathbb{Z}$-inequivalent monic polynomials with the property (6). A similar finiteness theorem was proved for the solutions of (7) by Birch and Merriman (1972), and independently by Győry (1973). Győry's proofs for (6) and (7) are effective. These results, in less precise form, were generalized in Győry (1978a) for the number field case, and in Győry (1982) in an ineffective form, for the finitely generated case, subject to the condition that the ground ring is integrally closed. These results have many applications, among others to power integral bases of ring extensions.

By using Győry's bounds on the solutions of unit equations in two unknowns, Győry (1976,1980a) and Győry and Papp (1978) generalized Baker's effective theorem on Thue equations to equations in arbitrarily many unknowns. They derived explicit bounds for the solutions of a class of decomposable form equations over number fields, including discriminant form equations and certain norm form equations.

Tijdeman (1976) used Baker's theory of logarithmic forms to give an explicit upper bound for the solutions of the Catalan equation

$$
\begin{equation*}
x^{m}-y^{n}=1 \text { in positive integers } x, y, m, n \text { with } m, n>1 \text { and } m n>4 . \tag{8}
\end{equation*}
$$

Further, when in equation (3) $m$ is also unknown and $f$ has at least two distinct zeros, Schinzel and Tijdeman (1976) gave an effective upper bound for $m$. In this case equation (3) is now called Schinzel-Tijdeman equation. It is interesting to note that the effective theorems of Tijdeman (1976) and Schinzel and Tijdeman (1976) had no previously ineffective versions.

For Thue equations, unit equations and hyper/superelliptic equations, analogous effective results were obtained by Mason $(1981,1983,1984)$ and others over function fields of characteristic 0 . The above-mentioned effective results over number fields and function fields were later improved and generalized by many people, and led to several further applications.

In Gyôry $(1983,1984 b)$ the author extended the effective finiteness theorems concerning Thue equations, discriminant equations and a class of decomposable form equations over number fields to similar such equations over restricted classes of finitely generated domains of characteristic 0 which may contain both algebraic and transcendental elements. To prove these extensions, Győry developed an effective specialization method to reduce the general equations under consideration to equations of the same type over number fields and function fields, and then used effective results concerning these reduced equations to derive effective bounds for the solutions of the initial equations.

Evertse and Győry (2013) refined the method of Győry, and proved effective finiteness theorems for unit equations in two unknowns in full generality, over arbitrary finitely generated domains of characteristic 0 over $\mathbb{Z}$. In fact, they obtained their results by combining Győry's techniques with work of Aschenbrenner (2004), concerning the effective resolution of systems of linear equations over polynomial rings $\mathbb{Z}\left[X_{1}, \ldots, X_{n}\right]$.

The general effective specialization method of Evertse and Győry led to effective finiteness results over finitely generated domains for several other classes of Diophantine equations, such as Thue equations, hyper/superelliptic equations and the Schinzel-Tijdeman equation (Bérczes, Evertse and Győry (2014)), a generalization of unit equations (Bérczes (2015a,b)) and the Catalan equation (Koymans (2016, 2017)). Further, generalizing another method of Győry (1973) and Győry and Papp (1978) applied over number fields, the present authors in Evertse and Győry (2017a,b) and in Sections 2.6 and 2.8 of this book obtained effective finiteness theorems for decomposable form equations and discriminant equations over finitely generated domains. This other method is not based on specialization, but instead uses a reduction of the equation under consideration to unit equations in two unknowns.

It is important to note that with the exception of discriminant equations and hyper- and superelliptic equations, both methods mentioned above provide quantitative results over finitely generated domains, giving effective bounds for the solutions. This is due to the effective and quantitative feature of the main tools from Chapters 4 to 8 .

Major open problems are to make effective the general finiteness theorems of Siegel (1929) on integral points of curves and of van der Poorten and Schlickewei (1982) and Evertse (1984) on multivariate unit equations over number fields. Such effective versions could be extended to the finitely generated case, using existing analogues over function fields and applying our general effective specialization method.

We now outline the contents of our book. In Chapter 1, we present the most general ineffective finiteness results over finitely generated domains for Thue equations, unit equations in two unknowns, a generalization of unit equations, hyper- and superelliptic equations, curves of genus $\geq 1$ with finitely many integral points, decomposable form equations, multivariate unit equations and discriminant equations. Further, except for curves of genus $\geq 1$ and multivariate unit equations, we cite the most general effective versions concerning the equations mentioned over number fields.

In Chapter 2, we state general effective finiteness theorems over finitely generated domains of characteristic 0 for unit equations in two unknowns, Thue equations, hyper- and superelliptic equations, the Schinzel-Tijdeman equation, the Catalan equation, decomposable form equations and discriminant equations. As was mentioned above, apart from discriminant equations, the other results give also effective bounds for the solutions.

Chapter 3is devoted to a short explanation of our general effective methods.

In Chapters 4 and 5 those effective results are collected on the above equations over number fields and function fields that are needed in Chapters 9 and 10, in the proofs of the general effective theorems stated in Chapter 2. We have skipped the complete proofs of the theorems in Chapters 4 and 5 , which are rather technical. Instead, we sketch the proofs in simplified forms, which give sufficient insight in the main ideas.

Chapters 6, 7 and 8 contain further important tools. In Chapter 6 we have collected results from effective commutative algebra, in Chapter 7 we give the detailed treatment of our effective specialization method, and in Chapter 8 we prove some useful results for'degree-height estimates,' which may be viewed as an analogue of the naiv height estimates of algebraic numbers for elements
of the algebraic closure of a finitely generated field.
Lastly, in Chapters 9 and 10, the results and methods from Chapters 4.8 are combined to prove the general effective results presented in Chapter 2

## Chapter 1

## Ineffective results for Diophantine equations over finitely generated domains

This book is about Diophantine equations where the solutions are taken from an integral domain of characteristic 0 that is finitely generated over $\mathbb{Z}$, that is, from a domain of the shape

$$
\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]=\left\{f\left(z_{1}, \ldots, z_{r}\right): f \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]\right\}
$$

whose quotient field is of characteristic 0 . The generators $z_{1}, \ldots, z_{r}$ may be either algebraic or transcendental over $\mathbb{Q}$.

For instance, let $K$ be a number field and $\mathcal{O}_{K}$ its ring of integers. Let $\left\{\omega_{1}, \ldots, \omega_{d}\right\}$ be a $\mathbb{Z}$-module basis of $\mathcal{O}_{K}$. Then $\mathcal{O}_{K}=\mathbb{Z}\left[\omega_{1}, \ldots, \omega_{d}\right]$.

More generally, let $K$ be a number field and with the notation introduced in Section 4.2, let $S$ be a finite set of places of $K$, consisting of all infinite places of $K$ and of the prime ideals $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{t}$ of $\mathcal{O}_{K}$. Then the ring of $S$ integers of $K$, denoted by $\mathcal{O}_{S}$, is given by the set of all elements $\alpha$ of $K$ such that there are non-negative integers $k_{1}, \ldots, k_{t}$ with $\alpha \mathfrak{p}_{1}^{k_{1}} \cdots \mathfrak{p}_{t}^{k_{t}} \subseteq \mathcal{O}_{K}$. In the particular case that $S$ consists only of the infinite places of $K$, the ring $\mathcal{O}_{S}$ is just equal to $\mathcal{O}_{K}$. We may express $\mathcal{O}_{S}$ otherwise as

$$
\mathcal{O}_{S}=\mathbb{Z}\left[\omega_{1}, \ldots, \omega_{d}, \pi^{-1}\right],
$$

where again $\left\{\omega_{1}, \ldots, \omega_{d}\right\}$ is a $\mathbb{Z}$-module basis of $\mathcal{O}_{K}$ and where $\pi \mathcal{O}_{K}=$ $\left(\mathfrak{p}_{1} \cdots \mathfrak{p}_{t}\right)^{h_{K}}$ with $h_{K}$ the class number of $K$. Thus, both the ring of integers
and the rings of $S$-integers of a number field are domains finitely generated over $\mathbb{Z}$, with algebraic generators.

In general, we will consider Diophantine equations over integral domains $\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]$ where some of the generators, say $z_{1}, \ldots, z_{q}$, are algebraically independent over $\mathbb{Q}$ and the other generators are algebraic over $\mathbb{Q}\left(z_{1}, \ldots, z_{q}\right)$.

In this chapter we present the most important ineffective finiteness theorems for integral solutions of various classes of Diophantine equations, including Thue equations, unit equations, hyper- and superelliptic equations, equations involving integral points on curves, decomposable form equations and discriminant equations. We consider these classes of equations in separate sections. For each class we state the finiteness results in their most general form, over an arbitrary integral domain of characteristic 0 that is finitely generated over $\mathbb{Z}$, and give an account of the earlier special cases leading to the general result. Over $\mathbb{Z}$ or more generally over the rings of integers or $S$ integers of number fields, these results were proved mostly by the powerful Thue-Siegel-Roth-Schmidt method, while in the finitely generated case the equations are reduced either to the number field and function field cases by means of some specialization arguments or to such equations for which the finiteness of the number of solutions is already proved; see e.g. Lang (1960), Győry (1982), Evertse and Győry (1988a,1988b) and van der Poorten and Schlickewei (1991). At the end of each section, we make a mention to the corresponding effective results over $\mathbb{Z}$ or over number fields whose general versions over finitely generated domains will be presented in Chapter 2

The above mentioned equations have been studied very extensively and they have many important generalizations, analogues and applications. For details, we refer e.g. to the books Lang (1962,1978,1983), Borevich and Shafarevich (1967), Mordell (1969), Baker (1975), Győry (1980b), Evertse (1983), Mason (1984), Shorey and Tijdeman (1986), Schmidt (1991), Sprindžuk (1993), Bombieri and Gubler (2006), Zannier (2009), Evertse and Győry (2015, 2017a), Bugeaud (2018) and the survey papers of Evertse, Győry, Stewart and Tijdeman (1988b) and Győry (1984a, 1992,2002).

### 1.1 Thue equations

Let $A$ denote an integral domain of characteristic 0 that is finitely generated over $\mathbb{Z}$. Let $K$ denote the quotient field of $A$, and fix an algebraic closure $\bar{K}$
of $K$. We first consider the equation

$$
\begin{equation*}
F(x, y)=\delta \quad \text { in } \quad x, y \in A \tag{1.1.1}
\end{equation*}
$$

over $A$, where $F(X, Y)$ is a binary form of degree $n$ with coefficients in $A$ and $\delta \in A \backslash\{0\}$.

The following result is a consequence of the more general Theorem 1.4.1, which will be stated below.

Theorem 1.1.1. Assume that $F$ has at least three pairwise non-proportional linear factors over $\bar{K}$. Then equation (1.1.1) has only finitely many solutions.

The condition in the theorem is obviously satisfied if $F$ has degree at least 3 and its discriminant is non-zero. This theorem cannot be extended to binary forms $F$ with fewer than three pairwise non-proportional linear factors; for instance the Pell equation $x^{2}-d y^{2}=1$ over $\mathbb{Z}$, where $d$ is a positive integer not being a square, has infinitely many solutions.

In the classical case $A=\mathbb{Z}$, Theorem 1.1.1 was proved by Thue (1909). In fact Thue proved it for irreducible $F$, but the general case can be easily reduced to the irreducible one. The proof of Thue's theorem is based on his result concerning approximations of algebraic numbers by rationals. After Thue, equations of the shape (1.1.1) are named Thue equations.

Thue's theorem has been generalized by many people. Siegel (1921) extended it to the case when $A$ is the ring of integers of a number field and Mahler (1933) to rings of the shape $\mathbb{Z}\left[\left(p_{1} \cdots p_{s}\right)^{-1}\right]$ where $p_{1}, \ldots, p_{s}$ are distinct primes. Parry (1950) gave a common generalization of the results of Siegel and Mahler to rings of $S$-integers of a number field. In the above general form, Theorem 1.1.1 is due to Lang (1960).

We would like to mention another equivalent formulation of Theorem 1.1.1. First, we recall a result of Mahler (1933). Let $F \in \mathbb{Z}[X, Y]$ be a binary form with at least three pairwise non-proportional linear factors over $\overline{\mathbb{Q}}$, and let $p_{1}, \ldots, p_{s}$ be distinct prime numbers. Then the equation

$$
\begin{equation*}
F(x, y)= \pm p_{1}^{z_{1}} \cdots p_{s}^{z_{s}} \text { in } x, y, z_{1}, \ldots, z_{s} \in \mathbb{Z} \text { with } \operatorname{gcd}(x, y)=1 \tag{1.1.2}
\end{equation*}
$$

has only finitely many solutions. If we drop the restriction $\operatorname{gcd}(x, y)=1$ we can construct infinite classes of solutions by multiplying $(x, y)$ with products of powers of $p_{1}, \ldots, p_{s}$. Thus, it is easily seen that Mahler's result can be translated as follows. Let $S=\left\{p_{1}, \ldots, p_{s}\right\}$ be a finite set of primes, $\mathbb{Z}_{S}=\mathbb{Z}\left[\left(p_{1}, \ldots, p_{s}\right)^{-1}\right]$ the corresponding ring of $S$-integers, and $\mathbb{Z}_{S}^{*}=$
$\left\{ \pm p_{1}^{z_{1}} \cdots p_{s}^{z_{s}}: z_{1}, \ldots, z_{s} \in \mathbb{Z}\right\}$ the group of units of $\mathbb{Z}_{S}$. Then the solutions of

$$
\begin{equation*}
F(x, y) \in \mathbb{Z}_{S}^{*} \text { in }(x, y) \in \mathbb{Z}_{S}^{2} \tag{1.1.3}
\end{equation*}
$$

lie in finitely many $\mathbb{Z}_{S}^{*}$-cosets, where a $\mathbb{Z}_{S}^{*}$-coset is a set of solutions of the shape $\left\{u \cdot\left(x_{0}, y_{0}\right): u \in \mathbb{Z}_{S}^{*}\right\}$, with $\left(x_{0}, y_{0}\right) \in \mathbb{Z}_{S}^{2}$ fixed.

We now generalize this last equation to arbitrary finitely generated domains of characteristic 0 that are finitely generated over $\mathbb{Z}$. Let $A$ be such a domain, denote by $A^{*}$ its unit group, i.e., group of invertible elements. Further, let $F \in A[X, Y]$ be a binary form and $\delta$ a non-zero element of $A$, and consider the following generalization of 1.1.3):

$$
\begin{equation*}
F(x, y) \in \delta A^{*} \text { in }(x, y) \in A^{2} \tag{1.1.4}
\end{equation*}
$$

Because of its connection with (1.1.2), equation (1.1.4) is called a ThueMahler equation. Just like above, we can divide the solutions $(x, y) \in A^{2}$ of (1.1.4) into $A^{*}$-cosets $A^{*}\left(x_{0}, y_{0}\right)=\left\{u \cdot\left(x_{0}, y_{0}\right): u \in A^{*}\right\}$.

The following assertion is equivalent to Theorem 1.1.1.
Theorem 1.1.2. Assume again that $F$ has at least three pairwise non-proportional linear factors over $\bar{K}$. Then equation (1.1.4) has only finitely many $A^{*}$-cosets of solutions.

Theorem 1.1.1 $\Rightarrow$ Theorem 1.1.2 Assume Theorem 1.1.1. According to a theorem of Roquette (1957), the unit group $A^{*}$ is finitely generated. Let $\left\{v_{1}, \ldots, v_{s}\right\}$ be a set of generators for $A^{*}$, and define $\mathcal{U}:=\left\{v_{1}^{m_{1}} \cdots v_{s}^{m_{s}}: m_{1}, \ldots, m_{s} \in\right.$ $\{0, \ldots, n-1\}\}$. Then every element of $A^{*}$ can be expressed as $u_{1} u_{2}^{n}$, where $u_{1} \in \mathcal{U}$ and $u_{2} \in A^{*}$. Clearly, if $(x, y) \in A^{2}$ satisfies (1.1.4), then $F(x, y)=$ $\delta u_{1} u_{2}^{n}$ for some $u_{1} \in \mathcal{U}, u_{2} \in A^{*}$, and so $F\left(x^{\prime}, y^{\prime}\right)=\delta u_{1}$ where $\left(x^{\prime}, y^{\prime}\right)=$ $u_{2}^{-1}(x, y)$. Hence every $A^{*}$-coset of solutions of (1.1.4) contains $\left(x^{\prime}, y^{\prime}\right)$ with $F\left(x^{\prime}, y^{\prime}\right)=\delta u_{1}$ with some $u_{1} \in \mathcal{U}$, and Theorem 1.1.1 implies that for each $u_{1} \in \mathcal{U}$ there are only finitely many possibilities for $\left(x^{\prime}, y^{\prime}\right)$. This implies Theorem 1.1.2.

Theorem 1.1.2 $\Rightarrow$ Theorem 1.1.1. Assume Theorem 1.1.2. Let $A^{*}\left(x_{0}, y_{0}\right)$ be one of the finitely many $A^{*}$-cosets of solutions of (1.1.4) and pick those solutions from it that satisfy (1.1.1). These solutions are all of the shape $u\left(x_{0}, y_{0}\right)$ with $u^{n}=F\left(x_{0}, y_{0}\right) / \delta$ and there are only finitely many of those. Hence (1.1.1) has only finitely many solutions.

Equation (1.1.1) has many further generalizations, see e.g. equation (1.4.1) in Section 1.4, equations (1.5.1), (1.5.2) and (1.5.4) in Section 1.5 and Evertse and Győry (2015, Chapter 9).

In the case $A=\mathbb{Z}$ the first general effective result for equation (1.1.1) was established by Baker (1968b). He gave an explicit upper bound for the solutions by means of his effective method based on lower bound for linear forms in logarithms. Coates (1969) extended Baker's result to the case of ground rings of the type $A=\mathbb{Z}\left[\left(p_{1} \cdots p_{s}\right)^{-1}\right]$, and later Kotov and Sprindzuk (1973) to the case when $A$ is the ring of $S$-integers of a number field. Győry (1983), using his effective specialization method, generalized the above results for a wide but special class of finitely generated domains which may contain both algebraic and transcendental elements. In Chapter 2, Theorem 2.3.1 gives an effective version of Theorem 1.1.1 in quantitative form over an arbitrary integral domain of characteristic 0 which is finitely generated over $\mathbb{Z}$. Its proof uses a precise effective version of Theorem 1.1.1 over rings of $S$-integers of number fields, see Theorem 4.4.1 in Chapter 4, as well as an effective version over function fields, see Theorem 5.4 .1 in Chapter 5 , which is a slight variation of a result of Mason (1981, 1984).

### 1.2 Unit equations in two unknowns

Let again $A$ be an integral domain of characteristic 0 which is finitely generated over $\mathbb{Z}$, and $K$ its quotient field. Further, let $a, b$ be non-zero elements of $K$. Consider the unit equation

$$
\begin{equation*}
a x+b y=1 \quad \text { in } \quad x, y \in A^{*}, \tag{1.2.1}
\end{equation*}
$$

where $A^{*}$ denotes the unit group of $A$, i.e. the multiplicative group of invertible elements of $A$.

By a theorem of Roquette (1957) the group $A^{*}$ is finitely generated. Lang (1960) proved the following general result.

Theorem 1.2.1. Equation (1.2.1) has only finitely many solutions.
The first finiteness result for equation (1.2.1) was implicitly proved by Siegel (1921) when $K$ is a number field and $A$ is the ring of integers of $K$. For the case when $A$ is of the type $\mathbb{Z}\left[\left(p_{1} \cdots p_{s}\right)^{-1}\right]$ with distinct primes $p_{1}, \ldots, p_{s}$, the finiteness of the number of solutions was obtained by Mahler
(1933), while a common generalization of the results of Siegel and Mahler follows from Parry (1950).

In fact, in Lang (1960) the following more general version of Theorem 1.2.1 is established.

Theorem 1.2.2. Let $K$ be a field of characteristic $0, a, b$ non-zero elements of $K$ and $\Gamma$ a finitely generated multiplicative subgroup of $K^{*}$. Then the equation

$$
\begin{equation*}
a x+b y=1 \quad \text { in } \quad x, y \in \Gamma \tag{1.2.2}
\end{equation*}
$$

has only finitely many solutions.
Proof. Using an argument due to Siegel (1921), the theorem can be easily reduced to Theorem 1.1.1. Indeed, suppose that equation (1.2.2) has infinitely many solutions. Let $n$ be an integer $\geq 3$. Since $\Gamma$ is finitely generated, the quotient group $\Gamma / \Gamma^{n}$ is finite. Hence there is a solution $x_{0}, y_{0}$ of (1.2.2) such that there are infinitely many solutions $x, y$ such that $x \in x_{0} \Gamma^{n}, y \in y_{0} \Gamma^{n}$. Each of these solutions $x, y$ can be written in the form $x=x_{0} u^{n}, y=y_{0} v^{n}$ with some $u, v \in \Gamma$. Denoting by $A$ the ring generated by $\Gamma$ over $\mathbb{Z}$, it follows that the Thue equation

$$
\left(a x_{0}\right) u^{n}+\left(b y_{0}\right) v^{n}=1
$$

has infinitely many solutions $u, v \in A$. This contradicts Theorem 1.1.1.
We note that conversely, Thue equations can be reduced to finitely many appropriate unit equations, see e.g. Evertse and Győry (2015). In other words, Thue equations and unit equations in two unknowns are in fact equivalent. This was (implicitly) pointed out by Siegel (1926).

Theorem 1.2 .2 has several generalizations, see e.g. Theorem $1.5 .4 \mathrm{in} \mathrm{Sec-}$ tion 1.5, Lang $(1960,1983)$ and Evertse and Győry (2015). Here we present one of them.

Lang (1960) extended his result concerning equation (1.2.2) to equations of the shape

$$
\begin{equation*}
F(x, y)=0 \quad \text { in } \quad x, y \in \Gamma, \tag{1.2.3}
\end{equation*}
$$

where $\Gamma$ is as above and $F \in A[X, Y]$ is a non-constant polynomial. He proved the following.

Theorem 1.2.3. Let $F \in A[X, Y]$ be a non-constant polynomial that is not divisible by any polynomial of the shape

$$
\begin{equation*}
X^{m} Y^{n}-\alpha \quad \text { or } \quad X^{m}-\alpha Y^{n} \tag{1.2.4}
\end{equation*}
$$

with $\alpha \in \Gamma$ and with non-negative integers $m, n$, not both zero. Then equation (1.2.3) has only finitely many solutions.

It is easy to see that the exceptions described in Theorem 1.2 .3 must be excluded.

Lang (1965a,1965b) conjectured that Theorem 1.2 .3 remains valid if one replaces $\Gamma$ by its division group $\bar{\Gamma}$, which consists of those $\gamma \in \bar{K}^{*}$ such that $\gamma^{k} \in \Gamma$ for some positive integer $k$. Hence, in this case the solutions $x, y$ do not necessarily belong to $K$. Lang's conjecture has been proved by Liardet $(1974,1975)$ who obtained the following.

Theorem 1.2.4. Let $F \in A[X, Y]$ be a non-constant polynomial that is not divisible by any polynomial of the shape (1.2.4) with $\alpha \in \bar{\Gamma}$ and with nonnegative integers $m, n$, not both zero. Then equation (1.2.3) has only finitely many solutions even in $x, y \in \bar{\Gamma}$.

The first general effective results for equation (1.2.1) over the ring of integers of algebraic number fields were proved in Győry (1972,1973,1974,1976), over rings of $S$-integers of an algebraic number field in Győry (1979), and independently, in a less precise form, in Kotov and Trelina (1979). Using Baker's method concerning linear forms is logarithms, effective upper bounds were given for the solutions. These bounds were improved later by several authors, see e.g. Bugeaud and Győry (1996a), Győry and Yu (2006) and Győry (2019).

Over algebraic number fields, Bombieri and Gubler (2006) gave an effective version of Lang's theorem on the equation (1.2.3), which was made explicit by Bérczes, Evertse, Győry and Pontreau (2009). These results are proved under a slightly stronger condition than (1.2.4), with $\alpha \in \bar{K}$ in place of $\alpha \in \Gamma$.

In the number field case, an effective version of Liardet's theorem for linear polynomials $F$ is due to Bérczes, Evertse and Győry (2009), and for the general case to Bérczes, Evertse, Győry and Pontreau (2009).

In Section 2.2, we present effective versions of Theorems 1.2.1 and 1.2.2 in quantitative form over arbitrary integral domain of characteristic 0 which is finitely generated over $\mathbb{Z}$, see Theorems 2.2.1 and 2.2.3. In its proof we use
the result of Győry and Yu over number fields mentioned above, as well as the Mason-Stothers abc-theorem for function fields (as in Mason (1984)), see Theorem 5.2 .2 in Chapter 5 . Further, we formulate some effective generalizations for equation (1.2.3), due to Bérczes (2015a, 2015b), see Theorems [2.2.4, 2.2.5

### 1.3 Hyper- and superelliptic equations

Now consider the equation

$$
\begin{equation*}
f(x)=\delta y^{m} \quad \text { in } \quad x, y \in A, \tag{1.3.1}
\end{equation*}
$$

where $A$ is again an integral domain of characteristic 0 which is finitely generated over $\mathbb{Z}, f \in A[X]$ is a polynomial of degree $n \geq 2, \delta \in A \backslash\{0\}$ and $m \geq 2$ integer. The equation (1.3.1) is called elliptic if $n=3, m=2$, hyperelliptic if $n \geq 3, m=2$, and superelliptic if $n \geq 2, m \geq 3$.

The following theorem follows from the general ineffective Theorem 1.4.1 of Lang.

Theorem 1.3.1. Suppose that in (1.3.1) $m$ or $n$ is at least 3 and that $f$ has no multiple zeros. Then (1.3.1) has only finitely many solutions.

Under the assumptions of Theorem 1.3.1 the affine curve $f(x)-\delta y^{m}=0$ has genus $\geq 1$. Thus Theorem 1.3 .1 is a consequence of the general Theorem 1.4.1 below on the finiteness of the number of intregral points on algebraic curves. The example of Pell equations shows that (1.3.1) may have infinitely many solutions if $m=2$ and $n=2$.

In the special case $A=\mathbb{Z}$, Mordell (1922a,1922b,1923) proved the finiteness of the numbers of solutions of elliptic equations for which the polynomial $f$ has no multiple zeros. In particular, this implies that for every non-zero integer $k$, the Mordell equation $x^{3}+k=y^{2}$ has only finitely many solutions. Mordell's finiteness results were extended by Siegel (1926) to hyperelliptic equations, by reducing such equations to unit equations. LeVeque (1964) considered (1.3.1) where $f$ may have multiple zeros, and gave a finiteness criterion for the equation (1.3.1) when $A$ is the ring of integers of a number field. The proofs of Mordell, Siegel and LeVeque are ineffective.

Over $\mathbb{Z}$, Baker $(1968 b, 1968 c, 1969)$ was the first to give effective upper bounds for the solutions of (1.3.1) in the case when $f$ has at least 3 simple zeros if $m=2$ and at least 2 simple zeros if $m \geq 3$. Brindza (1984) made

LeVeque's theorem effective and extended it to $S$-integral solutions from a number field.

Schinzel and Tijdeman (1976) considered the equation (1.3.1) in the more general situation when $m$ is also unknown. In the case that $A=\mathbb{Z}$ and that $f$ has at least 2 distinct zeros, they derived an effective upper bound for $m$. Equation (1.3.1) with $m$ also unknown is nowadays called the Schinzel-Tijdeman equation. All the effective results mentioned above depend on Baker's method.

In Chapter 2, we present effective versions of Theorem 1.3 .1 and the Schinzel-Tijdeman theorem in quantitative form, over an arbitrary integral domain of characteristic 0 which is finitely generated over $\mathbb{Z}$, see Theorems 2.4.1, 2.4.2. These results follow from similar effective results over number fields, see Theorems 4.5.1, 4.5.2, 4.5.3 and function fields, see Theorems 5.5.1, 5.5.2.

### 1.4 Curves with finitely many integral points

Let $K$ be a finitely generated extension of $\mathbb{Q}$, and $A$ a subring of $K$ which is finitely generated over $\mathbb{Z}$. The following finiteness theorem is of fundamental importance in Diophantine number theory.

Theorem 1.4.1. Let $F \in K[X, Y]$ be a polynomial irreducible over $\bar{K}$ such that the affine curve $F(x, y)=0$ is of genus $\geq 1$. Then this curve has only finitely many points with coordinates in $A$.

In other words, under the above assumptions the equation

$$
\begin{equation*}
F(x, y)=0 \quad \text { in } \quad x, y \in A \tag{1.4.1}
\end{equation*}
$$

has only finitely many solutions.
In the case when $K$ is a number field and $A$ its ring of integers this celebrated theorem was proved by Siegel (1929). Further, Siegel described the cases when the curve has genus 0 and has infinitely many points with coordinates in $A$. Mahler (1934) conjectured that a similar statement holds for rational points with coordinates having only finitely many fixed primes in their denominators, and proved this for curves of genus 1 . In the above general form Theorem 1.4 .1 is due to Lang (1960); see also Lang $(1962,1983)$. In this proof, Lang used a specialization argument, reducing Theorem 1.4.1to the case of number fields resp. function fields of one variable and then applied Siegel's theorem and its analogue over function fields from Lang (1960).

Confirming Mordell's (1922a) famous conjecture on rational points on curves, Faltings (1983) proved first for number fields $K$ and later for finitely generated extensions $K$ of $\mathbb{Q}$, cf. Faltings and Wüstholz (1984, page 205, Thm. 3), that if the above curve has genus $\geq 2$ then it has only finitely many points even with coordinates in $K$ as well. Except for the genus 1 case, Faltings' theorem contains Theorem 1.4.1.

All known proofs of Theorem 1.4 .1 and of Faltings' theorem are ineffective. As was mentioned in Sections 1.1 to 1.3, Theorem 1.4.1 has been made effective in a couple of important special cases. Further, in the case when $K$ is a number field, an effective version of Theorem 1.4.1 for genus 1 curves was obtained by Baker and Coates (1970).

It is a major open problem to give an effective version of Theorem 1.4.1 in full generality.

### 1.5 Decomposable form equations and multivariate unit equations

Let $K$ be a finitely generated extension field of $\mathbb{Q}$, and $F \in K\left[X_{1}, \ldots, X_{m}\right]$ a decomposable form in $m \geq 2$ variables, i.e., $F$ factorizes into linear forms over an extension of $K$, which we may choose to be a given algebraic closure $\bar{K}$ of $K$. Let $\delta \in K^{*}$ and let $A$ be a subring of $K$ that is finitely generated over $\mathbb{Z}$. As a generalization of the Thue equation we consider the decomposable form equation

$$
\begin{equation*}
F(\mathbf{x})=\delta \text { in } \mathbf{x}=\left(x_{1}, \ldots, x_{m}\right) \in A^{m} \tag{1.5.1}
\end{equation*}
$$

Let $\mathcal{L}_{0}$ be a maximal set of pairwise linearly independent linear factors of $F$. That is, we can express $F$ as $c \ell_{1}^{e_{1}} \cdots \ell_{n}^{e_{n}}$, where $\mathcal{L}_{0}=\left\{\ell_{1}, \ldots, \ell_{n}\right\}$, $c \in K^{*}$, and $e_{1}, \ldots, e_{n}$ are positive integers. For applications, it is convenient to consider the following generalization of equation (1.5.1). Let $\mathcal{L} \supseteq \mathcal{L}_{0}$ be a finite set of pairwise linearly independent linear forms in $X_{1}, \ldots, X_{m}$ with coefficients in $\bar{K}$, and consider now the equation

$$
F(\mathbf{x})=\delta \text { in } \mathbf{x}=\left(x_{1}, \ldots, x_{m}\right) \in A^{m} \text { with } \ell(\mathbf{x}) \neq 0 \text { for all } \ell \in \mathcal{L} . \text { 1.5.1a) }
$$

For $\mathcal{L}=\mathcal{L}_{0}$, equation (1.5.1 $)$ gives (1.5.1).
To state the main results we need some definitions. Given a non-zero linear subspace $V$ of the $K$-vector space $K^{m}$ and linear forms $\ell_{1}, \ldots, \ell_{r}$ in
$\bar{K}\left[X_{1}, \ldots, X_{m}\right]$, we say that $\ell_{1}, \ldots, \ell_{r}$ are linearly dependent on $V$ if there are $c_{1}, \ldots, c_{r} \in \bar{K}$, not all 0 , such that $c_{1} \ell_{1}+\cdots+c_{r} \ell_{r}$ vanishes identically on $V$. Otherwise, we say that $\ell_{1}, \ldots, \ell_{r}$ are linearly independent on $V$.

We say that a non-zero linear subspace $V$ of $K^{m}$ is $\mathcal{L}$-non-degenerate if $\mathcal{L}$ contains $r \geq 3$ linear forms $\ell_{1}, \ldots, \ell_{r}$ which are linearly dependent on $V$, while each pair $\ell_{i}, \ell_{j}(i \neq j)$ is linearly independent on $V$. Otherwise, the space $V$ is called $\mathcal{L}$-degenerate. That is, $V$ is $\mathcal{L}$-degenerate precisely if there are $\ell_{1}, \ldots, \ell_{r} \in \mathcal{L}$ such that $\ell_{1}, \ldots, \ell_{r}$ are linearly independent on $V$ while each other linear form $\ell \in \mathcal{L}$ is linearly dependent on $V$ to one of $\ell_{1}, \ldots, \ell_{r}$. In particular, $V$ is $\mathcal{L}$-degenerate if $V$ has dimension 1 .

Lastly, we call $V \mathcal{L}$-admissible if no linear form in $\mathcal{L}$ vanishes identically on $V$.

The following general finiteness criterion was proved by Evertse and Győry (1988b).

Theorem 1.5.1. The following two statements are equivalent:
(i) Every $\mathcal{L}$-admissible linear subspace of $K^{m}$ of dimension $\geq 2$ is $\mathcal{L}_{0}$ -non-degenerate;
(ii) For every subring $A$ of $K$ which is finitely generated over $\mathbb{Z}$ and for every $\delta \in K^{*}$, the equation (1.5.1 has only finitely many solutions.

For $\mathcal{L}=\mathcal{L}_{0}$, this theorem gives a finiteness criterion for equation (1.5.1). It relates a statement (cf. (ii)) about the finiteness of the number of solutions to a condition (cf. (i)) which can be formulated in terms of linear algebra. It can be shown that (i) is effectively decidable once $K, \mathcal{L}_{0}$ and $\mathcal{L}$ are given in some explicit form, see Evertse and Győry (2015, Theorem 9.1.1) for an equivalent formulation of (i) for which the effective decidability is clear.

In the case $m=2, \mathcal{L}=\mathcal{L}_{0}$, Theorem 1.5.1 gives immediately Theorem 1.1.1 on Thue equations. For a more general version of Theorem 1.5.1, see Evertse and Győry (2015, Chapter 9).

Decomposable form equations are of basic importance in Diophantine number theory. Besides Thue equations (when $m=2$ ), important classes of decomposable form equations are norm form equations, discriminant form equations and index form equations.

Let us start with norm form equations. Let $\alpha_{1}=1, \alpha_{2}, \ldots, \alpha_{m} \in \bar{K}$ and suppose they are linearly independent over $K$. Put $K^{\prime}:=K\left(\alpha_{1}, \ldots, \alpha_{m}\right)$. Assume that $K^{\prime}$ is of degree $n \geq 3$ over $K$. Putting $\ell(\mathbf{X})=\alpha_{1} X_{1}+\cdots+\alpha_{m} X_{m}$,
denote by $\ell^{(i)}(\mathbf{X})=\alpha_{1}^{(i)} X_{1}+\cdots+\alpha_{m}^{(i)} X_{m}, i=1, \ldots, n$, the conjugates of $\ell(\mathbf{X})$ with respect to $K^{\prime} / K$. Then

$$
N_{K^{\prime} / K}\left(\alpha_{1} X_{1}+\cdots+\alpha_{m} X_{m}\right):=\prod_{i=1}^{n} \ell^{(i)}(\mathbf{X})
$$

is a decomposable form of degree $n$ with coefficients in $K$. Such a form is called a norm form over $K$ (or with respect to $K^{\prime} / K$ ) and, for $\delta \in K^{*}$,

$$
\begin{equation*}
N_{K^{\prime} / K}\left(\alpha_{1} x_{1}+\cdots+\alpha_{m} x_{m}\right)=\delta \quad \text { in } \quad x_{1}, \ldots, x_{m} \in A \tag{1.5.2}
\end{equation*}
$$

a norm form equation.
Let $\mathcal{V}$ be the $K$-vector space generated by $\alpha_{1}, \ldots, \alpha_{m}$ in $K^{\prime}$. We say that $\mathcal{V}$ is degenerate if there exist a $\mu \in K^{\prime *}$ and an intermediate number field $K^{\prime \prime}$ with $K \subsetneq K^{\prime \prime} \subsetneq K^{\prime}$ such that $\mu K^{\prime \prime} \subseteq \mathcal{V}$. The following finiteness criterion is a consequence of Theorem 1.5.1, see Evertse and Győry (1988b).

Corollary 1.5.2. The following two statements are equivalent:
(i) $\mathcal{V}$ is non-degenerate;
(ii) For all $\delta \in K^{*}$ and all subrings $A$ of $K$ which are finitely generated over $\mathbb{Z}$, equation (1.5.2) has only finitely many solutions.

For $K=\mathbb{Q}, A=\mathbb{Z}$, Schmidt (1971) gave a criterion for equation (1.5.2) to have only finitely many solutions for every $\delta \in \mathbb{Q}^{*}$. Then Schmidt (1972) proved that all solutions of (1.5.2) over $\mathbb{Z}$ belong to finitely many so-called families of solutions. These results were later extended by Schlickewei (1977) to the case of arbitrary finitely generated subrings $A$ of $\mathbb{Q}$, and by Laurent (1984) to the above general case. As a generalization of Schmidt's (1972) result, Győry (1993) showed that all solutions of equation (1.5.1) belong to finitely many so-called wide families of solutions.

Next consider discriminant form equations. Let again $K^{\prime}=K\left(\alpha_{1}, \ldots, \alpha_{m}\right)$ be an extension of degree $n \geq 3$ of $K$, where now $1, \alpha_{1}, \ldots, \alpha_{m}$ are $K$ linearly independent elements of $K^{\prime}$. Let $\ell^{(i)}(\mathbf{X})=X_{0}+\alpha_{1}^{(i)} X_{1}+\cdots+$ $\alpha_{m}^{(i)} X_{m}, i=1, \ldots, n$, be the conjugates of $\ell(\mathbf{X})=X_{0}+\alpha_{1} X_{1}+\cdots+\alpha_{m} X_{m}$ with respect to $K^{\prime} / K$. Then the decomposable form

$$
D_{K^{\prime} / K}\left(\alpha_{1} X_{1}+\cdots+\alpha_{m} X_{m}\right)=\prod_{1 \leq i<j \leq n}\left(\ell^{(i)}(\mathbf{X})-\ell^{(j)}(\mathbf{X})\right)^{2}
$$

has its coefficients in $K$ and is independent of $X_{0}$. It is called a discriminant form, while, for $\delta \in K^{*}$,

$$
\begin{equation*}
D_{K^{\prime} / K}\left(\alpha_{1} x_{1}+\cdots+\alpha_{m} x_{m}\right)=\delta \quad \text { in } \quad x_{1}, \ldots, x_{m} \in A \tag{1.5.3}
\end{equation*}
$$

is called a discriminant form equation.
The following finiteness result is due to Győry (1982). It can be deduced from Theorem 1.5.1 as well.

Theorem 1.5.3. Under the above assumptions, equation (1.5.3) has only finitely many solutions.

This theorem and its various versions have several important applications, among others to index form equations and power integral bases; for references see e.g. Győry (1980b) and Evertse and Győry (2017a).

The above results concerning equations (1.5.1), (1.5.1a), (1.5.2) and (1.5.3) have been extended to equations of the form

$$
\begin{equation*}
F(\mathbf{x}) \in \delta A^{*} \quad \text { in } \quad \mathbf{x}=\left(x_{1}, \ldots, x_{m}\right) \in A^{m} . \tag{1.5.4}
\end{equation*}
$$

The set of solutions of equation 1.5.4 can be divided into $A^{*}$-cosets $\mathrm{x}_{0} A^{*}$, where $\mathbf{x}_{\mathbf{0}}=\left(x_{1}, \ldots, x_{m}\right)$ is a solution of (1.5.4). As was already mentioned, by a theorem of Roquette (1957) $A^{*}$ is finitely generated. Hence (1.5.4) can be reduced to finitely many equations of the form (1.5.1).

The proof of Theorem 1.5.1 and its variant concerning (1.5.4) depends on the following finiteness result on multivariate unit equations of the form

$$
\begin{equation*}
a_{1} x_{1}+\cdots+a_{m} x_{m}=1 \quad \text { in } \quad x_{1}, \ldots, x_{m} \in A^{*} \quad \text { resp. in } \quad \Gamma \text {, } \tag{1.5.5}
\end{equation*}
$$

where $K$ is a field of characteristic $0, a_{1}, \ldots, a_{m}$ are non-zero elements of $K$, $A$ is a subring of $K$ that is finitely generated over $\mathbb{Z}$, and $\Gamma$ is a finitely generated subgroup of $K^{*}$. A solution $x_{1}, \ldots, x_{m}$ of (1.5.5) is called degenerate if there is a vanishing subsum on the left hand side of (1.5.5). In this case (1.5.5) has obviously infinitely many solutions if $A^{*}$ resp. $\Gamma$ is infinite. The following theorem was proved by van der Poorten and Schlickewei (1982) and Evertse (1984) in the number field case, and by Evertse and Győry (1988a) and van der Poorten and Schlickewei (1991) in the finitely generated case.

Theorem 1.5.4. Equation (1.5.5) has only finitely many non-degenerate solutions.

As is pointed out in Evertse and Győry (1988b), Theorem 1.5 .4 and the implication (i) $\Rightarrow$ (ii) of Theorem 1.5.1 are equivalent statements; see also Evertse and Győry (2015, Chapter 9).

The above presented or mentioned results are all ineffective. In certain important cases they have effective versions. Concerning the discriminant form equation (1.5.3) over $\mathbb{Z}$, the first effective finiteness result was obtained by Győry (1976). This was extended to the number field case by Győry and Papp (1977) and Győry (1981a). Győry and Papp (1978) over $\mathbb{Z}$, and Győry (1981a) over arbitrary number fields established effective finiteness theorems for equations (1.5.1), (1.5.11] and (1.5.2), for some classes of decomposable forms and norm forms, including binary forms and discriminant forms. As was mentioned in Section 1.2, the first effective finiteness results for bivariate unit equations, i.e., equations (1.5.5) in $m=2$ unknowns over number fields were given by Győry $(1972,1973,1974)$.

Gyôry (1983) extended his effective results on equations (1.5.1), 1.5.2), (1.5.3) over number fields to a class of finitely generated ground domains over $\mathbb{Z}$ which may contain both algebraic and transcendental elements over $\mathbb{Q}$. In Chapter 2, we present a further extension, in slightly more general form, to the case of arbitraryground domains of characteristic 0 that are finitely generated over $\mathbb{Z}$, see Theorem 2.6.1. However, apart from the case of general discriminant form equations (1.5.3), it remains a major open problem to make Theorem 1.5.1, Corollary 1.5 .2 and Theorem 1.5 .4 effective in full generality.

### 1.6 Discriminant equations for polynomials and integral elements

Let again $A$ be an integral domain of characteristic 0 which is finitely generated over $\mathbb{Z}$ and $K$ its quotient field. Take a finite extension $G$ of $K$. Let $n \geq 2$ be an integer, $\delta$ a non-zero element of $A$ and consider the discriminant equation for polynomials
$D(f)=\delta$ in monic $f \in A[X]$ of degree $n$ having all its zeros in $G$.
Two monic polynomials $f, f^{\prime} \in A[X]$ are called strongly $A$-equivalent if $f^{\prime}(X)=f(X+a)$ for some $a \in A$. ${ }^{1}$ In this case $f$ and $f^{\prime}$ have the

[^0]same discriminant. Hence the solutions of (1.6.1) can be divided into strong $A$-equivalence classes.

Denote by $A_{K}$ the integral closure of $A$ in $K$.
Theorem 1.6.1. Let $n \geq 2$ be an integer and $A$ an integral domain of characteristic 0 , finitely generated over $\mathbb{Z}$, with quotient field $K$ such that the quotient $A$-module

$$
\begin{equation*}
\left(\frac{1}{n} A \cap A_{K}\right) / A \quad \text { is finite. } \tag{1.6.2}
\end{equation*}
$$

Further, let $G$ be a finite extension of $K$ and $\delta$ a non-zero element of $A$. Then the set of monic polynomials $f \in A[X]$ satisfying (1.6.1) is a union of finitely many strong $A$-equivalence classes.

This was proved in Evertse and Győry (2017b) in an effective form; see also Theorem 2.8.4 in Section 2.8.

The class of domains $A$ with (1.6.2) contains among others all finitely generated subrings of $\overline{\mathbb{Q}}$ and, more generally, all finitely generated domains over $\mathbb{Z}$ which are of characteristic 0 and are integrally closed. In the latter case, Győry (1982) proved the following more precise result, without fixing the degrees of the polynomials under consideration.

Theorem 1.6.2. Let A be an integrally closed integral domain of characteristic 0 which is finitely generated over $\mathbb{Z}$, and $G$ a finite extension of the quotient field of $A$. Then the set of solutions of (1.6.1) is a union of finitely many strong $A$-equivalence classes.

We don't know if condition (1.6.2) is the weakest possible. As is pointed out in Evertse and Győry (2017b), Theorem 1.6 .1 is not true for arbitrary finitely generated domains of characteristic 0 .

We also consider discriminant equations where the unknowns are elements of orders of finite étale $K$-algebras. Let $\Omega$ be a finite étale $K$-algebra, i.e., $\Omega=K[X] /(P)=K[\theta]$, where $P \in K[X]$ is some separable polynomial and $\theta:=X(\bmod P)$. If in particular $P$ is irreducible over $K$, then $\Omega$ is a finite extension field of $K$. Writing $[\Omega: K]:=\operatorname{dim}_{K} \Omega$, we have $[\Omega: K]=\operatorname{deg} P$. Let $\bar{K}$ be an algebraic closure of $K$. By a $K$-homomorphism from $\Omega$ to $\bar{K}$ we mean a non-trivial $K$-algebra homomorphism. There are precisely $n:=[\Omega: K] K$-homomorphisms from $\Omega$ to $\bar{K}$ which map $\theta$ to the $n$ distinct zeros of $P$ in $\bar{K}$. We denote these by $x \mapsto x^{(i)}(i=1, \ldots, n)$. The
discriminant of $\alpha \in \Omega$ over $K$ is given by

$$
D_{\Omega / K}(\alpha)=\prod_{1 \leq i<j \leq n}\left(\alpha^{(i)}-\alpha^{(j)}\right)^{2},
$$

where $\alpha^{(i)}$ denotes the image of $\alpha$ under $x \mapsto x^{(i)}$. This is an element of $K$. It is easy to see that $D_{\Omega / K}(\alpha+a)=D_{\Omega / K}(\alpha)$ for $\alpha \in \Omega, a \in K$. Further, $D_{\Omega / K}(\alpha)$ is different from zero if and only if $\Omega=K[\alpha]$.

Consider now discriminant equations for integral elements, of the shape

$$
\begin{equation*}
D_{\Omega / K}(\xi)=\delta \quad \text { in } \xi \in \mathcal{O}, \tag{1.6.3}
\end{equation*}
$$

where $\delta$ is a non-zero element of $A$, and $\mathcal{O}$ is an $A$-order of $\Omega$, i.e., an $A$ subalgebra of $\Omega$ which spans $\Omega$ as a $K$-vector space and which is finitely generated as an $A$-module. Then $\mathcal{O}$ is in fact an $A$-subalgebra of the integral closure of $A$ in $\Omega$. As was mentioned above, $A_{\Omega}$ is finitely generated as an $A$-module.

If $\xi \in \mathcal{O}$ is a solution of (1.6.3), then so is $\xi+a$ for every $a \in A$. Thus, the solution of (1.6.3) split into $A$-cosets $\xi+A=\{\xi+a: a \in A\}$.

The following theorem was established by Evertse and Győry (2017b) in an effective form; see also Corollary 2.8.3.

Theorem 1.6.3. Let $A$ be an integral domain of characteristic 0 which is finitely generated over $\mathbb{Z}$. Further, let $K$ be the quotient field of $A, \Omega$ a finite étale $K$-algebra, $\mathcal{O}$ an $A$-order in $\Omega$, and $\delta$ a non-zero element of $A$. Then the following two assertions are equivalent:
(i) The quotient $A$-module $(\mathcal{O} \cap K) / A$ is finite.
(ii) For every non-zero $\delta \in A$, the set of $\xi \in \mathcal{O}$ with (1.6.3) is a union of finitely many $A$-cosets.

The implication (ii) $\Rightarrow$ (i) is obvious. Suppose (i) does not hold. Pick $\xi_{0} \in$ $\mathcal{O}$ with $\Omega=K\left[\xi_{0}\right]$ and let $\delta:=D_{\Omega / K}\left(\xi_{0}\right)$. Then $\delta \neq 0$, the $\mathcal{O} \cap K$-coset $\xi_{0}+\mathcal{O} \cap K$ is contained in the set of solutions of (1.6.3), and this $\mathcal{O} \cap K$-coset is clearly the union of infinitely many $A$-cosets.

We note that $\mathcal{O} \cap K=A$ if $A$ is integrally closed. Hence Theorem 1.6.3 immediately gives the following.

Corollary 1.6.4. Let A be an integrally closed integral domain of characteristic 0 which is finitely generated over $\mathbb{Z}, K$ its quotient field, $\Omega$ a finite étale
$K$-algebra, $\mathcal{O}$ an $A$-order in $\Omega$, and $\delta$ a non-zero element of $A$. Then the set of $\xi \in \mathcal{O}$ with 1.6 .3 is a union of finitely many $A$-cosets.

As was mentioned above, $A_{\Omega}$ is finitely generated as an $A$-module. Taking for $\Omega$ a finite extension $L$ of $K$ and for $\mathcal{O}$ the integral closure $A_{L}$ of $A$ in $L$, we get the following important special case which is due to Győry (1982).

Corollary 1.6.5. Let $A$ be an integrally closed integral domain of characteristic 0 which is finitely generated over $\mathbb{Z}, K$ its quotient field, $L$ a finite extension of $K$ and $\delta \in A \backslash\{0\}$. Then the set of solutions of the equation

$$
\begin{equation*}
D_{L / K}(\xi)=\delta \quad \text { in } \quad \xi \in A_{L} \tag{1.6.4}
\end{equation*}
$$

is a union of finitely many $A$-cosets.

The following more general versions of equations (1.6.1) and (1.6.3) are also important for applications:
$D(f) \in \delta A^{*}$ in monic $f \in A[X]$ of degree $n \geq 2$
having all its zeros in $G$,
and

$$
\begin{equation*}
D_{\Omega / K}(\xi) \in \delta A^{*} \quad \text { in } \quad \xi \in \mathcal{O} . \tag{1.6.3a}
\end{equation*}
$$

The solutions of (1.6.11) can be partitioned into so-called $A$-equivalence classes, where two monic polynomials $f, f^{\prime} \in A[X]$ of degree $n$ are called $A$ equivalent if $f^{\prime}(X)=u^{n} f\left(u^{-1} X+a\right)$ for some $u \in A^{*}, a \in A$. Combining Theorem 1.6.1 with Roquette's (1957) theorem that $A^{*}$ is finitely generated, it follows that under the assumption (1.6.2) the polynomials $f$ with (1.6.1a lie only in finitely many $A$-equivalence classes. For integrally closed $A$, this finiteness result was proved by Győry (1982) in a more general form, without fixing the degree of the polynomials $f$ under consideration.

Similarly, the solutions of (1.6.3 ) can be divided into $A$-equivalence classes, where two elements $\alpha, \alpha^{\prime}$ of $\mathcal{O}$ are called $A$-equivalent if $\alpha^{\prime}=u \alpha+a$ with some $u \in A^{*}, a \in A$. Together with Roquette's theorem, Theorem $1.6 .3 \mathrm{im}-$ plies that under the condition (i) of Theorem 1.6.3, equation (1.6.3a) has only finitely many $A$-equivalence classes of solutions. In case of integrally closed $A$, this was obtained in Evertse and Győry (2017a, Chapter 5).

A further important application is as follows. If

$$
\begin{equation*}
\mathcal{O}=A[\xi] \tag{1.6.5}
\end{equation*}
$$

for some $\xi \in \mathcal{O}$ and $\xi^{\prime}$ is $A$-equivalent to $\xi$ then also $\mathcal{O}=A\left[\xi^{\prime}\right]$. The above result concerning (1.6.3a) implies that under the condition (i) of Theorem 1.6.3, the set of $\xi$ with (1.6.5) is a union of finitely many $A$-equivalence classes. For integrally closed $A$, see Evertse and Győry (2017a, Chapter 5), and if in addition $\Omega$ is a finite extension of $K$, see Győry (1982).

Over $\mathbb{Z}$ and more generally over number fields, the first finiteness results concerning equation (1.6.1), (1.6.1a), (1.6.3), 1.6.3 (1), 1.6.5) were proved by Győry, and in effective form. He proved in Győry (1973) for $A=\mathbb{Z}$ that given a non-zero $\delta \in \mathbb{Z}$, there are only finitely many strong $\mathbb{Z}$-equivalence classes of monic $f \in \mathbb{Z}[X]$ with discriminant $\delta$, and a full set of representatives of these equivalence classes can be effectively determined. Here neither the degree $n$, nor the splitting field $G$ of the polynomials $f$ is fixed. This result implied the first effective finiteness theorem for equation (1.6.4) with $A=\mathbb{Z}$. Further, in Győry (1976) it is proved in an effective form that if $L$ is a number field with ring of integers $O_{L}$ then there are only finitely many $\mathbb{Z}$-equivalence classes of $\alpha \in \mathcal{O}_{L}$ with $\mathcal{O}_{L}=\mathbb{Z}[\alpha]$.

It follows from finiteness results of Győry (1978a,1978b,1984b) that if $A$ is the ring of integers or $S$-integers of a number field then the finiteness results presented above on equations (1.6.1), (1.6.11], (1.6.3), (1.6.3 ${ }^{\text {i }}$, (1.6.5) are valid in effective form. Moreover, these versions of Theorem 1.6.2 remain true without fixing the number field $G$ or the degree $n$ of the polynomials $f$. Perhaps such a finiteness result without fixing $G$ can be extended to certain finitely generated integral domains of low transcendence degree. But extending this to arbitrary finitely generated domains over $\mathbb{Z}$ seems to be very hard.

For a class of finitely generated ground domains over $\mathbb{Z}$ which may contain both algebraic and transcendental elements over $\mathbb{Q}$, effective versions of Theorem 1.6 .2 and Corollary 1.6 .5 were obtained by Győry (1984b). These were extended, in slightly more general form, by Evertse and Győry (2017b) to the case of arbitrary finitely generated ground domains over $\mathbb{Z}$. These will be presented in Chapter 2, Section 2.8 .

## Chapter 2

## Effective results for Diophantine equations over finitely generated domains: the statements

In this chapter general effective finiteness theorems are presented for Diophantine equations over finitely generated integral domains of characteristic 0 , including unit equations, Thue equations, hyper- and superelliptic equations, the Schinzel-Tijdeman equation, the Catalan equation, decomposable form equations and discriminant equations. Apart from discriminant equations, the other theorems are established in quantitative form, providing effective bounds for the solutions. The results presented make it possible to solve, at least in principle, the equations under consideration. Their proofs are given in Chapters 9 and 10 .

### 2.1 Notation and preliminaries

To make sense of statements such as that a particular Diophantine equation can be solved effectively over a given finitely generated domain, we need an explicit representation for this domain, as well as for its elements. We start below with the necessary definitions. A more detailed treatment can be found in Chapter 6, and in particular in Section 6.3

Let $A=\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]$ be a finitely generated integral domain of characteristic 0 . Assume that $r>0$ and let

$$
\mathcal{I}:=\left\{f \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]: f\left(z_{1}, \ldots, z_{r}\right)=0\right\} .
$$

Then $\mathcal{I}$ is an ideal of $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ which by Hilbert's Basis Theorem is finitely generated, that is, we have

$$
\begin{equation*}
A \cong \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right] / \mathcal{I}, \quad \text { with } \quad \mathcal{I}=\left(f_{1}, \ldots, f_{M}\right) \tag{2.1.1}
\end{equation*}
$$

for some finite set of polynomials $f_{1}, \ldots, f_{M} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$. We call the tuple $\left(f_{1}, \ldots, f_{M}\right)$ an ideal representation for $A$. Recall that a necessary and sufficient condition for $A$ to be a domain of characteristic 0 is that $\mathcal{I}$ be a prime ideal of $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ with $\mathcal{I} \cap \mathbb{Z}=(0)$. Given a set of generators $\left(f_{1}, \ldots, f_{M}\right)$ for $\mathcal{I}$ this can be checked effectively for example using Aschenbrenner (2004, Lemma 4.8, Corollary 4.9) and the comments in Chapter 6 of the present work.

To perform computations in $A$ it will be necessary to be able to decide whether for any given $f \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ and any given ideal $\mathcal{I}=\left(f_{1}, \ldots, f_{M}\right)$ of $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ we have $f \in \mathcal{I}$, that is, whether there exist $g_{1}, \ldots, g_{M} \in$ $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ such that $f=g_{1} f_{1}+\cdots+g_{M} f_{M}$. An algorithm performing this task is called an ideal membership algorithm for $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$. Several such algorithms have been developed since the 1960s; we mention only the algorithm of Simmons (1970), and the more precise algorithm of Aschenbrenner (2004), which plays an important role in our work; see Corollary 6.1.6 in Chapter 6

Denote by $K$ the quotient field of $A$. For $\alpha \in A$, we call $f$ a representative for $\alpha$, or say that $f$ represents $\alpha$ if $f \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ and $\alpha=f\left(z_{1}, \ldots, z_{r}\right)$. With the notation (2.1.1) this means that $\alpha$ corresponds to the residue class $f \bmod \mathcal{I}$. Further, for $\alpha \in K$, we call $(f, g)$ a pair of representatives for $\alpha$, or say that $(f, g)$ represents $\alpha$ if $f, g \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right], g \notin \mathcal{I}$ and $\alpha=$ $f\left(z_{1}, \ldots, z_{r}\right) / g\left(z_{1}, \ldots, z_{r}\right)$. Note that $g \notin \mathcal{I}$ can be verified by means of an ideal membership algorithm for $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$. A representative for a tuple $\left(x_{1}, \ldots, x_{m}\right) \in A^{m}$ is a tuple $\left(\widetilde{x}_{1}, \ldots, \widetilde{x}_{m}\right)$ with elements from $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ such that $\widetilde{x}_{i}$ represents $x_{i}$, for $i=1, \ldots, m$. Finally, a representative for a polynomial $F$ with coefficients in $A$ is a polynomial $\widetilde{F}$ with coefficients in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ that represent the corresponding coefficients of $F$.

We say that the domain $A$ is effectively given if an ideal representation as in (2.1.1) for it is given. Further, we say that an element $\alpha$ of $A$, resp. of $K$, is effectively given/computable if a representative, resp. a pair of representatives, for $\alpha$ is given/can be computed.

Given (pairs of) representatives for two elements of $A$ or $K$, it is clear how to compute a (pair of) representative(s) for their sum, difference, product or quotient. Using an ideal membership algorithm for $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ we can
decide whether two given $f, f^{\prime} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ represent the same element of $A$ (i.e., whether $f-f^{\prime} \in \mathcal{I}$ ) and whether two given pairs $(f, g),\left(f^{\prime}, g^{\prime}\right)$ in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ represent the same element of $K$ (this amounts to $g \cdot g^{\prime} \notin \mathcal{I}$ and $f g^{\prime}-f^{\prime} g \in \mathcal{I}$ ).

Suppose we know somehow that a particular system of polynomial Diophantine equations

$$
\begin{equation*}
F_{1}(\mathbf{x})=0, \ldots, F_{s}(\mathbf{x})=0 \text { in } \mathbf{x}=\left(x_{1}, \ldots, x_{m}\right) \in A^{m} \tag{2.1.2}
\end{equation*}
$$

where $F_{1}, \ldots, F_{s} \in A\left[Y_{1}, \ldots, Y_{m}\right]$, has only finitely many solutions. Then determining the solutions of (2.1.2) effectively means finding a finite list of tuples $\widetilde{\mathbf{x}}=\left(\widetilde{x}_{1}, \ldots, \widetilde{x}_{m}\right) \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]^{m}$ that represent all the solutions in $A^{m}$ of (2.1.2) and such that no two tuples in the list represent the same solution.

Rather than merely showing that the set of solutions of (2.1.2) can be determined effectively, it is sometimes possible to obtain more precise quantitative statements by estimating the sizes of the coordinates of the tuples representing the solutions. In fact, we define the size of a non-zero polynomial $f \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ by

$$
s(f):=\max (1, \operatorname{deg} f, h(f))
$$

where $\operatorname{deg} f$ denotes the degree, that is, total degree, of $f$ and $h(f)$ the $\log$ arithmic height of $f$, that is the logarithm of the maximum of the absolute values of the coefficients of $f$. Further, we define $s(0):=1$. Clearly, there are only finitely many polynomials in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ with size below a given bound, and these can be determined effectively.
Proposition 2.1.1. Let the domain have ideal representation $f_{1}, \ldots, f_{M}$ as in (2.1.1), and let $F_{1}, \ldots, F_{s} \in A\left[Y_{1}, \ldots, Y_{m}\right]$ be given by representatives $\widetilde{F}_{1}, \ldots, \widetilde{F}_{s}$, which are polynomials in the variables $Y_{1}, \ldots, Y_{m}$ with coeffcients in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$. Suppose we can compute $C$ in terms of $f_{1}, \ldots, f_{M}$, $\widetilde{F}_{1}, \ldots, \widetilde{F}_{s}$ such that every solution of the system (2.1.2) has a representative $\widetilde{\mathbf{x}}=\left(\widetilde{x}_{1}, \ldots, \widetilde{x}_{m}\right)$ with

$$
\widetilde{x}_{i} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right], \quad s\left(\widetilde{x}_{i}\right) \leq C \text { for } i=1, \ldots, m
$$

Then we can effectively determine the solutions of (2.1.2).
Proof. We enumerate all $m$-tuples consisting of elements in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ of size at most $C$. By means of an ideal membership algorithm for $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$
we check for each of these tuples $\widetilde{\mathbf{x}}$ whether $\widetilde{F}_{i}(\widetilde{\mathbf{x}}) \in \mathcal{I}$ for $i=1, \ldots, s$ and make a list of the tuples passing this test. This list contains at least one representative for each solution of (2.1.2). Subsequently we check, for any two tuples $\widetilde{\mathbf{x}}_{1}, \widetilde{\mathbf{x}}_{2}$ from this list, whether there is an index $i$ such that the difference of their $i$-th coordinates is not in $\mathcal{I}$. If there is not such an index $i$, then $\widetilde{\mathbf{x}}_{1}, \widetilde{\mathbf{x}}_{2}$ represent the same solution of $(2.1 .2$ so we may remove one of them from our list. What remains is a list with precisely one representative for each solution.

In Sections 2.2 to 2.5 we present effective finiteness results in quantitative form, i.e., with bounds for the sizes for representatives of their solutions, for unit equations, a generalization of unit equations, Thue equations, hyperand superelliptic equations, the Schinzel-Tijdeman equation and the Catalan equation over finitely generated domains. These results have been proved by means of the effective method of Evertse and Győry (2013), reducing the equations to the number field and function field cases, applying effective specializations. As will be pointed out in Chapters 3 and 7 , this is an improved version of the effective specialization method of Győry $(1983,1984 b)$.

Sections $2.6-2.8$ are devoted to effective finiteness results concerning decomposable form equations, norm form equations and discriminant equations over finitely generated domains. Here, following Győry's method, the equations are reduced to unit equations and then the general effective results concerning unit equations are used. The proofs use several effective results from commutative algebra and some new, effective, so-called 'degree-height estimates' from Chapter 8 for elements of $\bar{K}$. In Section 2.9 we mention some Diophantine problems that can be solved effectively over number fields but for which as yet no effective analogue over finitely generated domains could be established. We recall that except for Section 2.5 dealing with the Catalan equation, the earlier less general effective theorems were already mentioned in Chapter 1, in the corresponding sections. They will be also referred to in Chapter 4 treating effective results over number fields. Hence, apart from Section 2.5, no mention will be made in this chapter on earlier effective results over number fields or special finitely generated domains.

Throughout this work we shall use $O(\cdot)$ to denote a quantity which is an effectively computable positive absolute contant times the expression between the parentheses. The constant may be different at each occurence of the $O$ symbol.

### 2.2 Unit equations in two unknowns

In what follows, $A$ will denote an integral domain of characteristic 0 that is finitely generated over $\mathbb{Z}$. We assume that $A \cong \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right] /\left(f_{1}, \ldots, f_{M}\right)$, where $f_{1}, \ldots, f_{M}$ are given elements of $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$.

We start with unit equations in two unknowns, these are equations of the form

$$
\begin{equation*}
a x+b y=c \quad \text { in } x, y \in A^{*} \tag{2.2.1}
\end{equation*}
$$

where $a, b, c$ are non-zero elements of $A$. Let $\widetilde{a}, \widetilde{b}, \widetilde{c}$ be representatives in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ for $a, b, c$, respectively.

The effective result below was established by Evertse and Győry (2013).
Theorem 2.2.1. Assume that $f_{1}, \ldots, f_{M}$ and $\widetilde{a}, \widetilde{b}, \widetilde{c}$ all have degree at most $d$ and logarithmic height at most $h$, where $d \geq 1, h \geq 1$. Then for each solution $x, y$ of (2.2.1), there are representatives $\widetilde{x}, \widetilde{x}^{\prime}, \widetilde{y}, \widetilde{y}^{\prime}$ of $x, x^{-1}, y, y^{-1}$, respectively, such that

$$
\mathrm{s}(\widetilde{x}), \mathrm{s}\left(\widetilde{x}^{\prime}\right), \mathrm{s}(\widetilde{y}), \mathrm{s}\left(\widetilde{y}^{\prime}\right) \leq \exp \left((2 d)^{\exp \mathrm{O}(r)} h\right)
$$

The exponential dependence of the upper bound on $d$ and $h$ is a consequence of the use of Baker's method in the proof of Theorem 4.3.1 on unit equations over number fields. The bad dependence on $r$ comes from the effective commutative algebra for polynomial rings over fields and over $\mathbb{Z}$, which is used in the specialization method of Evertse and Győry (2013); see also Chapters 6 to 9 .

We deduce the following effective version of Lang's Theorem 1.2.1
Corollary 2.2.2. Equation (2.2.1) has only finitely many solutions. Further, if $A$ and $a, b, c$ are effectively given, then all solutions of (2.2.1) can be determined effectively.
Proof of Corollary 2.2.2 Notice that equation (2.2.1) is equivalent to the system

$$
a x+b y=c, x \cdot x^{\prime}=1, y \cdot y^{\prime}=1 \quad \text { in } x, x^{\prime}, y, y^{\prime} \in A .
$$

Apply Proposition 2.1.1 with $C$ the upper bound occurring in Theorem 2.2.1

We present a variation on Theorem 2.2.1. Let $\gamma_{1}, \ldots, \gamma_{s}$ be multiplicately independent elements of $K^{*}$. We mention that Proposition 7.5.2 in Chapter

7 provides a method to check whether elements $\gamma_{1}, \ldots, \gamma_{s}$ given by pairs of representatives are multiplicatively independent; see also Lemma 7.2 from Evertse and Győry (2013). Let again $a, b, c$ be non-zero elements of $A$ and consider the equation

$$
\begin{equation*}
a \gamma_{1}^{u_{1}} \ldots \gamma_{s}^{u_{s}}+b \gamma_{1}^{v_{1}} \ldots \gamma_{s}^{v_{s}}=c \quad \text { in } \quad u_{1}, \ldots, u_{s}, v_{1}, \ldots, v_{s} \in \mathbb{Z} \tag{2.2.2}
\end{equation*}
$$

Theorem 2.2.3. Let $\widetilde{a}, \widetilde{b}, \widetilde{c}$ be representatives for $a, b, c$, and for $i=1, \ldots, s$ let $\left(g_{i, 1}, g_{i, 2}\right)$ be a pair of representatives for $\gamma_{i}$. Suppose that $f_{1}, \ldots, f_{M}, \widetilde{a}, \widetilde{b}, \widetilde{c}$ and $g_{i, 1}, g_{i, 2}(i=1, \ldots, s)$ all have degree at most $d$ and logarithmic height at most $h$, where $d \geq 1, h \geq 1$. Then for each solution $\left(u_{1}, \ldots, v_{s}\right)$ of (2.2.2 we have

$$
\max \left(\left|u_{1}\right|, \ldots,\left|u_{s}\right|,\left|v_{1}\right|, \ldots,\left|v_{s}\right|\right) \leq \exp \left((2 d)^{\exp \mathrm{O}(r+s)} h\right)
$$

An immediate consequence of Theorem 2.2.3 is that for given $f_{1}, \ldots, f_{M}$, $a, b, c$ and $\gamma_{1}, \ldots, \gamma_{s}$, all solutions of (2.2.2) can be determined effectively.

Since the unit group $A^{*}$ is finitely generated, equation (2.2.1) may be viewed as a special case of (2.2.2). But no general effective algorithm is known to find a finite system of generators for $A^{*}$, hence we cannot deduce an effective result for 2.2.1) from Theorem 2.2.3. In fact, in Chapter 9 we shall argue reversely, and prove Theorem 2.2.3 by combining Theorem 2.2.1 with an effective result on equations of the type $\gamma_{1}^{u_{1}} \ldots \gamma_{s}^{u_{s}}=\gamma_{0}$ in integers $u_{1}, \ldots, u_{s}$, where $\gamma_{0}, \gamma_{1}, \ldots, \gamma_{s} \in K^{*}$.

Finally, we mention that using the effective method of Evertse and Győry (2013), Bérczes (2015a,2015b) made effective in full generality the results of Lang resp. of Liardet presented in Section 1.2 on the equations

$$
\begin{array}{lll}
F(x, y)=0 & \text { in } & x, y \in A^{*}, \\
F(x, y)=0 & \text { in } & x, y \in \bar{\Gamma} \tag{2.2.4}
\end{array}
$$

where $\Gamma$ is a finitely generated multiplicative subgroup of $K^{*}, \bar{\Gamma}$ is its division group and $F \in A[X, Y]$. In Bérczes' results, $F$ has to satisfy a slightly stronger condition than in Theorems 1.2 .3 and 1.2.4, that is, $F \in A[X, Y]$ is a non-constant polynomial that is not divisible by any polynomial of the form

$$
X^{m} Y^{n}-\alpha \quad \text { or } \quad X^{m}-\alpha Y^{n}
$$

with $\alpha \in \bar{K}$ and with non-negative integers $m, n$, not both zero. As was
pointed out by Bérczes, this condition can be checked effectively once pairs of representatives for the coefficients of $F$ are given.

Assume that $f_{1}, \ldots, f_{M}$ and a set of representatives for the coefficients of $F$ have degree at most $d$ and logarithmic height at most $h$, with $d>1, h>1$. Further, denote by $N$ the total degree of $F$. Bérczes (2015a, Thm. 2.1) proved in a more precise form the following.

Theorem 2.2.4. Under the above assumptions, there is an effectively computable number $C_{1}$ depending only on $r, d, h$ and $N$ such that for every solution $x, y \in A^{*}$ of (2.2.3) there are representatives $\widetilde{x}, \widetilde{x}^{\prime}, \widetilde{y}, \widetilde{y}^{\prime}$ of $x, x^{-1}, y, y^{-1}$, respectively, such that

$$
s(\widetilde{x}), s\left(\widetilde{x}^{\prime}\right), s(\widetilde{y}), s\left(\widetilde{y}^{\prime}\right) \leq C_{1} .
$$

Let the generators $\gamma_{1}, \ldots, \gamma_{s}$ of $\Gamma$ be given by pairs of representatives $\left(g_{1}, h_{1}\right), \ldots,\left(g_{s}, h_{s}\right)$. Assume now that $f_{1}, \ldots, f_{M}, g_{1}, h_{1}, \ldots, g_{s}, h_{s}$ and a set of representatives for the coefficients of $F$ have degree at most $d$ and logarithmic height at most $h$, with $d, h>1$. Then Bérczes (2015b, Thm. 2.1) obtained the following.

Theorem 2.2.5. There is an effectively computable number $C_{2}$ depending only on $r, s, d, h$ and $N$ such that for every solution $x, y$ of (2.2.4) in $\bar{\Gamma}$ we have

$$
x^{k}=\gamma_{1}^{k_{1, x}} \cdots \gamma_{s}^{k_{s, x}}, \quad y^{k}=\gamma_{1}^{k_{1, y}} \cdots \gamma_{s}^{k_{s, y}}
$$

where $k, k_{1, x}, \ldots, k_{s, x}, k_{1, y}, \ldots, k_{s, y}$ are integers with $k \geq 1$ and

$$
k,\left|k_{1, x}\right|, \ldots,\left|k_{s, x}\right|,\left|k_{1, y}\right|, \ldots,\left|k_{s, y}\right| \leq C_{2} .
$$

Theorems 2.2.4 and 2.2.5 imply in an effective form the finiteness of the number of solutions of equations (2.2.3) and (2.2.4). We have not included the rather technical proofs of Theorems 2.2.4 and 2.2.5 in our book.

### 2.3 Thue equations

As before, $A$ denotes an integral domain of characteristic 0 that is finitely generated over $\mathbb{Z}$, and we assume that $A \cong \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right] / \mathcal{I}$, where $\mathcal{I}=$ $\left(f_{1}, \ldots, f_{M}\right)$ with $f_{1}, \ldots, f_{M}$ given elements of $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$. We consider
the Thue equation

$$
\begin{equation*}
F(x, y)=\delta \text { in }(x, y) \in A^{2} \tag{2.3.1}
\end{equation*}
$$

over $A$, where

$$
F(X, Y)=a_{0} X^{n}+a_{1} X^{n-1} Y+\cdots+a_{n} Y^{n} \in A[X, Y]
$$

is a binary form of degree $n \geq 3$ with non-zero discriminant $D_{F}$, i.e., with $n$ pairwise non-proportional linear factors, and $\delta \in A \backslash\{0\}$. We choose representatives

$$
\widetilde{a_{0}}, \widetilde{a_{1}}, \ldots, \widetilde{a_{n}}, \widetilde{\delta} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]
$$

of $a_{1}, \ldots, a_{n}, \delta$ respectively, where $\widetilde{\delta} \notin \mathcal{I}$ and the discriminant $D_{\widetilde{F}}$ of $\widetilde{F}:=$ $\sum_{j=0}^{n} \widetilde{a_{j}} X^{n-j} Y^{j}$ is not in $\mathcal{I}$. These conditions on $\widetilde{\delta}$ and $D_{\widetilde{F}}$ can be checked by means of an ideal membership algorithm for $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$.

The next theorem is due to Bérczes, Evertse and Győry (2014).
Theorem 2.3.1. Assume that $f_{1}, \ldots, f_{M}$ and $\widetilde{a_{0}}, \ldots, \widetilde{a_{n}}, \widetilde{\delta}$ all have degree at most $d$ and logarithmic height at most $h$, where $d \geq 1, h \geq 1$. Then every solution $(x, y)$ of equation (2.3.1) has a representative $(\widetilde{x}, \widetilde{y})$ such that

$$
\begin{equation*}
\mathrm{s}(\widetilde{x}), \mathrm{s}(\widetilde{y}) \leq \exp \left\{n!(n d)^{\exp O(r)} h\right\} . \tag{2.3.2}
\end{equation*}
$$

Combining Theorem 2.3.1 with Proposition 2.1.1 one immediately obtains the following:
Corollary 2.3.2. Equation (2.3.1) has only finitely many solutions. Further, if $A, a_{1}, \ldots, a_{n}$ and $\delta$ are effectively given, then all solutions of (2.3.1) can be determined effectively.

### 2.4 Hyper- and superelliptic equations, the SchinzelTijdeman equation

We keep our notation that $A$ is an integral domain of characteristic 0 that is finitely generated over $\mathbb{Z}$, satisfying $A \cong \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right] / \mathcal{I}$, where $\mathcal{I}=$ $\left(f_{1}, \ldots, f_{M}\right)$ with $f_{1}, \ldots, f_{M}$ given elements of $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$. We now consider the equation

$$
\begin{equation*}
F(x)=\delta y^{m} \text { in } x, y \in A, \tag{2.4.1}
\end{equation*}
$$

where

$$
F(X)=a_{0} X^{n}+a_{1} X^{n-1}+\cdots+a_{n} \in A[X],
$$

$\delta \in A \backslash\{0\}$ and $F$ has $n$ distinct roots in an algebraic closure of the quotient field of $A$, i.e., $a_{0}$ as well as the discriminant of $F$ are different from zero. We choose representatives

$$
\widetilde{a_{0}}, \widetilde{a_{1}}, \ldots, \widetilde{a_{n}}, \widetilde{\delta} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]
$$

for $a_{0}, a_{1}, \ldots, a_{n}, \delta$, respectively, where $\widetilde{\delta}, \widetilde{a_{0}}$ and the discriminant of $\widetilde{F}:=$ $\sum_{j=0}^{n} \widetilde{a_{j}} X^{n-j}$ are not in $\mathcal{I}$. We assume that

$$
\text { either } \quad m=2 \text { and } n \geq 3 \quad \text { or } \quad m \geq 3 \text { and } n \geq 2 .
$$

The following theorems are due to Bérczes, Evertse and Győry (2014).

Theorem 2.4.1. Assume that $f_{1}, \ldots, f_{M}$ and $\widetilde{a}_{0}, \ldots, \widetilde{a}_{n}, \widetilde{\delta}$ have degree at most $d$ and logarithmic height at most $h$, where $d \geq 1, h \geq 1$. Then every solution of equation (2.4.1) has representatives $\widetilde{x}, \widetilde{y}$ such that

$$
\begin{equation*}
\mathrm{s}(\widetilde{x}), \mathrm{s}(\widetilde{y}) \leq \exp \left\{m^{3}(n d)^{\exp \mathrm{O}(r)} h\right\} . \tag{2.4.2}
\end{equation*}
$$

Combined with Proposition 2.1.1, this provides a method to determine effectively the solutions of (2.4.1), i.e., it provides an effective version of Theorem 1.3.1 concerning hyperelliptic/superelliptic equations.

Our next result concerns the Schinzel-Tijdeman equation

$$
\begin{equation*}
F(x)=\delta y^{m} \quad \text { in } \quad x, y \in A, \quad m \in \mathbb{Z}_{\geq 2} . \tag{2.4.3}
\end{equation*}
$$

Keeping the above notation, we have

Theorem 2.4.2. Assume that in (2.4.3) $F$ has degree $n \geq 2$ and non-zero discriminant. Let $x, y \in A, m \geq 2$ integer be a solution of (2.4.3). Then with the same notation as in Theorem 2.4.1] we have

$$
\begin{align*}
& m \leq \exp \left\{(n d)^{\exp \mathrm{O}(r)} h\right\} \\
& \quad \text { if } y \in \overline{\mathbb{Q}}, y \neq 0, \text { and } y \text { is not a root of unity, }  \tag{2.4.4}\\
& m \leq(n d)^{\exp \mathrm{O}(r) \quad \text { if } y \notin \overline{\mathbb{Q}} .} \tag{2.4.5}
\end{align*}
$$

### 2.5 The Catalan equation

As before, $A$ is an integral domain of characteristic 0 that is finitely generated over $\mathbb{Z}$, satisfying $A \cong \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right] / \mathcal{I}$, where $\mathcal{I}=\left(f_{1}, \ldots, f_{M}\right)$ with $f_{1}, \ldots, f_{M}$ given elements of $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$. Consider the Catalan equation

$$
\begin{align*}
x^{m}-y^{n}=1 & \text { in } x, y \in A \backslash\{0\} \\
& \text { and } m, n \in \mathbb{Z} \text { with } m, n>1 \text { and } m n>4 . \tag{2.5.1}
\end{align*}
$$

In contrast with the other equations from this chapter, we cite here the most important earlier results concerning equation (2.5.1). In the classical case $A=\mathbb{Z}$, Catalan (1844) conjectured that $3^{2}-2^{3}=1$ is the only solution of the equation in positive integers $x, y, m, n$ with $m, n>1$. In this case Tijdeman (1976), using Baker's method, gave an effectively computable, but very large upper bound for the solutions of equation (2.5.1). Brindza, Győry and Tijdeman (1986) and Brindza (1987) generalized Tijdeman's result for the case when $x, y$ are integers resp. $S$-integers of a given number field, and Brindza (1993) further generalized this for the case of the restricted class of finitely generated ground domains $A$ considered in Győry $(1983,1984 b)$.

Mihailescu (2004) used methods from pure algebraic number theory to prove Catalan conjecture over $\mathbb{Z}$.

Strengthening earlier results of $\operatorname{Brindza}(1987,1993)$ on equation (2.5.1), Koymans $(2016,2017)$ proved the following theorem using the method of Evertse and Győry (2013).

Theorem 2.5.1. Assume that $f_{1}, \ldots, f_{M}$ have degree at most $d$ and logarithmic height at mos $h$, where $d \geq 1, h \geq 1$. Let $x, y, m, n$ be a solution of (2.5.1) such that $x, y$ are not roots of unity. Then

$$
\begin{align*}
& \max (m, n) \leq \exp \exp \exp \left\{(2 d)^{\exp \mathrm{O}(r)} h\right\} \quad \text { if } x, y \in \overline{\mathbb{Q}},  \tag{2.5.2}\\
& \max (m, n) \leq(2 d)^{\exp \mathrm{O}(r)} \quad \text { if } x, y \notin \overline{\mathbb{Q}}, \tag{2.5.3}
\end{align*}
$$

It is easy to see that in Theorem 2.5.1 the conditions $m, n>1, m n>4$ are necessary.

By combining Theorem 2.5.1 with Theorem 2.4.1, it follows immediately that equation (2.5.1) has only finitely many solutions with $x, y$ not roots of unity and, combined with Proposition 2.1.1, we obtain that from a given ideal representation $\left(f_{1}, \ldots, f_{M}\right)$, all solutions can be determined effectively.

In his master's thesis, Koymans (2016) proved an analogue of Theorem
2.5.1 over finitely generated domains of positive characteristic.

### 2.6 Decomposable form equations

Let again $A$ be an integral domain of characteristic 0 which is finitely generated over $\mathbb{Z}$ such that $A \cong \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right] / \mathcal{I}$ with $\mathcal{I}=\left(f_{1}, \ldots, f_{M}\right)$ for some given polynomials $f_{1}, \ldots, f_{M} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$. We denote by $K$ the quotient field of $A$, and by $\bar{K}$ an algebraic closure of $K$.

Pick linear forms

$$
\begin{equation*}
\ell_{i}=\alpha_{i, 1} X_{1}+\cdots+\alpha_{i, m} X_{m} \in \bar{K}\left[X_{1}, \ldots, X_{m}\right] \quad(i=1, \ldots, n) \tag{2.6.1}
\end{equation*}
$$

in $m \geq 2$ variables. We allow that some of these linear forms are equal. Let $F=\ell_{1} \cdots \ell_{n}$ be their product, $\delta \in \bar{K}^{*}$, and consider the decomposable form equation

$$
\begin{equation*}
F(\mathbf{x})=\ell_{1}(\mathbf{x}) \cdots \ell_{n}(\mathbf{x})=\delta \quad \text { in } \mathbf{x}=\left(x_{1}, \ldots, x_{m}\right) \in A^{m} \tag{2.6.2}
\end{equation*}
$$

We do not require that $F$ have its coefficients in $K$. Subject to certain conditions on $\ell_{1}, \ldots, \ell_{n}$ and $\mathbf{x}$, we will formulate an effective finiteness result on equation (2.6.2) in a quantitative form. To this end, we introduce some notation, and explain the conditions imposed on the $\ell_{i}$.

Let us first introduce the $A$-module

$$
\begin{equation*}
\mathcal{Z}_{A, F}=\left\{\mathbf{x} \in A^{m}: \ell_{1}(\mathbf{x})=\cdots=\ell_{m}(\mathbf{x})=0\right\} . \tag{2.6.3}
\end{equation*}
$$

Clearly, if $\mathbf{x}$ is a solution of (2.6.2), then so is $\mathbf{x}+\mathbf{y}$ for every $\mathbf{y} \in \mathcal{Z}_{A, F}$. Hence the set of solutions of (2.6.2) falls apart into $\mathcal{Z}_{A, F^{-}}$cosets $\mathbf{x}+\mathcal{Z}_{A, F},{ }^{1}$ and we want to determine representatives for these cosets.

In case that $\operatorname{rank}\left\{\ell_{1}, \ldots, \ell_{n}\right\}=m$ we have $\mathcal{Z}_{A, F}=\{0\}$ and the $\mathcal{Z}_{A, F^{-}}$ cosets are just single solutions. Had we been interested in non-effective finiteness results only, the generalization to the case $\operatorname{rank}\left\{\ell_{1}, \ldots, \ell_{n}\right\}<m$ and $\mathcal{Z}_{A, F} \neq\{\mathbf{0}\}$ would not have been necessary, but for certain effective applications this turned out to be of importance.

Given $\mathbf{x}=\left(x_{1}, \ldots, x_{m}\right) \in A^{m}$, a representative for $\mathbf{x}$ is a tuple $\widetilde{\mathbf{x}}=$

[^1]$\left(\widetilde{x_{1}}, \ldots, \widetilde{x_{m}}\right)$ with
$$
\widetilde{x_{i}} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right], x_{i}=\widetilde{x_{i}}\left(z_{1}, \ldots, z_{r}\right) \text { for } i=1, \ldots, m .
$$

We define the size of this tuple by

$$
\begin{aligned}
s(\widetilde{\mathbf{x}}) & :=\max \left(s\left(\widetilde{x_{1}}\right), \ldots, s\left(\widetilde{x_{m}}\right)\right) \\
& =\max \left(1, \operatorname{deg} \widetilde{x_{1}}, h\left(\widetilde{x_{1}}\right), \ldots, \operatorname{deg} \widetilde{x_{m}}, h\left(\widetilde{x_{m}}\right)\right) .
\end{aligned}
$$

Slightly diverging from its usual meaning, by a representative for a $\mathcal{Z}_{A, F^{-}}$ coset we shall mean a tuple $\widetilde{\mathbf{x}} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]^{m}$ representing any element from this coset. Thus, in order to effectively determine a full system of representives for the $\mathcal{Z}_{A, F}$-cosets of solutions, it suffices to compute a number $C$ such that each of these cosets has a representative $\widetilde{\mathbf{x}} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]^{m}$ with $s(\widetilde{\mathbf{x}}) \leq C$.

Next, we need some measures for elements of $\bar{K}$. Let $\alpha \in \bar{K}$. We denote by $\operatorname{deg}_{K} \alpha$ the degree of $\alpha$ over $K$. A tuple of representatives for $\alpha$ is a tuple $\left(g_{0}, \ldots, g_{n}\right)$, where $n=\operatorname{deg}_{K} \alpha$, and where $g_{0}, \ldots, g_{n} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$, $g_{0} \notin \mathcal{I}$, such that

$$
X^{n}+\frac{g_{1}\left(z_{1}, \ldots, z_{r}\right)}{g_{0}\left(z_{1}, \ldots, z_{r}\right)} X^{n-1}+\cdots+\frac{g_{n}\left(z_{1}, \ldots, z_{r}\right)}{g_{0}\left(z_{1}, \ldots, z_{r}\right)}
$$

is the monic minimal polynomial of $\alpha$ over $K$. If $\alpha \in K$, then a tuple of representatives for $\alpha$ is up to sign a pair of representatives for $\alpha$, as introduced before. We say that $\left(g_{0}, \ldots, g_{n}\right)$ has degree at most $d$ and logarithmic height at most $h$, if each $g_{i}(i=0, \ldots, n)$ has total degree at most $d$ and logarithmic height at most $h$.

In order to formulate our effective results, we adopt some terminology from Győry and Papp (1978) and Győry (1981a, 1982a, 1983). Let $\mathcal{L}=$ $\left(\ell_{1}, \ldots, \ell_{n}\right)$ be the system of linear forms from (2.6.1). As said before, these linear forms need not be pairwise distinct. We define the triangular graph $\mathcal{G}(\mathcal{L})$ of $\mathcal{L}$ as follows:
$\mathcal{G}(\mathcal{L})$ has vertex system $\mathcal{L}$;
$\ell_{i}$ and $\ell_{j}$ with $i \neq j$ are connected by an edge if either $\ell_{i}, \ell_{j}$ are linearly dependent over $\bar{K}$ or they are linearly independent over $\bar{K}$ and there is $q \neq i, j$ such that $\ell_{i}, \ell_{j}, \ell_{q}$ are linearly dependent over $\bar{K}$.

Let $\mathcal{L}_{1}, \ldots, \mathcal{L}_{k}$ denote the vertex systems of the connected components of $\mathcal{G}(\mathcal{L})$. When $k=1$, we say that $\mathcal{L}$ or $F$ is triangularly connected; see Győry and Papp (1978). For $j=1, \ldots, k$, denote by $[\mathcal{L} j]$ the $\bar{K}$-vector space generated by the linear forms from $\mathcal{L}_{j}$ and assume that

$$
\begin{equation*}
\left[\mathcal{L}_{1}\right] \cap \cdots \cap\left[\mathcal{L}_{k}\right] \neq(0) \tag{2.6.5}
\end{equation*}
$$

This is in general a serious restriction, which is not satisfied by most systems $\mathcal{L}$. In fact, it is much stronger than the criterion from Theorem 1.5.1.

In what follows we want to consider solutions $\mathbf{x} \in A^{m}$ of (2.6.2) such that

$$
\begin{equation*}
\text { there is } \ell \in\left[\mathcal{L}_{1}\right] \cap \cdots \cap\left[\mathcal{L}_{k}\right] \text { with } \ell(\mathbf{x}) \neq 0 . \tag{2.6.6}
\end{equation*}
$$

This is the set of solutions of (2.6.2) to which our effective method can be applied. Here the linear form $\ell$ may vary with $\mathbf{x}$. We should note here that if $\mathbf{x} \in A^{m}$ satisfies (2.6.6), then so does every element of the $\mathcal{Z}_{A, F}$-coset $\mathrm{x}+\mathcal{Z}_{A, F}$.

In the case that $\mathcal{L}$ is triangularly connected, i.e., $k=1$, we have $\mathcal{L}_{1}=\mathcal{L}=$ $\left(\ell_{1}, \ldots, \ell_{n}\right)$ hence 2.6 .5 is satisfied. Further, if $\delta \neq 0$ then every solution of (2.6.2) automatically satisfies (2.6.6).

We are now ready to state our results. We first formulate a quantitative result, and then a corollary giving an effective finiteness statement.

Theorem 2.6.1. Suppose the following:

- the given generators $f_{1}, \ldots, f_{M}$ of $\mathcal{I}$ have degree at most $d$ and logarithmic height at most $h$;
- $\delta$ and the coefficients of $\ell_{1}, \ldots, \ell_{n}$ all have tuples of representatives of degree at most $d$ and logarithmic height at most $h$;
- the coefficients of $\ell_{1}, \ldots, \ell_{n}$ all have degree at most $\nu$ over $K$;
$-\left[\mathcal{L}_{1}\right] \cap \cdots \cap\left[\mathcal{L}_{k}\right] \neq(0)$.
Then every $\mathcal{Z}_{A, F}$-coset of $\mathbf{x} \in A^{m}$ such that

$$
\begin{equation*}
F(\mathbf{x})=\delta \text {, there is } \ell \in\left[\mathcal{L}_{1}\right] \cap \cdots \cap\left[\mathcal{L}_{k}\right] \text { with } \ell(\mathbf{x}) \neq 0 \tag{2.6.7}
\end{equation*}
$$

is represented by $\widetilde{\mathbf{x}} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]^{m}$ with

$$
\begin{equation*}
s(\widetilde{\mathbf{x}}) \leq \exp \left(\left(2 m n \cdot \nu^{\nu m n} d\right)^{\exp O(r)} h\right) \tag{2.6.8}
\end{equation*}
$$

We deduce from this an effective finiteness result. Assume that $A$ and its quotient field $K$ are given effectively. A finite extension $G$ of $K$ is said to be
given effectively, if it is given in the form $K[X] /(P)$, where $P$ is an effectively given irreducible monic polynomial in $K[X]$. We note that for a given polynomial $P \in K[X]$ it can be decided effectively whether it is irreducible, see for instance Theorem 6.2 .3 in Section 6 . We may write $G=K(\theta)$ where $\theta:=X$ $(\bmod P)$. Thus, elements of $G$ can be expressed uniquely as $\sum_{i=0}^{g-1} a_{i} \theta^{i}$ with $a_{0}, \ldots, a_{g-1} \in K$, where $g$ denotes the degree of $G$ over $K$. We say that an element of $G$ is given/can be determined effectively if the corresponding $a_{0}, \ldots, a_{g-1}$ are given/can be determined effectively.

Corollary 2.6.2. There are only finitely many $\mathcal{Z}_{A, F}$-cosets of $\mathrm{x} \in A^{m}$ with (2.6.7). Moreover, if $\delta$ and the coefficients of $\ell_{1}, \ldots, \ell_{n}$ all belong to a finite extension $G$ of $K$ and if $A, K, G, \delta$ and the coefficients of $\ell_{1}, \ldots, \ell_{n}$ are given effectively, then one can determine effectively a set, consisting of precisely one representative $\widetilde{\mathbf{x}} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]^{m}$ for each of these cosets.

The essence of the proofs of Theorem 2.6.1 and Corollary 2.6 .2 is that thanks to the condition 2.6.6, equation (2.6.7) can be reduced to a finite system of unit equations in two unknowns, however with units from a subring $A^{\prime} \supset A$ of $G$ that is finitely generated over $\mathbb{Z}$. Then Theorem 2.6.1 and Corollary 2.6.2 are deduced by applying Theorem 2.2.1 with $A^{\prime}$ instead of $A$. In the course of the proof of Theorem 2.6.1 we use so-called 'degree-height estimates' for elements of $\bar{K}$, see Chapter 8 . The proofs of Theorems 2.6.1 and Corollary 2.6.2 are given in Section 10.1.

In this generality Theorem 2.6.1 and Corollary 2.6 .2 are new. The finiteness statement of Corollary 2.6 .2 was first proved in Győry (1982), but with slightly stronger conditions instead of (2.6.5) and (2.6.6): instead of (2.6.5) Győry assumed that $X_{m} \in\left[\mathcal{L}_{1}\right] \cap \cdots \cap\left[\mathcal{L}_{k}\right]$, and instead of (2.6.6) he assumed that $x_{m} \neq 0$; see also Evertse and Győry (2015, Chapter 9).

Gyôry (1983) established a quantitative result comparable to Theorem 2.6.1 but only for a restricted class of finitely generated integral domains $A$; see also Győry (1984a). Over number fields, more precise quantitative versions were obtained in Győry (1998) and Győry and Yu (2006).

We now discuss some applications. Let $F \in A[X, Y]$ be a non-zero binary form and $\delta$ a non-zero element of $A$, and consider again the Thue equation

$$
\begin{equation*}
F(x, y)=\delta \text { in }(x, y) \in A^{2} \tag{2.3.1}
\end{equation*}
$$

We can factorize $F$ as

$$
F=\ell_{1} \cdots \ell_{n} \text { with linear forms } \ell_{i} \in \bar{K}[X, Y] .
$$

Assume that at least three among the linear forms $\ell_{1}, \ldots, \ell_{n}$ are pairwise nonproportional over $\bar{K}$. Then it is easily verified that $\mathcal{L}=\left(\ell_{1}, \ldots, \ell_{n}\right)$ is triangularly connected and that $\mathcal{Z}_{A, F}=\{\mathbf{0}\}$. Further, let $\delta$ be a non-zero element of $A$. Theorem 2.6.1 and Corollary 2.6.2 imply the following variation on Theorem 2.3.1, with worse bounds.

Corollary 2.6.3. Assume that the given generators $f_{1}, \ldots, f_{M}$ of $\mathcal{I}$ have degree at most d and logarithmic height at most $h$, and that $\delta$ and the coefficients of $F$ have representatives in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ of degree at most $d$ and logarithmic height at most $h$, where $d \geq 1$ and $h \geq 1$.

Then each solution $(x, y) \in A^{2}$ of (2.3.1) is represented by a pair $(\widetilde{x}, \widetilde{y})$ with

$$
\begin{equation*}
\left.\widetilde{x}, \widetilde{y} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right], \quad s(\widetilde{x}), s(\widetilde{y})\right) \leq \exp \left(\left(n^{n^{2}} d\right)^{\exp O(r)} h\right) \tag{2.6.9}
\end{equation*}
$$

Consequently, the solutions $(x, y) \in A^{2}$ of (2.3.1) can be determined effectively.

The next application is to a system of double Pell equations

$$
\begin{equation*}
\gamma_{1} x_{1}^{2}-\gamma_{2} x_{2}^{2}=\beta_{1,2}, \quad \gamma_{1} x_{1}^{2}-\gamma_{3} x_{3}^{2}=\beta_{1,3} \text { in }\left(x_{1}, x_{2}, x_{3}\right) \in A^{3} \tag{2.6.10}
\end{equation*}
$$

where $\gamma_{1}, \gamma_{2}, \gamma_{3}, \beta_{1,2}, \beta_{1,3} \in A$ with

$$
\begin{equation*}
\gamma_{1} \gamma_{2} \gamma_{3} \beta_{1,2} \beta_{1,3}\left(\beta_{1,2}-\beta_{1,3}\right) \neq 0 \tag{2.6.11}
\end{equation*}
$$

From the two equations in (2.6.10) it follows that

$$
\gamma_{2} x_{2}^{2}-\gamma_{3} x_{3}^{2}=\beta_{1,3}-\beta_{1,2}
$$

and thus,

$$
\begin{equation*}
F\left(x_{1}, x_{2}, x_{3}\right)=\delta, \tag{2.6.12}
\end{equation*}
$$

where

$$
\begin{equation*}
\delta=\beta_{1,2} \beta_{1,3}\left(\beta_{1,3}-\beta_{1,2}\right) \tag{2.6.13}
\end{equation*}
$$

and

$$
\begin{align*}
F= & \left(\gamma_{1} X_{1}^{2}-\gamma_{2} X_{2}^{2}\right)\left(\gamma_{1} X_{1}^{2}-\gamma_{3} X_{3}^{2}\right)\left(\gamma_{2} X_{2}^{2}-\gamma_{3} X_{3}^{2}\right) \\
= & \left(\sqrt{\gamma_{1}} X_{1}+\sqrt{\gamma_{2}} X_{2}\right)\left(\sqrt{\gamma_{1}} X_{1}-\sqrt{\gamma_{2}} X_{2}\right) . \\
& \cdot\left(\sqrt{\gamma_{1}} X_{1}+\sqrt{\gamma_{3}} X_{3}\right)\left(\sqrt{\gamma_{1}} X_{1}-\sqrt{\gamma_{3}} X_{2}\right) . \\
& \cdot\left(\sqrt{\gamma_{2}} X_{2}+\sqrt{\gamma_{3}} X_{3}\right)\left(\sqrt{\gamma_{2}} X_{2}-\sqrt{\gamma_{3}} X_{3}\right) \tag{2.6.14}
\end{align*}
$$

with appropriate choices for the square roots. It is easy to verify that the linear factors of $F$ form a triangularly connected system and that $\mathcal{Z}_{A, F}=\{\mathbf{0}\}$. So Theorem 2.6.1 can be applied. This leads to the following:

Corollary 2.6.4. Assume that the given generators $f_{1}, \ldots, f_{M}$ of $\mathcal{I}$ have degree at most $d$ and logarithmic height at most $h$, and that $\gamma_{i}(i=0,1,2), \beta_{1,2}$, $\beta_{1,3}$ have representatives in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ of degree at most $d$ and logarithmic height at most $h$, where $d \geq 1, h \geq 1$. Assume (2.6.11).

Then each solution $\left(x_{1}, x_{2}, x_{3}\right) \in A^{3}$ of 2.6.10) is represented by a triple $\left(\widetilde{x_{1}}, \widetilde{x_{2}}, \widetilde{x_{3}}\right)$ with

$$
\begin{equation*}
\widetilde{x_{i}} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right], \quad s\left(\widetilde{x_{i}}\right) \leq \exp \left((2 d)^{\exp O(r)} h\right) \text { for } i=1,2,3 \tag{2.6.15}
\end{equation*}
$$

Consequently, the solutions $\left(x_{1}, x_{2}, x_{3}\right) \in A^{3}$ of 2.6.10) can be determined effectively.

### 2.7 Norm form equations

We keep the notation that $A$ is an integral domain of characteristic 0 which is finitely generated over $\mathbb{Z}$ such that $A \cong \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right] / \mathcal{I}$ with $\mathcal{I}=\left(f_{1}, \ldots, f_{M}\right)$ for some given polynomials $f_{1}, \ldots, f_{M} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$. As before, we denote by $K$ the quotient field of $A$, and by $\bar{K}$ an algebraic closure of $K$.

Let $\alpha_{1}=1, \alpha_{2}, \ldots, \alpha_{m} \in \bar{K}(m \geq 2)$ be linearly independent over $K$. Consider the norm form equation

$$
\begin{equation*}
N_{K^{\prime} / K}\left(\alpha_{1} x_{1}+\cdots+\alpha_{m} x_{m}\right)=\delta \quad \text { in }\left(x_{1}, \ldots, x_{m}\right) \in A^{m}, \tag{2.7.1}
\end{equation*}
$$

where $K^{\prime}=K\left(\alpha_{1}, \ldots, \alpha_{m}\right)$ and $\delta$ is a non-zero element of $K$. In Section 10.2 we deduce the following from Theorem 2.6.1 and Corollary 2.6.2:

Theorem 2.7.1. Let $f_{1}, \ldots, f_{M}$ have degree at most d and logarithmic height at most $h$, and let $\delta, \alpha_{1}, \ldots, \alpha_{m}$ be represented by tuples of degree at most
$d$ and logarithmic height at most $h$. Suppose that $\left[K^{\prime}: K\right]=n \geq 3$, that $\alpha_{1}, \ldots, \alpha_{m}$ are linearly independent over $K$ and that $\alpha_{m}$ is of degree $\geq 3$ over $K\left(\alpha_{1}, \ldots, \alpha_{m-1}\right)$. Then each solution $\mathbf{x}=\left(x_{1}, \ldots, x_{m}\right) \in A^{m}$ of (2.7.1) with $x_{m} \neq 0$ is represented by $\widetilde{\mathbf{x}} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]^{m}$ with

$$
s(\widetilde{\mathbf{x}}) \leq \exp \left(\left(n^{m n^{2}} d\right)^{\exp O(r)} h\right)
$$

Corollary 2.7.2. Suppose again that $\alpha_{1}, \ldots, \alpha_{m}$ are linearly independent over $K$ and that $\alpha_{m}$ is of degree $\geq 3$ over $K\left(\alpha_{1}, \ldots, \alpha_{m-1}\right)$. Then equation (2.7.1) has only finitely many solutions with $x_{m} \neq 0$. Moreover, if $A, K, K^{\prime}$ and $\alpha_{1}, \ldots, \alpha_{m}$ and $\delta$ are given effectively, then all solutions of (2.7.1) with $x_{m} \neq 0$ can be determined effectively.

The norm form in (2.7.1) can be expressed as

$$
\begin{align*}
& N_{K^{\prime} / K}\left(\alpha_{1} X_{1}+\cdots+\alpha_{m} X_{m}\right)=\ell_{1} \cdots \ell_{n} \\
& \quad \text { with } \ell_{i}=\alpha_{1}^{(i)} X_{1}+\cdots+\alpha_{m}^{(i)} X_{m} \quad(i=1, \ldots, m) . \tag{2.7.2}
\end{align*}
$$

Let $\mathcal{L}=\left(\ell_{1}, \ldots, \ell_{n}\right), \mathcal{G}(\mathcal{L})$ the triangular graph of $\mathcal{L}$, and $\mathcal{L}_{1}, \ldots, \mathcal{L}_{k}$ the vertex systems of the connected components of $\mathcal{L}$. The essential observation in the deduction of Theorem 2.7.1 and Corollary 2.7.2 is that by the assumptions on $\alpha_{1}, \ldots, \alpha_{m}$ we have $X_{m} \in\left[\mathcal{L}_{1}\right] \cap \cdots \cap\left[\mathcal{L}_{m}\right]$, see Section 10.2 . So the solutions of (2.7.1) with $x_{m} \neq 0$ satisfy (2.6.6). Further, letting $F$ denote the decomposable form from (2.7.2), we have $\mathcal{Z}_{A, F}=\{0\}$ since $\alpha_{1}, \ldots, \alpha_{m}$ are linearly independent over $K$.

The finiteness statement of Corollary 2.7.2 was proved by Győry (1982) in full generality. In Győry (1983), he also established the effectivity statement of this corollary, but only for a restricted class of integral domains $A$. In Győry (1983) it is pointed out that Corollary 2.7.2 does not remain valid in general if we lower the bound 3 concerning the degree of $\alpha_{m}$ over $K\left(\alpha_{1}, \ldots, \alpha_{m-1}\right)$. Further, under the assumptions of Corollary 2.7.2, equation (2.7.1) may have infinitely many solutions $\left(x_{1}, \ldots, x_{m}\right)$ with $x_{m}=0$.

The following result is an easy consequence of Corollary 2.7.2.

Corollary 2.7.3. Suppose that in (2.7.1) $\alpha_{i+1}$ is of degree $\geq 3$ over $K\left(\alpha_{1}, \ldots, \alpha_{i}\right)$ for $i=1, \ldots, m-1$. Then (2.7.1 has only finitely many solutions. Moreover, if $A, K, K^{\prime}, \alpha_{1}, \ldots, \alpha_{m}$ and $\delta$ are given effectively, then all solutions of (2.7.1) can be determined effectively.

### 2.8 Discriminant form equations and discriminant equations

Let $\Omega$ be a finite étale $K$-algebra. We represent $\Omega$ in the form $K[X] /(P)$ where $P \in K[X]$ is monic and separable. We view $K$ as a subfield of $\Omega$. The degree $[\Omega: K]:=\operatorname{dim}_{K} \Omega$ is equal to $\operatorname{deg} P$. We say that $\Omega$ is given effectively if $P$ is given effectively. The separability of $P$ can be checked for instance by determining the factorization of $P$ into irreducible factors, see for instance Theorem 6.2 .3 . By the choice of our representation we have $\Omega=K[\theta]$, where $\theta:=X(\bmod P)$. Elements of $\Omega$ can be expressed uniquely as $\sum_{i=0}^{n-1} a_{i} \theta^{i}$ with $a_{0}, \ldots, a_{n-1} \in K$, where $n=[\Omega: K]$. We say that an element of $\Omega$ is given/can be determined effectively if $a_{0}, \ldots, a_{n-1}$ are given/can be determined effectively. Denote by $G$ the splitting field of $P$ over $K$. Then there are precisely $n K$-algebra homomorphisms from $\Omega$ to $G$, denoted by $\alpha \mapsto \alpha^{(i)}$ for $i=1, \ldots, n$, mapping $\theta$ to the $n$ distinct zeros of $P$ in $G$. One can verify that if $\alpha \in \Omega$, then

$$
\begin{equation*}
\alpha \in K \Longleftrightarrow \alpha^{(1)}=\cdots=\alpha^{(n)} . \tag{2.8.1}
\end{equation*}
$$

Assume that $[\Omega: K]=n \geq 2$. Let $\mathcal{M} \subset \Omega$ be a finitely generated $A$-module, i.e., there are $\omega_{1}, \ldots, \omega_{m} \in \mathcal{M}$ such that

$$
\mathcal{M}=\left\{\sum_{i=1}^{m} b_{i} \omega_{i}: b_{1}, \ldots, b_{m} \in A\right\} .
$$

We do not require that $\mathcal{M}$ is free over $A$. We say that $\mathcal{M}$ is given effectively if such $\omega_{1}, \ldots, \omega_{m}$ are given effectively. Further, we say that an element $\alpha$ of $\mathcal{M}$ is given/can be determined effectively, if $b_{1}, \ldots, b_{m} \in A$ are given/can be determined effectively such that $\alpha=\sum_{i=1}^{m} b_{i} \omega_{i}$.

We consider the discriminant equation for elements of $\mathcal{M}$

$$
\begin{equation*}
D_{\Omega / K}(\xi)=\prod_{1 \leq i<j \leq n}\left(\xi^{(i)}-\xi^{(j)}\right)^{2}=\delta \text { in } \xi \in \mathcal{M}, \tag{2.8.2}
\end{equation*}
$$

where $\delta \in K^{*}$.
Assertion (2.8.1) implies that if $\xi$ is a solution of (2.8.2), then so is $\xi+\eta$ for every $\eta \in \mathcal{M} \cap K$. Hence the set of solutions of (2.8.2) is a union of $\mathcal{M} \cap K$-cosets $\xi+\mathcal{M} \cap K:=\{\xi+\eta: \eta \in \mathcal{M} \cap K\}$. Our aim is to determine a full system of representatives for these cosets.

We can give an equivalent formulation of (2.8.2) in terms of discriminant form equations. Choosing a set of $A$-module generators $\left\{\omega_{1}, \ldots, \omega_{m}\right\}$ for $\mathcal{M}$ (this need not be an $A$-basis), we can express a solution $\xi \in \mathcal{M}$ of (2.8.2) as $\sum_{i=1}^{m} x_{i} \omega_{i}$ with $x_{1}, \ldots, x_{m} \in A$. Thus, (2.8.2) translates into the discriminant form equation

$$
\begin{array}{r}
D_{\Omega / K}\left(x_{1} \omega_{1}+\cdots+x_{m} \omega_{m}\right)=\prod_{1 \leq i<j \leq n}\left(\sum_{k=1}^{m} x_{k}\left(\omega_{k}^{(i)}-\omega_{k}^{(j)}\right)\right)^{2}=\delta \\
\quad \text { in } \mathbf{x}=\left(x_{1}, \ldots, x_{m}\right) \in A^{m} \tag{2.8.3}
\end{array}
$$

which is a decomposable form equation. Let

$$
\begin{equation*}
\mathcal{Z}_{A, D}:=\left\{\mathbf{x}=\left(x_{1}, \ldots, x_{m}\right) \in A^{m}: \sum_{i=1}^{m} x_{i} \omega_{i} \in K\right\} . \tag{2.8.4}
\end{equation*}
$$

Then the set of solutions in $A^{m}$ of (2.8.3) is a union of $\mathcal{Z}_{A, D}$-cosets $\mathbf{x}+\mathcal{Z}_{A, D}$. By a representative for a $\mathcal{Z}_{A, D}$-coset, we mean a tuple $\widetilde{\mathbf{x}} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]^{m}$ that is a representative for an element of this coset.

In Section 10.3, we deduce the following result from Theorem 2.6.1. The essential observation is that $D_{\Omega / K}\left(X_{1} \omega_{1}+\cdots+X_{m} \omega_{m}\right)$ is a decomposable form whose linear factors form a triangularly connected system.

Theorem 2.8.1. Assume that $f_{1}, \ldots, f_{M}$ have degree at most $d$ and logarithmic height at most $h$ and that $\delta$ and $\omega_{i}^{(j)}(i=1, \ldots, m, j=1, \ldots, n)$ have tuples of representatives of degree at most d and logarithmic height at most $h$. Then every $\mathcal{Z}_{A, D}$-coset of solutions of (2.8.3) has a representative $\widetilde{\mathbf{x}} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]^{m}$ with

$$
s(\widetilde{\mathbf{x}}) \leq \exp \left(\left(n^{m n^{4}} d\right)^{\exp O(r)} h\right)
$$

Recall that a finitely generated $A$-module $\mathcal{M} \subset \Omega$ is given effectively once a finite set of $A$-module generators $\left\{\omega_{1}, \ldots, \omega_{m}\right\}$ for $\mathcal{M}$ is given effectively. According to the definitions, determining a full system of representatives for the $\mathcal{M} \cap K$-cosets of solutions of (2.8.2) means the same as determining a full system of representatives for the $\mathcal{Z}_{A, D}$-cosets of (2.8.3). This leads to the following consequence.

Corollary 2.8.2. Let $\Omega$ be a finite étale $K$-algebra with $[\Omega: K] \geq 2, \mathcal{M} \subset \Omega$ a finitely generated $A$-module and $\delta \in K^{*}$. Then equation (2.8.2 has only
finitely many $\mathcal{M} \cap K$-cosets of solutions. Moreover, if $A, \Omega, \delta$ and $\mathcal{M}$ are given effectively, then one can determine effectively a set, consisting of precisely one element from each of these cosets.

We mention here that once $\mathcal{M}$ is given effectively and $\alpha \in \Omega$ is given effectively, then it can be decided whether $\alpha \in \mathcal{M}$. Further, given $\alpha, \alpha^{\prime} \in \mathcal{M}$ one can decide whether $\alpha-\alpha^{\prime} \in K$, see Corollary 6.3.8.

We consider the special case that $\mathcal{M}=\mathcal{O}$ is an $A$-order in $\Omega$, i.e., $\mathcal{O}$ is a subring of $\Omega$ such that $A \subseteq \mathcal{O} \subseteq \Omega, K \mathcal{O}=\Omega$ and $\Omega$ is finitely generated as an $A$-module.

By an $A$-coset we mean a set $\xi+A:=\{\xi+a: a \in A\}$.
Corollary 2.8.3. Let $\delta \in A \backslash\{0\}$, and let $\mathcal{O}$ be an $A$-order in $\Omega$ such that the quotient $A$-module

$$
\begin{equation*}
(\mathcal{O} \cap K) / A \quad \text { is finite } . \tag{2.8.5}
\end{equation*}
$$

Then the set of $\xi \in \mathcal{O}$ with

$$
D_{\Omega / K}(\xi)=\delta
$$

is a union of finitely many $A$-cosets. Moreover, if $A, \Omega, \delta$ and $\mathcal{O}$ are given effectively, then one can determine effectively a set, consisting of precisely one element from each of these cosets.

Corollary 2.8 .3 is an easy consequence of Corollary 2.8.2. It will be deduced in Section 10.3. In the deduction of Corollary 2.8.3 we use Corollary 6.3 .9 from Chapter 6. Since for this latter result we do not have a quantitative version at our disposal, we were not able to deduce a quantitative version of Corollary 2.8.3 similar to Theorem 2.8.1.

We mention here that if $\mathcal{O}$ is given effectively then it can be decided effectively whether it is an $A$-order, and whether it satisfies 2.8 .5 , see Corollary 6.3.9. Further, for any two given $\xi_{1}, \xi_{2} \in \mathcal{O}$ it can be decided whether $\xi_{1}-\xi_{2} \in A$, see Corollary 6.3 .8 and Theorem 6.3.2.

Corollary 2.8.2 has further consequences, among others for index form equations; we refer to Győry (1982) for ineffective finiteness results and Győry $(1983,1984 b)$ for effective results over a class of finitely generated domains over $\mathbb{Z}$.

We now consider another type of discriminant equation. Let $A$ be as above a finitely generated integral domain over $\mathbb{Z}$ of characteristic 0 and with quotient field $K$, let $n \geq 2$ be an integer, $\delta$ a non-zero element of $A$, and $G$ a finite
extension of $K$. We consider the discriminant equation for polynomials

$$
\begin{array}{ll}
D(f)=\delta & \text { in monic polynomials } f \in A[X] \\
& \text { of degree } n \text { having all their zeros in } G . \tag{2.8.6}
\end{array}
$$

As in Section 1.6, two monic polynomials $f, f^{\prime} \in A[X]$ are called strongly $A$-equivalent if there is $a \in A$ such that $f^{\prime}(X)=f(X+a)$. We recall that strongly $A$-equivalent polynomials have the same discriminant, and so the solutions of equation (2.8.6 split into strong $A$-equivalence classes.

Assuming that A is effectively given in the above sense, we say that a polynomial with coefficients in $A$ or $K$ is given/can be determined effectively if its coefficients are given/can be determined effectively.

Denote by $A_{K}$ the integral closure of $A$ in $K$. The following theorem is an effective version of Theorem 1.6.1. In Evertse and Győry (2017b) this result was deduced directly from a general effective result on unit equations. In Section 10.3 we give another proof, taking Corollary 2.6 .2 as a starting point.

Theorem 2.8.4. Let $n \geq 2$ be an integer and $A$ an integral domain of characteristic 0 , finitely generated over $\mathbb{Z}$ with quotient field $K$ such that the quotient $A$-module

$$
\begin{equation*}
\left(\frac{1}{n} A \cap A_{K}\right) / A \quad \text { is finite. } \tag{2.8.7}
\end{equation*}
$$

Further, let $G$ be a finite extension of $K$ and $\delta$ a non-zero element of $A$. Then the set of monic polynomials $f \in A[X]$ with (2.8.6 is a union of finitely many strong $A$-equivalence classes.

Moreover, for any effectively given $n, A, G, \delta$ as above, a set, consisting of precisely one element from each of these classes can be determined effectively.

For any effectively given integral domain $A$ of characteristic 0 which is finitely generated over $\mathbb{Z}$ it can be decided effectively whether it satisfies (2.8.7), see Corollary 6.3.7 or Evertse and Győry (2017b).

The proof of Theorem 2.8.4 uses both Corollary 2.6.2 and Theorem 6.3.6 and Corollary 6.3 .9 from Chapter 6. Since for the latter two results we do not have quantitative versions at our disposal, we were not able to deduce a quantitative version of Theorem 2.8.4 with estimates for the sizes of the coefficients of a polynomial solution $f$ of 2.8 .6 from each strong $A$-equivalent class.

For integrally closed domains $A$, Theorem 2.8.4 gives the following result of Evertse and Győry (2017a). We note that for an effectively given domain $A$ of characteristic 0 which is finitely generated over $\mathbb{Z}$ it can be decided whether it is integrally closed; see e.g. Theorem 10.7.17 in Evertse and Győry (2017a) and the references given there.

Corollary 2.8.5. Let $n \geq 2$ be an integer, $A$ an integrally closed integral domain of characteristic 0 which is finitely generated over $\mathbb{Z}$ and $G$ a finite extension of the quotient field of $A$. Then the solutions of (2.8.6) lie in finitely many strong $A$-equivalence classes. If moreover $A, G, \delta$ are given effectively, then a set, consisting of precisely one element from each of these classes can be determined effectively.

The finiteness part of this corollary was proved in a more general, but ineffective form in Győry (1982), without fixing the degree of the polynomials under consideration; see also Theorem 1.6 .2 above. For a restricted class of integral domains $A$ containing transcendental elements, the effective part was proved in Győry (1984b).

### 2.9 Open problems

Let again $A$ be a domain of characteristic 0 that is finitely generated over $\mathbb{Z}$, $K$ its quotient field, and $\bar{K}$ an algebraic closure of $K$.

Let $F \in A[X, Y]$ be a binary form of degree $n$ having at least three pairwise non-proportional linear factors over $\bar{K}$ and let $\delta \in A \backslash\{0\}$. In Section 1.1 it was explained that the Thue-Mahler equation

$$
\begin{equation*}
F(x, y) \in \delta A^{*} \text { in }(x, y) \in A^{2} \tag{2.9.1}
\end{equation*}
$$

has at most finitely many $A^{*}$-cosets of solutions. As was explained there, if $\left\{v_{1}, \ldots, v_{s}\right\}$ is a set of generators for $A^{*}$, and $\mathcal{U}=\left\{v_{1}^{m_{1}} \cdots v_{s}^{m_{s}}: m_{1}, \ldots, m_{s} \in\right.$ $\{0, \ldots, n-1\}\}$, every $A^{*}$-coset of solutions of (2.9.1) contains a pair $(x, y) \in$ $A^{2}$ with

$$
F(x, y)=\delta u_{1} \text { for some } u_{1} \in \mathcal{U}
$$

By applying the known results on Thue equations to the latter, it follows that (2.9.1) has only finitely many $A^{*}$-cosets of solutions.

Unfortunately, as yet no method is known that on input an arbitrary domain $A$ of characteristic 0 that is finitely generated over $\mathbb{Z}$, computes a set
of generators for $A^{*}$. Consequently, it is therefore an open problem how to determine effectively the $A^{*}$-cosets of solutions of (2.9.1), for arbitrary $A$.

It should be mentioned that for a restricted class of domains $A$ it is possible to compute a set of generators for $A^{*}$ and thus, the $A^{*}$-cosets of solutions of (2.9.1), for instance for localizations of polynomial rings $A=$ $\mathcal{O}_{S}\left[X_{1}, \ldots, X_{q}, 1 / g\right]$, where $\mathcal{O}_{S}$ is the ring of $S$-integers in a number field and $g \in \mathcal{O}_{S}\left[X_{1}, \ldots, X_{q}\right] \backslash\{0\}$, see Evertse and Győry (2017a, Lemma 10.6.2).

The same can be said about decomposable form equations in $m \geq 3$ unknowns

$$
F(\mathbf{x}) \in \delta A^{*} \text { in } \mathbf{x} \in A^{m},
$$

where $F$ is a decomposable form of degree $n$. The solutions of the latter equation can be divided into $A^{*}$-cosets $A^{*} \mathbf{x}_{0}=\left\{u \cdot \mathbf{x}_{0}: u \in A^{*}\right\}$. Completely similarly as for Thue-Mahler equations, one can reduce the above equation to finitely many equations of the form

$$
F(\mathbf{x})=\delta u_{1} \text { with } u_{1} \in \mathcal{U}
$$

Again, for arbitrary $A$ this reduction can not be made effective since we cannot compute a set of generators for $A^{*}$. The same applies to discriminant equations for polynomials and integral elements,

$$
\begin{align*}
& D_{\Omega / K}(\xi) \in \delta A^{*} \text { in } \xi \in \mathcal{M},  \tag{2.9.2}\\
& D(f) \in \delta A^{*}, \tag{2.9.3}
\end{align*}
$$

respectively, where the solutions of the latter equations are monic polynomials having their zeros in a given finite extension $G$ of $K$.

Lastly, we would like to discuss some open problems related to monogeneity of orders. Let $A, K, \bar{K}$ be as above, $\Omega$ a finite étale $K$-algebra with $[\Omega: K]=n \geq 2$, and $\mathcal{O}$ an $A$-order in $\Omega$ such that the quotient $A$-module $(\mathcal{O} \cap K) / A$ is finite. Consider the 'equation'

$$
\begin{equation*}
A[\xi]=\mathcal{O} \text { in } \xi \in \mathcal{O} \tag{2.9.4}
\end{equation*}
$$

We call two elements $\xi$ and $\xi^{\prime}$ of $\mathcal{O}$ A-equivalent if $\xi^{\prime}=u \xi+a$ for some $u \in A^{*}$ and $a \in A$. Clearly, the set of solutions of 2.9.4 is a union of $A$ equivalence classes. It is as yet an open problem to effectively determine these for arbitrary finitely generated domains $A$. Below, we will discuss some of the obstacles. For more details we refer to Evertse and Győry (2017a,

Chaps. 5,10).
First we observe that $\xi$ is a solution of (2.9.4) if and only if $\left\{1, \xi, \ldots, \xi^{n-1}\right\}$ is an $A$-basis of $\mathcal{O}$. So for (2.9.4) to be solvable it is necessary that $\mathcal{O}$ be free. Suppose this is the case, and let $\left\{\omega_{1}, \ldots, \omega_{n}\right\}$ be an $A$-basis for $\mathcal{O}$. Define the discriminant of this basis,

$$
\delta:=D_{\Omega / K}\left(\omega_{1}, \ldots, \omega_{n}\right)=\left(\operatorname{det}\left(\omega_{i}^{(j)}\right)_{i, j=1, \ldots, n}\right)^{2}
$$

Recall that the discriminant of an $A$-basis of $\mathcal{O}$ is uniquely determined up to multiplication with a factor from $A^{*}$ (see Evertse and Győry (2017a, subsection 5.4.4). Thus, $\xi \in \mathcal{O}$ satisfies (2.8.2) if and only if

$$
\begin{equation*}
D_{\Omega / K}(\xi)=D_{\Omega / K}\left(1, \xi, \ldots, \xi^{n-1}\right) \in \delta A^{*} \tag{2.9.5}
\end{equation*}
$$

Similarly as mentioned above, from a set of generators for $A^{*}$ we can compute a finite set $\mathcal{U} \subset A^{*}$, such that every element of $A^{*}$ can be expressed as $u_{1} \cdot u_{2}^{n(n-1)}$ with $u_{1} \in \mathcal{U}, u_{2} \in A^{*}$. Then for every solution $\xi \in \mathcal{O}$ of (2.9.4), hence (2.9.5) there are $u_{1} \in \mathcal{U}, u_{2} \in A^{*}$ such that $\xi^{\prime}=u_{2}^{-1} \xi$ satisfies

$$
\begin{equation*}
D_{\Omega / K}\left(\xi^{\prime}\right)=\delta u_{1} \tag{2.9.6}
\end{equation*}
$$

By Corollary 2.8.3, the solutions $\xi^{\prime} \in \mathcal{O}$ of 2.9.6 lie in finitely many $A$ cosets. Hence the solutions of (2.9.4) lie in finitely many $A$-equivalence classes.

As yet we do not know how to solve the following problems for arbitrary domains $A$ finitely generated over $\mathbb{Z}$ with quotient field $K$ of characteristic 0 . The first problem is to decide whether a given $A$-order $\mathcal{O}$ in a given finite étale $K$-algebra $\Omega$ is a free $A$-module and if so, to determine an $A$-basis for it. The second problem, as mentioned above, is to compute a set of generators for $A^{*}$, needed to get the set $\mathcal{U}$.

These two problems can be solved, and thus, the $A$-equivalence classes of solutions of (2.9.4) can be computed, for the class of domains $A$ mentioned above, i.e., those of the shape $\mathcal{O}_{S}\left[X_{1}, \ldots, X_{q}, g^{-1}\right]$, where $\mathcal{O}_{S}$ is the ring of $S$-integers in some number field and $g$ is a non-zero element of $\mathcal{O}_{S}\left[X_{1}, \ldots, X_{q}\right]$, see Evertse and Győry (2017a, Chap. 10).

## Chapter 3

## A brief explanation of our effective methods over finitely generated domains

There are two effective methods for solving Diophantine equations over finitely generated integral domains over $\mathbb{Z}$ of characteristic 0 , which may contain both algebraic and transcendental elements over $\mathbb{Q}$. The first one, reducing equations to the number field and function field cases by means of effective specializations, was introduced by Győry $(1983,1984 b)$ for a restricted class of finitely generated integral domains over $\mathbb{Z}$ of characteristic 0 . This can be regarded as an effective version of Lang's method (1960) mentioned in Section 1.4. Gyôry's method was later refined and extended by Evertse and Győry (2013) to arbitrary finitely generated integral domains of characteristic 0 ; see also Bérczes, Evertse and Győry (2014), Evertse and Győry (2015) and Chapters 7 and 9 of the present book.

Over number fields, the second effective method, reducing equations in two unknowns to unit equations, was extended by Győry to equations in an arbitrary number of unknowns, including discriminant equations and important classes of decomposable form equations; see e.g. Győry (1973,1976,1980b) and Győry and Papp (1978). This was generalized in Győry (1982) in an ineffective way, and in Evertse and Győry (2017a,2017b) and Chapter 10 of this book in an effective form to the case of arbitrary finitely generated integral domains over $\mathbb{Z}$ of characteristic 0 .

We note that both effective methods have quantitative versions as well which provide effective bounds for the solutions of the equations under con-
sideration.
In this chapter we briefly explain the two effective methods and illustrate their applications to Diophantine equations. Detailed presentations, quantitative versions and applications are given in Chapters 2 and 7 to 10 .

### 3.1 Sketch of the effective specialization method

First we briefly outline the first method of Györy $(1983,1984 b)$ which enabled him to obtain effective finiteness results for some important classes of Diophantine equations over a class of finitely generated domains of characteristic 0 . The core of the method is to reduce the equations under consideration to equations of the same type over function fields and over number fields by means of an effective specialization procedure, and then to apply the existing effective results over number fields and over function fields. Győry applied his method to discriminant equations and decomposable form equations, including Thue equations, index form equations and some norm form equations. Later, this method was applied by Brindza (1989) and Végső (1994) to hyperand superelliptic equations, and by Brindza (1993) to the Catalan equation over the class of domains considered by Győry.

Evertse and Győry (2013) refined Győry's method and extended it to arbitrary finitely generated domains. We now present their general method and compare it with Győry's. For convenience, we use here the notation of Chapters 7 to 10

Let

$$
A=\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]
$$

be a finitely generated integral domain of characteristic 0 with quotient field $K$ of characteristic 0 . Denote by $q$ the transcendence degree of $K$. We consider only the case that $q>0$ since otherwise, $K$ is algebraic over $\mathbb{Q}$, and no specialization argument is needed. We assume without loss of generality that $\left\{z_{1}, \ldots, z_{q}\right\}$ is a transcendence basis of $K$ over $\mathbb{Q}$. Since $z_{1}, \ldots, z_{q}$ may be viewed as polynomial variables, we write henceforth $X_{i}$ for $z_{i}$, for $i=1, \ldots, q$. Let

$$
A_{0}:=\mathbb{Z}\left[X_{1}, \ldots, X_{q}\right], \quad K_{0}:=\mathbb{Q}\left(X_{1}, \ldots, X_{q}\right) .
$$

Then $K=K_{0}\left(z_{q+1}, \ldots, z_{r}\right)$ is a finite extension of $K_{0}$. Given $\alpha \in A_{0}$, we denote by $\operatorname{deg} \alpha, h(\alpha)$ the total degree and logarithmic height of $\alpha$. Recall that $A_{0}$ is a unique factorization domain with unit group $A_{0}^{*}=\{ \pm 1\}$, hence
any finite set $a_{1}, \ldots, a_{r}$ of non-zero elements of $A_{0}$ has an up to sign unique greatest common divisor $\operatorname{gcd}\left(a_{1}, \ldots, a_{r}\right)$ such that every element of $A_{0}$ that divides $a_{1}, \ldots, a_{r}$ in fact divides their gcd.

We first describe the approach of Győry $(1983,1984 \mathrm{~b})$. Take $w \in A$ such that $K=K_{0}(w)$ and $w$ has minimal polynomial $\mathcal{F}(X)=X^{D}+\mathcal{F}_{1} X^{D-1}+$ $\cdots+\mathcal{F}_{D}$ over $K_{0}$ with coefficients in $A_{0}$. For every $\alpha \in K$ there are up to sign unique $P_{\alpha, 0}, \ldots, P_{\alpha, D-1}, Q_{\alpha} \in A_{0}$ such that $\operatorname{gcd}\left(P_{\alpha, 0}, \ldots, P_{\alpha, D-1}, Q_{\alpha}\right)=1$ and

$$
\begin{equation*}
\alpha=Q_{\alpha}^{-1} \sum_{j=0}^{D-1} P_{\alpha, j} w^{j} . \tag{3.1.1}
\end{equation*}
$$

Now define the measures

$$
\overline{\operatorname{deg}} \alpha:=\max \left(\operatorname{deg} P_{\alpha, 0}, \ldots, \operatorname{deg} P_{\alpha, D-1}, \operatorname{deg} Q_{\alpha}\right)
$$

and

$$
\bar{h}(\alpha):=\max \left(h\left(P_{\alpha, 0}\right), \ldots, h\left(P_{\alpha, D-1}\right), h\left(Q_{\alpha}\right)\right) .
$$

Of course, there are only finitely many elements in $K$ with bounded $\overline{\mathrm{deg}}$-value and $\bar{h}$-value and, if the bounds are given, the elements under consideration can be effectively determined.

Next, let $g$ be the product $\prod_{i=q+1}^{r} Q_{z_{i}}$ of the denominators of $z_{q+1}, \ldots, z_{r}$ in their representations of the form (3.1.1). Then $g \in A_{0} \backslash\{0\}$. Suppose that

$$
\begin{aligned}
& \max \left(\operatorname{deg} \mathcal{F}_{1}, \ldots, \operatorname{deg} \mathcal{F}_{D}, \operatorname{deg} g\right) \leq d_{0}, \\
& \max \left(h\left(\mathcal{F}_{1}\right), \ldots, h\left(\mathcal{F}_{D}\right), h(g)\right) \leq h_{0}
\end{aligned}
$$

Instead of the domain $A$, Győry $(1983,1984 b)$ considers the solutions of the equations under consideration in the overring $B$ of $A$ defined by

$$
A \subseteq B:=A_{0}\left[w, g^{-1}\right]
$$

and gives explicit upper bounds for the $\overline{\mathrm{deg}}$-values and $\bar{h}$-values of the solutions, depending on $d_{0}, h_{0}$ and on appropriate parameters of the equations. This implies in an effective form the finiteness of the number of solutions of the equations with coordinates in $B$. What remains is to select from these the solutions with coordinates in $A$. Győry was able to do this only if $A$ is given in a special manageable form. This is the case e.g. if $A=A_{0}, A=B$ with the above $B$ or if $A$ is an $A_{0}$-module with a given basis, say $\left\{z_{q+1}, \ldots, z_{r}\right\}$.

For arbitrary $A$, he could prove only the finiteness of the number of solutions.
We note that Győry $(1983,1984 b)$ established effective results in the socalled relative case as well, when $A$ is a finitely generated domain over a field of characteristic 0 instead of $\mathbb{Z}$, and he gave explicit upper bounds for the $\overline{\operatorname{deg}}$-values of the solutions.

We now outline the general effective method of Evertse and Győry (2013) which can be applied to the case of arbitrary finitely generated domains $A$ over $\mathbb{Z}$. Further, we point out the refinements compared with Győry's method.

Evertse and Győry (2013) use the representation for $A=\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]$ that we introduced in Section 2.1. That is, let $\mathcal{I}$ be the ideal

$$
\mathcal{I}=\left\{f \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]: f\left(z_{1}, \ldots, z_{r}\right)=0\right\}
$$

Then $\mathcal{I}$ is finitely generated, and so

$$
A \cong \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right] / \mathcal{I}, \quad \mathcal{I}=\left(f_{1}, \ldots, f_{M}\right)
$$

for certain polynomials $f_{1}, \ldots, f_{M}$. We call $\left(f_{1}, \ldots, f_{M}\right)$ an ideal representation for $A$ and say that $A$ is effectively given if such a representation is given. Further, $\alpha \in A$ is said to be effectively given/computable if a representative of $\alpha$ in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$, say $\widetilde{\alpha}$, is given/computable such that $\alpha=$ $\widetilde{\alpha}\left(z_{1}, \ldots, z_{r}\right)$. For $\alpha \in K$, we call $(a, b)$ a pair of representatives for $\alpha$ if $a, b \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right], b \notin \mathcal{I}$ and $\alpha=a\left(z_{1}, \ldots, z_{r}\right) / b\left(z_{1}, \ldots, z_{r}\right)$.

We collect here in simplified form those lemmas/propositions from Chapter 7 which together constitute our general specialization method, and give a brief explanation how these can be used in Chapter 9 to prove the effective results formulated in Chapter 2 . We assume again that $z_{i}=X_{i}(i=1, \ldots, q)$ form a transcendence basis of $K$ and keep the notation $A_{0}=\mathbb{Z}\left[X_{1}, \ldots, X_{q}\right]$, $K_{0}=\mathbb{Q}\left(X_{1}, \ldots, X_{q}\right)$. We recall that the notation $O(r)$, introduced in Chapter 2, denotes any expression of the type 'effectively computable absolute constant times $r^{\prime}$, where at each occurrence of $O(r)$ the constant may be different.

The following lemma can be regarded as a modified and more explicit version of Győry's result on the overring $B$ of $A$. To allow for more flexibility in applications, we have extended Győry's result a little bit further, and prescribe that certain elements of $K^{*}$ are units of $B$. Thus, let $\mathcal{A}$ be a possibly empty finite subset of $K^{*}$ and for $\alpha \in \mathcal{A}$, let $\left(a_{\alpha}, b_{\alpha}\right)$ be a pair of representatives for $\alpha$.

Let $d_{1} \geq d \geq 1, h_{1} \geq h \geq 1$, and assume that

$$
\left.\begin{array}{l}
\operatorname{deg} f_{i} \leq d, h\left(f_{i}\right) \leq h \text { for } i=1, \ldots, M  \tag{3.1.2}\\
\operatorname{deg} a_{\alpha}, \operatorname{deg} b_{\alpha} \leq d_{1}, h\left(a_{\alpha}\right), h\left(b_{\alpha}\right) \leq h_{1} \text { for } \alpha \in \mathcal{A}
\end{array}\right\}
$$

Lemma 3.1.1. There are $w, g$ with $w \in A, g \in A_{0} \backslash\{0\}$ such that

$$
A \subseteq B:=A_{0}\left[w, g^{-1}\right], \quad \mathcal{A} \subset B^{*},
$$

such that $w$ has minimal polynomial $\mathcal{F}(X)=X^{D}+\mathcal{F}_{1} X^{D-1}+\cdots+\mathcal{F}_{D}$ over $K_{0}$ of degree $D \leq d^{r-q}$ with

$$
\mathcal{F}_{i} \in A_{0}, \quad \operatorname{deg} \mathcal{F}_{i} \leq(2 d)^{\exp O(r)}, \quad h\left(\mathcal{F}_{i}\right) \leq(2 d)^{\exp O(r)} h \quad \text { for } i=1, \ldots, D
$$

and such that
$\operatorname{deg} g \leq(k+1)\left(2 d_{1}\right)^{\exp O(r)}, h(g) \leq(k+1)\left(2 d_{1}\right)^{\exp O(r)} h_{1} \quad$ where $k:=|\mathcal{A}|$.
Proof. This is a combination of Corollary 3.4 and Lemma 3.6 of Evertse and Győry (2013); see also Propositions 7.2.5 and 7.2.7 from Chapter 7 . One has to take $g:=\prod_{i=q+1}^{r} Q_{z_{i}} \cdot \prod_{\alpha \in \mathcal{A}} Q_{\alpha} Q_{\alpha^{-1}}$.

The next lemma is new in the method of Evertse and Győry (2013). It plays an important role in the extension of Győry's method to the case of arbitrary finitely generated domains.

Lemma 3.1.2. Let $\alpha \in A \backslash\{0\}$.
(i) Let $\widetilde{\alpha} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ be a representative for $\alpha$. Put

$$
d_{2}:=\max (d, \operatorname{deg} \widetilde{\alpha}), h_{2}:=\max (h, h(\widetilde{\alpha})) .
$$

Then

$$
\overline{\operatorname{deg}} \alpha \leq\left(2 d_{2}\right)^{\exp O(r)}, \bar{h}(\alpha) \leq\left(2 d_{2}\right)^{\exp O(r)} h_{2}
$$

(ii) Put

$$
d_{2}^{\prime}:=\max (d, \overline{\operatorname{deg}} \alpha), h_{2}^{\prime}:=\max (h, \bar{h}(\alpha))
$$

Then $\alpha$ has a representative $\widetilde{\alpha} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ such that

$$
\operatorname{deg} \widetilde{\alpha} \leq\left(2 d_{2}^{\prime}\right)^{\exp O\left(r \log ^{*} r\right)} h_{2}^{\prime}, h(\widetilde{\alpha}) \leq\left(2 d_{2}^{\prime}\right)^{\exp O\left(r \log ^{*} r\right)} h_{2}^{\prime r+1}
$$

Proof. This is a combination of Lemmas 3.5 and 3.7 of Evertse and Győry
(2013); see also Lemmas 7.2.6 and 7.3.1 in Chapter 7. The proofs of these lemmas depend heavily on work of Aschenbrenner (2004).

Both Győry $(1983,1984 b)$ and Evertse and Győry (2013) embed the Diophantine equations under consideration into appropriate function fields in a fixed algebraic closure $\bar{K}_{0}$ of $K_{0}$. The next lemma relates $\overline{\operatorname{deg}} \alpha$ to the function field height of $\alpha$ in such a function field. Let $\alpha \mapsto \alpha^{(j)}(j=1, \ldots, D)$ denote the $K_{0}$-isomorphic embeddings of $K$ in $\bar{K}_{0}, j=1, \ldots, D$. For $i=$ $1, \ldots, q$, let $\mathbb{k}_{i}$ be the algebraic closure of $\mathbb{Q}\left(X_{1}, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{q}\right)$ in $\bar{K}_{0}$, and let $L_{i}=\mathbb{k}_{i}\left(X_{i}, w^{(1)}, \ldots, w^{(D)}\right)$. Thus $K$ may be viewed as a subfield of $L_{1}, \ldots, L_{q}$. We recall that the height of $\alpha \in K$ relative to $L_{i} / \mathbb{k}_{i}$ is defined as

$$
H_{L_{i}}(\alpha):=\sum_{v \in \mathcal{M}_{L_{i}}} \max (0,-v(\alpha)),
$$

where $\mathcal{M}_{L_{i}}$ denotes the set of normalized discrete valuations on $L_{i}$ that are trivial on $\mathbb{k}_{i}$. Put $\Delta_{i}:=\left[L_{i}: \mathbb{k}_{i}\left(X_{i}\right)\right]$.

A slightly different version of the following lemma was implicitly proved in Györy (1983,1984b) with dependence on $d_{0}, h_{0}$ instead of $d, h$. We recall that $d, h$ are given by (3.1.2).
Lemma 3.1.3. Let $\alpha \in K^{*}$. Then

$$
\overline{\operatorname{deg}} \alpha \leq(2 d)^{\exp O(r)}+r \cdot d^{r} \max _{i, j} \Delta_{i}^{-1} H_{L_{i}}\left(\alpha^{(j)}\right)
$$

and

$$
\max _{i, j} \Delta_{i}^{-1} H_{L_{i}}\left(\alpha^{(j)}\right) \leq 2 d^{r} \overline{\operatorname{deg}} \alpha+(2 d)^{\exp O(r)}
$$

where the maxima are taken over $i=1, \ldots, q, j=1, \ldots, D$.
Proof. The first assertion is a consequence of Lemma 4.4 of Evertse and Győry (2013) and Lemma 3.1.1 above. The second assertion is Lemma 4.4 of Bérczes, Evertse and Győry (2014); see also Lemmas 7.3.3 and 7.3.4 in Chapter 7.

The main idea of the specialization method is to construct ring homomorphisms from an overring of $A$ to $\overline{\mathbb{Q}}$, with which one can reduce the equations under consideration over $A$ to the number field case. We use a refined version, due to Evertse and Győry (2013), of the ring homomorphisms from Győry (1983,1984b). Take the overring $B$ from Lemma 3.1.1. Define

$$
\mathcal{T}:=\Delta_{\mathcal{F}} \mathcal{F}_{D} \cdot g
$$

where $\Delta_{\mathcal{F}}$ denotes the discriminant of $\mathcal{F}$. Clearly $\mathcal{T} \in A_{0} \backslash\{0\}$ and, by Lemma 3.1.1, the additivity of the total degree and the 'almost additivity' of the logarithmic height for products of polynomials we have

$$
\operatorname{deg} \mathcal{T} \leq(k+1)\left(2 d_{1}\right)^{\exp O(r)}, h(\mathcal{T}) \leq(k+1)\left(2 d_{1}\right)^{\exp O(r)} h_{1}
$$

where $d_{1}, h_{1}$ are given by (3.1.2).
Any $\mathbf{u}=\left(u_{1}, \ldots, u_{q}\right) \in \mathbb{Z}^{q}$ gives rise to a ring homomorphism $\varphi_{\mathbf{u}}$ : $A_{0}=\mathbb{Z}\left[X_{1}, \ldots, X_{q}\right] \rightarrow \mathbb{Z}$ by substituting $u_{i}$ for $X_{i}$ for $i=1, \ldots, q$. We write $\alpha(\mathbf{u}):=\varphi_{\mathbf{u}}(\alpha)$ for $\alpha \in A_{0}$. The map $\varphi_{\mathbf{u}}$ can be extended to $B$ in the following way. Choose $\mathbf{u} \in \mathbb{Z}^{q}$ such that

$$
\mathcal{T}(\mathbf{u}) \neq 0
$$

Let $\mathcal{F}_{\mathbf{u}}:=X^{D}+\mathcal{F}_{1}(\mathbf{u}) X^{D-1}+\cdots+\mathcal{F}_{D}(\mathbf{u})$. By our choice of $\mathbf{u}$, the polynomial $\mathcal{F}_{\mathbf{u}}$ has non-zero discriminant, hence it has $D$ distinct zeros, say $w_{1}(\mathbf{u}), \ldots, w_{D}(\mathbf{u}) \in \overline{\mathbb{Q}}$, which are all non-zero since $\mathcal{F}_{D}(\mathbf{u}) \neq 0$. Further, $g(\mathbf{u}) \neq 0$. Hence each substitution

$$
X_{1} \mapsto u_{1}, \ldots, X_{q} \mapsto u_{q}, w \mapsto w_{j}(\mathbf{u}), \quad(j=1, \ldots, D)
$$

defines a ring homomorphism $\varphi_{\mathbf{u}, j}: B \rightarrow \overline{\mathbb{Q}}$. We write $\alpha_{j}(\mathbf{u}):=\varphi_{\mathbf{u}, j}(\alpha)$ for $\alpha \in B, j=1, \ldots, D$. It follows from $\alpha_{i} \in B^{*}$ that

$$
\begin{equation*}
\alpha_{i, j}(\mathbf{u}) \neq 0 \text { for } i=1, \ldots, k, j=1, \ldots, D \tag{3.1.3}
\end{equation*}
$$

The image $\varphi_{\mathbf{u}, j}(B)$ of $B$ is contained in the algebraic number field $K_{\mathbf{u}, j}:=$ $\mathbb{Q}\left(w_{j}(\mathbf{u})\right)$ with $\left[K_{\mathbf{u}, j}: \mathbb{Q}\right] \leq D \leq d^{r-q}$.

As usual, we denote by $h(\xi)$ the absolute logarithmic height of $\xi \in \overline{\mathbb{Q}}$. For $\mathbf{u}=\left(u_{1}, \ldots, u_{q}\right) \in \mathbb{Z}^{q}$, we write $|\mathbf{u}|=\max \left(\left|u_{1}\right|, \ldots,\left|u_{q}\right|\right)$. An earlier version of the lemma below was proved in Győry $(1983,1984 b)$ with different bounds, which depend on $d_{0}, h_{0}$ instead of $d, h$.

Lemma 3.1.4. Let $d, h, d_{1}, h_{1}$ be given by (3.1.2). Then for $\alpha \in B \backslash\{0\}$ we have the following:
(i) Let $\mathbf{u} \in \mathbb{Z}^{q}$ with $\mathcal{T}(\mathbf{u}) \neq 0$ and let $j \in\{1, \ldots, D\}$. Then

$$
h\left(\alpha_{j}(\mathbf{u})\right) \leq \bar{h}(\alpha)+(2 d)^{\exp O(r)}(h+(\overline{\operatorname{deg}} \alpha+1) \log \max (1,|\mathbf{u}|)) .
$$

(ii) There exists $\mathbf{u} \in \mathbb{Z}^{q}, j \in\{1, \ldots, D\}$ such that

$$
\begin{aligned}
& |\mathbf{u}| \leq \max \left(\overline{\operatorname{deg}} \alpha,(k+1)\left(2 d_{1}\right)^{\exp O(r)}\right), \mathcal{T}(\mathbf{u}) \neq 0 \\
& \bar{h}(\alpha) \leq\left(2 d_{1}\right)^{\exp O(r)}\left((k+1+\overline{\operatorname{deg}} \alpha)^{q+5}\left(h_{1}+h\left(\alpha_{j}(\mathbf{u})\right)\right)\right.
\end{aligned}
$$

Proof. This is a modification of Lemmas 5.6 and 5.7 from Evertse and Győry (2013); see also Lemmas 7.4.6 and 7.4.7 in Chapter 7.

### 3.2 Illustration of the application of the effective specialization method to Diophantine equations

We briefly illustrate how to apply the specialization method to Diophantine equations over finitely generated domains. As an example, consider the Thue equation

$$
\begin{equation*}
F(x, y)=\delta \text { in } x, y \in A, \tag{3.2.1}
\end{equation*}
$$

where $F$ is a binary form of degree $\geq 3$ in $A[X, Y]$ with non-zero discriminant and where $\delta \in A \backslash\{0\}$.
Step 1. Let $x, y \in A$ be a solution of (3.2.1). Having upper bounds for the degrees and heights of representatives of $\delta$ and the coefficients of $F$, Lemma 3.1.2 gives effective upper bounds for the $\overline{\mathrm{deg}}$-values and $\bar{h}$-values of $\delta$ and the coefficients of $F$. Then, by means of Lemma 3.1.3 one gets effective upper bounds for the $H_{L_{i}}$-values of the conjugates of $\delta$ and the coefficients of $F$ over $K_{0}$. Applying effective results of Schmidt (1978), Mason (1984) or Theorem 5.4.1 from Chapter 5 on Thue equations over function fields, one can derive effective upper bounds for $H_{L_{i}}\left(x^{(j)}\right), H_{L_{i}}\left(y^{(j)}\right)$ for all $i, j$ and subsequently, effective upper bounds for $\overline{\operatorname{deg}} x, \overline{\operatorname{deg}} y$ from Lemma 3.1.3.

Step 2. Next, let the set $\mathcal{A}$ from Lemma 3.1.1 consist of $\delta$ and the discriminant of $F$. Choose $\mathbf{u} \in \mathbb{Z}^{q}$ such that $|\mathbf{u}| \leq \max \left(\bar{d},\left(2 d_{1}\right)^{\exp O(r)}\right), \mathcal{T}(\mathbf{u}) \neq 0$, and subject to these conditions, $H:=\max \left(h\left(x_{j}(\mathbf{u})\right), h\left(y_{j}(\mathbf{u})\right)\right)$ is maximal; here $\bar{d}$ denotes the maximum of the deg-values of $x, y, \delta$ and the coefficients of $F$. Let $F_{\mathbf{u}, j}$ be the binary form obtained by applying $\varphi_{\mathbf{u}, j}$ to the coefficients of $F$. It follows from (3.1.3) and from our choice of $\mathcal{A}$ that $\delta_{j}(\mathbf{u}) \neq 0$ and the discriminant of $F_{\mathbf{u}, j}$ is also different from zero.

Step 3. Clearly

$$
F_{\mathbf{u}, j}\left(x_{j}(\mathbf{u}), y_{j}(\mathbf{u})\right)=\delta_{j}(\mathbf{u}) .
$$

We can now apply the explicit result of Győry and Yu (2006) on Thue equations, see also Theorem 4.4.1 in Chapter 4 to obtain an effective upper bound for $H$. Then Lemma 3.1.4, (ii) gives an effective upper bound for $\bar{h}(x), \bar{h}(y)$. Finally, Lemma 3.1.2 yields an effective upper bound $C$ for the degrees and heights, i.e. for the sizes of certain representatives $\tilde{x}, \tilde{y}$ of $x, y$.

Step 4. This last step makes it possible to effectively determine all the solutions of equation (3.2.1). Adapting the proof of Proposition 2.1.1 from Chapter 2 to equation (3.2.1), we can enumerate all pairs $\tilde{x}, \tilde{y}$ from $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ of size at most $C$. Using an ideal membership algorithm for $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$, see Section 6.1, we can check for each of these pairs $\tilde{x}, \tilde{y}$ whether $\tilde{F}(\tilde{x}, \tilde{y})-$ $\tilde{\delta} \in \mathcal{I}$, where $F$ denotes a binary form with coefficients in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ that represents the corresponding coefficients of $F$. Then we can make a list of all pairs passing this test. This list contains at least one representative for each solution of (3.2.1). Subsequently, we can check, for any two pairs $\tilde{x}_{1}, \tilde{y}_{1}$ and $\tilde{x}_{2}, \tilde{y}_{2}$ from this list whether they represent the same solution of 3.2.1) by checking if $\tilde{x}_{1}-\tilde{x}_{2}, \tilde{y}_{1}-\tilde{y}_{2} \in \mathcal{I}$. If this is the case, we remove one of these pairs from our list. This finally results in a list with precisely one representative for each solution.

Remark. The above procedure applies also to unit equations, hyper- and superelliptic equations, see Chapter 9 , as well as to discriminant equations and decomposable form equations, including discriminant form equations, index form equations and some norm form equations, cf. Győry ( $1983,1984 b$ ), because there are effective function field and number field results for these equations. As for unit equations $a x+b y=c$ in $x, y \in A^{*}$ one may apply the above method to systems of equations $a x+b y=c, x \cdot x^{\prime}=1, y \cdot y^{\prime}=1$ in $x, y, x^{\prime}, y^{\prime} \in A$. Then, in Step 4 above, the general version of Proposition 2.1.1 concerning systems of equations must be used.

### 3.3 Sketch of the method reducing equations to unit equations

Lang (1960) was the first to emphasize the importance of unit equations of the form

$$
\begin{equation*}
a x+b y=1 \text { in } x, y \in A^{*} \tag{3.3.1}
\end{equation*}
$$

where $A$ is a finitely generated integral domain over $\mathbb{Z}$, and $a, b$ are nonzero elements of the quotient field $K$ of $A$. Generalizing the results of Siegel (1921) and others obtained over number fields, he proved that equation 3.3.1) has only finitely many solutions. These results imply the finiteness of the number of solutions of some other classical equations in two unknowns; see e.g. Lang (1962).

In the number field case when $K$ is a number field and $A$ the ring of integers or a ring of $S$-integers of $K$, Győry $(1974,1979)$ gave explicit bounds for the solutions of (3.3.1). He applied his results to get the first effective bounds for the solutions of polynomial Diophantine equations in an arbitrary number of unknowns, including discriminant equations and a wide class of decomposable form equations; see e.g. Győry (1974,1980a,b), Győry and Papp (1978). For ineffective generalizations for the case of arbitrary finitely generated domains over $\mathbb{Z}$, see Győry (1982).

Lang's ineffective finiteness theorem on equation (3.3.1) was made effective in quantitative form in Evertse and Győry (2013); see also Theorem 2.2.1 and Corollary 2.2.2. It is applied in an effective way to discriminant equations in Evertse and Győry (2017a, 2017b), and to a wider class of decomposable form equations in quantitative form, in Chapter 10 of the present book.

In this section we briefly outline the method of reducing the above-mentioned equations to unit equations. In fact the equations are reduced to socalled connected systems of unit equations. We illustrate in some special cases and in simplified form how to apply the effective theorem of Evertse and Győry (2013) on equation (3.3.1) to decomposable form equations and discriminant equations via systems of unit equations. The general theorems concerning decomposable form equations and discriminant equations and their proofs can be found in Chapter 2 and Chapter 10, respectively. As will be pointed out in Subsection 3.3.3, the proofs of the general, quantitative versions concerning decomposable form equations are more complicated and need several further tools from Chapter 8 .

### 3.3.1 Effective finiteness result for systems of unit equations

Let $A=\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]$ be a finitely generated integral domain with $A \supseteq \mathbb{Z}$, $r>0$ and with quotient field $K$. Let $n \geq 3$ be an integer, and let $I_{1}, \ldots, I_{k}$ be subsets of $\{1,2, \ldots, n\}$ with

$$
2 \leq\left|I_{j}\right| \leq 3 \text { for } j=1, \ldots, k, \quad I_{1} \cup \cdots \cup I_{k}=\{1, \ldots, n\}
$$

where $|\mathcal{S}|$ denotes the cardinality of a set $\mathcal{S}$. Many Diophantine problems can be reduced to systems of unit equations of the form

$$
\begin{equation*}
\sum_{i \in I_{1}} \lambda_{1, i} \delta_{i}=0, \ldots, \sum_{i \in I_{k}} \lambda_{k, i} \delta_{i}=0 \text { in }\left(\delta_{1}, \ldots, \delta_{n}\right) \in\left(A^{*}\right)^{n} \tag{3.3.2}
\end{equation*}
$$

where the coefficients $\lambda_{j, i}$ are non-zero elements of $K$. For $k=1$, this is a homogeneous unit equation in at most three unknowns.

Denote by $\mathcal{G}$ the graph whose vertex set is $\{1, \ldots, n\}$ and whose edges are the pairs $\left\{i, i^{\prime}\right\}$ belonging to the same set $I_{j}$, for some $j$ with $1 \leq j \leq k$. The system of unit equations (3.3.2) is said to be connected if the graph $\mathcal{G}$ is connected.

Over number fields resp. over finitely generated domains, various versions of the theorem below were explicitly or implicitly used, mostly in quantitative form, in papers of Győry, including Győry (1976,1980b,1982,1983,1984b,1990), and Győry and Papp (1978). Theorem 3.3.1 is in fact the core of Győry's approach reducing certain important classes of equations to systems of unit equations in two unknowns and then, over number fields, applying effective results concerning unit equations.

From Corollary 2.2.2, due to Evertse and Győry (2013), we deduce the following.

Theorem 3.3.1. Suppose that the system of equations (3.3.2) is connected. Then up to a proportional factor from $A^{*}$, 3.3.2) has only finitely many solutions. Further, if $A$ and the coefficients $\lambda_{j, i}$ in (3.3.2) are given effectively, then all solutions can be determined effectively.

The finiteness assertion is a special case of Theorem 4 of Győry (1990) which holds in the more general form when in (3.3.2) $2 \leq\left|I_{j}\right| \leq n$ holds for $j=1, \ldots, k$, and only those solutions $\left(\delta_{1}, \ldots, \delta_{n}\right)$ are considered for which the equations in (3.3.2) have no proper vanishing subsums. Obviously, this assumption is necessary for the finiteness. Over finitely generated domains, the second assertion of Theorem 3.3.1 is new.

We note that a quantitative variant of Theorem 3.3.1 can be obtained by using Theorem 2.2.1 instead of Corollary 2.2.2.

Proof of Theorem 3.3 .1 (sketch). Let $\left(\delta_{1}, \ldots, \delta_{n}\right)$ be a solution of equation (3.3.2). We show that for each $i, i^{\prime} \in\{1,2, \ldots, n\}, \delta_{i} / \delta_{i^{\prime}}$ can take only finitely many values from $A^{*}$ and, if $A$ and the $\lambda_{i, j}$ are effectively given, all these values can be effectively determined. This immediately implies Theorem 3.3.1.

By assumption, system (3.3.2) is connected and $I_{1} \cup \ldots \cup I_{k}=\{1, \ldots, n\}$. Hence it is easy to see that there are $j_{1}, \ldots, j_{\ell}$ in $\{1, \ldots, k\}$ with the following three properties:

- $I_{j_{1}} \cup \ldots \cup I_{j_{\ell}}=\{1, \ldots, n\} ;$
- for $t=1, \ldots, \ell$ the system of equations

$$
\begin{equation*}
\sum_{i \in I_{j_{1}}} \lambda_{j_{1}} \delta_{i}=0, \ldots, \sum_{i \in I_{j_{t}}} \lambda_{j_{t} i} \delta_{i}=0 \tag{3.3.3}
\end{equation*}
$$

to be solved in $\delta_{i} \in A^{*}$ for $i \in I_{j_{1}} \cup \ldots \cup I_{j_{t}}$, is connected;

- for $t=1, \ldots, \ell-1, I_{j_{t+1}}$ has at least one element not contained in $I_{j_{1}} \cup \ldots \cup I_{j_{t}}$.

Then for $t=1, \ldots, \ell-1$, system (3.3.3) and the $j_{t+1}$-th equation have a common unknown.

For $t=1$, our claim is a consequence of Corollary 2.2.2. Then we can proceed by induction on $t$, and our theorem follows.

In the next two subsections, we illustrate how to apply Theorem 3.3.1 to decomposable form equations and discriminant equations. For convenience, we prove our effective finiteness results in simplified, qualitative form. The precise general, quantitative statements and their proofs can be found in Chapters 2 and 10 , respectively.

### 3.3.2 Reduction of decomposable form equations to unit equations

Consider now the decomposable form equation

$$
\begin{equation*}
F\left(x_{1}, \ldots, x_{m}\right)=\delta \text { in } x_{1}, \ldots, x_{m} \in A, \tag{3.3.4}
\end{equation*}
$$

where $\delta \in A \backslash\{0\}$, and $F$ is a decomposable form of degree $n \geq 3$ with coefficients in $A$ which factorizes into linear factors, say

$$
\ell_{i}=\alpha_{i, 1} X_{1}+\cdots+\alpha_{i, m} X_{m} \quad(i=1, \ldots, n)
$$

Put $\mathcal{L}=\left\{\ell_{1}, \ldots, \ell_{n}\right\}$. Suppose that rank $\mathcal{L}=m$ and that $\mathcal{L}$ contains at least three pairwise linearly independent linear forms.

For simplicity, we assume that

$$
\begin{equation*}
\delta \in A^{*} \text {, the coefficients of } \ell_{1}, \ldots, \ell_{n} \text { all belong to } A \text {. } \tag{3.3.5}
\end{equation*}
$$

Denote by $\mathcal{G}(\mathcal{L})$ the graph with vertex system $\mathcal{L}$ in which $\ell_{i}$ and $\ell_{j}$ are connected by an edge if $\ell_{i}, \ell_{j}$ are linearly dependent over $K$ or they are linearly independent and there is a $q \notin\{i, j\}$ such that $\lambda_{i} \ell_{i}+\lambda_{j} \ell_{j}+\lambda_{q} \ell_{q}=0$ for some non-zero $\lambda_{i}, \lambda_{j}, \lambda_{q} \in K$.

The following proposition can be deduced from Theorem 3.3.1
Proposition 3.3.2. Suppose that $\mathcal{G}(\mathcal{L})$ is connected. Then, under the assumptions (3.3.5), equation (3.3.4) has only finitely many solutions. Moreover, if $A, \delta$ and the coefficients $\alpha_{i, j}$ are effectively given, all solutions of (3.3.4) can be effectively determined.

This is a special case of Corollary 2.6 .2 on decomposable form equations.
In the proof sketch of Proposition 3.3.2 given below, we combine the main arguments of Győry and Papp (1978) over number fields with some effective results from Chapter 6 over finitely generated domains.

Proof (sketch). Let $\mathbf{x}=\left(x_{1}, \ldots, x_{m}\right) \in A^{m}$ be a solution of equation (3.3.4). It follows from (3.3.5) that $\ell_{i}(\mathbf{x})$ is a unit in $A$, say $\ell_{i}(\mathbf{x})=\delta_{i}, i=1, \ldots, n$. By assumption the graph $\mathcal{G}(\mathcal{L})$ is connected. Consider all pairs $i, j$ for which $\ell_{i}, \ell_{j}$ are connected by an edge in $\mathcal{G}(\mathcal{L})$. Then for each such $i, j$ we have

$$
\begin{equation*}
\lambda_{i} \delta_{i}+\lambda_{j} \delta_{j}=0 \text { or } \lambda_{i} \delta_{i}+\lambda_{j} \delta_{j}+\lambda_{q} \delta_{q}=0 \tag{3.3.6}
\end{equation*}
$$

with non-zero elements $\lambda_{i}, \lambda_{j} \lambda_{q}$ of $K$. Here we may assume that $\lambda_{i}, \lambda_{j}, \lambda_{q} \in$ $A \backslash\{0\}$.

Since $\mathcal{G}(\mathcal{L})$ is connected, the system of unit equations (3.3.6) is also connected. By applying Theorem 3.3.1 to this system of equations we get

$$
\delta_{i}=\varepsilon \delta_{i}^{\prime} \text { for } i=1, \ldots, n,
$$

where $\varepsilon \in A^{*}$ is still an unknown and $\delta_{i}^{\prime}$ can take only finitely many values for $i=1, \ldots, n$. But it follows from (3.3.4) that $\varepsilon^{n}=\delta / \delta_{1}^{\prime} \ldots \delta_{n}^{\prime}$, whence $\varepsilon$, and hence $\delta_{i}$ can take only finitely many values for each $i$. Finally, in view of the assumption $\operatorname{rank} \mathcal{L}=m$, from the systems of equations

$$
\begin{equation*}
\ell_{i}(\mathbf{x})=\delta_{i} i=1, \ldots, n \tag{3.3.7}
\end{equation*}
$$

we obtain the finiteness of the number of solutions $\mathbf{x}$.
Now assume that $A, \delta$ and the coefficients $\alpha_{i j}$ are effectively given. Then appropriate values for $\lambda_{i}, \lambda_{j}$ and $\lambda_{q}$ can be determined effectively from the coefficients of $\ell_{i}, \ell_{j}$ and $\ell_{q}$. By Theorem 3.3.1 $\delta_{1}^{\prime}, \ldots, \delta_{n}^{\prime}$ can be computed effectively. As was seen above, $\varepsilon$ is a zero of the polynomial $X^{n}-\delta / \delta_{1}^{\prime} \ldots \delta_{n}^{\prime}$. Hence it can be determined by using Theorem6.6.3. Further, from (3.3.7) we can determine $\mathrm{x} \in K^{m}$ for each possible value of $\delta_{1}, \ldots, \delta_{n}$. Finally, it can be decided by Theorem 6.3.3 whether the x so obtained is an element of $A^{n}$, and can be checked if $\mathbf{x}=\left(x_{1}, \ldots, x_{m}\right)$ is a solution of (3.3.4).

### 3.3.3 Quantitative version

As before, let $A=\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]$ be an integral domain of characteristic 0 , $\mathcal{I}$ the ideal of polynomials in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ vanishing at $\left(z_{1}, \ldots, z_{r}\right), K$ the quotient field of $A$, and $\bar{K}$ an algebraic closure of $K$.

In the previous subsection we considered the decomposable form equation (3.3.4) over $A$. For simplicity, we assumed that (3.3.5) holds, i.e., that $\delta \in A^{*}$ and that the coefficients $\alpha_{i, j}$ of the linear factors of $F$ are elements of $A$. Then (3.3.4) leads to a system of unit equations over $A$. However, in our general Theorem 2.6.1 and Corollary 2.6 .2 this is not the case, the coefficients $\alpha_{i, j}$ of the linear factors belong to a finite extension $G$ of $K$. In this case the equation 3.3.4 can be reduced to a finite system of unit equations in two unknowns, but with units from a subring $A^{\prime} \supset A$ of $G$ that is finitely generated over $\mathbb{Z}$. Then Theorem 2.6.1 can be deduced by using Theorem 2.2.1 with $A^{\prime}$ instead of $A$. To do so, in the proof of Theorem 2.6.1 we use so-called 'degree-height estimates' for the elements of $\bar{K}$.

As an analogue of the naive height (height of the minimal polynomial over $\mathbb{Z}$ ) of an algebraic number, we introduce in Chapter 8 the notion of the 'degree-height estimate' for the elements of $\bar{K}$. Given a monic polynomial $P \in K[X]$, we call $\left(p_{0}, \ldots, p_{n}\right)$ a tuple of representatives for $P$ if
$p_{0}, \ldots, p_{n} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right], p_{0} \notin \mathcal{I}$ and

$$
P(X)=X^{n}+\frac{p_{1}\left(z_{1}, \ldots, z_{r}\right)}{p_{0}\left(z_{1}, \ldots, z_{r}\right)} X^{n-1}+\cdots+\frac{p_{n}\left(z_{1}, \ldots, z_{r}\right)}{p_{0}\left(z_{1}, \ldots, z_{r}\right)} .
$$

We write

$$
P \prec\left(d^{*}, h^{*}\right)
$$

if $P$ has a tuple of representatives $\left(p_{0}, \ldots, p_{n}\right)$ with total degree $\operatorname{deg} p_{i} \leq$ $d^{*}$ and logarithmic height $h\left(p_{i}\right) \leq h^{*}$ for $i=0, \ldots, n$, and call $\left(d^{*}, h^{*}\right)$ a degree-height estimate for $P$. If $\alpha \in \bar{K}$ and $P_{\alpha}$ denotes the monic minimal polynomial of $\alpha$ over $K$, we define a tuple of representatives for $\alpha$ to be a tuple of representatives for $P_{\alpha}$. We write

$$
\alpha \prec\left(d^{*}, h^{*}\right) \quad \text { if } \quad P_{\alpha} \prec\left(d^{*}, h^{*}\right)
$$

and call $\left(d^{*}, h^{*}\right)$ a degree-height estimate for $\alpha$.
In Chapter 8, whose results are new, we give a degree-height estimate for $\beta \in \bar{K}$ in terms of degree-height estimates for $\alpha_{1}, \ldots, \alpha_{m} \in \bar{K}$, if $\beta$ is related to the $\alpha_{i}$ by $P\left(\beta, \alpha_{1}, \ldots, \alpha_{m}\right)=0$ for some given $P \in \mathbb{Z}\left[X_{0}, X_{1}, \ldots, X_{m}\right]$. Such estimates can be used in the proof of Theorem 2.6.1 to construct in an effective way a finitely generated domain $A^{\prime} \supset A$ in $G$ and certain scalar multiples $\ell_{i}^{\prime}$ of $\ell_{i}$ for $i=1, \ldots, n$ such that $\ell_{1}^{\prime}(\mathbf{x}), \ldots, \ell_{n}^{\prime}(\mathbf{x})$ are units of $A^{\prime}$ for any solution x of (3.3.4). Then one can follow a quantitative version of the arguments of the proof of Proposition 3.3.2 above and can use Theorem 2.2.1 with $A^{\prime}$ instead of $A$ as well as some estimates from Chapter 8 to prove Theorem 2.6.1 in the case when $\mathcal{G}(\mathcal{L})$ is connected. In the general case when $\mathcal{G}(\mathcal{L})$ is not connected, some further argument is needed from Step 4 of the proof of Theorem 2.6.1.

### 3.3.4 Reduction of discriminant equations to unit equations

In the remaining part of this chapter, let again $A$ be a finitely generated integral domain over $\mathbb{Z}$. Let $n \geq 2$ be an integer, $\delta \in A \backslash\{0\}$ and consider the discriminant equation

$$
\begin{equation*}
D(f)=\delta \text { in monic polynomials } f \in A[X] \text { of degree } n . \tag{3.3.8}
\end{equation*}
$$

We recall that the monic polynomials $f, f^{\prime} \in A[X]$ are called strongly $A$ equivalent if $f^{\prime}(X)=f(X+a)$ for some $a \in A$. Then $f$ and $f^{\prime}$ have the same degree and same discriminant.

For simplicity, here we restrict ourselves to the special situation when

$$
\left.\begin{array}{l}
\delta \in A^{*}, A \text { is integrally closed and all zeros of } f \text { belong to }  \tag{3.3.9}\\
\text { the quotient field } K \text { of } A \text { (and hence to } A \text { ). }
\end{array}\right\}
$$

We deduce from Theorem 3.3.1 the following proposition.
Proposition 3.3.3. Under the assumptions (3.3.9), the solutions of equations (3.3.8) lie in finitely many strong A-equivalence classes of solutions. Moreover, if $A$ and $\delta$ are effectively given, a full set of representatives of these equivalence classes can be effectively determined.
This is a special case of Corollary 2.8.5.
The first proof of Proposition 3.3.3. (reducing directly to unit equations; sketch) Let $A, \delta$ be as above and let $f \in A[X]$ be a monic polynomial of degree $n \geq 2$ with zeros $\alpha_{1}, \ldots, \alpha_{n}$ and with the properties (3.3.8), (3.3.9). Then we have

$$
\begin{equation*}
D(f)=\prod_{1 \leq i<j \leq n}\left(\alpha_{i}-\alpha_{j}\right)^{2}=\delta \in A^{*} \tag{3.3.10}
\end{equation*}
$$

where $\alpha_{1}, \ldots, \alpha_{n}$ are the zeros of $f$ in $A$. This implies that

$$
\delta_{i, j}:=\alpha_{i}-\alpha_{j} \in A^{*} \text { for each } i, j \text { with } 1 \leq i<j \leq n .
$$

First suppose that $n \geq 3$. Consider the system of unit equations

$$
\delta_{i, j}+\delta_{j, q}+\delta_{q, i}=0 \text { in } \delta_{i, j}, \delta_{j, q}, \delta_{q, i} \in A^{*}
$$

for distinct $i, j, q \in\{1, \ldots, n\}$. We show that this system of equations is connected. Indeed, for $\delta_{i, j}, \delta_{i^{\prime}, j^{\prime}}$, with $(i, j) \neq\left(i^{\prime}, j^{\prime}\right)$ and $i \neq j, i^{\prime} \neq j^{\prime}$, we have

$$
\delta_{i, j}+\delta_{j, i^{\prime}}+\delta_{i^{\prime}, i}=0, \quad \delta_{j, i^{\prime}}+\delta_{i^{\prime}, j^{\prime}}+\delta_{j^{\prime}, j}=0
$$

if $i^{\prime} \neq j$, and

$$
\delta_{i, j}+\delta_{i^{\prime}, j^{\prime}}+\delta_{j^{\prime}, i}=0
$$

if $i^{\prime}=j$. Hence by Theorem 3.3.1 we get

$$
\begin{equation*}
\delta_{i j}=\varepsilon \delta_{i j}^{\prime} \text { for any distinct } i, j \tag{3.3.11}
\end{equation*}
$$

with a common factor $\varepsilon \in A^{*}$ and with $\delta_{i, j}^{\prime} \in A^{*}$ which may take only finitely many values. This is obviously true for $n=2$ as well. Furthermore, if $A$ and $\delta$ are effectively given, by Theorem 3.3.1 the $\delta_{i, j}^{\prime}$ can also be effectively determined.

Now (3.3.10) and (3.3.11) give

$$
\varepsilon^{n(n-1)}=\delta \prod_{1 \leq i<j \leq n}\left(1 / \delta_{i, j}^{\prime}\right)^{2}
$$

This implies that there are only finitely many $\varepsilon, \delta_{i, j} \in A^{*}$ and polynomials $f^{\prime}(X)=\prod_{i=1}^{n}\left(X-\delta_{i, 1}\right) \in A[X]$ under consideration. Further, if $A$ and $\delta$ are effectively given, then using Theorem 6.2 .3 we can effectively determine the zeros in $A$ of the polynomials $X^{n(n-1)}-\theta$ for all $\theta:=\delta \prod_{1 \leq i<j \leq n}\left(1 / \delta_{i, j}^{\prime}\right)^{2}$ in question. Consequently, all $\varepsilon, \delta_{i, j}$ and $f^{\prime}(X)$ can be effectively determined. But $f^{\prime}(X)=f\left(X+\alpha_{1}\right)$, i.e. $f$ is strongly $A$-equivalent to $f^{\prime}$. Finally, from among the $f^{\prime}$ one can easily select a maximal set of pairwise strongly $A$ inequivalent polynomials $f$ satisfying (3.3.8) and (3.3.9).

Proposition 3.3.3 can also be deduced from Proposition 3.3.2 on decomposable form equations, following the arguments of the proof of Theorem 4.8.1. However, we recall that the proof of Proposition 3.3.2 is also based on effective results on unit equations.

The second proof of Proposition 3.3.3. (reducing to decomposable form equations; sketch) Let again $A, \delta$ be as above, and let $f \in A[X]$ be a monic polynomial of degree $n \geq 2$ with zeros $\alpha_{1}, \ldots, \alpha_{n}$ and with the properties (3.3.8) and (3.3.9). Then we have again 3.3.10). Writing now $x_{i}:=\alpha_{i}-\alpha_{1}$ for $i=2, \ldots, n$, we have $x_{i} \in A$ for each $i$. Putting

$$
F\left(X_{2}, \ldots, X_{n}\right)=X_{2} \ldots X_{n} \prod_{2 \leq i<j \leq n}\left(X_{i}-X_{j}\right),
$$

(3.3.10) implies

$$
\begin{equation*}
F\left(x_{2}, \ldots, x_{n}\right)= \pm \delta_{0} \text { in } x_{2}, \ldots, x_{n} \in A, \tag{3.3.12}
\end{equation*}
$$

where $\delta_{0}^{2}=\delta$ and, if (3.3.12) is solvable, $\delta_{0} \in A^{*}$ must hold. The decomposable form $F$ is of degree $n(n-1) / 2$ and it is easily seen that for $n \geq 3$, it satisfies the assumptions of Proposition 3.3.2. Hence by Proposition 3.3.2 the equation (3.3.12) has only finitely many solutions $\left(x_{2}, \ldots, x_{n}\right)$. Further,
if $A$ and $\delta$ are effectively given, then by Theorem 6.2.3, the quantity $\delta_{0}$ can also be effectively determined and Proposition 3.3.2 gives that all solutions $\left(x_{2}, \ldots, x_{n}\right)$ can be effectively found. For $n=2$ the same assertion holds because in this case $x_{2}= \pm \delta_{0}$. Now we can argue as in the above proof to show that $f^{\prime}(X)=f\left(X+\alpha_{1}\right)$ is strongly $A$-equivalent to $f$ and is effectively computable. Finally, our proof can be completed as above in the first proof.

### 3.4 Comparison of our two effective methods

Comparing our effective methods over finitely generated domains over $\mathbb{Z}$, it is easy to observe that the 'unit equation' method, reducing equations to appropriate systems of unit equations, is technically less complicated, at least in the qualitative case. The other 'effective specialization' method, involving effective specializations, is more complicated to apply. To some classes of equations, for example to Thue equations, discriminant equations and decomposable form equations, both methods can be applied, while in case of unit equations, hyper- and superelliptic equations, the Schinzel-Tijdeman equation and the Catalan equation only the 'effective specialization' method applies.

It is interesting to note that over number fields, the superelliptic equations can be reduced to systems of unit equations via Thue equations. However, this reduction uses Lemma 4.5 .4 below (which in turn uses estimates for class numbers, regulators and fundamental units) for which there is no analogue over arbitrary finitely generated domains. Hence the reduction of superelliptic equations to unit equations cannot be extended to arbitrary finitely generated domains.

To illustrate both methods, we used above the 'effective specialization' method for Thue equations and the 'unit equation' method for decomposable form equations and discriminant equations.

## Chapter 4

## Effective results over number fields

In our first general effective method, equations over finitely generated domains are reduced to equations of the same type over number fields and over function fields. Then the best known or best applicable effective results over number fields / function fields can be applied to bound the solutions of the initial equations over finitely generated domains.

Our second effective method reduces, if possible, equations to unit equations in two unknowns. Such equations are e.g. Thue equations, discriminant equations and certain other decomposable form equations. Then using explicit bounds for unit equations, one can derive explicit bounds for the solutions of the initial equations, as well.

In Chapter 9 , we apply our first method to unit equations in two unknowns, Thue equations, hyper- and superelliptic equations, the Schinzel-Tijdeman equation and the Catalan-equation over finitely generated domains. The second method will be extended in Chapter 10 from the number field case to the finitely generated situation, reducing decomposable form equations and discriminant equations to unit equations in two unknowns over finitely generated domains. We note that in the general case the results concerning decomposable form equations are new.

In Sections 4.3 to 4.6 of the present chapter we present the best applicable explicit bounds for the solutions of those equations over number fields which are considered in Chapter 9. Although in Chapter 10 we shall not need equations over number fields, for convenience of reader, in Sections 4.7 and 4.8 we have included the best explicit results for decomposable form equations and discriminant form equations as well over number fields.

To avoid long and complicated computations but emphasize the role of the ingredients, we shall sketch the proofs of less precise versions of the presented results over number fields.

### 4.1 Notation and preliminaries

First we introduce some notation and recall some basic facts on number fields. For further details we refer e.g. to Evertse and Győry (2015, Chapter 1).

Let $L$ be an algebraic number field. Denote by $d, \mathcal{O}_{L}, \mathcal{M}_{L}, D_{L}, h_{L}, r$ and $R_{L}$ its degree, ring of integers, set of places, discriminant, class number, unit rank and regulator, respectively. The set $\mathcal{M}_{L}$ consists of real infinite places, these are the embeddings $\sigma: L \hookrightarrow \mathbb{R}$, complex infinite places, these are the pairs of conjugate complex embeddings $\{\sigma, \bar{\sigma}: L \hookrightarrow \mathbb{C}\}$, and finite places, these are the prime ideals of $\mathcal{O}_{L}$. We denote by $S_{\infty}$ the set of all infinite places, i.e., both real and complex, of $L$. To every $v \in \mathcal{M}_{L}$ we associate a normalized absolute value $|\cdot|_{v}$ such that for $\alpha \in L$ we have

$$
\begin{array}{ll}
|\alpha|_{v}:=|\sigma(\alpha)| & \text { if } v=\sigma \text { is real; } \\
|\alpha|_{v}:=|\sigma(\alpha)|^{2}=|\bar{\sigma}(\alpha)|^{2} & \text { if } v=\{\sigma, \bar{\sigma}\} \text { is complex; } \\
|\alpha|_{v}:=N(\mathfrak{p})^{-\operatorname{ord}_{\mathfrak{p}}(\alpha)} & \text { if } v=\mathfrak{p} \text { is a prime ideal of } \mathcal{O}_{L} .
\end{array}
$$

Here $N(\mathfrak{p}):=\left|\mathcal{O}_{L} / \mathfrak{p}\right|$ denotes the absolute norm of $\mathfrak{p}$, and $\operatorname{ord}_{\mathfrak{p}}(\alpha)$ denotes the exponent of $\mathfrak{p}$ in the unique prime ideal factorization of $[\alpha]$ (i.e., the fractional ideal generated by $\alpha$ ), with $\operatorname{ord}_{\mathfrak{p}}(0)=\infty$. The absolute values defined above satisfy the product formula

$$
\prod_{v \in \mathcal{M}_{L}}|\alpha|_{v}=1 \quad \text { for } \quad \alpha \in L^{*} .
$$

If $M$ is a finite extension of $L$ and $V, v$ are places of $M, L$, respectively, we say that $V$ lies above $v$ or $v$ below $V$, notation $V \mid v$, if the restriction of $|\cdot|_{V}$ to $L$ is a power of $|\cdot|_{v}$. In case of finite places $V, v$ this means that $V=\mathfrak{P}$, $v=\mathfrak{p}$ are prime ideals in $M, L$, respectively with $\mathfrak{P} \supset \mathfrak{p}$.

Given $v \in \mathcal{M}_{L}$, we denote by $L_{v}$ the completion of $L$ at $v$ and by $\overline{L_{v}}$ an algebraic closure of $L_{v}$. The absolute value $|\cdot|_{v}$ has a unique extension to $\overline{L_{v}}$ that we denote also by $|\cdot|_{v}$. If $v$ is real, then $\overline{L_{v}}=\mathbb{C}$, and $|\cdot|_{v}$ is the ordinary absolute value on $\mathbb{C}$; thus it satisfies the triangle inequality. If $v$ is complex then $|\cdot|_{v}$ is the square of the ordinary absolute value on $\mathbb{C}$ and thus, $|\cdot|_{v}^{1 / 2}$
satisfies the triangle inequality. If $v$ is finite then $|\cdot|_{v}$ satisfies the ultrametric inequality. Summarizing, for $\alpha, \beta \in \overline{L_{v}}$ we have

$$
\left.\begin{array}{ll}
|\alpha+\beta|_{v} \leq|\alpha|_{v}+|\beta|_{v} & \text { if } v \text { is real, }  \tag{4.1.1}\\
|\alpha+\beta|_{v}^{1 / 2} \leq|\alpha|_{v}^{1 / 2}+|\beta|_{v}^{1 / 2} & \text { if } v \text { is complex, } \\
|\alpha+\beta|_{v} \leq \max \left(|\alpha|_{v},|\beta|_{v}\right) & \text { if } v \text { is finite. }
\end{array}\right\}
$$

From this general fact we deduce the following useful inequality. We define the quantities $s(v)\left(v \in \in \mathcal{M}_{L}\right)$ as follows:

$$
s(v):= \begin{cases}1 & \text { if } v \text { is real, }  \tag{4.1.2}\\ 2 & \text { if } v \text { is complex } \\ 0 & \text { if } v \text { is finite }\end{cases}
$$

Lemma 4.1.1. Let $v$ be a place of $L, m$ a positive integer, and $\alpha \in \overline{L_{v}}$ such that $|\alpha|_{v} \leq(2 m)^{-s(v)}$. Then

$$
\left|(1+\alpha)^{m}-1\right|_{v} \leq(2 m)^{s(v)}|\alpha|_{v}
$$

Proof. Assume without loss of generality that $\alpha \neq 0$. Then

$$
A:=\frac{(1+\alpha)^{m}-1}{\alpha}=\sum_{k=1}^{m}\binom{m}{k} \alpha^{k-1} .
$$

If $v$ is finite then by the ultrametric inequality, $|A|_{v} \leq 1$. Assume that $v$ is infinite. Then $|\cdot|^{1 / s(v)}$ is the ordinary absolute value on $\mathbb{C}$ and thus, by the triangle inequality,

$$
|A|_{v}^{1 / s(v)} \leq \sum_{k=1}^{m}\binom{m}{k}|\alpha|_{v}^{(k-1) / s(v)} \leq \sum_{k=1}^{m} m^{k}|\alpha|_{v}^{(k-1) / s(v)} \leq 2 m .
$$

The lemma follows.

Let $\alpha \in \overline{\mathbb{Q}}$ and choose a number field $L$ such that $\alpha \in L$. The absolute multiplicative height of $\alpha$ is defined by

$$
H(\alpha):=\prod_{v \in \mathcal{M}_{L}} \max \left(1,|\alpha|_{v}\right)^{1 /[L: \mathbb{Q}]},
$$

while its absolute logarithmic height or briefly height is given by

$$
h(\alpha):=\log H(\alpha)=\frac{1}{[L: \mathbb{Q}]} \sum_{v \in \mathcal{M}_{L}} \log \max \left(1,|\alpha|_{v}\right) .
$$

These notions are independent of the choice of $L$.
The denominator of $\alpha \in \overline{\mathbb{Q}}^{*}$, denoted by den $\alpha$, is defined as the smallest positive rational integer $d_{0}$ for which $d_{0} \alpha$ is an algebraic integer.

The logarithmic height has the following important properties:

$$
\left\{\begin{array}{l}
h\left(\alpha_{1} \cdots \alpha_{k}\right) \leq \sum_{i=1}^{k} h\left(\alpha_{i}\right) \quad \text { for } \alpha_{1}, \ldots, \alpha_{k} \in \overline{\mathbb{Q}} ;  \tag{4.1.3}\\
h\left(\alpha_{1}+\cdots+\alpha_{k}\right) \leq \log k+\sum_{i=1}^{k} h\left(\alpha_{i}\right) \text { for } \alpha_{1}, \ldots, \alpha_{k} \in \overline{\mathbb{Q}} ; \\
h\left(\alpha^{m}\right)=|m| h(\alpha) \text { for } \alpha \in \overline{\mathbb{Q}}^{*}, m \in \mathbb{Z} ; \\
h(\zeta \alpha)=h(\alpha) \text { for } \alpha \in \overline{\mathbb{Q}} \text { and } \zeta \text { a root of unity; } \\
h(\alpha) \geq \frac{\log \operatorname{den} \alpha}{\operatorname{deg} \alpha} \text { for } \alpha \in \overline{\mathbb{Q}}^{*} .
\end{array}\right.
$$

Further, we need the following more advanced result.

Lemma 4.1.2. Let $\alpha$ be an algebraic number of degree $d \geq 1$ which is not equal to 0 or to a root of unity. Then

$$
h(\alpha) \geq m(d):=\left\{\begin{array}{l}
\log 2 \quad \text { if } d=1 \\
2 / d(\log 3 d)^{3} \quad \text { if } d \geq 2
\end{array}\right.
$$

Proof. See Voutier (1996). Asymptotically, this lower bound is not the most optimal but it is most convenient for our purposes. See, e.g. Dobrowolski (1979) for an estimate which is still the best in terms of $d$.

For a number field $L$ and $v \in \mathcal{M}_{L}$, the $v$-adic norm of a vector $\mathbf{x}=$ $\left(x_{1}, \ldots, x_{n}\right) \in \overline{L_{v}}$ is defined by

$$
|\mathbf{x}|_{v}:=\max \left(\left|x_{1}\right|_{v}, \ldots,\left|x_{n}\right|_{v}\right) .
$$

Let $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right) \in \overline{\mathbb{Q}}^{n}$ and choose an algebraic number field $L$ such that $\mathbf{x} \in L^{n}$. Then the multiplicative height and homogeneous multiplicative
height of x are defined by

$$
H(\mathbf{x}):=\left(\prod_{v \in \mathcal{M}_{L}} \max \left(1,|\mathbf{x}|_{v}\right)\right)^{1 / d}, H^{\mathrm{hom}}(\mathbf{x}):=\left(\prod_{v \in \mathcal{M}_{L}}|\mathbf{x}|_{v}\right)^{1 / d}
$$

respectively, where $d=[L: \mathbb{Q}]$. As in the case $n=1$ above, these heights are independent of the choice of $L$. We define the corresponding logarithmic heights by

$$
h(\mathbf{x}):=\log H(\mathbf{x}), \quad h^{\mathrm{hom}}(\mathbf{x}):=\log H^{\mathrm{hom}}(\mathbf{x})
$$

respectively. For instance,

$$
\begin{align*}
& h(\mathbf{x})=\log \max \left(\left|x_{1}\right|, \ldots,\left|x_{n}\right|\right), \quad h^{\text {hom }}(\mathbf{x})=\log \left(\frac{\max \left(\left|x_{1}\right|, \ldots,\left|x_{n}\right|\right)}{\operatorname{gcd}\left(x_{1}, \ldots, x_{n}\right)}\right) \\
& \text { for } \mathbf{x}=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{Z}^{n} \backslash\{\mathbf{0}\} . \tag{4.1.4}
\end{align*}
$$

From the definitions it is clear that

$$
\begin{equation*}
\max _{1 \leq i \leq n} h\left(x_{i}\right) \leq h(\mathbf{x}) \leq \sum_{i=1}^{n} h\left(x_{i}\right) \quad \text { for } \mathbf{x}=\left(x_{1}, \ldots, x_{n}\right) \in \overline{\mathbb{Q}}^{n} \tag{4.1.5}
\end{equation*}
$$

Further,

$$
\begin{equation*}
h^{\mathrm{hom}}(\lambda \mathbf{x})=h^{\mathrm{hom}}(\mathbf{x}) \quad \text { for } \mathbf{x} \in \overline{\mathbb{Q}}^{n}, \lambda \in \overline{\mathbb{Q}}^{*} \tag{4.1.6}
\end{equation*}
$$

This is shown by applying the product formula with any number field $L$ containing $\lambda$ and the coordinates of $\mathbf{x}$.

Let again $L$ be a number field. For a polynomial $P \in L\left[X_{1}, \ldots, X_{n}\right]$ and for $v \in \mathcal{M}_{L}$ we define

$$
|P|_{v}:=\left|\mathbf{x}_{P}\right|_{v}
$$

where $\mathbf{x}_{P}$ is a vector consisting of the non-zero coefficients of $P$. We will need the following estimate.

Lemma 4.1.3. Let $P_{1}, \ldots, P_{q} \in L\left[X_{1}, \ldots, X_{n}\right]$ be polynomials in $n$ variables and $P:=P_{1} \cdots P_{q}$ their product. Suppose that the partial degrees of $P$ have sum at most $D$. If $v \in \mathcal{M}_{L}$ we have

$$
2^{-n D s(v)} \prod_{j=1}^{q}\left|P_{j}\right|_{v} \leq|P|_{v} \leq 2^{n D s(v)} \prod_{j=1}^{q}\left|P_{j}\right|_{v} \text { if } v \text { is infinite }
$$

where $s(v)=1$ if $v$ is real and $s(v)=2$ if $v$ is complex, while

$$
|P|_{v}=\prod_{j=1}^{q}\left|P_{j}\right|_{v} \text { if } v \text { is finite. }
$$

Proof. See Bombieri and Gubler (2006), Lemmas 1.6.11 and 1.6.3.
We deduce some consequences. For a polynomial $P \in \mathbb{\mathbb { Q }}\left[X_{1}, \ldots, X_{n}\right]$ we define

$$
h(P):=h\left(\mathbf{x}_{P}\right), \quad h^{\mathrm{hom}}(P):=h^{\mathrm{hom}}\left(\mathbf{x}_{P}\right) .
$$

Corollary 4.1.4. Let $P_{1}, \ldots, P_{q} \in \overline{\mathbb{Q}}\left[X_{1}, \ldots, X_{n}\right]$ be non-zero polynomials and $P:=P_{1} \cdots P_{q}$ their product. Suppose that the partial degrees of $P$ have sum at most $D$. Then

$$
\left|h^{\mathrm{hom}}(P)-\sum_{j=1}^{q} h^{\mathrm{hom}}\left(P_{j}\right)\right| \leq D \log 2 .
$$

Proof. Pick a number field $L$ containing the coefficients of $P_{1}, \ldots, P_{q}$, take the logarithms of the inequalities and identity from Lemma 4.1.3 and sum over $v \in \mathcal{M}_{L}$.

Corollary 4.1.5. Let $P(X)=\left(X-\alpha_{1}\right) \cdots\left(X-\alpha_{n}\right) \in \overline{\mathbb{Q}}[X]$. Then

$$
\left|h(P)-\sum_{i=1}^{n} h\left(\alpha_{i}\right)\right| \leq n \log 2 .
$$

Proof. Immediate consequence of Corollary 4.1.4.
We have the following estimate for the inhomogeneous heights of polynomials with rational integer coefficients.

Corollary 4.1.6. Let $P_{1}, \ldots, P_{q} \in \mathbb{Z}\left[X_{1}, \ldots, X_{n}\right]$ and $P:=P_{1} \cdots P_{q}$ their product. Suppose that the partial degrees of $P$ have sum at most $D$. Then

$$
\left|h(P)-\sum_{i=1}^{q} h\left(P_{i}\right)\right| \leq D \log 2 .
$$

Proof. Use that $h(Q)=\log |Q|_{\infty}$ if $Q$ is a polynomial with coefficients in $\mathbb{Z}$ and apply Lemma 4.1.3.

In addition to the above, we will frequently use the following simple estimates for heights and lengths of polynomials with integer coefficients. Given a polynomial $Q$ with integer coefficients, its height $H(Q)$ is the maximum of the absolute values of its coefficients, while its length $L(Q)$ is the sum of the absolute values of its coefficients. Denote by $n(Q)$ the number of non-zero coefficients of $Q$. Then for $Q \in \mathbb{Z}\left[X_{1}, \ldots, X_{m}\right]$ we have

$$
\begin{equation*}
H(Q) \leq L(Q) \leq n(Q) H(Q) \leq(\underset{m}{\operatorname{deg} Q+m}) H(Q) \leq 2^{\operatorname{deg} Q+m} H(Q) \tag{4.1.7}
\end{equation*}
$$

where as usual we denote by $\operatorname{deg} Q$ the total degree of $Q$. Further, for $Q_{1}, Q_{2} \in$ $\mathbb{Z}\left[X_{1}, \ldots, X_{m}\right]$ we have

$$
\begin{equation*}
L\left(Q_{1}+Q_{2}\right) \leq L\left(Q_{1}\right)+L\left(Q_{2}\right), \quad L\left(Q_{1} Q_{2}\right) \leq L\left(Q_{1}\right) L\left(Q_{2}\right) \tag{4.1.8}
\end{equation*}
$$

(see e.g., Waldschmidt (2000, p. 76)). From these inequalities we deduce the following:

Lemma 4.1.7. Let $P_{1}, \ldots, P_{q} \in \mathbb{Z}\left[X_{1}, \ldots, X_{n}\right]$ and $F \in \mathbb{Z}\left[X_{1}, \ldots, X_{q}\right]$ be non-zero polynomials. Then for the composed polynomial $G:=F\left(P_{1}, \ldots, P_{q}\right)$ we have

$$
\begin{aligned}
& \operatorname{deg} G \leq \operatorname{deg} F \cdot \max _{1 \leq i \leq q} \operatorname{deg} P_{i}, \\
& L(G) \leq L(F)\left(\max _{1 \leq i \leq q} L\left(P_{i}\right)\right)^{\operatorname{deg} F}, \\
& H(G) \leq n(F) H(F)\left(\max _{1 \leq i \leq q} n\left(P_{i}\right) H\left(P_{i}\right)\right)^{\operatorname{deg} F} .
\end{aligned}
$$

Proof. The estimate for $\operatorname{deg} G$ is obvious. The estimate for $L(G)$ follows directly from (4.1.8), and then the estimate for $H(G)$ follows from (4.1.7).

As above, $L$ is an algebraic number field of degree $d$. Let $S$ be a finite set of places of $L$ which contains the set $S_{\infty}$ of infinite places. Denote by $s$ the cardinality of $S$. We have $d \leq 2 s$. Recall that the ring of $S$-integers $\mathcal{O}_{S}$ is defined as

$$
\mathcal{O}_{S}=\left\{\alpha \in L:|\alpha|_{v} \leq 1 \quad \text { for } \quad v \in \mathcal{M}_{L} \backslash S\right\} .
$$

For $S=S_{\infty}$, this is just the ring of integers $\mathcal{O}_{L}$ in $L$. Denote by $\mathcal{O}_{L}^{*}$ and, more generally, by $\mathcal{O}_{S}^{*}$ the unit groups of $\mathcal{O}_{L}$ and $\mathcal{O}_{S}$, respectively. By the Dirichlet-Chevalley-Weil $S$-unit theorem, that is, the extension to $S$-units of

Dirichlet's unit theorem, the group $\mathcal{O}_{S}^{*}$ has rank $s-1$. This means that there are $\varepsilon_{1}, \ldots, \varepsilon_{s-1} \in \mathcal{O}_{S}^{*}$ such that every $\varepsilon \in \mathcal{O}_{S}^{*}$ can be expressed uniquely as

$$
\varepsilon=\zeta \varepsilon_{1}^{b_{1}} \cdots \varepsilon_{s-1}^{b_{s-1}}
$$

where $\zeta$ is a root of unity in $L$ and $b_{1}, \ldots, b_{s-1}$ are rational integers. Such a system $\left\{\varepsilon_{1}, \ldots, \varepsilon_{s-1}\right\}$ is called a fundamental system of $S$-units and for $S=$ $S_{\infty}$, a fundamental system of units in $L$. Notice that $\operatorname{rank} \mathcal{O}_{L}^{*}=r_{1}+r_{2}-1=$ : $r$, where $r_{1}$ is the number of real places, and $r_{2}$ the number of complex places of $L$.

Pick $s-1$ places $v_{1}, \ldots, v_{s-1}$ from $S$, i.e., we omit one place. The $S$ regulator is defined by

$$
R_{S}:=\left|\operatorname{det}\left(\log \left|\varepsilon_{i}\right|_{v_{j}}\right)_{i, j=1, \ldots, s-1}\right|
$$

This quantity is non-zero, and independent of the choice of $\varepsilon_{1}, \ldots, \varepsilon_{s-1}$ and of the choice $v_{1}, \ldots, v_{s-1}$ from $S$. In the case that $S=S_{\infty}$, the $S$-regulator $R_{S}$ is equal to the regulator $R_{L}$ of $L$.

If $S$ contains finite places, i.e., prime ideals of $\mathcal{O}_{L}$, we denote by $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{t}$ the prime ideals in $S$, and we put

$$
\begin{array}{lll}
P_{S}:=\max \left(N\left(\mathfrak{p}_{1}\right), \ldots, N\left(\mathfrak{p}_{t}\right)\right), & Q_{S}:=N\left(\mathfrak{p}_{1} \cdots \mathfrak{p}_{t}\right) & \text { if } S \supsetneq S_{\infty}, \\
P_{S_{\infty}}:=1, & Q_{S_{\infty}}:=1 . \tag{4.1.9}
\end{array}
$$

The $S$-regulator $R_{S}$ and the regulator $R_{L}$ are related by

$$
\begin{equation*}
R_{S}=h_{S} R_{L} \prod_{i=1}^{t} \log N\left(\mathfrak{p}_{i}\right) \tag{4.1.10}
\end{equation*}
$$

where $h_{S}$ is a (positive) divisor of the class number $h_{L}$. We have

$$
\begin{equation*}
h_{L} R_{L} \leq\left|D_{L}\right|^{1 / 2}\left(\log ^{*}\left|D_{L}\right|\right)^{d-1} \tag{4.1.11}
\end{equation*}
$$

see Louboutin (2000) and Győry and Yu (2006), while on the other hand,

$$
\begin{equation*}
R_{L}>0.2052, \tag{4.1.12}
\end{equation*}
$$

see Friedman (1989). From (4.1.10) and 4.1.11) we infer

$$
\begin{equation*}
R_{S} \leq\left|D_{L}\right|^{1 / 2}\left(\log ^{*}\left|D_{L}\right|\right)^{d-1}\left(\log P_{S}\right)^{t} \tag{4.1.13}
\end{equation*}
$$

while in the opposite direction we have, by (4.1.10) and (4.1.12),

$$
R_{S} \geq \begin{cases}(\log 2)(\log 3)^{s-2} & \text { if } \quad K=\mathbb{Q}, s=|S| \geq 3 \\ 0.2052(\log 2)^{s-2} & \text { if } \quad K \neq \mathbb{Q}, s \geq 3\end{cases}
$$

For $\alpha \in L^{*}$, the fractional ideal $[\alpha]$ generated by $\alpha$ can be written uniquely as a product of two fractional ideals $\mathfrak{a}_{1} \cdot \mathfrak{a}_{2}$, where $\mathfrak{a}_{1}$ is composed of $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{t}$ and $\mathfrak{a}_{2}$ is composed solely of prime ideals different from $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{t}$. The $S$ norm of $\alpha$ is now defined as $N_{S}(\alpha):=N\left(\mathfrak{a}_{2}\right)$. Another expression for the $S$-norm is

$$
N_{S}(\alpha)=\prod_{v \in S}|\alpha|_{v} .
$$

Combining this with $h(\alpha)=d^{-1} \sum_{v \in S} \log \max \left(1,|\alpha|_{v}\right)$ for $\alpha \in \mathcal{O}_{S}$, we derive the very useful inequality

$$
\begin{equation*}
\log N_{S}(\alpha) \leq d h(\alpha) \leq s \max _{v \in S} \log |\alpha|_{v} \text { for } \alpha \in \mathcal{O}_{S} \backslash\{0\} \tag{4.1.14}
\end{equation*}
$$

In case that $\alpha$ is an $S$-unit we obtain, by applying (4.1.14) to $\alpha^{-1}$,

$$
\begin{equation*}
\min _{v \in S} \log |\alpha|_{v} \leq-\frac{d}{s} h(\alpha) \text { for } \alpha \in \mathcal{O}_{S}^{*} \tag{4.1.15}
\end{equation*}
$$

In many of our estimates we need an effective version of the Dirichlet-Chevalley-Weil $S$-unit theorem. We state a suitable version below. Let again $L$ be a number field of degree $d$ and denote by $r$ the rank of $\mathcal{O}_{L}^{*}$. Further, let as before $S$ be a finite set of places of $L$ containing the infinite places and $s$ the cardinality of $S$. Define the constants

$$
\begin{aligned}
& c_{1}:=\frac{((s-1)!)^{2}}{2^{s-2} d^{s-1}} \text { for } s \geq 2 \\
& c_{2}:=116 e \cdot \frac{((s-1)!)^{2} \sqrt{s-2}}{2^{s}} \cdot \log ^{*} d \text { for } s \geq 3
\end{aligned}
$$

and

$$
\begin{aligned}
& c_{3}:=\left\{\begin{array}{l}
\frac{(s-1)!)^{2}}{2^{s-2} \log 2} \text { for } s \geq 2, d=1, \\
\frac{(s-1)!)^{2}}{2^{s-1}} \cdot(\log (3 d))^{3} \quad \text { for } s \geq 2, d \geq 2,
\end{array}\right. \\
& c_{4}:=\left\{\begin{array}{l}
0 \text { for } r=0, \\
1 / d \text { for } r=1, \\
29 e \cdot r!r \sqrt{r-1} \cdot \log d \quad \text { for } r \geq 2 .
\end{array}\right.
\end{aligned}
$$

Proposition 4.1.8. There exists a fundamental system $\left\{\varepsilon_{1}, \ldots, \varepsilon_{s-1}\right\}$ of $S$ units in $L$ such that
(i) $\prod_{i=1}^{s-1} h\left(\varepsilon_{i}\right) \leq c_{1} R_{S}$,
(ii) $\max _{1 \leq i \leq s-1} h\left(\varepsilon_{i}\right) \leq c_{2} R_{S}$ if $s \geq 3$,
(iii) if $v_{1}, \ldots, v_{s-1}$ are any distinct places from $S$, then the absolute values of the entries of the inverse matrix of $\left(\log \left|\varepsilon_{i}\right|_{v_{j}}\right)_{i, j=1, \ldots, s-1}$ do not exceed $c_{3}$.

Proof. (i) and (iii) were proved in Bugeaud and Győry (1996a, 1996b). The inequality (ii) is an improvement, at least in terms of $s$, of the corresponding statements of Bugeaud and Győry (1996a,1996b) and Bugeaud (1998). The main tool in the proof of (i) and (iii) is Minkowski's theorem on the successive minima of symmetric convex bodies from the geometry of numbers. Assertion (ii) was proved by combining (i) with a similar type of result as Lemma4.1.2. See also Evertse and Győry (2015), Prop. 4.3.9 for a proof of (i),(ii),(iii).

We shall also need the following lemma.
Proposition 4.1.9. For every non-zero $\alpha \in \mathcal{O}_{S}$ and for every integer $n \geq 1$ there exists an $\varepsilon \in \mathcal{O}_{S}^{*}$ such that

$$
h\left(\varepsilon^{n} \alpha\right) \leq \frac{1}{d} \log N_{S}(\alpha)+n\left(c_{4} R_{L}+\frac{h_{L}}{d} \log Q_{S}\right) .
$$

Proof. See e.g. Győry and Yu (2006), Lemma 3 or Evertse and Győry (2015), Proposition 4.3.12. The basic idea is as follows. The vectors $\left(\log \left|\varepsilon^{n}\right|_{v}\right)_{v \in S}$
( $\varepsilon \in \mathcal{O}_{S}^{*}$ ) form a full lattice in the vector space of $\left(x_{v}\right)_{v \in S} \in \mathbb{R}^{s}$ with $\sum_{v \in S} x_{v}=$ 0 , hence every point from this vector space is within bounded distance from a point from this lattice. The lemma follows from an explicit bound for this distance.

We finish this section with some estimates for discriminants. As before, we denote by $D_{L}$ the discriminant of a number field $L$.

Lemma 4.1.10. Suppose that $L$ is the compositum of the algebraic number fields $L_{1}, \ldots, L_{k}$. Then $D_{L}$ divides $D_{L_{1}}^{\left[L: L_{1}\right]} \cdots D_{L_{k}}^{\left[L: L_{k}\right]}$ in $\mathbb{Z}$.

Proof. See Stark (1974).
Lemma 4.1.11. Let $L$ be a number field of degree d, let $G \in L[X]$ be a polynomial without multiple zeros which factorizes over an extension of $L$ as $a_{0}\left(X-\vartheta_{1}\right) \cdots\left(X-\vartheta_{n}\right)$, and let $M:=L\left(\vartheta_{1}, \ldots, \vartheta_{k}\right)$ with $1 \leq k \leq n$. Then for the discriminant of $M$ we have the estimate

$$
\left|D_{M}\right| \leq\left(n e^{h(G)}\right)^{2 k n^{k} d}\left|D_{L}\right|^{n^{k}}
$$

In the case that $k=1$ we have the sharper estimate

$$
\left|D_{M}\right| \leq n^{(2 n-1) d} e^{\left(2 n^{2}-2\right) h(G)}\left|D_{L}\right|^{n}
$$

Proof. This is Lemma 4.1 of Bérczes, Evertse and Győry (2013).
Throughout our work we shall use $A \ll_{a, b, \ldots} B$ or $B>_{a, b, \ldots} A$ to denote that $|A|$ is less than $c$ times $B$, where $c$ is an effectively computable positive number, depending only on $a, b, \ldots$, which may be different at each occurrence of the symbol $\ll$. Further, we define

$$
\log ^{*} u:=\max \{1, \log u\} \quad \text { for } \quad u>0, \log ^{*} 0:=1
$$

### 4.2 Effective estimates for linear forms in logarithms

In this section we present some results from Baker's effective theory of logarithmic forms that are used in Section 4.3 to 4.6. We formulate, without proof, some consequences of the best known effective estimates for linear
forms in logarithms, due to Matveev (2000) in the complex case and Yu (2007) in the $\mathfrak{p}$-adic case.

We first give a brief introduction. For the moment, $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and algebraic numbers are supposed to belong to $\overline{\mathbb{Q}}$. Here and below log denotes any fixed determination of the logarithm.

Gel'fond (1934) and Schneider (1934) proved independently of each other that if $\alpha$ and $\beta$ are algebraic numbers such that $\alpha \neq 0,1$ and $\beta$ is not rational, then $\alpha^{\beta}:=\exp (\beta \log \alpha)$ is transcendental for any choice of $\log \alpha$. An equivalent formulation of the Gel'fond-Schneider theorem is that if $\alpha_{1}, \alpha_{2}$ are non-zero algebraic numbers such that $\log \alpha_{1}$ and $\log \alpha_{2}$ are linearly independent over $\mathbb{Q}$ for any choice of the logarithms, then they are linearly independent over $\overline{\mathbb{Q}}$. Gel'fond (1935) gave a non-trivial effective lower bound for $\left|\beta_{1} \log \alpha_{1}+\beta_{2} \log \alpha_{2}\right|$, where $\beta_{1}, \beta_{2}$ denote algebraic numbers, not both 0 , and $\alpha_{1}, \alpha_{2}$ denote algebraic numbers different from 0 and 1 such that $\log \alpha_{1} / \log \alpha_{2}$ is not rational.

In his celebrated series of papers, Baker (1966,1967a,1967b,1968a) made a significant breakthrough by generalizing the Gel'fond-Schneider theorem to arbitrarily many logarithms. Baker $\left(1966,1967\right.$ b) proved that if $\alpha_{1}, \ldots, \alpha_{n}$ denote non-zero algebraic numbers such that $\log \alpha_{1}, \ldots, \log \alpha_{n}$ are linearly independent over $\mathbb{Q}$, then $1, \log \alpha_{1}, \ldots, \log \alpha_{n}$ are linearly independent over $\overline{\mathbb{Q}}$. Further, Baker (1967a,1967b,1968a) gave non-trivial lower bounds for $\left|\beta_{1} \log \alpha_{1}+\cdots+\beta_{n} \log \alpha_{n}\right|$, where $\alpha_{1}, \ldots, \alpha_{n}$ are non-zero algebraic numbers such that $\log \alpha_{1}, \ldots, \log \alpha_{n}$ are linearly independent over $\mathbb{Q}$ and $\beta_{1}, \ldots, \beta_{n}$ are algebraic numbers, not all 0 .

Baker's general effective estimates led to significant applications in number theory. Later, many improvements, generalizations and applications were established by Baker and others. For comprehensive accounts of Baker's theory, analogues for $\mathfrak{p}$-adic and elliptic logarithms and algebraic groups and extensive bibliographies, the reader can consult Baker $(1975,1988)$, Baker and Masser (1977), Lang (1978), Feldman and Nesterenko (1998), Waldschmidt (2000), Wüstholz (2002), Baker and Wüstholz (2007) and Bugeaud (2018).

For applications to Diophantine equations, the effective estimates of Baker (1968a,1968b) in which $\beta_{1}, \ldots, \beta_{n}$ are rational integers proved to be particularly useful. Using his estimates in this special case, Baker (1968b,1968c, 1969) gave the first explicit upper bounds for the solutions of Thue equations, Mordell equations and super- and hyperelliptic equations over $\mathbb{Z}$. For further applications of Baker's theory to Diophantine problems, we refer to Baker (1975), Győry (1980b,2002), Sprindzuk (1982,1993), Shorey and Tijdeman (1986),

Bilu (1995), Smart (1998), Evertse and Győry (2015,2017a), Bugeaud (2018), and to the references given there.

Since the 1990's, some other effective methods have also been developed for Diophantine equations by various authors, including Bombieri (1993), Bombieri and Cohen (1997,2003), Bugeaud (1998), Bennett and Skinner (2004), Siksek (2013), Murty and Pasten (2013), Pasten (2017), von Känel (2014), von Känel and Matschke (2016), Kim (2017), Poonen (2019), Le Fourn (2019), Győry (2019), Triantafillou (2020) and Freitas, Kraus and Siksek (2020a,b). See also Evertse and Győry (2015, Sect.4.5). However, at present Baker's theory is the most suitable to derive effective bounds for the solutions of our equations over number fields.

We now state some consequences of the best known results, due to Matveev (2000) and Yu (2007), from Baker's theory. These are the main tools in the next sections, in the proofs for unit equations, hyper- and superelliptic equations and the Catalan equation. We note that in the complex case Matveev (2000) gives lower bounds for linear forms in logarithms. For applications, it will be more convenient to consider some consequences concerning

$$
\Lambda=\alpha_{1}^{b_{1}} \cdots \alpha_{n}^{b_{n}}-1
$$

where $\alpha_{1}, \ldots, \alpha_{n}\left(n(\geq 2)\right.$ are non-zero algebraic numbers and where $b_{1}, \ldots, b_{n}$ are rational integers, not all zero.

Throughout this section, $L$ is a number field containing $\alpha_{1}, \ldots, \alpha_{n}$, and $d$ denotes the degree of $L$. Set

$$
B^{*}:=\max \left\{\left|b_{1}\right|, \ldots,\left|b_{n}\right|\right\}
$$

and let $A_{1}, \ldots, A_{n}$ be reals with

$$
A_{i} \geq \max \left\{d h\left(\alpha_{i}\right), \pi\right\}, \quad i=1, \ldots, n .
$$

We state two propositions that are consequences of results of Matveev (2000).

Proposition 4.2.1. Suppose $\Lambda \neq 0, b_{n}= \pm 1$ and B satisfies

$$
B \geq \max \left\{B^{*}, 2 e A_{n} \max \left(n \pi / \sqrt{2}, A_{1}, \ldots, A_{n-1}\right)\right\}
$$

Then

$$
\log |\Lambda|>-c_{5}(n, d) A_{1} \cdots A_{n} \log \left(B /\left(\sqrt{2} A_{n}\right)\right)
$$

where

$$
c_{5}(n, d)=\min \left\{1.451(30 \sqrt{2})^{n+4}(n+1)^{5.5}, \pi 2^{6.5 n+27}\right\} d^{2} \log (e d)
$$

Proof. This is Proposition 4 in Győry and Yu (2006). It is an easy consequence of Corollary 2.3 of Matveev (2000).

We put $\chi:=1$ if $L$ is real and $\chi:=2$ otherwise. Let now

$$
A_{i}^{\prime}:=d h\left(\alpha_{i}\right)+\pi, \quad i=1, \ldots, n
$$

Proposition 4.2.2. Suppose that $\Lambda \neq 0$, and that $B^{\prime}$ satisfies

$$
B^{\prime} \geq \max \left\{1, \max \left(\left|b_{i}\right| A_{i}^{\prime} / A_{n}^{\prime} ; 1 \leq i \leq n\right)\right\}
$$

Then we have

$$
\log |\Lambda|>-c_{6}(n, d) A_{1}^{\prime} \cdots A_{n}^{\prime} \log \left(e(n+1) B^{\prime}\right)
$$

where

$$
c_{6}(n, d)=2 \pi \min \left\{\frac{1}{\chi}\left(\frac{1}{2} e(n+1)\right)^{\chi} 30^{n+4}(n+1)^{3.5}, 2^{6 n+26}\right\} d^{2} \log (e d) .
$$

Proof. This is Lemma 3.6 in Koymans (2017). It is easily deduced from Corollary 2.3 of Matveev (2000).

Consider again $\Lambda$ defined as above. Let $B$ and $B_{n}$ be real numbers satisfying

$$
B \geq \max \left\{\left|b_{1}\right|, \ldots,\left|b_{n}\right|\right\}, \quad B \geq B_{n} \geq\left|b_{n}\right|
$$

Denote by $\mathfrak{p}$ a prime ideal in $\mathcal{O}_{L}$ lying above the prime number $p$ and by $e_{\mathfrak{p}}$ and $f_{\mathfrak{p}}$ the ramification index and residue class degree of $\mathfrak{p}$, respectively. Thus $N(\mathfrak{p})=p^{f_{p}}$.

The following result is due to Yu (2007).

Proposition 4.2.3. Assume that $\operatorname{ord}_{\mathfrak{p}} b_{n} \leq \operatorname{ord}_{\mathfrak{p}} b_{i}$ for $i=1, \ldots, n$, and set

$$
A_{i}^{\prime \prime}:=\max \left\{h\left(\alpha_{i}\right), 1 /\left(16 e^{2} d^{2}\right)\right\}, \quad i=1, \ldots, n .
$$

If $\Lambda \neq 0$, then for any real $\delta$ with $0<\delta \leq 1 / 2$ we have

$$
\operatorname{ord}_{\mathfrak{p}} \Lambda<c_{7}(n, d) e_{\mathfrak{p}}^{n} \frac{N(\mathfrak{p})}{(\log N(\mathfrak{p}))^{2}} \max \left\{A_{1}^{\prime \prime} \cdots A_{n}^{\prime \prime} \log M, \frac{\delta B}{B_{n} c_{8}(n, d)}\right\}
$$

where

$$
\begin{aligned}
& c_{7}(n, d)=(16 e d)^{2(n+1)} n^{3 / 2} \log (2 n d) \log (2 d), \\
& c_{8}(n, d)=(2 d)^{2 n+1} \log (2 d) \log ^{3}(3 d)
\end{aligned}
$$

and

$$
M=\left(B_{n} / \delta\right) c_{9}(n, d) N(\mathfrak{p})^{n+1} A_{1}^{\prime \prime} \cdots A_{n-1}^{\prime \prime}
$$

with

$$
c_{9}(n, d)=2 e^{(n+1)(6 n+5)} d^{3 n} \log (2 d) .
$$

Proof. This is the second consequence of the Main Theorem in Yu (2007).

As before, $L$ denotes a number field of degree $d$ and $\mathcal{M}_{L}$ its set of places, $\alpha_{1}, \ldots, \alpha_{n}$ are $n(\geq 2)$ non-zero elements of $L$, and $b_{1}, \ldots, b_{n}$ rational integers, not all zero. Let $\Lambda$ be defined again as above, and put

$$
\begin{aligned}
\Omega & :=\prod_{i=1}^{n} \max \left\{h\left(\alpha_{i}\right), m(d)\right\}, \\
B & :=\max \left\{3,\left|b_{1}\right|, \ldots,\left|b_{n}\right|\right\}
\end{aligned}
$$

where $m(d)$ is the lower bound from Lemma 4.1.2. For a place $v \in \mathcal{M}_{L}$, we write

$$
N(v):=\left\{\begin{array}{l}
2 \quad \text { if } v \text { is infinite } \\
N(\mathfrak{p}) \quad \text { if } v=\mathfrak{p} \text { is finite }
\end{array}\right.
$$

The following proposition is in fact a combination of some inequalities of Matveev (2000) and Yu (2007).

Proposition 4.2.4. Suppose that $\Lambda \neq 0$. Then for $v \in \mathcal{M}_{L}$ we have

$$
\log |\Lambda|_{v}>-c_{10}(n, d) \frac{N(v)}{\log N(v)} \Omega \log B
$$

where $c_{10}(n, d)=12(16 e d)^{3 n+2}\left(\log ^{*} d\right)^{2}$.

Proof. This is Proposition 3.10 in Bérczes, Evertse and Győry (2013). For $v$ infinite, it is deduced from Corollary 2.3 of Matveev (2000), while for $v$ finite, from the first consequence of the Main Theorem on p. 190 of Yu (2007). We have used Lemma 4.1.2 to incorporate the small values of $B$.

## 4.3 $S$-unit equations

Keeping the notation introduced in Section 4.1, $L$ is a number field, $S$ a finite set of places of $L$ containing the infinite places, $\mathcal{O}_{S}$ the ring of $S$-integers of $L$ and $\mathcal{O}_{S}^{*}$ the unit group of $\mathcal{O}_{S}$, that is the group of $S$-units in $L$.

Let $\alpha$ and $\beta$ be non-zero elements of $L$ with

$$
\max \{h(\alpha), h(\beta)\} \leq H
$$

where, for technical reasons, we assume that $H \geq \max (1, \pi / d)$. Consider the $S$-unit equation

$$
\begin{equation*}
\alpha x+\beta y=1 \quad \text { in } \quad x, y \in \mathcal{O}_{S}^{*} . \tag{4.3.1}
\end{equation*}
$$

For $S=S_{\infty}$, this is called a(n ordinary) unit equation.
The first effective finiteness theorems for equation (4.3.1) were proved for $S=S_{\infty}$ by Győry ( 1973,1974 ), and for general $S$ by Győry (1979) and independently, in a less precise form, by Kotov and Trelina (1979). In the proofs, Baker's theory of logarithmic forms was used. The results in Győry $(1974,1979)$ and Kotov and Trelina $(1979)$ are quantitative, providing explicit upper bounds for the heights of the solutions $x, y$ of (4.3.1).

In Chapter 9, we shall use the following theorem, due to Győry and Yu (2006). Here, $d$ denotes the degree of $L, s$ denotes the cardinality of $S, P_{S}=$ $\max \left(1, N\left(\mathfrak{p}_{1}\right), \ldots, N\left(\mathfrak{p}_{t}\right)\right)$ where $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{t}$ are the prime ideals in $S$, and $R_{S}$ denotes the $S$-regulator.

Theorem 4.3.1. All solutions $x, y$ of equation (4.3.1) satisfy

$$
\max \{h(x), h(y)\} \leq c_{11} P_{S} R_{S}\left(1+\left(\log ^{*} R_{S}\right) / \log ^{*} P_{S}\right) H
$$

where

$$
c_{11}=s^{2 s+3.5} \cdot 2^{7 s+27}(\log 2 s) d^{2(s+1)}\left(\log ^{*}(2 d)\right)^{3} .
$$

In the proof of Theorem 4.3.1 the main tools are Propositions 4.2.1, 4.2.3
(logarithmic forms estimates) and a variation on Proposition 4.1.8 (the effective $S$-unit theorem).

Combining the method of proof with his new approach, Le Fourn (2019) has recently improved Theorem 4.3 .1 with $P_{S}$ replaced by $P_{S}^{\prime}$ which denotes the third largest norm of prime ideals from $S$. For a further improvement, see Győry (2019). However, these improvements would not give better bounds in Chapter 9 .

To avoid long and complicated computations but emphasize the role of the main tools, we shall sketch the proof of the following less precise version of Theorem 4.3.1.

Let $x, y$ be a solution of equation (4.3.1). Then

$$
\begin{equation*}
\max \{h(x), h(y)\}<_{L, S} H . \tag{4.3.2}
\end{equation*}
$$

Here and below the positive constants implied by $<_{L, S}$ depend only on $L$ and $S$ and are effectively computable.

For a complete proof of Theorem 4.3.1, the reader can consult Győry and Yu (2006).

Sketch of the proof of (4.3.2). Let $x, y$ be a solution of (4.3.1). We may assume that $h(x) \geq h(y)$. Further, the case $s=1$ being trivial, we assume that $s \geq 2$.

Let $\left\{\varepsilon_{1}, \ldots, \varepsilon_{s-1}\right\}$ be a fundamental system of $S$-units with the properties specified in Proposition 4.1.8. Then $y$ can be written in the form

$$
\begin{equation*}
y=\zeta \varepsilon_{1}^{b_{1}} \cdots \varepsilon_{s-1}^{b_{s-1}} \tag{4.3.3}
\end{equation*}
$$

where $\zeta$ is a root of unity in $L$ and $b_{1}, \ldots, b_{s-1}$ are rational integers. Set

$$
B:=\max \left\{3,\left|b_{1}\right|, \ldots,\left|b_{s}\right|\right\}
$$

and let $v_{1}, \ldots, v_{s}$ be distinct places from $S$. Then it follows from (4.3.3) that

$$
\log |y|_{v_{j}}=\sum_{i=1}^{s-1} b_{i} \log \left|\varepsilon_{i}\right|_{v_{j}}, \quad j=1, \ldots, s-1
$$

Together with Cramer's rule and (iii) of Proposition 4.1 .8 this implies that

$$
\begin{equation*}
B<_{L, S} h(y)<_{L, S} B \tag{4.3.4}
\end{equation*}
$$

Set $\alpha_{s}=\zeta \beta, b_{s}=1$ and

$$
\Lambda=\varepsilon_{1}^{b_{1}} \cdots \varepsilon_{s-1}^{b_{s-1}} \alpha_{s}^{b_{s}}-1 .
$$

Let $v \in S$ for which $|x|_{v}$ is minimal. Then, using (4.3.1) and (4.1.15), we deduce that

$$
\begin{equation*}
\log |\Lambda|_{v}=\log |\alpha x|_{v} \leq-\frac{d}{s} h(x)+d H \tag{4.3.5}
\end{equation*}
$$

First assume that $v$ is infinite. We may assume that

$$
B \geq c(L, S) H
$$

with an appropriate, effectively computable positive number $c(L, S)$ depending only on $L$ and $S$, since otherwise (4.3.1) and (4.3.4) would give immediately (4.3.2). Applying Proposition 4.2.1 to $\log |\Lambda|_{v}$ and using Lemma 4.1.8, (4.3.4) and $h(x) \geq h(y)$, we infer that

$$
\log |\Lambda|_{v}>_{L, S}\left(-H \log \left(\frac{h(x)}{H}\right)\right)
$$

Together with (4.3.5) this gives 4.3.2).
Next assume that $v$ is finite and corresponds to the prime ideal $\mathfrak{p}$. Then it follows that

$$
\begin{equation*}
\log |\alpha x|_{v}=-\left(\operatorname{ord}_{\mathfrak{p}} \Lambda\right) \log N(\mathfrak{p}) \tag{4.3.6}
\end{equation*}
$$

We apply now Proposition 4.2.3 to $\operatorname{ord}_{\mathfrak{p}} \Lambda$ with the choice $\delta=\frac{c(L, S)}{2} \cdot \frac{H}{B} \leq \frac{1}{2}$. Then, using Proposition 4.1.8, inequality (4.3.4) and $h(y) \leq h(x)$, we get

$$
\operatorname{ord}_{\mathfrak{p}} \Lambda<_{L, S} H \log \left(\frac{h(x)}{H}\right) .
$$

Together with (4.3.5) and (4.3.6) this gives (4.3.2).

The following consequence of Theorem 4.3.1, due to Győry and Yu (2006), will be very useful.

Corollary 4.3.2. Let $\alpha_{1}, \alpha_{2}, \alpha_{3}$ be non-zero elements in $L$ with logarithmic
heights at most $H(\geq 2)$. Then for every solution $x_{1}, x_{2}, x_{3}$ of

$$
\begin{align*}
& \alpha_{1} x_{1}+\alpha_{2} x_{2}+\alpha_{3} x_{3}=0 \\
& \quad \text { in } x_{k} \in \mathcal{O}_{S} \backslash\{0\} \text { with } N_{S}\left(x_{k}\right) \leq N \quad \text { for } k=1,2,3 \tag{4.3.7}
\end{align*}
$$

there is an $\varepsilon \in \mathcal{O}_{S}^{*}$ such that

$$
\max _{1 \leq k \leq 3} h\left(\varepsilon x_{k}\right) \leq 2.001 c_{11} P_{S} R_{S}\left(1+\left(\log ^{*} R_{S}\right) /\left(\log ^{*} P_{S}\right)\right) \mathcal{N}
$$

where

$$
\mathcal{N}=c_{4} R_{L}+\frac{h_{L}}{d} \log Q_{S}+H+\frac{1}{d} \log N
$$

with the constants $c_{11}$ from Theorem 4.3.1 and $c_{4}$ from Proposition 4.1.9

We prove a weaker version of Corollary 4.3.2, that is, we show that for every solution $\left(x_{1}, x_{2}, x_{3}\right)$ of (4.3.7), there is $\varepsilon \in \mathcal{O}_{S}^{*}$ such that

$$
\begin{equation*}
\max _{1 \leq k \leq 3} h\left(\varepsilon x_{k}\right)<_{L, S} \max (H, \log N) . \tag{4.3.8}
\end{equation*}
$$

Proof of (4.3.8). Put $H^{*}:=\max (H, \log N)$. Pick a solution $\left(x_{1}, x_{2}, x_{3}\right)$ of 4.3.7. By Proposition 4.1.9, for $k=1,2,3$ there are $\mu_{k} \in \mathcal{O}_{S} \backslash\{0\}$ with $h\left(\mu_{k}\right)<_{L, S} \log N$ and $\varepsilon_{k} \in \mathcal{O}_{S}^{*}$, such that $x_{k}=\mu_{k} \varepsilon_{k}$. This gives

$$
\alpha_{1} \mu_{1} \varepsilon_{1}+\alpha_{2} \mu_{2} \varepsilon_{2}+\alpha_{3} \mu_{3} \varepsilon_{3}=0
$$

or equivalently,

$$
\beta_{1} \cdot \frac{\varepsilon_{1}}{\varepsilon_{3}}+\beta_{2} \cdot \frac{\varepsilon_{2}}{\varepsilon_{3}}=1 \text { where } \beta_{1}=-\frac{\alpha_{1} \mu_{1}}{\alpha_{3} \mu_{3}}, \beta_{2}=-\frac{\alpha_{1} \mu_{1}}{\alpha_{3} \mu_{3}} .
$$

By the height properties (4.1.3) we have $h\left(\beta_{1}\right), h\left(\beta_{2}\right)<_{L, S} H^{*}$, and so, by 4.3.2),

$$
h\left(\varepsilon_{1} / \varepsilon_{3}\right), h\left(\varepsilon_{2} / \varepsilon_{3}\right)<_{L, S} H^{*}
$$

Now taking $\varepsilon:=\varepsilon_{1}^{-1}$ and invoking the height properties (4.1.3), we readily obtain 4.3.8.

### 4.4 Thue equations

As before, $L$ is a number field and $S$ a finite set of places of $L$, containing all infinite places. Let $F(X, Y) \in L[X, Y]$ be a binary form of degree $n \geq 3$ with at least three pairwise non-proportional linear factors over $\bar{L}$, a fixed algebraic closure of $L$. Further, let $\delta$ be a non-zero element of $L$ and consider the Thue equation over $\mathcal{O}_{S}$,

$$
\begin{equation*}
F(x, y)=\delta \quad \text { in } \quad x, y \in \mathcal{O}_{S} . \tag{4.4.1}
\end{equation*}
$$

For a polynomial $P$ with algebraic coefficients, we denote by $h(P)$ the maximum of the absolute logarithmic heights of its coefficients.

In the classical case $L=\mathbb{Q}, \mathcal{O}_{S}=\mathbb{Z}$, the first effective upper bound for the heights of the solutions of equation (4.4.1) was given by Baker (1968b), using one of his effective estimates concerning linear forms in logarithms. Later, Baker's effective result has been improved and generalized by several people; for references, see e.g. Shorey and Tijdeman (1986), Sprindžuk (1993), Győry (2002), Evertse and Győry (2015) and Bugeaud (2018).

For convenience, choose $L$ such that $F$ factors into linear forms over $L$. Then the best known bound to date for the solutions of equation (4.4.1) is due to Győry and Yu (2006). As before, $d$ denotes the degree of $L$, while $s$ denotes the cardinality of $S$, the quantities $P_{S}, Q_{S}$ are defined by (4.1.9), and $R_{S}$ denotes the $S$-regulator.

Theorem 4.4.1. Assume that $F$ splits into linear factors over L. Then all solutions $x$, $y$ of equation (4.4.1) satisfy

$$
\begin{aligned}
& \max \{h(x), h(y)\} \\
& \quad \leq c_{12} P_{S} R_{S}\left(1+\frac{\log ^{*} R_{S}}{\log ^{*} P_{S}}\right) \cdot\left(c_{4} R_{L}+\frac{h_{L}}{d} \log Q_{S}+2 n d H_{1}+H_{2}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
& H_{1}:=\max (1, h(F)), \quad H_{2}=\max (1, h(\delta)), \\
& c_{12}:=250 n^{6} s^{2 s+3.5} \cdot 2^{7 s+29}(\log 2 s) d^{2 s+4}\left(\log ^{*}(2 d)\right)^{3}
\end{aligned}
$$

and $c_{4}$ is the constant from Proposition 4.1.9 i.e.,

$$
c_{4}=0 \text { if } r=0, \quad 1 / d \text { if } r=1, \quad 29 e \cdot r!r \sqrt{r-1} \cdot \log d \text { if } r \geq 2 .
$$

This is Corollary 3 of Győry and Yu (2006). It is a special case of Theorem 3 of Győry and Yu (2006) concerning decomposable form equations.

In terms of $S$, a better bound has been recently obtained in Győry (2019), replacing $P_{S}$ by $P_{S}^{\prime}$; see the remark after Theorem 4.3.1. However, this improvement would not lead to a better bound in Chapter 9 .

We shall outline a proof of the following less precise version of Theorem 4.4.1, under the same assumptions as in Theorem 4.4.1.

All solutions $x, y$ of equation (4.4.1) satisfy

$$
\begin{equation*}
\max \{h(x), h(y)\}<_{n, L, S} \max \left(H_{1}, H_{2}\right) . \tag{4.4.2}
\end{equation*}
$$

We recall that here and below, constants implied by $<_{a_{1}, \ldots, a_{r}}$ are positive and effectively computable, and depend only on the parameters $a_{1}, \ldots, a_{r}$ in the subscript.

Sketch of the proof of (4.4.2). We start with some remarks. There is an $a \in \mathbb{Z}$ with $1 \leq a \leq n$ such that $F(1, a) \neq 0$. Consider the binary form $G(X, Y):=$ $F(X, a X+Y)$ in which the coefficient of $X^{n}$ is $F(1,0) \neq 0$ and the heights of the coefficients of $G$ are at most $<_{n} H_{1}$. Denote by $g_{0}$ the product of the denominators of the coefficients of $G$. Then we can write

$$
\begin{aligned}
g_{0} G(X, Y) & =a_{0} X^{n}+a_{1} X^{n-1} Y+\cdots+a_{n} Y^{n} \\
& =a_{0}\left(X-\alpha_{1} Y\right) \cdots\left(X-\alpha_{n} Y\right),
\end{aligned}
$$

where $a_{0}, \ldots, a_{n}$ are already integers in $L$ with heights $<_{n, d} H_{1}$. Further, at least three from among $\alpha_{1}, \ldots, \alpha_{n}$, say $\alpha_{1}, \alpha_{2}, \alpha_{3}$, are pairwise distinct.

Let $x, y$ be a solution of 4.4.1). Then

$$
\begin{equation*}
x^{\prime}=a_{0} x, \quad y^{\prime}=-a x+y \tag{4.4.3}
\end{equation*}
$$

is a solution of the equation

$$
\begin{equation*}
\left(x^{\prime}-\alpha_{1}^{\prime} y^{\prime}\right) \cdots\left(x^{\prime}-\alpha_{n}^{\prime} y^{\prime}\right)=\delta^{\prime} \tag{4.4.4}
\end{equation*}
$$

where $\delta^{\prime}=g_{0} a_{0}^{n-1} \delta \in \mathcal{O}_{S}$ and $\alpha_{i}^{\prime}=a_{0} \alpha_{i}$ for $i=1, \ldots, n$. We have

$$
\left(\alpha_{i}^{\prime}\right)^{n}+a_{1} a_{0}\left(\alpha_{i}^{\prime}\right)^{n-1}+\cdots+a_{n} a_{0}^{n-1}=0
$$

which implies that $\alpha_{i}^{\prime}$ is an integer in $L$ and $h\left(\alpha_{i}^{\prime}\right)<_{n, d} H_{1}$ for each $i$. Further,

$$
\begin{equation*}
h\left(\delta^{\prime}\right)<_{n, d} \max \left(H_{1}, H_{2}\right) . \tag{4.4.5}
\end{equation*}
$$

Putting $\delta_{i}^{\prime}=x^{\prime}-\alpha_{i}^{\prime} y^{\prime}$, we have $\delta_{i}^{\prime} \in \mathcal{O}_{S} \backslash\{0\}$. It follows from (4.4.4) that $\delta_{i}^{\prime}$ divides $\delta^{\prime}$ in $\mathcal{O}_{S}$. Therefore

$$
\log N_{S}\left(\delta_{i}^{\prime}\right) \leq \log N_{S}\left(\delta^{\prime}\right)<_{n, d} \max \left(H_{1}, H_{2}\right)
$$

The linear forms $X-\alpha_{i}^{\prime} Y$ are pairwise non-proportional for $i=1,2,3$. Hence there are non-zero integers $\alpha_{1}, \alpha_{2}, \alpha_{3}$ in $L$ such that

$$
\alpha_{1} \delta_{1}^{\prime}+\alpha_{2} \delta_{2}^{\prime}+\alpha_{3} \delta_{3}^{\prime}=0
$$

and $h\left(\alpha_{i}\right)<_{n, d} H_{1}$ for $i=1,2,3$. Then, by (4.3.8), which was proved in the previous section, there is an $\varepsilon \in \mathcal{O}_{S}^{*}$ such that

$$
h\left(\varepsilon \delta_{i}^{\prime}\right)<_{n, L, S} \max \left(H_{1}, H_{2}\right) \quad \text { for } \quad i=1,2,3 .
$$

But then from

$$
\left(\varepsilon x^{\prime}\right)-\alpha_{i}^{\prime}\left(\varepsilon y^{\prime}\right)=\varepsilon \delta_{i}^{\prime}, \quad i=1,2
$$

it follows that $h\left(\varepsilon x^{\prime}\right), h\left(\varepsilon y^{\prime}\right)<_{n, L, S} \max \left(H_{1}, H_{2}\right)$ and thus $h\left(x^{\prime} / y^{\prime}\right)<_{n, L, S}$ $\max \left(H_{1}, H_{2}\right)$. Using this together with (4.4.4), (4.4.5) we obtain $h\left(x^{\prime}\right), h\left(y^{\prime}\right)<_{n, L, S}$ $\max \left(H_{1}, H_{2}\right)$, and finally, a combination with (4.4.3) gives (4.4.2).

### 4.5 Hyper- and superelliptic equations, the SchinzelTijdeman equation

Again, $S$ is a finite set of places of a number field $L$, containing all infinite places. Let

$$
f(X)=a_{0} X^{n}+a_{1} X^{n-1}+\cdots+a_{n} \in \mathcal{O}_{S}[X]
$$

be a polynomial of degree $n \geq 2$ with non-zero discriminant, $\delta \in \mathcal{O}_{S} \backslash\{0\}$, $m \geq 2$ an integer, and consider the hyperelliptic equation

$$
\begin{equation*}
f(x)=\delta y^{2} \quad \text { in } \quad x, y \in \mathcal{O}_{S}, \tag{4.5.1}
\end{equation*}
$$

where $n \geq 3$, and the superelliptic equation

$$
\begin{equation*}
f(x)=\delta y^{m} \quad \text { in } \quad x, y \in \mathcal{O}_{S}, \tag{4.5.2}
\end{equation*}
$$

where $n \geq 2$ and $m \geq 3$.
For $L=\mathbb{Q}, S=S_{\infty}$ the first explicit upper bounds for the solutions of (4.5.1) and (4.5.2) were obtained by Baker (1968b, 1968c,1969). Over $\mathbb{Z}$, Schinzel and Tijdeman (1976) were the first to consider equation (4.5.2) in the more general situation when also the exponent $m$ is unknown, and they gave an effective upper bound for $m$. Quantitative improvements and generalizations were later obtained by many authors, including Brindza (1984), who gave an effective upper bound for the solutions $x, y$ of (4.5.1) and (4.5.2) under the most general condition, where $f$ is allowed to have multiple roots. For further references, see Shorey and Tijdeman (1986), Sprindžuk (1993), Győry (2002), Evertse and Győry (2015) and Bugeaud (2018).

The following best known explicit results are due to Bérczes, Evertse and Győry (2013). As before, $d$ denotes the degree of $L, s$ the cardinality of $S$, and the quantities $P_{S}, Q_{S}$ are defined by (4.1.9). Further, we put

$$
\begin{equation*}
\widehat{h}:=\frac{1}{d} \sum_{v \in \mathcal{M}_{L}} \log \max \left(1,|\delta|_{v},\left|a_{0}\right|_{v}, \ldots,\left|a_{n}\right|_{v}\right) . \tag{4.5.3}
\end{equation*}
$$

Theorem 4.5.1. All solutions $x, y$ of equation (4.5.1) satisfy

$$
h(x), h(y) \leq c_{13}\left|D_{L}\right|^{8 n^{3}} Q_{S}^{20 n^{3}} e^{50 n^{4} d \widehat{h}}
$$

where $c_{13}=(4 n s)^{212 n^{4} s}$.
Theorem 4.5.2. All solutions $x, y$ of equation (4.5.2) satisfy

$$
h(x), h(y) \leq c_{14}\left|D_{L}\right|^{2 m^{2} n^{2}} Q_{S}^{3 m^{2} n^{2}} e^{8 m^{2} n^{3} d \widehat{h}},
$$

where $c_{14}=(6 n s)^{14 m^{3} n^{3} s}$.
Finally, consider the Schinzel-Tijdeman equation

$$
\begin{equation*}
f(x)=\delta y^{m} \quad \text { in } \quad x, y \in \mathcal{O}_{S}, \quad m \in \mathbb{Z}_{\geq 2} \tag{4.5.4}
\end{equation*}
$$

where $m \geq 2$ is also unknown.

Theorem 4.5.3. All solutions of (4.5.4) such that $y \neq 0$ and $y$ is not a root of unity satisfy

$$
m \leq c_{15}\left|D_{L}\right|^{6 n} P_{S}^{n^{2}} e^{11 n d \widehat{h}},
$$

where $c_{15}=\left(10 n^{2} s\right)^{40 n s}$.
The above theorems are Theorems 2.1, 2.2 and 2.3 in Bérczes, Evertse and Győry (2013).

Let $f(X)$ be as in Theorem 4.5.1 (when $n \geq 3$ ) or in Theorem 4.5.2 (when $n \geq 2$ ) with non-zero discriminant, and $G$ the splitting field of $f$ over $L$. Then

$$
f(X)=a_{0}\left(X-\alpha_{1}\right) \cdots\left(X-\alpha_{n}\right) \quad \text { with } \quad \alpha_{1}, \ldots, \alpha_{n} \in G .
$$

Let $M_{i}=L\left(\alpha_{i}\right)$ and denote by $T_{i}$ the set of places of $M_{i}$ lying above the places of $S$ and by $\mathcal{O}_{T_{i}}$ the ring of $T_{i}$-integers in $M_{i}, i=1, \ldots, n$.

We state without proof the following crucial lemma.
Lemma 4.5.4. Let $x, y \in \mathcal{O}_{S}$ be a solution of (4.5.1) (when $m=2$ ) or (4.5.2) (when $m \geq 3$ ) with $y \neq 0$. Then for $i=1, \ldots, n$ there are $\gamma_{i}, \xi_{i}$ with

$$
x-\alpha_{i}=\gamma_{i} \xi_{i}^{m}, \quad \xi_{i} \in \mathcal{O}_{T_{i}}, \quad \gamma_{i} \in M_{i}^{*}
$$

and

$$
h\left(\gamma_{i}\right) \leq c_{16} e^{2 n d \widehat{h}}\left|D_{L}\right|^{n}\left(80(d n)^{d n+2}+\frac{1}{d} \log Q_{S}\right),
$$

where $c_{16}=m\left(n^{3} d\right)^{n d}$.
For a proof, see (ii) in Lemma 4.2 of Bérczes, Evertse and Győry (2013). The first version of this lemma in a less general and ineffective form was proved by Siegel (1926), and in a less general but effective form by Baker (1969). The proof uses the ideal factorization of $\left[x-\alpha_{i}\right]$ in $\mathcal{O}_{T_{i}}$, estimates for the class number of $M_{i}$ and an analogue of Proposition 4.1.9 for $\mathcal{O}_{T_{i}}$.

Below we sketch proofs of less explicit versions of Theorems 4.5.1, 4.5.2, 4.5.3. We first prove the following weaker version of Lemma 4.5.4:

Let $x, y \in \mathcal{O}_{S}$ be as in the statement of Lemma 4.5.4 Thenfor $i=1, \ldots, n$ there are $\gamma_{i}, \xi_{i}$ with

$$
\begin{equation*}
x-\alpha_{i}=\gamma_{i} \xi_{i}^{m}, \quad \xi_{i} \in \mathcal{O}_{T_{i}} . \gamma_{i} \in M_{i}^{*}, h\left(\gamma_{i}\right) \ll_{n, L, S} m \cdot c(n, d)^{\widehat{h}} . \tag{4.5.5}
\end{equation*}
$$

Here and in the remainder of this section, $c(n, d)$ will denote effectively computable numbers exceeding 1 that depend only on $n$ and $d$, and that may be different at each occurrence.

Proof. It suffices to prove (4.5.5) for $i=1$. We write $M, T$ for $M_{1}, T_{1}$. We denote by $\left[\beta_{1}, \ldots, \beta_{r}\right]$ the fractional ideal of $\mathcal{O}_{T}$ generated by $\beta_{1}, \ldots, \beta_{r} \in$ $M$. Let $g(X)=\left(X-a_{0} \alpha_{2}\right) \cdots\left(X-a_{0} \alpha_{n}\right)=a_{0}^{n} f\left(X / a_{0}\right) /\left(X-a_{0} \alpha_{1}\right)$. Then $g \in \mathcal{O}_{T}[X]$. We first prove that there are ideals $\mathfrak{C}, \mathfrak{A}$ of $\mathcal{O}_{T}$ such that

$$
\begin{equation*}
\left[a_{0}\left(x-\alpha_{1}\right)\right]=\mathfrak{C} \cdot \mathfrak{A}^{m}, \quad \mathfrak{C} \supseteq\left[\left(\delta a_{0} g\left(a_{0} \alpha_{1}\right)\right)^{m-1}\right] . \tag{4.5.6}
\end{equation*}
$$

First observe that there is $h \in \mathcal{O}_{T}[X]$ such that

$$
g(X)-g\left(a_{0} \alpha_{1}\right)=\left(X-a_{0} \alpha_{1}\right) h(X) .
$$

Substituting $a_{0} x$ for $X$ we infer

$$
\begin{equation*}
g\left(a_{0} \alpha_{1}\right) \in\left[a_{0}\left(x-\alpha_{1}\right), g\left(a_{0} x\right)\right] . \tag{4.5.7}
\end{equation*}
$$

Further,

$$
\begin{equation*}
a_{0}^{n} \delta y^{m}=a_{0}\left(x-\alpha_{1}\right) g\left(a_{0} x\right) . \tag{4.5.8}
\end{equation*}
$$

For a prime ideal $\mathfrak{P}$ of $\mathcal{O}_{T}$ (or equivalently, a prime ideal of $\mathcal{O}_{K}$ outside $T$ ), denote by $\operatorname{ord}_{\mathfrak{P}}(\beta)$ the exponent of $\mathfrak{P}$ in the prime ideal factorization of $[\beta]$. For every prime ideal of $\mathcal{O}_{T}$, write $\operatorname{ord}_{\mathfrak{P}}\left(a_{0}\left(x-\alpha_{1}\right)\right)=v_{\mathfrak{F}}+m w_{\mathfrak{P}}$ with $v_{\mathfrak{P}}$, $w_{\mathfrak{F}}$ non-negative integers and $0 \leq v_{\mathfrak{F}} \leq m-1$. The above two relations imply that $v_{\mathfrak{P}}=0$ for those prime ideals $\mathfrak{P}$ that do not divide $\delta a_{0} g\left(a_{0} \alpha_{1}\right)$. Now clearly, 4.5.6 holds with

$$
\mathfrak{C}=\prod_{\mathfrak{P} \mid \delta a_{0} g\left(a_{0} \alpha_{1}\right)} \mathfrak{P}^{v_{\mathfrak{Y}}}, \quad \mathfrak{A}=\prod_{\mathfrak{P}} \mathfrak{P}^{w_{\mathfrak{Y}}} .
$$

We now proceed to prove (4.5.5). An ideal $\mathfrak{B}$ of $\mathcal{O}_{T}$ can be expressed uniquely as $\mathfrak{B}^{*} \mathcal{O}_{T}$, where $\mathfrak{B}^{*}$ is an ideal of $\mathcal{O}_{K}$ composed of prime ideals outside $T$. We define $N_{T} \mathfrak{B}:=\left|\mathcal{O}_{K} / \mathfrak{B}^{*}\right|$. For instance by Lang (1994, pp. 119,120 ), there is non-zero $\xi \in \mathfrak{A}^{*}$ with $\left|N_{L / \mathbb{Q}}(\xi)\right| \leq\left|D_{M}\right|^{1 / 2}\left|\mathcal{O}_{K} / \mathfrak{A}^{*}\right|$. This translates into $N_{T}(\xi) \leq\left|D_{M}\right|^{1 / 2} N_{T} \mathfrak{A}$, i.e., $[\xi]=\mathfrak{B A}$ where $\mathfrak{B}$ is an ideal of $O_{T}$ with $N_{T} \mathfrak{B} \leq\left|D_{M}\right|^{1 / 2}$. Similarly, there exists $\gamma \in L$ with $[\gamma]=\mathfrak{D C}$,
where $\mathfrak{D}$ is an ideal of $O_{T}$ with $N_{T} \mathfrak{D} \leq\left|D_{M}\right|^{1 / 2}$. As a consequence, we have

$$
a_{0}(x-\alpha)=\frac{\gamma}{\gamma^{\prime}} \xi^{m}
$$

where $\gamma, \gamma^{\prime} \in O_{T}$, and

$$
\left[\gamma^{\prime}\right]=\mathfrak{D} \mathfrak{B}^{m}
$$

Using (4.5.6) and the choice of $\mathfrak{B}, \mathfrak{D}$, we get

$$
N_{T}(\gamma) \leq\left|D_{M}\right|^{1 / 2} N_{T}\left(\delta a_{0} g\left(a_{0} \alpha_{1}\right)\right)^{m-1}, \quad N_{T}\left(\gamma^{\prime}\right) \leq\left|D_{M}\right|^{(m+1) / 2}
$$

By Lemma 4.1.11 we have $\left|D_{M}\right|<_{L} c(n, d)^{\widehat{h}}$. Combined with (4.1.3), (4.1.14), this shows

$$
N_{T}(\gamma), N_{T}\left(\gamma^{\prime}\right)<_{L, n, S} c(n, d)^{m \widehat{h}}
$$

By (4.1.11) we have $R_{M}<_{L} c(n, d)^{\widehat{h}}$. Using this together with Proposition 4.1.9, we infer that there are $T$-units $\eta, \eta^{\prime} \in O_{T}^{*}$ such that

$$
h\left(\gamma \eta^{m}\right), h\left(\gamma^{\prime} \eta^{\prime m}\right)<_{n, L, S} m \cdot c(n, d)^{\widehat{h}}
$$

Putting

$$
\gamma_{1}:=a_{0}^{-1} \gamma \gamma^{\prime-1}\left(\eta \eta^{\prime-1}\right)^{m}, \quad \xi_{1}=\eta^{\prime} \eta^{-1} \xi
$$

we obtain $x-\alpha_{1}=\gamma_{1} \xi_{1}^{m}$, where $\gamma_{1} \in M^{*}$ with $h\left(\gamma_{1}\right)<_{n, L, S} m \cdot c(n, d)^{\widehat{h}}$ and $\xi_{1} \in \mathcal{O}_{T}$. This proves (4.5.5).

We now sketch the proof of the following weaker version of Theorem 4.5.2

All solutions $x, y \in \mathcal{O}_{S}$ of (4.5.2) with $y \neq 0$ satisfy

$$
\begin{equation*}
\max \{h(x), h(y)\}<_{m, n, L, S} c(n, d)^{m^{2} \widehat{h}} \tag{4.5.9}
\end{equation*}
$$

Sketch of the proof of (4.5.9). Let $m \geq 3$ and let $x, y \in \mathcal{O}_{S}$ be a solution of $f(x)=\delta y^{m}$ with $y \neq 0$. Then we have $x-\alpha_{i}=\gamma_{i} \xi_{i}^{m}$ with $\gamma_{i}, \xi_{i}$ as in (4.5.5), $i=1, \ldots, n$. Let $N=L\left(\alpha_{1}, \alpha_{2}, \sqrt[m]{\gamma_{1} / \gamma_{2}}, \varrho\right)$, where $\varrho$ is a primitive $m$-th root of unity. Let $T$ be the set of places of $N$, lying above the places from $S$, and $\mathcal{O}_{T}$ the ring of $T$-integers in $N$.

Then

$$
\begin{equation*}
\gamma_{1} \xi_{1}^{m}-\gamma_{2} \xi_{2}^{m}=\alpha_{2}-\alpha_{1}, \quad \xi_{1}, \xi_{2} \in \mathcal{O}_{T} \tag{4.5.10}
\end{equation*}
$$

The left-hand side is a binary form in $\xi_{1}, \xi_{2}$ with non-zero discriminant which splits into linear factors over $N$. We are going to apply Theorem 4.4.1 with $N, T$ instead of $L, S$. Notice that $[N: L] \leq m^{2} n^{2}$. A repeated application of Lemma 4.1.11 gives a discriminant estimate

$$
\left|D_{N}\right|<_{m, n, L, S} c(n, d)^{m^{2} \widehat{h}}
$$

and together with (4.1.11), (4.1.13), this implies the estimates

$$
h_{N}, R_{T}<_{m, n, L, S} c(n, d)^{m^{2} \widehat{h}}
$$

By applying Theorem 4.4.1 to (4.5.10) and inserting these bounds, we get

$$
\begin{equation*}
h\left(\xi_{1}\right) \ll_{m, n, L, S} c(n, d)^{m^{2} \widehat{h}} \max \left(H_{1}, H_{2}\right), \tag{4.5.11}
\end{equation*}
$$

where

$$
H_{1}:=\max \left\{1, h\left(\gamma_{1}\right), h\left(\gamma_{2}\right)\right\}, \quad H_{2}:=\max \left\{1, h\left(\alpha_{2}-\alpha_{1}\right)\right\} .
$$

Further, we have $H_{2}<_{n} \widehat{h}$ and by (4.5.5),

$$
\begin{equation*}
H_{1}<_{m, n, L, S} c(n, d)^{\widehat{h}} . \tag{4.5.12}
\end{equation*}
$$

Using
$h(x) \leq \log 2+h\left(\alpha_{1}\right)+h\left(\gamma_{1}\right)+m h\left(\xi_{1}\right), \quad h(y) \leq m^{-1}(h(\delta)+h(f)+n h(x))$
and combining these with (4.5.11) and (4.5.12), inequality (4.5.9) follows.

In the proof of Theorem 4.5.1 we shall use the following.
Lemma 4.5.5. Let $\gamma_{1}, \gamma_{2}, \gamma_{3}, \beta_{12}, \beta_{13}$ be non-zero elements of $L$ such that

$$
\begin{aligned}
& \quad \beta_{12} \neq \beta_{13}, \quad \sqrt{\gamma_{1} / \gamma_{2}}, \quad \sqrt{\gamma_{1} / \gamma_{3}} \in L \\
& h\left(\gamma_{i}\right) \leq H_{1} \quad \text { for } \quad i=1,2,3, \quad h\left(\beta_{12}\right), h\left(\beta_{13}\right) \leq H_{2} .
\end{aligned}
$$

Then for the solutions $x_{1}, x_{2}, x_{3}$ of the system of equations

$$
\begin{equation*}
\gamma_{1} x_{1}^{2}-\gamma_{2} x_{2}^{2}=\beta_{12}, \quad \gamma_{1} x_{1}^{2}-\gamma_{3} x_{3}^{2}=\beta_{13} \quad \text { in } \quad x_{1}, x_{2}, x_{3} \in \mathcal{O}_{S} \tag{4.5.13}
\end{equation*}
$$

we have

$$
\begin{align*}
& \max _{1 \leq i \leq 3} h\left(x_{i}\right) \\
& \quad \leq c_{17} P_{S} R_{S}\left(1+\frac{\log ^{*} R_{S}}{\log ^{*} P_{S}}\right)\left(R_{L}+\frac{h_{L}}{d} \log Q_{S}+d H_{1}+H_{2}\right) \tag{4.5.14}
\end{align*}
$$

where $c_{17}=s^{2 s+4} 2^{7 s+60} d^{2 s+d+2}$.
Proof. This is Proposition 3.12 of Bérczes, Evertse and Győry (2013). The idea of the proof is to reduce (4.5.13) to the decomposable form equation

$$
F\left(x_{1}, x_{2}, x_{3}\right)=\delta \quad \text { in } \quad x_{1}, x_{2}, x_{3} \in \mathcal{O}_{S},
$$

where

$$
\delta=\beta_{12} \beta_{13} \beta_{23} \quad \text { with } \quad \beta_{23}=\beta_{13}-\beta_{12}
$$

and

$$
F\left(X_{1}, X_{2}, X_{3}\right)=\prod_{1 \leq i<j \leq 3}\left(\gamma_{i} X_{i}^{2}-\gamma_{j} X_{j}^{2}\right) .
$$

Here $F$ is a decomposable form of degree 6 with splitting field $L$, whose linear factors form a triangularly connected system; see Section 2.6. To this equation one can apply Theorem 3 of Győry and Yu (2006), which is a quantitative number field version of Theorem 2.6.1, to obtain (4.5.14). In Section 4.7 we have stated a special case of this result of Győry and Yu (Theorem 4.7.1) and at the end of Section 4.7 we have included a sketch of its proof.

We sketch now the proof of the following less explicit version of Theorem 4.5.1.

All solutions $x, y \in \mathcal{O}_{S}$ of equation (4.5.1) with $y \neq 0$ satisfy

$$
\begin{equation*}
\max \{h(x), h(y)\}<_{n, L, S} c(n, d)^{\widehat{h}} . \tag{4.5.15}
\end{equation*}
$$

Sketch of the proof of (4.5.15). Let $x, y \in \mathcal{O}_{S}$ be a solution of the equation $f(x)=\delta y^{2}$ with $y \neq 0$. Then we have $x-\alpha_{i}=\gamma_{i} \xi_{i}^{2}(i=1, \ldots, n)$ with the $\gamma_{i}, \xi_{i}$ as in (4.5.5). Let

$$
N=L\left(\alpha_{1}, \alpha_{2}, \alpha_{3}, \sqrt{\gamma_{1} / \gamma_{3}}, \sqrt{\gamma_{2} / \gamma_{3}}\right)
$$

let $T$ be the set of places of $N$ lying above the places from $S$, and let $\mathcal{O}_{T}$ be
the ring of $T$-integers in $N$. Then

$$
\begin{equation*}
\gamma_{1} \xi_{1}^{2}-\gamma_{2} \xi_{2}^{2}=\alpha_{2}-\alpha_{1}, \quad \gamma_{1} \xi_{1}^{2}-\gamma_{3} \xi_{3}^{2}=\alpha_{3}-\alpha_{1}, \quad \xi_{1}, \xi_{2} \in \mathcal{O}_{T} \tag{4.5.16}
\end{equation*}
$$

Notice that $[N: L] \leq 4 n^{3}$. Further, a repeated application of Lemma 4.1.11 gives $\left|D_{N}\right|<_{n, L, S} c(n, d)^{\widehat{h}}$ and together with (4.1.11), 4.1.13), this implies the estimates

$$
h_{N}, R_{T}<_{n, L, S} c(n, d)^{\widehat{h}} .
$$

By applying Lemma 4.5.5 to 4.5.16) with $N, T$ instead of $L, S$, inserting the estimates for $h_{N}, R_{T}$ and following the same computations as above, (4.5.15) follows.

Finally, we shall sketch the proof of a less explicit version of Theorem 4.5 .3

Let $L, S$ be as above, let $f(X) \in \mathcal{O}_{S}[X]$ be a polynomial of degree $n \geq$ 2 with non-zero discriminant and consider the Schinzel-Tijdeman equation (4.5.4), where both $x, y \in \mathcal{O}_{S}$ and $m \geq 2$ are unknowns.

All solutions $x, y \in \mathcal{O}_{S}, m \in \mathbb{Z}_{\geq 3}$ of (4.5.4) such that $y \neq 0$ and $y$ is not a root of unity satisfy

$$
\begin{equation*}
m<_{n, L, S} c(n, d)^{\widehat{h}} . \tag{4.5.17}
\end{equation*}
$$

We start with some preliminaries and a lemma. Let again $f(X)=a_{0}(X-$ $\left.\alpha_{1}\right) \cdots\left(X-\alpha_{n}\right)$. For $i=1, \ldots, n$, let $M_{i}=L\left(\alpha_{i}\right)$, and denote by $d_{M_{i}}, h_{M_{i}}, R_{M_{i}}$ the degree, class number and regulator of $M_{i}$. Further, let $T_{i}$ be the set of places of $M_{i}$ lying above the places in $S$ and denote by $t_{i}$ the cardinality of $T_{i}$ and by $R_{T_{i}}$ the $T_{i}$-regulator of $M_{i}$. By Lemma 4.1.11 and (4.1.11), (4.1.13) we have the estimates

$$
\begin{equation*}
h_{M_{i}}, R_{T_{i}}<{ }_{n, L, S} c(n, d)^{\widehat{h}} . \tag{4.5.18}
\end{equation*}
$$

The group of $T_{i}$-units is finitely generated and, by Proposition 4.1.8, we can choose a fundamental system of $T_{i}$-units $\eta_{i, 1}, \ldots, \eta_{i, t_{i}-1}$ such that

$$
\begin{equation*}
\prod_{j=1}^{t_{i}-1} h\left(\eta_{i, j}\right), \max _{1 \leq j \leq t_{i}-1} h\left(\eta_{i, j}\right)<_{n, L, S} c(n, d)^{\widehat{h}} . \tag{4.5.19}
\end{equation*}
$$

Lemma 4.5.6. Let $x, y \in \mathcal{O}_{S}$ and $m \geq 3$ be a solution of equation (4.5.4)
such that $y \neq 0$ and $y$ is not a root of unity. Then for $i=1,2$ there are $\gamma_{i}, \xi_{i} \in M_{i}^{*}$, and integers $b_{i, 1}, \ldots, b_{i, t_{i}-1}$ of absolute value at most $m / 2$ such that

$$
\begin{align*}
& \left(x-\alpha_{i}\right)^{h_{M_{1}} h_{M_{2}}}=\eta_{i, 1}^{b_{i, 1}} \cdots \eta_{i, t_{i}-1}^{b_{i, t}-1} \gamma_{i} \xi_{i}^{m}  \tag{4.5.20}\\
& h\left(\gamma_{i}\right) \leq\left(2 n^{3} s\right)^{6 n s}\left|D_{L}\right|^{2 n} e^{4 n d \widehat{h}}\left(\widehat{h}+\log ^{*} P_{S}\right) \tag{4.5.21}
\end{align*}
$$

This is Lemma 5.1 in Bérczes, Evertse and Győry (2013). We prove the following less precise result with instead of (4.5.21) the estimate

$$
\begin{equation*}
h\left(\gamma_{i}\right)<_{n, L, S} c(n, d)^{\widehat{h}} \text { for } i=1,2 \tag{4.5.22}
\end{equation*}
$$

Proof. We prove this only for $i=1$. We write $M, T$ for $M_{1}, T_{1}$ and use again [.] to denote ideals in $\mathcal{O}_{T}$. Let again $g(X)=\left(X-a_{0} \alpha_{1}\right) \cdots\left(X-a_{0} \alpha_{n}\right)$. By (4.5.8), (4.5.7) we have for every prime ideal $\mathfrak{P}$ of $\mathcal{O}_{T}$,

$$
\begin{aligned}
& \operatorname{ord}_{\mathfrak{P}}\left(a_{0}^{n} \delta\right)+\operatorname{mord}_{\mathfrak{P}}(y)=\operatorname{ord}_{\mathfrak{P}}\left(a_{0}\left(x-\alpha_{1}\right)\right)+\operatorname{ord}_{\mathfrak{P}}\left(g\left(a_{0} x\right)\right), \\
& \operatorname{ord}_{\mathfrak{P}}\left(a_{0}\left(x-\alpha_{1}\right)\right) \leq \operatorname{ord}_{\mathfrak{P}}\left(g\left(a_{0} \alpha_{1}\right)\right), \text { or } \operatorname{ord}_{\mathfrak{P}}\left(g\left(a_{0} x\right)\right) \leq \operatorname{ord}_{\mathfrak{P}}\left(g\left(a_{0} \alpha_{1}\right)\right),
\end{aligned}
$$

hence
$\operatorname{ord}_{\mathfrak{P}}\left(a_{0}\left(x-\alpha_{1}\right)\right)=v+m w$ with $v, w \in \mathbb{Z},|v| \leq \operatorname{ord}_{\mathfrak{P}}\left(\delta a_{0}^{n} g\left(a_{0} \alpha_{1}\right)\right), w \geq 0$.
It follows that there are ideals $\mathfrak{C}_{1}, \mathfrak{C}_{2}, \mathfrak{A}$ of $\mathcal{O}_{T}$ such that

$$
\left[a_{0}\left(x-\alpha_{1}\right)\right]=\mathfrak{C}_{1} \mathfrak{C}_{2}^{-1} \mathfrak{A}^{m}, \mathfrak{C}_{1}, \mathfrak{C}_{2} \supseteq\left[a_{0}^{n} \delta g\left(a_{0} \alpha_{1}\right)\right] .
$$

Raising to the $h_{M_{1}} h_{M_{2}}$-th power to make all ideals principal, and invoking (4.5.18), we infer that there are $\theta_{1}, \theta_{2}, \xi \in \mathcal{O}_{T}$ such that

$$
\begin{align*}
& {\left[\left(a_{0}\left(x-\alpha_{1}\right)\right)^{h_{M_{1}} h_{M_{2}}}\right]=\left[\theta_{1} \theta_{2}^{-1} \xi^{m}\right]} \\
& \log N_{T}\left(\theta_{j}\right) \leq h_{M_{1}} h_{M_{2}} \log N_{T}\left(a_{0}^{n} \delta g\left(a_{0} \alpha_{1}\right)\right)<_{n, L, S} c(n, d)^{\widehat{h}} \text { for } j=1,2 . \tag{4.5.23}
\end{align*}
$$

Proposition 4.1.9 and again (4.5.18) imply that for $j=1,2$ there is $\varepsilon_{j} \in \mathcal{O}_{T}^{*}$ such that $\theta_{j}^{\prime}:=\varepsilon_{j} \theta_{j}$ satisfies

$$
h\left(\theta_{j}^{\prime}\right)<_{n, L, S} c(n, d)^{\widehat{h}} .
$$

Combined with (4.5.23), this gives

$$
\left(a_{0}\left(x-\alpha_{1}\right)\right)^{h_{M_{1}} h_{M_{2}}}=\eta \theta_{1}^{\prime} \theta_{2}^{\prime-1} \xi^{m}
$$

with $\eta \in \mathcal{O}_{T}^{*}$. By the $S$-unit Theorem, we can express $\eta$ as

$$
\eta=\zeta \eta_{1,1}^{b_{1,1}} \cdots \eta_{1, t_{1}-1}^{b_{1, t_{1}-1}} \cdot \varepsilon^{m}
$$

with $\zeta$ a root of unity, $b_{1, k}\left(k=1, \ldots, t_{1}-1\right)$ integers of absolute value at most $m / 2$, and $\varepsilon \in \mathcal{O}_{T}^{*}$. Now (4.5.20), (4.5.22) hold with $\gamma_{1}:=\zeta \theta_{1}^{\prime} \theta_{2}^{\prime-1}, \xi_{1}:=\varepsilon \xi$,

Sketch of the proof of (4.5.17). Let $x, y \in \mathcal{O}_{S}$ and $m \geq 3$ be a solution of equation (4.5.4) such that $y \neq 0$ and $y$ is not a root of unity. We assume that

$$
\begin{equation*}
h(x) \gg_{n, L, S} c(n, d)^{\widehat{h}} . \tag{4.5.24}
\end{equation*}
$$

This is no loss of generality for if $h(x)<_{n, L, S} c(n, d)^{\widehat{h}}$, then (4.5.4) and Lemma 4.1.2 imply 4.5.17). In the course of our proof, the constant $c(n, d)$ in 4.5.24) will be chosen sufficiently large to make all our estimates work.

Let $M=L\left(\alpha_{1}, \alpha_{2}\right), d_{M}=[M: \mathbb{Q}], T$ the set of places of $M$ lying above the places from $S$ and $t$ the cardinality of $T$. Put

$$
\Lambda:=1-\left(\frac{x-\alpha_{1}}{x-\alpha_{2}}\right)^{h_{M_{1}} h_{M_{2}}} .
$$

By (4.1.14), there is $w \in T$ such that $\log \left|x-\alpha_{2}\right|_{w} \geq \frac{d_{M}}{t} h\left(x-\alpha_{2}\right)$. Now a combination of Lemma 4.1.1, (4.5.18) and (4.5.24) gives

$$
\begin{align*}
\log |\Lambda|_{w} & <_{n, L, S}\left(h_{K_{1}} h_{K_{2}}\right)^{2}+\log \left|1-\frac{x-\alpha_{1}}{x-\alpha_{2}}\right|_{w} \\
& <_{n, L, S} c(n, d)^{\widehat{h}}+\log \frac{\left|\alpha_{1}-\alpha_{2}\right|_{w}}{\left|x-\alpha_{2}\right|_{w}}<_{n, L, S}(-h(x)) . \tag{4.5.25}
\end{align*}
$$

To obtain a lower bound for $\log |\Lambda|_{w}$ we substitute the identity

$$
\left(\frac{x-\alpha_{1}}{x-\alpha_{2}}\right)^{h_{K_{1}} h_{K_{2}}}=\frac{\gamma_{1}}{\gamma_{2}} \eta_{1,1}^{b_{1,1}} \cdots \eta_{1, t_{1}-1}^{b_{1, t_{1}-1}} \cdot \eta_{2,1}^{-b_{2,1}} \cdots \eta_{2, t_{2}-1}^{-b_{2, t}-1}\left(\frac{\xi_{1}}{\xi_{2}}\right)^{m}
$$

from (4.5.20) and then apply Proposition 4.2.4. Notice that by 4.5.19) and
(4.5.22) we have

$$
\begin{aligned}
& h\left(\xi_{1} / \xi_{2}\right) \lll n, L, S \\
& m^{-1}\left(h\left(\gamma_{1} / \gamma_{2}\right)+c(n, d)^{\widehat{h}} m+h\left(\frac{x-\alpha_{1}}{x-\alpha_{2}}\right)\right) \\
& \ll n, L, S \\
& c(n, d)^{\hat{h}}\left(1+m^{-1} h(x)\right) .
\end{aligned}
$$

By inserting this, as well as the bounds from (4.5.22) and (4.5.19) into the lower bound from Proposition 4.2.4, we obtain

$$
\begin{equation*}
\log |\Lambda|_{w} \gg_{n, L, S}\left(-c(n, d)^{\widehat{h}}\left(1+m^{-1} h(x)\right) \log m\right) . \tag{4.5.26}
\end{equation*}
$$

Comparing (4.5.25) and (4.5.26) and combining this with (4.5.24), estimate (4.5.17) easily follows.

### 4.6 The Catalan equation

Consider now the Catalan equation in the following generalized form

$$
\begin{align*}
& x^{m} \pm y^{n}=1 \quad \text { in } x, y \in \mathcal{O}_{S}, m, n \in \mathbb{Z} \\
& \quad \text { with } x, y \text { not roots of unity and } m, n>1, m n>4, \tag{4.6.1}
\end{align*}
$$

where again $S$ is a finite set of places, containing all infinite places, of a number field $L$. As was mentioned in Section 2.5, in the classical case $L=\mathbb{Q}$, $\mathcal{O}=\mathbb{Z}$, Tijdeman (1976) proved that the solutions $x, y, m, n$ are bounded above by an effectively computable absolute constant. His proof relies on Baker's theory of logarithmic forms.

Brindza, Győry and Tijdeman (1986) generalized Tijdeman's proof for equations (4.6.1) where $\mathcal{O}_{S}=\mathcal{O}_{L}$ is the ring of integers of $L$. They showed that in this case the heights of the solutions of (4.6.1) can be effectively bounded above by a number which depends only on $L$. Brindza (1987) further generalized this to equations (4.6.1) where $S$ is an arbitary finite set of places. The following theorem, due to Koymans (2016), is a more explicit version of Brindza's result. Again, $d$ denotes the degree of $L, s$ denotes the cardinality of $S$, the quantities $P_{S}, Q_{S}$ are defined by (4.1.9), and $R_{S}$ denotes the $S$-regulator.

Theorem 4.6.1. Suppose that in (4.6.1) $m$ and $n$ are primes. Then all solu-
tions of (4.6.1) satisfy

$$
\max \{m, n\}<\left(s P_{S}^{2}\right)^{c_{1}^{*} s P_{S}}\left|D_{L}\right|^{6 P_{S}} P_{S}^{P_{S}^{2}}=: C
$$

where $c_{1}^{*}$ is an effectively computable positive absolute constant, and

$$
\max \{h(x), h(y)\}<(C \cdot s)^{C^{6}}\left(\left|D_{L}\right| Q_{S}\right)^{C^{4}}
$$

Furthermore, if in 4.6.1 $m$ and $n$ are arbitrary positive integers, we have

$$
\max \{m, n\}<(C \cdot s)^{C^{6}}\left(\left|D_{L}\right| Q_{S}\right)^{C^{4}}
$$

The proof of Theorem4.6.1 is a generalization of the proof given for ordinary rings of integers in Brindza, Győry and Tijdeman (1986). The main tools are Propositions 4.2.2 and 4.2.3 from Baker's theory of logarithmic forms, while also Theorems 4.3.1, 4.5.1 and 4.5.2, Proposition 4.1.8 play an important role.

Following the proof of Koymans, we shall sketch the proof of the following less precise version of Theorem 4.6.1. In all statements below, constants implied by the Vinogradov symbols $<_{L, S}$ and $>_{L, S}$ are effectively computable and depend only on $L$ and $S$.

Suppose that in 4.6.1) $m$ and $n$ are prime. Then all solutions of (4.6.1) satisfy

$$
\begin{equation*}
\max \{m, n\}<_{L, S} 1 \tag{4.6.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\max \{h(x), h(y)\}<_{L, S} 1 \tag{4.6.3}
\end{equation*}
$$

Furthermore, if in 4.6.1) $m$ and $n$ are arbitrary positive integers, we have

$$
\begin{equation*}
\max \{m, n\}<_{L, S} 1 \tag{4.6.4}
\end{equation*}
$$

We mention here that inequality (4.6.3) follows at once from (4.6.2) and Theorems 4.5.1 and 4.5.2. Further, if $m, n$ are arbitrary positive integers instead of just primes, we pick prime divisors $m^{\prime}, n^{\prime}$ of $m$ and $n$, respectively, estimate $m^{\prime}$ and $n^{\prime}$ by means of (4.6.2), then estimate $h\left(x^{\prime}\right), h\left(y^{\prime}\right)$ where $x^{\prime}:=x^{m / m^{\prime}}$ and $y^{\prime}:=y^{n / n^{\prime}}$ by means of 4.6.3), and subsequently obtain
(4.6.4) by applying Voutier's inequality Lemma 4.1.2. So it suffices to prove (4.6.2) under the assumption that $m$ and $n$ are primes. This will be assumed henceforth.

Another important ingredient of the proof of Theorem 4.6.1 is the following result of Koymans (2017), which is of interest in itself.

Lemma 4.6.2. If $p, x_{1}, x_{2}, y$ are such that $p$ is a prime, $x_{1}, x_{2} \in \mathcal{O}_{S}^{*}, y \in \mathcal{O}_{S}$ with $y \neq 0$ and $y \notin \mathcal{O}_{S}^{*}$ and

$$
\begin{equation*}
x_{1}+x_{2}=y^{p} \tag{4.6.5}
\end{equation*}
$$

then

$$
\begin{equation*}
p \leq(2 s)^{c_{2}^{*} s} P_{S}^{2} R_{S}^{4} \tag{4.6.6}
\end{equation*}
$$

with an effectively computable positive absolute contant $c_{2}^{*}$.
Proof. This is Theorem 4.2 in Koymans (2017). It is a generalization of Lemma 6 in Brindza, Győry and Tijdeman (1986). Koymans' proof is a more modern and simplified version of that of Theorem 9.3 in Shorey and Tijdeman (1986). We briefly sketch a proof of the weaker inequality

$$
\begin{equation*}
p \lll L, S 1, \tag{4.6.7}
\end{equation*}
$$

which is in fact sufficient for (4.6.2).
Below, we write $\ll, \gg$ for $<_{L, S},>_{L, S}$. Choose a fundamental system of $S$-units $\left\{\eta_{1}, \ldots, \eta_{s-1}\right\}$ of $\mathcal{O}_{S}^{*}$, where $s:=|S|$. Let $x_{1}, x_{2}, y, p$ be as in Lemma 4.6.2. Then

$$
x_{1}=\zeta_{1} \eta_{1}^{a_{1}} \cdots \eta_{s-1}^{a_{s-1}}, \quad x_{2}=\zeta_{2} \eta_{1}^{b_{1}} \cdots \eta_{s-1}^{b_{s-1}}
$$

where $\zeta_{1}, \zeta_{2}$ are roots of unity and the $a_{i}, b_{i}$ are integers. We may and shall assume that

$$
\begin{equation*}
0 \leq b_{i}<p \text { for } i=1, \ldots, s-1 \tag{4.6.8}
\end{equation*}
$$

indeed, these inequalities can be achieved after multiplying $x_{1}, x_{2}, y^{p}$ with a suitable $p$-th power of an $S$-unit. From this assumption together with the lower bound for $h(y)$ arising from Lemma 4.1.2 we deduce that for $v \in S$,

$$
\left.\left|\sum_{i=1}^{s-1} a_{i} \log \right| \eta_{i}\right|_{v}\left|=|\log | x_{1}\right|_{v} \mid \ll p \max \left(1,\left.|\log | y\right|_{v} \mid\right) \ll p h(y),
$$

and then, on multiplying the vector $\left(a_{1}, \ldots, a_{s-1}\right)$ with the inverse of any $(s-1) \times(s-1)$-submatrix of $\left(\log \left|\eta_{i}\right|_{v}\right)_{i=1, \ldots, s-1, v \in S}$,

$$
\max \left(\left|a_{1}\right|, \ldots,\left|a_{s-1}\right|\right) \ll p h(y)
$$

The next step is to derive upper and lower bounds for $\left|x_{2} y^{-p}\right|_{v}=\left|1-x_{1} y^{-p}\right|_{v}$ for $v \in S$. Here we assume without loss of generality that $p>P_{S}$ (the maximum of the norms of the finite places in $S$ ). First suppose that $v$ is an infinite place. Then by Proposition 4.2.2 with $n=s, A_{s}^{\prime} \ll h(y), b_{s}=p$, $A_{i}^{\prime} \ll 1, b_{i} \ll p h(y)$ for $i<s$,

$$
\begin{align*}
\log \left|x_{2} y^{-p}\right|_{v} & =\log \left|1-x_{1} y^{-p}\right|_{v}=\log \left|1-\zeta_{1} \eta_{1}^{a_{1}} \cdots \eta_{s-1}^{a_{s-1}} y^{-p}\right|_{v} \\
& \gg(y) \log p \tag{4.6.9}
\end{align*}
$$

If $v$ is finite, then $|p|_{v}=1$ since $p$ is a prime exceeding $P_{S}$. Now we can apply Proposition 4.2.3 with $A_{n}^{\prime \prime} \ll h(y), \delta=\frac{1}{2}, B_{n}=p, B \ll p h(y)$, and obtain again (4.6.9). By (4.1.14) there is $v \in S$ such that $\log |y|_{v} \geq \frac{d}{s} h(y)$. Then by (4.6.9) we have

$$
\log \left|x_{2}\right|_{v}-\frac{d}{s} p h(y) \gg-h(y) \log p,
$$

while $\log \left|x_{2}\right|_{v} \ll p$ by 4.6.8). Assuming $p \gg 1$ as we may, we infer $h(y) \ll$ 1. Since $x_{2} \in \mathcal{O}_{S}^{*}$ we have $\prod_{v \in S}\left|x_{2}\right|_{v}=1$, and since $y \in \mathcal{O}_{S}, y \neq 0$ and $y \notin \mathcal{O}_{S}^{*}$ we have $\prod_{v \in S}|y|_{v} \geq 2$. Using these inequalities and summing (4.6.9) over $v \in S$, we obtain

$$
-p \log 2 \gg-\log p,
$$

which implies (4.6.7).

Sketch of the proof of (4.6.2). The proof will be carried out in several steps. Again, constants implied by Vinogradov symbols $\ll, \gg$ will be effectively computable and depend only on $L, S$. We fix a solution $(x, y, m, n)$ of (4.6.1), with $m$ and $n$ primes.

## Step 1. Simplifications.

In view of Theorem 4.5.3 we may assume that $m, n$ are primes exceeding $P_{S}$ (the maximum of the norms of the prime ideals in $S$ ). If $x$ and $y$ are $S$-units, then we obtain $h\left(x^{m}\right), h\left(y^{n}\right) \ll 1$ from Theorem 4.3.1 and subsequently $m, n \ll 1$ from Lemma 4.1.2. If exactly one of $x, y$, say $x$, is an $S$-unit, then by (4.6.7) we have $n \ll 1$ and subsequently also $m \ll 1$ by The-
orem 4.5.3. So we may assume that neither of $x, y$ is an $S$-unit. This implies also that in the course of the proof we may assume that $h(x), h(y) \gg 1$. For by enlarging $S$ with a finite number of places we can achieve that all non-zero elements $z$ of $L$ with $h(z) \ll 1$ are $S$-units. So by the above arguments, if $h(x) \ll 1$ or $h(y) \ll 1$ then $m, n \ll 1$ follows. Lastly, we may assume that $m \neq n$. For suppose that $m=n$ is a prime. Then $u=x^{m}, v=-x y$ satisfy $u(u \pm 1)=v^{m}, v$ is non-zero and not an $S$-unit, and so $m \ll 1$ by Theorem 4.5.3.

So summarizing, it suffices to deal with the equation

$$
\begin{array}{ll}
x^{m}+y^{n}=1 \quad & \text { in } x, y \in \mathcal{O}_{S} \text { with } x, y \neq 0, x, y \notin \mathcal{O}_{S}^{*},  \tag{4.6.10}\\
& \text { and primes } m, n \text { with } m>n>P_{S} .
\end{array}
$$

Henceforth, $(x, y, m, n)$ will be a fixed solution of 4.6.10, and we may assume that $h(x), h(y)>C$ for any effectively computable constant depending on $L$ and $S$ of our choice.

Step 2. A special case.
Assume that

$$
\begin{equation*}
(x-1)^{m}+(y-1)^{n}=0 \tag{4.6.11}
\end{equation*}
$$

The exponent $n$ is not an $S$-unit, since $n>P_{S}$. Let $\mathfrak{q}$ be a prime ideal of $\mathcal{O}_{S}$ dividing $n$. Then (4.6.11) together with (4.6.10) implies $(x-1)^{m} \equiv x^{m}(\bmod \mathfrak{q})$. This shows that $x$ and $x-1$ are coprime with $n$. Moreover, $m$ divides the order of the unit group of $\mathcal{O}_{S} / \mathfrak{q}$, which is smaller than $n^{d}$.

Let $a, b$ be integers with $a m+b n=1$. Then by (4.6.11) we have $x=1+\varepsilon^{n}$ and $y=1-\varepsilon^{m}$ where $\varepsilon=(x-1)^{b}(1-y)^{a}$. Clearly, $\varepsilon \in \mathcal{O}_{S}$. A substitution into 4.6.10 gives

$$
\left(1+\varepsilon^{n}\right)^{m}+\left(1-\varepsilon^{m}\right)^{n}-1=0
$$

implying

$$
\begin{equation*}
1+\sum_{k=1}^{m-1}\binom{m}{k} \varepsilon^{n k}+\sum_{k=1}^{n-1}\binom{n}{k}(-1)^{k} \varepsilon^{m k}=0 \tag{4.6.12}
\end{equation*}
$$

As observed above, we may assume that $h(x), h(y)>3$, say, which implies that $\varepsilon$ is not a root of unity. Hence there is a place $v$ of $L$ with $|\varepsilon|_{v}<1$. This place $v$ cannot be finite since otherwise, the left-hand side of 4.6.12 would
have $v$-adic absolute value 1 . So $\varepsilon$ is an $S$-unit, and (4.1.15) implies that there is an infinite place $v$ of $K$ with $\log |\varepsilon|_{v} \leq-\frac{d}{s} h(\varepsilon)$. There is an embedding $\sigma: L \hookrightarrow \mathbb{C}$ such that $|\cdot|_{v}=|\sigma(\cdot)|^{s(v)}$ with $s(v)=1$ or 2 . So by Lemma 4.1.2, we have $|\sigma(\varepsilon)| \leq c(d)^{-1}$, where $c(d)>1$ is effectively computable and depends only on $d$. Using $m<n^{d}$ we obtain the following lower bound for the absolute value of the quantity obtained by applying $\sigma$ to the left-hand side of (4.6.12):

$$
1-\sum_{k=1}^{m-1} n^{d k} c(d)^{-n k}-\sum_{k=1}^{n-1} n^{k} c(d)^{-m k} .
$$

If $n \gg 1$ then $n^{d} c(d)^{-n}<\frac{1}{4}$, say, and this lower bound is strictly positive, contradicting (4.6.12). So under the hypothesis (4.6.11) we conclude $n \ll 1$ and then also $m \ll 1$ since $m<n^{d}$.

Having disposed of the case 4.6.11, we assume henceforth that

$$
\begin{equation*}
(x-1)^{m}+(y-1)^{n} \neq 0 . \tag{4.6.13}
\end{equation*}
$$

## Step 3. Ideal arithmetic.

We denote by $\left[\alpha_{1}, \ldots, \alpha_{k}\right]$ the fractional ideal of $\mathcal{O}_{S}$ generated by $\alpha_{1}, \ldots, \alpha_{k} \in$ $L$. Fix a system of fundamental $S$-units $\eta_{1}, \ldots, \eta_{s-1}$ of $\mathcal{O}_{S}^{*}$. Since

$$
[x-1] \cdot\left[\frac{x^{m}-1}{x-1}\right]=[y]^{n}, \quad\left[x-1, \frac{x^{m}-1}{x-1}\right] \supseteq[m],
$$

we have an ideal factorization $[x-1]=\mathfrak{a}_{1} \mathfrak{a}_{2}^{-1} \cdot \mathfrak{b}^{n}$, where $\mathfrak{a}_{1}, \mathfrak{a}_{2}, \mathfrak{b}$ are ideals of $\mathcal{O}_{S}$ with $\mathfrak{a}_{1}, \mathfrak{a}_{2} \supseteq[m]$. By raising to the $h_{L}$-th power we get a factorization $[x-1]^{h_{L}}=\left[\theta_{0}\right] \cdot[\omega]^{n}$ with $\omega \in \mathcal{O}_{S} \backslash\{0\}$ and $\theta_{0} \in L^{*}$ with numerator and denominator of $\theta_{0}$ dividing $m^{h_{L}}$. Using Propositions 4.1.8 and 4.1.9 one infers that $\theta_{0}$ and $\omega$ can be chosen in such a way, that

$$
\begin{equation*}
(x-1)^{h_{L}}=\eta_{1}^{u_{1}} \cdots \eta_{s-1}^{u_{s-1}} \theta_{0} \omega^{n}, \tag{4.6.14}
\end{equation*}
$$

where the $u_{i}$ are rational integers with $0 \leq u_{i}<n$ for $i=1, \ldots s-1$ and where $\omega \in \mathcal{O}_{S} \backslash\{0\}$ and $\theta_{0} \in K^{*}$ with

$$
\begin{equation*}
h\left(\theta_{0}\right) \ll \log m . \tag{4.6.15}
\end{equation*}
$$

Similarly, we can write

$$
\begin{equation*}
(1-y)^{h_{L}}=\eta_{1}^{v_{1}} \cdots \eta_{s-1}^{v_{s-1}} \tau_{0} \sigma^{m} \tag{4.6.16}
\end{equation*}
$$

with rational integers $v_{i}$ such that $0 \leq v_{i}<m$ for $i=1, \ldots, s-1$, and with $\sigma \in \mathcal{O}_{S} \backslash\{0\}$ and $\tau_{0} \in K^{*}$ such that

$$
\begin{equation*}
h\left(\tau_{0}\right) \ll \log n \tag{4.6.17}
\end{equation*}
$$

Step 4. First bounds for $m$ and $n$.
We show that

$$
\begin{align*}
& m \ll h(y) \log m,  \tag{4.6.18}\\
& n \ll h(x) \log m . \tag{4.6.19}
\end{align*}
$$

Notice that (4.6.19) follows from (4.6.18) via $|m h(x)-n h(y)| \ll 1$. We prove (4.6.18). Let $\Lambda_{1}:=1-\frac{(-y)^{n}}{x^{m}}=\frac{1}{x^{m}}$. By (4.1.14) there is $v \in S$ such that

$$
\log \left|\Lambda_{1}\right|_{v}=-m \log |x|_{v} \leq-\frac{m d}{s} h(x)
$$

while Proposition 4.2.4 produces a lower bound

$$
\log \left|\Lambda_{1}\right|_{v} \gg-h(x) h(y) \log m .
$$

By combining the two bounds we get (4.6.18).
Step 5. $A$ bound for $n$.
We now show that

$$
\begin{equation*}
n \ll(\log m)^{4} \tag{4.6.20}
\end{equation*}
$$

We assume without loss of generality that

$$
\begin{equation*}
n>(\log m)^{4}, \quad m \gg 1 \tag{4.6.21}
\end{equation*}
$$

Then by (4.6.19) we have

$$
\begin{equation*}
h(x) \gg(\log m)^{3} . \tag{4.6.22}
\end{equation*}
$$

We deal with the expression

$$
\Lambda_{2}:=\frac{(1-y)^{n h_{L}}}{(x-1)^{m h_{L}}}-1
$$

and derive an upper and lower bound for $\left|\Lambda_{2}\right|_{v}$ for appropriate $v$. We assume for the moment that $\Lambda_{2} \neq 0$. By (4.1.14) there is $v \in S$ such that

$$
\log |x|_{v} \geq \frac{d}{s} h(x)
$$

For this $v$ we have $\log |x|_{v} \gg(\log m)^{3}$. Now by Lemma 4.1.1, (4.6.10) and (4.6.22), we have

$$
\begin{aligned}
\log \left|\frac{(1-y)^{n h_{L}}}{y^{n h_{L}}}-1\right|_{v} & \ll \log n-\log |y|_{v} \ll \log n-\log |x|_{v} \\
& \ll \log m-\frac{d}{s} h(x) \ll-h(x) .
\end{aligned}
$$

Likewise,

$$
\log \left|\frac{x^{m h_{L}}}{(x-1)^{m h_{L}}}-1\right|_{v} \ll \log m-\log |x-1|_{v} \ll-h(x),
$$

and lastly, $\log \left|y^{n h_{L}} x^{-m h_{L}}-1\right|_{v} \ll-m \log |x|_{v} \ll-m h(x)$. So

$$
\begin{equation*}
\log \left|\Lambda_{2}\right|_{v}=\log \left|\frac{(1-y)^{n h_{L}}}{y^{n h_{L}}} \cdot \frac{x^{m h_{L}}}{(x-1)^{m h_{L}}} \cdot y^{n h_{L}} x^{-m h_{L}}-1\right|_{v} \ll-h(x) . \tag{4.6.23}
\end{equation*}
$$

By inserting (4.6.14), (4.6.16), using the inequalities (4.6.15), (4.6.17) and applying Proposition 4.2.4, we get
$\log \left|\Lambda_{2}\right|_{v}=\log \left|\eta_{1}^{n v_{1}-m u_{1}} \cdots \eta_{s-1}^{n v_{s-1}-m u_{s-1}} \tau_{0}^{n} \theta_{0}^{-m}(\sigma / \omega)^{m n}\right|_{v} \gg-(\log m)^{3} H_{0}$,
where $H_{0}:=\max (h(\sigma), h(\omega))$. From (4.6.14), (4.6.15), (4.6.21) and (4.6.19), we infer

$$
\begin{aligned}
h(\omega) & \ll n^{-1}\left(n+h\left(\theta_{0}\right)+h(x-1)\right) \ll n^{-1}(n+\log m+h(x)) \\
& \ll 1+\frac{h(x)}{n} \ll \frac{\log m}{n} \cdot h(x)
\end{aligned}
$$

while (4.6.16), 4.6.17) and 4.6.18) give $h(\sigma) \ll h(y) \cdot(\log m) / m \ll h(x)$. $(\log m) / n$. So altogether, $H_{0} \ll h(x) \cdot(\log m) / n$. Now a combination with (4.6.23) and (4.6.24) gives $n \ll(\log m)^{4}$.

We still have to deal with the case $\Lambda_{2}=0$. We now consider instead

$$
\Lambda_{3}:=\frac{(1-y)^{n}}{(x-1)^{m}}-1
$$

By (4.6.13) we have $\Lambda_{3} \neq 0$. By precisely the same argument as above we get (4.6.23) with $\Lambda_{3}$ instead of $\Lambda_{2}$. Further, $\Lambda_{3}=\zeta-1$ for a root of unity $\zeta \neq 1$ of $L$, so we certainly have the analogue of (4.6.24). Again we obtain $n \ll(\log m)^{4}$.

## Step 6. Finishing the proof.

Let

$$
\Lambda_{4}:=\frac{x^{m h_{L}}}{(1-y)^{n h_{L}}}-1 .
$$

Assume for the moment that $\Lambda_{4} \neq 0$. By (4.1.14), we can choose $v \in S$ such that $\log |y|_{v} \geq \frac{d}{s} h(y)$. By an argument similar as in Step 5 we get

$$
\begin{align*}
\log \left|\Lambda_{4}\right|_{v} & =\log \left|\frac{x^{m h_{L}}}{y^{n h_{L}}} \cdot \frac{y^{n h_{L}}}{(y-1)^{n h_{L}}}-1\right|_{v} \ll \log n-\log |y|_{v} \ll-h(y) \\
& \ll-\frac{m}{n} h(x) \tag{4.6.25}
\end{align*}
$$

By virtue of 4.6.16) we can write

$$
\Lambda_{4}=\eta_{1}^{d_{1}} \cdots \eta_{s-1}^{d_{s-1}} \tau_{0}^{-n}\left(\frac{x^{h_{L}}}{\sigma^{n}}\right)^{m}
$$

with rational integers $d_{i}$ such that $\left|d_{i}\right|<m n$ for $i=1, \ldots, s-1$. Notice that by (4.6.16), (4.6.17), (4.6.20) we have

$$
h\left(x^{h_{L}} \sigma^{-n}\right) \ll h(x)+\frac{n}{m}(h(1-y)+m+\log n) \ll h(x)+n .
$$

By Proposition 4.2.4, inserting this upper bound and 4.6.18, we obtain

$$
\log \left|\Lambda_{4}\right|_{v} \gg-(h(x)+n) \log n \log m
$$

Together with (4.6.25) and $n \ll(\log m)^{4}$ this implies $m \ll 1$.
We still have to deal with the case $\Lambda_{4}=0$, i.e., $x^{m h_{L}}=(1-y)^{n h_{L}}$. This
implies $\left(1-y^{n}\right)^{h_{L}}=(1-y)^{n h_{L}}$, whence

$$
1-y^{n}=\xi(1-y)^{n}
$$

where $\xi$ is a $h_{L}$-th root of unity in $L$. We show that $n \ll 1$. Suppose without loss of generality that $n>h_{L}$. Then in fact, $n$ is coprime with $h_{L}$, so $\xi=\zeta^{n}$ for some root of unity $\zeta$ of $L$, and thus,

$$
(\zeta(1-y))^{n}+y^{n}=1 .
$$

As observed in Step 1., this implies $n \ll 1$. Then, using the fact $g(y)=0$ for the polynomial $g(X):=(1-X)^{n h_{L}}-\left(1-X^{n}\right)^{h_{L}}$, one can deduce that $h(y) \ll 1$. Combined with 4.6.10) this gives $h\left(x^{m}\right) \ll 1$, and then an application of Lemma 4.1.2 finally leads to $m \ll 1$.

### 4.7 Decomposable form equations

Keeping the above notation, let again $L$ be a number field, and $S$ a finite set of places of $L$ containing the set $S_{\infty}$ of infinite places. Consider now the decomposable form equation

$$
\begin{equation*}
F(\mathbf{x})=\delta \quad \text { in } \quad \mathbf{x}=\left(x_{1}, \ldots, x_{m}\right) \in \mathcal{O}_{S}^{m} \tag{4.7.1}
\end{equation*}
$$

where $\delta \in L \backslash\{0\}$ and $F(\mathbf{X})=F\left(X_{1}, \ldots, X_{m}\right)$ is a decomposable form of degree $n \geq 3$ in $m \geq 2$ variables which factorizes into linear forms over $L$. As in Chapter 2, we write

$$
\begin{equation*}
F=\ell_{1} \cdots \ell_{n} \tag{4.7.2}
\end{equation*}
$$

where $\ell_{1}, \ldots, \ell_{n}$ are linear forms in the variables $X_{1}, \ldots, X_{m}$ with coefficients in $L$, and denote by $\mathcal{L}_{F}$ the system $\left(\ell_{1}, \ldots, \ell_{n}\right)$. Suppose that $\mathcal{L}_{F}$ has at least three pairwise non-proportional linear forms. Let $\mathcal{G}\left(\mathcal{L}_{F}\right)$ denote the triangular graph of $\mathcal{L}_{F}$ as defined by (2.6.4), i.e., $\mathcal{G}\left(\mathcal{L}_{F}\right)$ has vertex system $\mathcal{L}_{F}$, andh $\ell_{i}$ and $\ell_{j}$ with $i \neq j$ are connected by an edge if $\ell_{i}, \ell_{j}$ are linearly dependent over $L$ or they are linearly independent but there is $q \notin\{i, j\}$ such that $\lambda_{i} \ell_{i}+\lambda_{j} \ell_{j}+\lambda_{q} \ell_{q}=0$ for some non-zero $\lambda_{i}, \lambda_{j}, \lambda_{q} \in L$. Let $\mathcal{L}_{1}, \ldots, \mathcal{L}_{k}$ be the vertex systems of the connected components of $\mathcal{G}\left(\mathcal{L}_{F}\right)$, and denote by $\left[\mathcal{L}_{j}\right]$ the $L$-vector space spanned by the linear forms from $\mathcal{L}_{j}$. To be in accordance with earlier work to which we will refer, we suppose that $F$ in (4.7.1)
satisfies conditions slightly stronger than (2.6.5), i.e.,

$$
\begin{align*}
& \mathcal{L}_{F} \text { has rank } m,  \tag{4.7.3}\\
& X_{m} \in\left[\mathcal{L}_{1}\right] \cap \cdots \cap\left[\mathcal{L}_{k}\right] ; \tag{4.7.4}
\end{align*}
$$

of course (4.7.4) is automatically satisfied if (4.7.3) holds and $k=1$.
In the next theorem we use the earlier notation, i.e., $d, r, h_{L}$, and $R_{L}$ denote the degree, unit rank, class number and regulator of $L, s$ is the cardinality of $S, R_{S}$ the $S$-regulator of $L$ and $P_{S}$ and $Q_{S}$ the largest norm and the product of the norms of the prime ideals corresponding to the finite places in $S$, with the convention that $P_{S_{\infty}}=Q_{S_{\infty}}=1$. Recall that by the height of an algebraic number we always meant the absolute logarithmic height.

The following theorem was proved by Győry and Yu (2006) in a slightly more general form.

Theorem 4.7.1. Let $F \in L\left[X_{1}, \ldots, X_{m}\right]$ be a decomposable form of degree $n \geq 3$ that factorizes into linear forms over $L$ such that $\mathcal{L}_{F}$ satisfies the conditions (4.7.3), (4.7.4). Suppose that the logarithmic heights of the coefficients of the linear forms in $\mathcal{L}_{F}$ do not exceed $H_{1}(\geq 1)$. Further, let $\delta \in L \backslash\{0\}$ with logarithmic height at most $H_{2}(\geq 1)$. With the above notation, all solutions $\mathbf{x}=\left(x_{1}, \ldots, x_{m}\right) \in \mathcal{O}_{S}^{m}$ of (4.7.1) with $x_{m} \neq 0$ if $k>1$, satisfy

$$
\begin{align*}
& \max _{1 \leq i \leq m} h\left(x_{i}\right) \leq c_{18} P_{S} R_{S}\left(1+\frac{\log ^{*} R_{S}}{\log ^{*} P_{S}}\right) \times \\
& \times\left(c_{4} R_{L}+\frac{h_{L}}{d} \log Q_{S}+m n d H_{1}+H_{2}\right) \tag{4.7.5}
\end{align*}
$$

with $c_{18}=50 m(m+1)(n-1) c_{11}$, where $c_{4}, c_{11}$ denote the constants specified in Proposition 4.1.9 and Theorem 4.3.1

For $m=2$, this gives Theorem 4.4.1 on Thue equations with a somewhat different bound.

Recently, in terms of $S$ the bound in (4.7.5) has been improved in Győry (2019), replacing e.g. $P_{S}$ by $P_{S}^{\prime}$; see Section 4.3 . However, this does not play any role in our work.

Next suppose that $L$ is a finite normal extension of degree $\geq 3$ of a number field $K$. Let $\alpha_{1}=1, \alpha_{2}, \ldots, \alpha_{m}$ be linearly independent elements of $L$ over $K$ such that $K^{\prime}:=K\left(\alpha_{1}, \ldots, \alpha_{m}\right)$ is of degree $n \geq 3$ over $K$. Consider the
norm form equation

$$
\begin{equation*}
N_{K^{\prime} / K}\left(\alpha_{1} x_{1}+\cdots+\alpha_{m} x_{m}\right)=\delta \quad \text { in } \quad x_{1}, \ldots, x_{m} \in \mathcal{O}_{S}, \tag{4.7.6}
\end{equation*}
$$

where $N_{K^{\prime} / K}\left(\alpha_{1} X_{1}+\cdots+\alpha_{m} X_{m}\right)=\prod_{i=1}^{n} \ell^{(i)}(\mathbf{X})$, with $\ell^{(i)}(\mathbf{X})=\alpha_{1}^{(i)} X_{1}+$ $\cdots+\alpha_{m}^{(i)} X_{m}, i=1, \ldots, n$, the conjugates of $\ell(\mathbf{X})$ with respect to $K^{\prime} / K$. With the above notation, for $k>1$ Theorem 4.7.1 implies the following.

Corollary 4.7.2. Suppose that $\alpha_{m}$ is of degree $\geq 3$ over $K\left(\alpha_{1}, \ldots, \alpha_{m-1}\right)$, the heights of $\alpha_{2}, \ldots, \alpha_{m}$ do not exceed $H_{1}(\geq 1)$ and $h(\delta)$ is at most $H_{2}(\geq 1)$. Then all solutions $\left(x_{1}, \ldots, x_{m}\right) \in \mathcal{O}_{S}^{m}$ of equation (4.7.6) with $x_{m} \neq 0$ satisfy (4.7.5).

It is not difficult to show that under the conditions of Corollary 4.7.2 the norm form $N_{K^{\prime} / K}\left(\alpha_{1} X_{1}+\cdots+\alpha_{m} X_{m}\right)$ satisfies the conditions concerning $F$ in Theorem 4.7.1, see e.g. Győry (1981a) or the proof of Corollary 2.7.2. Hence Corollary 4.7.2 is a consequence of Theorem 4.7.1. Corollary 4.7.2 has a further consequence for equation (4.7.6), corresponding to Corollary 2.7.3.

Let now $1, \alpha_{1}, \ldots, \alpha_{m}$ be linearly independent elements of $L$ over $K$ with heights at most $H(\geq 1)$. Assume again that $K^{\prime}:=K\left(\alpha_{1}, \ldots, \alpha_{m}\right)$ is of degree $n \geq 3$ over $K$. In the discriminant form equation

$$
\begin{equation*}
D_{K^{\prime} / K}\left(\alpha_{1} x_{1}+\cdots+\alpha_{m} x_{m}\right)=\delta \quad \text { in } \quad x_{1}, \ldots, x_{m} \in \mathcal{O}_{S} \tag{4.7.7}
\end{equation*}
$$

the discriminant form $D_{K^{\prime} / K}\left(\alpha_{1} X_{1}+\cdots+\alpha_{m} X_{m}\right)$ satisfies the assumptions concerning $F$ in Theorem 4.7.1 with $k=1$ and with $n(n-1)$ in place of $n$. Further, the coefficients of the linear factors of the discriminant form have heights at most $2 H_{1}+\log 2$. Suppose again that $h(\delta)$ does not exceed $H_{2}(\geq 1)$.

Corollary 4.7.3. Under the above assumptions, all solutions of equation (4.7.7) satisfy (4.7.5) with $n$ replaced by $n(n-1)$ and $H_{1}$ by $2 H_{1}+\log 2$.

Similarly to Corollary 2.8.2, our Corollary 4.7.3 has applications to index form equations, integral elements of given discriminant and simple integral ring extensions of $\mathcal{O}_{S}$; for details see e.g. Gyôry (1981a,1998).

As was mentioned in Section 1.5, for equation (4.7.7) over $\mathbb{Z}$ the first effective result was obtained by Győry (1976) in quantitative form. This was extended by Győry and Papp (1977) to the case of rings of integers of number fields. For $S=S_{\infty}$ and $k=1$ Győry and Papp (1978), while for arbitrary $S$ and $k \geq 1$ Győry (1981a) proved the above results with weaker effective bounds. Improvements were later obtained among others by Bugeaud
and Győry (1996), Bugeaud (1998) for equation 4.7.6) and by Győry (1998), Győry and Yu (2006) and Győry (2019) for equation (4.7.1).

As was seen in Chapters 1 and 2 , equation (4.7.1) as well as (4.7.6), (4.7.7) can be extended to equations of the type

$$
\begin{equation*}
F\left(x_{1}, \ldots, x_{m}\right)=\eta \delta \quad \text { in } \quad \eta \in \mathcal{O}_{S}^{*}, x_{1}, \ldots, x_{m} \in \mathcal{O}_{S} \tag{4.7.8}
\end{equation*}
$$

where we may assume that $\delta$ and the coefficients of $F$ are contained in $\mathcal{O}_{S}$. If $\eta, \mathbf{x}_{0}=\left(x_{1}, \ldots, x_{m}\right)$ is a solution of (4.7.8) then so is $\varepsilon^{n} \eta, \varepsilon \mathbf{x}_{0}$ for any $S$-unit $\varepsilon$. However, for each solution $\eta, \mathbf{x}_{0}$ of (4.7.8) there is an $\varepsilon \in \mathcal{O}_{S}^{*}$ such that $\varepsilon \mathbf{x}_{0}$ is a solution of the equation corresponding to (4.7.1) with $\delta$ replaced by $\varepsilon^{n} \eta \delta$ whose height can be explicitly bounded above by Proposition 4.1.9. Then the above results give an explicit upper bound for $\max _{1 \leq i \leq m} h\left(\varepsilon x_{i}\right)$.

We now sketch the proof of the following less explicit version of Theorem 4.7.1.

Under the assumptions of Theorem 4.7.1 all solution $\mathbf{x}=\left(x_{1}, \ldots, x_{m}\right) \in$ $\mathcal{O}_{S}^{m}$ of equation (4.7.1) with $x_{m} \neq 0$ if $k>1$ satisfy

$$
\begin{equation*}
\max _{1 \leq i \leq m} h\left(x_{i}\right)<_{m, n, L, S} \max \left(H_{1}, H_{2}\right) . \tag{4.7.9}
\end{equation*}
$$

Sketch of the proof of (4.7.9). We set $H:=\max \left(H_{1}, H_{2}\right)$. Constants implied by $\ll$ will be effectively computable and depend on $m, n, L, S$ only. We frequently use the elementary height properties listed in (4.1.3) without mention.

Let $F, \delta$ be as in (4.7.1) and (4.7.2). Let the positive rational integer $a$ be the product of the denominators of the coefficients of $\ell_{1}, \ldots, \ell_{n}$ and put $\ell_{i}^{\prime}:=a \ell_{i}(i=1, \ldots, n), F^{\prime}:=\ell_{1}^{\prime} \cdots \ell_{n}^{\prime}$ and $\delta^{\prime}:=a^{n} \delta$. Then equation (4.7.1) is equivalent to

$$
\begin{equation*}
F^{\prime}(\mathrm{x})=\ell_{1}^{\prime}(\mathrm{x}) \cdots \ell_{n}^{\prime}(\mathrm{x})=\delta^{\prime} \tag{4.7.10}
\end{equation*}
$$

Further,
the coefficients of $\ell_{1}^{\prime}, \ldots, \ell_{n}^{\prime}$ and $\delta^{\prime}$ have logarithmic heights $\ll H$.

To the system $\mathcal{L}_{F^{\prime}}=\left(\ell_{1}^{\prime}, \ldots, \ell_{n}^{\prime}\right)$ we can attach a graph $\mathcal{G}\left(\mathcal{L}_{F^{\prime}}\right)$ similar to $\mathcal{G}\left(\mathcal{L}_{F}\right)$, and we denote the vertex systems of its connected components by $\mathcal{L}_{1}^{\prime}, \ldots, \mathcal{L}_{k}^{\prime}$. Then it satisfies conditions analogous to (4.7.3), (4.7.4).

Pick a solution $\mathrm{x}=\left(x_{1}, \ldots, x_{m}\right) \in \mathcal{O}_{S}^{m}$ of 4.7.1) or equivalently, (4.7.10)
with $x_{m} \neq 0$ if $k>1$. Let $\left\{\ell_{i}^{\prime}, \ell_{j}^{\prime}\right\}$ be an edge of $\mathcal{G}\left(\mathcal{L}_{F^{\prime}}\right)$. Assume that $\ell_{i}^{\prime}, \ell_{j}^{\prime}$ are linarly independent. Then there are $q \notin\{i, j\}$, and non-zero $\lambda_{i}^{\prime}, \lambda_{j}^{\prime}, \lambda_{q}^{\prime} \in L$, such that $\lambda_{i}^{\prime} \ell_{i}^{\prime}+\lambda_{j} \ell_{j}^{\prime}+\lambda_{q} \ell_{q}^{\prime}=0$. This leads to

$$
\begin{equation*}
\lambda_{i}^{\prime} \ell_{i}^{\prime}(\mathbf{x})+\lambda_{j}^{\prime} \ell_{j}^{\prime}(\mathbf{x})+\lambda_{q} \ell_{q}^{\prime}(\mathbf{x})=0 \tag{4.7.12}
\end{equation*}
$$

The coefficients $\lambda_{i}^{\prime}, \lambda_{j}^{\prime}, \lambda_{q}^{\prime}$ can be chosen as $2 \times 2$-determinants with entries from the coefficients of $\ell_{i}^{\prime}, \ell_{j}^{\prime}, \ell_{q}^{\prime}$. So by 4.7.11,

$$
h\left(\lambda_{i}^{\prime}\right), h\left(\lambda_{j}^{\prime}\right), h\left(\lambda_{q}^{\prime}\right) \ll H .
$$

Further, $\ell_{i}^{\prime}(\mathbf{x}), \ell_{l}^{\prime}(\mathbf{x}), \ell_{j}^{\prime}(\mathbf{x})$ divide $\delta^{\prime}$ in $\mathcal{O}_{S}$, so

$$
\log N_{S}\left(\ell_{i}^{\prime}(\mathbf{x})\right), \log N_{S}\left(\ell_{j}^{\prime}(\mathbf{x})\right), \log N_{S}\left(\ell_{q}^{\prime}(\mathbf{x})\right) \leq \log N_{S}\left(\delta^{\prime}\right) \ll H
$$

Now an application of (4.3.8) yields that there is $\varepsilon \in \mathcal{O}_{S}^{*}$ such that

$$
h\left(\varepsilon \ell_{i}^{\prime}(\mathbf{x})\right), h\left(\varepsilon \ell_{j}^{\prime}(\mathbf{x})\right) \ll H .
$$

In case that $\ell_{i}^{\prime}, \ell_{j}^{\prime}$ are linearly dependent this is trivially true; so this holds for each edge $\left\{\ell_{i}^{\prime}, \ell_{j}^{\prime}\right\}$ of $\mathcal{G}\left(\mathcal{L}_{F^{\prime}}\right)$. Thus, we obtain

$$
\begin{equation*}
h\left(\frac{\ell_{i}^{\prime}(\mathbf{x})}{\ell_{j}^{\prime}(\mathbf{x})}\right) \ll H \quad \text { for each edge }\left\{\ell_{i}^{\prime}, \ell_{j}^{\prime}\right\} \text { of } \mathcal{G}\left(\mathcal{L}_{F^{\prime}}\right) \tag{4.7.13}
\end{equation*}
$$

Now let $\ell_{i}^{\prime}, \ell_{j}^{\prime}$ belong to the same connected component of $\mathcal{G}\left(\mathcal{L}_{F^{\prime}}\right)$. Then there is a path from $\ell_{i}^{\prime}$ to $\ell_{j}^{\prime}$, i.e., a sequence of edges $\left\{\ell_{i}^{\prime}, \ell_{i_{1}}^{\prime}\right\},\left\{\ell_{i_{1}}^{\prime}, \ell_{i_{2}}^{\prime}\right\}, \ldots,\left\{\ell_{i_{t}}^{\prime}, \ell_{j}^{\prime}\right\}$ of $\mathcal{G}\left(\mathcal{L}_{F^{\prime}}\right)$. Taking $t$ minimal we have $t \leq n$. Writing

$$
\frac{\ell_{i}^{\prime}(\mathbf{x})}{\ell_{j}^{\prime}(\mathbf{x})}=\frac{\ell_{i}^{\prime}(\mathbf{x})}{\ell_{i_{1}}^{\prime}(\mathbf{x})} \cdot \frac{\ell_{i_{1}}^{\prime}(\mathbf{x})}{\ell_{i_{2}}^{\prime}(\mathbf{x})} \cdots \frac{\ell_{i_{t}}^{\prime}(\mathbf{x})}{\ell_{j}^{\prime}(\mathbf{x})}
$$

and invoking (4.7.13), we obtain

$$
\begin{array}{r}
h\left(\frac{\ell_{i}^{\prime}(\mathbf{x})}{\ell_{j}^{\prime}(\mathbf{x})}\right) \ll H \text { for each } \ell_{i}^{\prime}, \ell_{j}^{\prime}  \tag{4.7.14}\\
\quad \text { in the same connected component of } \mathcal{G}\left(\mathcal{L}_{F^{\prime}}\right) .
\end{array}
$$

For the moment, let $k>1$. We want to extend (4.7.14) to all pairs $\ell_{i}^{\prime}, \ell_{j}^{\prime}$, not necessarily belonging to the same connected component. Let $i \in\{1, \ldots, n\}$
and let $\mathcal{L}_{j}^{\prime}$ be the connected component of $\ell_{i}^{\prime}$. According to (4.7.4), we have

$$
X_{m}=\sum_{\ell_{u}^{\prime} \in \mathcal{L}_{j}^{\prime}} \gamma_{u} \ell_{u}^{\prime}
$$

with coefficients $\gamma_{u} \in L$. Taking such an expression with a minimal number of non-zero coefficients, the $\gamma_{u}$ are quotients of determinants of order at most $m$, whose entries are from the coefficients of the $\ell_{u}^{\prime} \in \mathcal{L}_{j}^{\prime}$. So by 4.7.11), $h\left(\gamma_{u}\right) \ll H$. Now we have a relation

$$
\frac{x_{m}}{\ell_{i}^{\prime}(\mathbf{x})}=\sum_{\ell_{u}^{\prime} \in \mathcal{L}_{j}^{\prime}} \gamma_{u} \frac{\ell_{u}^{\prime}(\mathbf{x})}{\ell_{i}^{\prime}(\mathbf{x})}
$$

and so by (4.7.14),

$$
h\left(\frac{x_{m}}{\ell_{i}^{\prime}(\mathbf{x})}\right) \ll H
$$

This holds for $i=1, \ldots, n$. Since we assumed $k>1$ we have $x_{m} \neq 0$. Thus we conclude

$$
h\left(\frac{\ell_{i}^{\prime}(\mathbf{x})}{\ell_{j}^{\prime}(\mathbf{x})}\right) \ll H \quad \text { for each } i, j \in\{1, \ldots, n\} .
$$

We proved this assuming $k>1$, but for $k=1$ this is immediate from (4.7.14). Now from (4.7.10) we infer

$$
\ell_{i}^{\prime}(\mathbf{x})^{n}=\delta^{\prime} \prod_{j=1}^{n} \frac{\ell_{i}^{\prime}(\mathbf{x})}{\ell_{j}^{\prime}(\mathbf{x})}
$$

and thus,

$$
\begin{equation*}
h\left(\ell_{i}^{\prime}(\mathbf{x})\right) \ll H \quad \text { for } i=1, \ldots, n . \tag{4.7.15}
\end{equation*}
$$

In view of (4.7.3) we may assume that $\ell_{1}^{\prime}, \ldots, \ell_{m}^{\prime}$ are linearly independent. By Cramer's rule, we can express each $x_{i}$ as a quotient $\Delta_{i} / \Delta$, where $\Delta$ is the determinant of order $m$ whose $j$-th column consists of the coefficients of $\ell_{j}^{\prime}$, for $j=1, \ldots, m$, and where $\Delta_{i}$ is obtained from $\Delta$ by replacing the $i$-th row of the latter by $\ell_{1}^{\prime}(\mathrm{x}), \ldots, \ell_{m}^{\prime}(\mathrm{x})$. Now (4.7.15) in combination with (4.7.11) gives (4.7.9). This completes our proof.

### 4.8 Discriminant equations

Let $K$ be a number field, $D_{K}$ its discriminant, $\mathcal{M}_{K}$ the set of places of $K, T$ a finite subset of $\mathcal{M}_{K}$ containing the set of infinite places $T_{\infty}$, and $L$ a finite normal extension of $K$ with the parameters $d, r, h_{L}$ and $R_{L}$ specified above. Let $S$ denote the set of extensions to $L$ of the places in $T$, with $s, P_{S}, Q_{S}$ and $R_{S}$ as in Section 4.1 .

If $f$ is a monic polynomial with coefficients in $\mathcal{O}_{T}$, the ring of $T$-integers of $K$, and $f^{\prime}(X)=f(X+a)$ for some $a \in \mathcal{O}_{T}$, then the discriminants $D(f), D\left(f^{\prime}\right)$ coincide. As before, such polynomials $f, f^{\prime}$ are called strongly $\mathcal{O}_{T}$-equivalent.

For a polynomial $P(X)=X^{n}+a_{1} X^{n-1}+\cdots+a_{n} \in K[X]$, we put

$$
h(P):=\frac{1}{[K: \mathbb{Q}]} \sum_{v \in \mathcal{M}_{K}} \log \max \left(1,\left|a_{1}\right|_{v}, \ldots,\left|a_{n}\right|_{v}\right) .
$$

From Theorem 4.7.1 one can deduce the following.
Theorem 4.8.1. Let $f \in \mathcal{O}_{T}[X]$ be a monic polynomial of degree $n \geq 2$ with zeros in $L$ such that

$$
\begin{equation*}
D(f)=\delta, \tag{4.8.1}
\end{equation*}
$$

where $\delta$ is a non-zero element of $K$ with height not exceeding $H(\geq 1)$. Then $f$ is strongly $\mathcal{O}_{T}$-equivalent to a polynomial $f^{\prime} \in \mathcal{O}_{T}[X]$ for which

$$
\begin{equation*}
h\left(f^{\prime}\right) \leq n\left((n+1) C+\log \left|D_{K}\right|+1\right) . \tag{4.8.2}
\end{equation*}
$$

Here

$$
C:=c_{19} P_{S} R_{S}\left(1+\log ^{*} R_{S} / \log ^{*} P_{S}\right)\left(c_{4} R_{L}+\frac{h_{L}}{d} \log Q_{S}+n^{3}+H\right)
$$

with $c_{19}=50 n^{3} c_{11}$, where $c_{4}, c_{11}$ denote the constants specified in Proposition 4.1.9 and Theorem 4.3.1.

This theorem is an improvement of Theorem 3 of Győry (1998). In the proof of Theorem 4.8.1 one can follow the arguments of the deduction of Theorem 3 from Theorem 1 of Győry (1998). We note that Theorem 4.8.1 could be directly deduced, with a slightly different bound, from Theorem 4.3.1 concerning $S$-unit equations.

Theorem 4.8.1 has several consequences.
Let again $K$ and $L$ be number fields with the above properties and parameters such that there is a number field $K^{\prime}$ with $K \subset K^{\prime} \subseteq L$ and with $n=\left[K^{\prime}: K\right] \geq 2$. Note that if $\xi \in K^{\prime}$, then every element $\xi^{\prime}$ of the $\mathcal{O}_{T}$-coset $\xi+\mathcal{O}_{T}=\left\{\xi+a: a \in \mathcal{O}_{T}\right\}$ satisfies $D_{K^{\prime} / K}\left(\xi^{\prime}\right)=D_{K^{\prime} / K}(\xi)$.

Corollary 4.8.2. Let $\delta$ be a non-zero element of $K$ with height at most $H(\geq$ 1). Then for every $\xi \in K^{\prime}$ such that

$$
\begin{equation*}
D_{K^{\prime} / K}(\xi)=\delta, \quad \xi \text { is integral over } \mathcal{O}_{T} \tag{4.8.3}
\end{equation*}
$$

there are $\xi^{\prime} \in K^{\prime}, a \in \mathcal{O}_{T}$ such that

$$
\begin{equation*}
h\left(\xi^{\prime}\right) \leq(n+1) C+\log \left|D_{K}\right|, \xi=\xi^{\prime}+a \tag{4.8.4}
\end{equation*}
$$

with the above $C$.
This is an improvement of Theorem 15 of Győry (1984b). It could be easily deduced from Theorem 4.8.1, but only with a slightly weaker bound. To obtain Corollary 4.8.2 in the present form it suffices to apply the proof of Theorem 4.8.1 to the minimal polynomial, say $f$, of the $\xi$ under consideration. Then $D(f)=\delta$, and following the proof of Theorem 4.8.1 it follows that $f$ is strongly $\mathcal{O}_{T}$-equivalent to a polynomial $f^{\prime}$ which has a zero $\xi^{\prime} \in \xi+\mathcal{O}_{T}$ such that (4.8.4) holds.

In the classical case $K=\mathbb{Q}, T=T_{\infty}$, the first effective results for equations (4.8.1) and 4.8 .3 ) were proved by Györy $(1973,1974)$ in quantitative forms, without fixing the splitting field of the polynomials $f$ resp. the number field $L$ containing the algebraic numbers $\xi$. For general $K$ and $T$, and various applications, see Győry $(1976,1978 b, 1981 \mathrm{c}, 1984 \mathrm{~b}, 1998)$ and Evertse and Győry (2017).

As was mentioned in Section 1.6, the following more general versions of equation (4.8.1) in $f$ and equation (4.8.3) in $\xi$ have also important applications:

$$
\begin{align*}
D(f) \in \delta \mathcal{O}_{T}^{*} & \text { in monic } f \in \mathcal{O}_{T}[X] \text { of degree } n \geq 2 \\
& \text { having all its zeros in } \mathcal{O}_{S} \tag{4.8.5}
\end{align*}
$$

and

$$
\begin{equation*}
D_{K^{\prime} / K}(\xi) \in \delta \mathcal{O}_{T}^{*} \quad \text { in } \xi \in K^{\prime}, \text { integral over } \mathcal{O}_{T}, \tag{4.8.6}
\end{equation*}
$$

where $\mathcal{O}_{T}^{*}$ is the unit group of $\mathcal{O}_{T}$.
We recall that the monic polynomials $f, f^{\prime} \in \mathcal{O}_{T}[X]$ of degree $n$ are called $\mathcal{O}_{T}$-equivalent if $f^{\prime}(X)=\varepsilon^{n} f\left(\varepsilon^{-1} X+a\right)$ for some $\varepsilon \in \mathcal{O}_{T}^{*}, a \in \mathcal{O}_{T}$. If $f$ satisfies (4.8.5), so does $f^{\prime}$. Using Proposition 4.1.9, equation (4.8.5) can be reduced to finitely many equations of the form (4.8.1). Then by means of Theorem 4.8.1 one can prove that each $\mathcal{O}_{T}$-equivalence class of solutions of (4.8.5) has a representative with explicitly bounded height. A similar effective result can be proved for equation (4.8.6) by using Corollary 4.8.2. In this case two elements $\xi, \xi^{\prime}$ of $K^{\prime}$, integral over $\mathcal{O}_{T}$, are said to be $\mathcal{O}_{T}$-equivalent if $\xi^{\prime}=\varepsilon \xi+a$ with some $\varepsilon \in \mathcal{O}_{T}^{*}, a \in \mathcal{O}_{T}$. This latter result has an important application to the equation

$$
\begin{equation*}
\mathcal{O}=\mathcal{O}_{T}[\xi] \quad \text { in } \xi \in K^{\prime}, \text { integral over } \mathcal{O}_{T}, \tag{4.8.7}
\end{equation*}
$$

where $\mathcal{O}$ denotes the integral closure of $\mathcal{O}_{T}$ in $K^{\prime}$. If 4.8.7) holds for some $\xi \in \mathcal{O}$ and $\xi^{\prime}$ is $\mathcal{O}_{T}$-equivalent to $\xi$ then $\mathcal{O}=\mathcal{O}_{T}\left[\xi^{\prime}\right]$. The above-mentioned result concerning (4.8.6 implies in an effective and quantitative form that every $\xi$ satisfying (4.8.7) is $\mathcal{O}_{T}$-equivalent to an $\xi^{\prime} \in \mathcal{O}_{T}$ whose height can be explicitly bounded. For these and related results, see Győry (1981b,1984b) and Evertse and Győry (2017a).

We sketch the proof of the following less explicit version of Theorem 4.8.1

Under the assumption of Theorem 4.8.1 every solution $f$ of equation (4.8.1) is strongly $\mathcal{O}_{T}$-equivalent to a monic polynomial $f^{\prime} \in \mathcal{O}_{T}[X]$ for which

$$
\begin{equation*}
h\left(f^{\prime}\right)<_{n, K, L, S} H . \tag{4.8.8}
\end{equation*}
$$

In the proof of 4.8.8 we use the next division with remainder lemma. Denote by $\mathcal{O}_{K}$ the ring of integers of $K$.
Lemma 4.8.3. Let $n \geq 2$ be an integer and let $\beta \in \mathcal{O}_{T}$. Then there is an $\alpha \in \mathcal{O}_{K}$ such that

$$
\alpha \equiv \beta(\bmod n)
$$

and

$$
h(\alpha) \leq \log \left([K: \mathbb{Q}] \cdot n\left|D_{K}\right|^{1 / 2}\right) .
$$

Proof. This is a special case of Lemma 6 of Evertse and Győry (1991).
Sketch of the proof of (4.8.8). Assume that $\alpha_{1}, \ldots, \alpha_{n}$ are the zeros of $f$ in
$L$. Denote by $\mathcal{O}_{S}$ the ring of $S$-integers in $L$. Writing $x_{i}=\alpha_{i}-\alpha_{1}$ for $i=$
$1, \ldots, n$, we have $\alpha_{i} \in \mathcal{O}_{S}$ and $x_{i} \in \mathcal{O}_{S}$. Further, putting

$$
F\left(X_{2}, \ldots, X_{n}\right)=X_{2} \cdots X_{n} \prod_{2 \leq i<j \leq n}\left(X_{i}-X_{j}\right),
$$

(4.8.1) implies

$$
\begin{equation*}
F\left(x_{2}, \ldots, x_{n}\right)= \pm \delta_{0} \text { with } x_{2}, \ldots, x_{n} \in \mathcal{O}_{S} \tag{4.8.9}
\end{equation*}
$$

where $\delta_{0} \in \mathcal{O}_{S} \backslash\{0\}$ and $\delta_{0}^{2}=\delta$. We have $h\left(\delta_{0}\right) \leq \frac{1}{2} H$. The decomposable form $F$ is of degree $n(n-1) / 2$, and it is easy to verify that for $n \geq 3$ it satisfies the assumptions of Theorem 4.7.1 with $k=1$. Hence by the less precise version (4.7.9) we deduce from (4.8.9) that both for $n=2$ and for $n \geq 3$

$$
\begin{equation*}
\max _{2 \leq i \leq n} h\left(x_{i}\right)<_{n, L, S} H=C^{\prime} \tag{4.8.10}
\end{equation*}
$$

holds.
The sum $a_{0}=\alpha_{1}+\cdots+\alpha_{n}$ is contained in $\mathcal{O}_{T}$. Setting $\beta=-\left(x_{1}+\right.$ $\cdots+x_{n}$ ), it follows from (4.8.10) that $h(\beta) \leq(n-1 / 2) C^{\prime}$. Further, we have $n \alpha_{1}-a_{0}=\beta$. By Lemma 4.8.3 there is an $a_{1} \in \mathcal{O}_{K}$ such that $a_{1} \equiv$ $a_{0}(\bmod n)$ in $\mathcal{O}_{T}$ and $h\left(a_{1}\right) \leq \log \left(d n\left|D_{K}\right|^{1 / 2}\right)$. Set $\alpha_{1}^{\prime}=\left(\beta+a_{1}\right) / n$. Then $h\left(\alpha_{1}^{\prime}\right) \leq(n-1 / 3) C^{\prime}+\log \left|D_{K}\right|$. Further, $\alpha_{1}=a+\alpha_{1}^{\prime}$ with some $a \in \mathcal{O}_{T}$ and $\alpha_{1}^{\prime} \in \mathcal{O}_{S}$. Finally, with the notation $\alpha_{i}^{\prime}=x_{i}+\alpha_{1}^{\prime}$ we get $\alpha_{i}^{\prime}=\alpha_{i}-a$ and

$$
h\left(\alpha_{i}^{\prime}\right) \leq(n+1) C^{\prime}+\log \left|D_{K}\right| \quad \text { for } i=1, \ldots, n .
$$

Put $f^{\prime}(X):=\prod_{i=1}^{n}\left(X-\alpha_{i}\right)$. Then $f^{\prime}(X)=f(X+a)$, while Corollary 4.1.5 gives our height estimate 4.8.8).

## Chapter 5

## Effective results over function fields

As was mentioned above, $S$-unit equations, Thue-equations, hyper- and superelliptic equations and the Catalan equation over finitely generated domains will be reduced in Chapter 9 to equations of the same type over number fields and over function fields. In this chapter we formulate the best bounds to date for the heights of the integral solutions of the reduced equations over function fields, and sketch the main ideas of their proofs. In contrast with the number field case, these bounds in the function field case do not imply the finiteness of the number of solutions.

### 5.1 Notation and preliminaries

We start with some notation and definitions and with recalling some preliminary results over function fields. For further details we refer to Mason (1984) and Evertse and Győry (2015), Chapter 2.

Let $\mathbb{k}$ be an algebraically closed field of characteristic 0 and $K$ an algebraic function field in one variable over $\mathbb{k}$, that is, a finitely generated extension of $\mathbb{k}$ of transcendence degree 1 . Recall that $K$ is a finite extension of the rational function field $\mathbb{k}(z)$ for any $z \in K \backslash \mathbb{k}$. By a valuation on $K$ over $\mathbb{k}$ we mean a normalized, discrete valuation on $K$ that is trivial on $\mathbb{k}$, that is, a
surjective map $v: K \mapsto \mathbb{Z} \cup\{\infty\}$ such that

$$
\begin{aligned}
v(\alpha) & =\infty \Longleftrightarrow \alpha=0 \\
v(\alpha \beta) & =v(\alpha)+v(\beta), \quad v(\alpha+\beta) \geq \min (v(\alpha), v(\beta)) \quad \text { for } \quad \alpha, \beta \in K ; \\
v(\alpha) & =0 \quad \text { for } \quad \alpha \in \mathbb{k}^{*} .
\end{aligned}
$$

We denote by $\mathcal{M}_{K}$ the set of valuations on $K$ over $\mathbb{k}$.
For a finite extension $L$ of $K$, we say that a valuation $w$ on $L$ lies above a valuation $v$ on $K$, notation $w \mid v$, if the restriction of $w$ to $K$ is a multiple of $v$. In this case, we have $w(\alpha)=e_{w \mid v} v(\alpha)$ for $\alpha \in K$, where $e_{w \mid v}$ is a positive integer, called the ramification index of $w$ over $v$.

The valuations over $\mathbb{k}$ on the field of rational functions $\mathbb{k}(z)$, with $z$ transcendental over $\mathbb{k}$, can be described easily as follows. Let $z_{a}:=z-a$ if $a \in \mathbb{k}$, and $z_{\infty}:=z^{-1}$. For every $a \in \mathbb{k} \cup\{\infty\}$, we may expand $\alpha \in \mathbb{k}(z)$ as a formal Laurent series $\sum_{m=n(\alpha)}^{\infty} a_{m}(\alpha) z_{a}^{m}$ with $a_{m}(\alpha) \in \mathbb{k}$ for all $m$ and $a_{n(\alpha)}(\alpha) \neq 0$. Then $\operatorname{ord}_{a}$ defined by $\operatorname{ord}_{a}(\alpha):=n(\alpha)$ defines a valuation on $\mathbb{k}(z)$. In particular, $\operatorname{ord}_{\infty}(\alpha)=-\operatorname{deg}(\alpha)$ for $\alpha \in \mathbb{k}[z]$. The valuations $\operatorname{ord}_{a}$ $(a \in \mathbb{k} \cup\{\infty\})$ provide all valuations on $\mathbb{k}(z)$ over $\mathbb{k}$.

Let $K$ be a function field in one variable, and $L$ a finite extension of $K$. Clearly, every valuation on $L$ lies above a valuation on $K$. We explain how to construct, for a given valuation $v$ on $K$, the valuations on $L$ that lie above $v$. As a special case, we may choose $z \in K \backslash \mathbb{k}$, take the valuations on $\mathbb{k}(z)$ just described, and construct from these the valuations on $K$.

Let $v$ be a valuation on $K$. Take a local parameter $z_{v} \in K$ of $v$, i.e., with $v\left(z_{v}\right)=1$. Then the completion $K_{v}$ of $K$ at $v$ is (up to isomorphism) just the field of Laurent series $\mathbb{k}\left(\left(z_{v}\right)\right)$, and the algebraic closure $\overline{K_{v}}$ of $K_{v}$ is the field of Puiseux series in $z_{v}$, i.e., $\sum_{m=n}^{\infty} a_{m} z_{v}^{m / e}$ with $n \in \mathbb{Z}, e \in \mathbb{Z}_{>0}$ and $a_{m} \in \mathbb{k}$ for all $m \geq n$, where for every positive integer $e$ we have fixed an $e$-th root of $z_{v}$. There are precisely $[L: K] K$-invariant isomorphic embeddings $L \hookrightarrow \overline{K_{v}}$, given by

$$
\begin{equation*}
\alpha \mapsto \sum_{m=n_{i}(\alpha)}^{\infty} a_{i m}(\alpha) \zeta_{i}^{j m} z_{v}^{m / e_{i}}\left(i=1, \ldots, g, j=0, \ldots, e_{i}-1\right), \tag{5.1.1}
\end{equation*}
$$

where $e_{1}, \ldots, e_{g}$ are positive integers with $e_{1}+\cdots+e_{g}=[L: K], \zeta_{i}$ is some fixed $e_{i}$-th root of unity, $a_{i m}(\alpha) \in \mathbb{k}$ for all $m \geq n_{i}(\alpha)$, and $a_{i, n_{i}(\alpha)} \neq 0$. We can now define for $i=1, \ldots, g$ a valuation $w_{i}$ on $L$ by $w_{i}(\alpha):=n_{i}(\alpha)$ for $\alpha \in L^{*}$ and $w_{i}(0)=\infty$. These are precisely the valuations on $L$ lying above
$v$. Writing $w$ for $w_{i}$, the integer $e_{i}$ is just the ramification index $e_{w \mid v}$. Thus, we have

$$
\begin{equation*}
\sum_{w \mid v} e_{w \mid v}=[L: K] \text { for } v \in \mathcal{M}_{K}, \tag{5.1.2}
\end{equation*}
$$

where the sum is taken over all valuations on $L$ lying above $v$. We easily deduce from this the sum formula for $K$,

$$
\begin{equation*}
\sum_{v \in \mathcal{M}_{K}} v(\alpha)=0 \quad \text { for } \alpha \text { in } K^{*} . \tag{5.1.3}
\end{equation*}
$$

Indeed, this is clear if $\alpha \in \mathbb{k}^{*}$. Let $\alpha \in K^{*} \backslash \mathbb{k}^{*}$. By using the above description of the valuations on $\mathbb{k}(\alpha)$, with $\alpha$ replacing $z$, one easily gets $\sum_{u \in \mathcal{M}_{\mathbb{k}(\alpha)}} u(\alpha)=$ 0 , and together with (5.1.2), with $\mathbb{k}(\alpha), K$ instead of $K, L$, identity (5.1.3) follows.

Denote by $g_{K / \mathbb{k}}$ the genus of $K$ over $\mathbb{k}$. In case that $z$ is transcendental over $\mathbb{k}$ one has

$$
\begin{equation*}
g_{\mathbf{k}(z) / \mathbb{k}}=0 . \tag{5.1.4}
\end{equation*}
$$

We can relate the genus of a finite extension $L$ of $K$ to that of $K$ by means of the Riemann-Hurwitz formula

$$
\begin{equation*}
2 g_{L / \mathbb{k}}-2=[L: K]\left(2 g_{K / \mathbb{k}}-2\right)+\sum_{v \in \mathcal{M}_{K}} \sum_{w \mid v}\left(e_{w \mid v}-1\right) . \tag{5.1.5}
\end{equation*}
$$

Let $S$ be a finite subset of $\mathcal{M}_{K}$. We call $\alpha \in K$ an $S$-integer if $v(\alpha) \geq 0$ for all $v \in \mathcal{M}_{K} \backslash S$, and an $S$-unit if $v(\alpha)=0$ for all $v \in \mathcal{M}_{K} \backslash S$. The $S$ integers form a ring in $K$, denoted by $\mathcal{O}_{S}$, and the $S$-units a multiplicative group, denoted by $\mathcal{O}_{S}^{*}$.

We define the height $H_{K}(\alpha)$ of $\alpha \in K$ relative to $K / \mathbb{k}$ by

$$
H_{K}(\alpha):=-\sum_{v \in \mathcal{M}_{K}} \min (0, v(\alpha)) .
$$

It is clear that $H_{K}(\alpha) \geq 0$ for $\alpha \in K$ and $H_{K}(\alpha)=0$ if and only if $\alpha \in \mathbb{k}$. If $L$ is a finite extension of $K$ then by (5.1.2),

$$
H_{L}(\alpha)=[L: K] \cdot H_{K}(\alpha) \text { for } \alpha \in K .
$$

From the description of the valuations on a field of rational functions one easily deduces that $H_{\mathbb{k}(\alpha)}(\alpha)=1$ if $\alpha \notin \mathbb{k}$. So in particular,

$$
\begin{equation*}
H_{K}(\alpha)=[K: \mathbb{k}(\alpha)] \text { for } \alpha \in K \backslash \mathbb{k} . \tag{5.1.6}
\end{equation*}
$$

It is easy to see that

$$
\begin{equation*}
H_{K}\left(\alpha^{m}\right)=|m| H_{K}(\alpha) \quad \text { for } \alpha \in K^{*}, m \in \mathbb{Z} \tag{5.1.7}
\end{equation*}
$$

(where for negative $m$ one has to employ the sum formula) and

$$
\begin{equation*}
H_{K}(\alpha+\beta) \leq H_{K}(\alpha)+H_{K}(\beta), \quad H_{K}(\alpha \beta) \leq H_{K}(\alpha)+H_{K}(\beta) \tag{5.1.8}
\end{equation*}
$$

for all $\alpha, \beta \in K$. Further

$$
\begin{align*}
H_{K}(\alpha) & =\frac{1}{2}\left(H_{K}(\alpha)+H_{K}\left(\alpha^{-1}\right)\right) \\
& =\frac{1}{2} \sum_{v \in \mathcal{M}_{K}}|v(\alpha)| \geq \frac{1}{2}|T| \quad \text { for } \alpha \in K^{*}, \tag{5.1.9}
\end{align*}
$$

where $T$ denotes the set of valuations $v \in \mathcal{M}_{K}$ for which $v(\alpha) \neq 0$.
We define the height of a vector $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right) \in K^{n}$ relative to $K / \mathbb{k}$ by

$$
H_{K}(\boldsymbol{\alpha}):=H_{K}\left(\alpha_{1}, \ldots, \alpha_{n}\right)=-\sum_{v \in \mathcal{M}_{K}} \min (0, v(\boldsymbol{\alpha})),
$$

where $v(\boldsymbol{\alpha})=\min _{i}\left(v\left(\alpha_{i}\right)\right)$ is the $v$-value of $\boldsymbol{\alpha}$. If $L$ is a finite extension of $K$, then

$$
H_{L}(\boldsymbol{\alpha})=[L: K] H_{K}(\boldsymbol{\alpha}) \quad \text { for } \boldsymbol{\alpha} \in K^{n} \backslash\{\mathbf{0}\} .
$$

We note that

$$
\begin{equation*}
H_{K}\left(\alpha_{i}\right) \leq H_{K}(\boldsymbol{\alpha}) \leq H_{K}\left(\alpha_{1}\right)+\cdots+H_{K}\left(\alpha_{n}\right) \quad(i=1, \ldots, n) \tag{5.1.10}
\end{equation*}
$$

The homogeneous height of $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right) \in K^{n} \backslash\{\mathbf{0}\}$, relative to $K / \mathbb{k}$ is defined as

$$
H_{K}^{\mathrm{hom}}(\boldsymbol{\alpha}):=-\sum_{v \in \mathcal{M}_{K}} v(\boldsymbol{\alpha}) .
$$

It is clear that

$$
H_{K}^{\mathrm{hom}}(\boldsymbol{\alpha}) \leq H_{K}(\boldsymbol{\alpha}) .
$$

By the sum formula, we have

$$
\begin{equation*}
H_{K}^{\mathrm{hom}}(\lambda \boldsymbol{\alpha})=H_{K}^{\mathrm{hom}}(\boldsymbol{\alpha}) \quad \text { for } \quad \lambda \in K^{*} . \tag{5.1.11}
\end{equation*}
$$

For instance, let $p_{1}, \ldots, p_{n} \in \mathbb{k}[z]$ with $\operatorname{gcd}\left(p_{1}, \ldots, p_{n}\right)=1$. Then

$$
\begin{equation*}
H_{\mathfrak{k}(z)}^{\mathrm{hom}}\left(p_{1}, \ldots, p_{n}\right)=\max \left(\operatorname{deg} p_{1}, \ldots, \operatorname{deg} p_{n}\right) . \tag{5.1.12}
\end{equation*}
$$

Further, if $L$ is a finite extension of $K$, then, for $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right) \in K^{n} \backslash\{0\}$, we have

$$
\begin{equation*}
H_{L}^{\mathrm{hom}}(\boldsymbol{\alpha})=[L: K] H_{K}^{\mathrm{hom}}(\boldsymbol{\alpha}) \tag{5.1.13}
\end{equation*}
$$

Since

$$
H_{K}^{\mathrm{hom}}\left(\alpha_{1}, \ldots, \alpha_{n}\right)=H_{K}\left(\alpha_{1} / \alpha_{i}, \ldots, \alpha_{n} / \alpha_{i}\right) \quad \text { for all } i \text { with } \alpha_{i} \neq 0,
$$

we deduce from (5.1.10) that, for $\alpha_{1} \neq 0$,

$$
\begin{gather*}
H_{K}\left(\alpha_{i} / \alpha_{1}\right) \leq H_{K}^{\mathrm{hom}}\left(\alpha_{1}, \ldots, \alpha_{n}\right) \leq \sum_{j=1}^{n} H_{K}\left(\alpha_{j}\right)+(n-2) H_{K}\left(\alpha_{i}\right), \\
\text { for } i=1, \ldots, n . \tag{5.1.14}
\end{gather*}
$$

Further, for a polynomial $F \in K[X]$, its height $H_{K}(F)$ resp. its homogeneous height $H_{K}^{\text {hom }}(F)$ and its $v$-value $v(F)$ are defined by the height resp. homogeneous height and the $v$-value of a vector whose coordinates are the coefficients of $F$. Clearly, for monic polynomials the two heights coincide, while in general,

$$
H_{K}^{\mathrm{hom}}(F) \leq H_{K}(F)
$$

For any two polynomials $F, G$ in $K[X]$, we have

$$
\begin{equation*}
v(F G)=v(F)+v(G) \text { for } v \in \mathcal{M}_{K}, \quad H_{K}^{\mathrm{hom}}(F G)=H_{K}^{\mathrm{hom}}(F)+H_{K}^{\mathrm{hom}}(G) \tag{5.1.15}
\end{equation*}
$$

If a non-zero polynomial $F(X)$ factorizes in $K$ as $f_{0}\left(X-\alpha_{1}\right) \cdots\left(X-\alpha_{n}\right)$
then by 5.1 .15 ) and the sum formula, applied to $f_{0}$, we obtain

$$
\begin{equation*}
H_{K}\left(f_{0}\right) \leq H_{K}(F) \quad \text { and } \quad H_{K}^{\mathrm{hom}}(F)=\sum_{i=1}^{n} H_{K}\left(\alpha_{i}\right) \geq \max _{1 \leq i \leq n} H_{K}\left(\alpha_{i}\right) . \tag{5.1.16}
\end{equation*}
$$

Lemma 5.1.1. Let

$$
F=f_{0} X^{n}+f_{1} X^{n-1}+\cdots+f_{n} \in K[X]
$$

be a polynomial with $f_{0} \neq 0$ and with non-zero discriminant. Let $L$ be the splitting field of $F$ over $K$. Then

$$
g_{L / \mathbf{k}} \leq[L: K]\left(g_{K / \mathbf{k}}+n H_{K}(F)\right) .
$$

In particular, if $K=\mathbb{k}(z)$ and $f_{0}, \ldots, f_{n} \in \mathbb{k}[z]$, we have

$$
g_{L / \mathbb{k}} \leq[L: K] n \max \left(\operatorname{deg} f_{0}, \ldots, \operatorname{deg} f_{n}\right)
$$

Proof. Lemma 4.2 of Bérczes, Evertse and Győry (2014) gives a slightly better estimate with $H_{K}^{\text {hom }}(F)$ instead of $H_{K}(F)$. The second assertion is due to Schmidt (1978).

In the next sections we present explicit upper bounds for the heights of the solutions of $S$-unit equations, the Catalan equation, Thue equations, hyperand superelliptic equations.

We denote by $|S|$ the cardinality of a set $S$.

## $5.2 S$-unit equations

Let again $K$ be a function field in one variable over an algebraically closed field $\mathbb{k}$ of characteristic 0 , and $S$ a finite, non-empty set of valuations on $K$ over $\mathbb{k}$, of cardinality at least 2 . Consider the $S$-unit equation

$$
\begin{equation*}
\alpha x+\beta y=1 \quad \text { in } \quad x, y \in \mathcal{O}_{S}^{*} \backslash \mathbb{k}^{*}, \tag{5.2.1}
\end{equation*}
$$

where $\alpha, \beta \in K^{*}$. For $\alpha=\beta=1$, the following theorem is due to Mason (1983). The general case, under the assumption $\alpha x \notin \mathbb{k}^{*}$, was established in Evertse and Győry (2015). We prove Theorem 5.2.1 without this assumption.

Theorem 5.2.1. Every solution $x, y$ of equation (5.2.1) satisfies

$$
\begin{equation*}
\max \left(H_{K}(x), H_{K}(y)\right) \leq 5 H+|S|+2 g_{K / \mathbf{k}}-2 \tag{5.2.2}
\end{equation*}
$$

where $H=\max \left(H_{K}(\alpha), H_{K}(\beta)\right)$.
Independently of Mason, Győry (1983) proved a version of Theorem 5.2.1 with larger explicit coefficients of $|S|$ and $g_{K / \mathbb{k}}$. We note that this weaker version would also be sufficient for application in Chapter 9 . Theorem 5.2.1 is a consequence of the following theorem of Mason $(1983,1984)$. It is a generalization of an earlier result of Stothers (1981).

Theorem 5.2.2. Let $S$ be a finite, non-empty subset of $\mathcal{M}_{K}$, and let $x_{1}, x_{2}, x_{3}$ be non-zero elements of $K$ with $x_{1}+x_{2}+x_{3}=0$ such that $v\left(x_{1}\right)=v\left(x_{2}\right)=$ $v\left(x_{3}\right)$ for every $v$ in $\mathcal{M}_{K} \backslash S$. Then either $x_{1} / x_{2}$ lies in $\mathbb{k}$, or

$$
\begin{equation*}
H_{K}\left(x_{1} / x_{2}\right) \leq|S|+2 g_{K / \mathbf{k}}-2 . \tag{5.2.3}
\end{equation*}
$$

Proof. We do not give Mason's proof based on derivations, but instead another well-known proof based on the Riemann-Hurwitz formula. We assume without loss of generality that $x_{1} / x_{2} \notin \mathbb{k}$ and that $S$ is precisely the set of all $v \in \mathcal{M}_{K}$ such that $v\left(x_{1}\right), v\left(x_{2}\right), v\left(x_{3}\right)$ are distinct. Let $z:=x_{1} / x_{2}$. Then $z$ and $1+z$ are $S$-units. We can write $S$ as a disjoint union $S_{0} \cup S_{-1} \cup S_{\infty}$ where

$$
\begin{aligned}
S_{0} & :=\left\{v \in \mathcal{M}_{K}: v(z)>0\right\}, \quad S_{-1}:=\left\{v \in \mathcal{M}_{K}: v(z+1)>0\right\}, \\
S_{\infty} & :=\left\{v \in \mathcal{M}_{K}: v(z)<0\right\} .
\end{aligned}
$$

Note that for $a=0,-1, \infty, S_{a}$ is precisely the set of valuations on $K$ lying above the valuation $\operatorname{ord}_{a}$ on $\mathbb{k}(z)$. So by the Riemann-Hurwitz formula (5.1.5), and by (5.1.4) and (5.1.2),

$$
\begin{aligned}
& 2 g_{K / \mathbb{k}}-2=[K: \mathbb{k}(z)]\left(2 g_{\mathbb{k}(z) / \mathbb{k}}-2\right)+\sum_{a \in \mathbb{k} \cup\{\infty\}} \sum_{v \mid a}\left(e_{v \mid a}-1\right) \\
& \quad \geq-2[K: \mathbb{k}(z)]+\sum_{a \in\{0,-1, \infty\}} \sum_{v \mid a}\left(e_{v \mid a}-1\right) \\
& \quad=-2[K: \mathbb{k}(z)]+\sum_{a \in\{0,-1, \infty\}}\left([K: \mathbb{k}(z)]-\left|S_{a}\right|\right)=[K: \mathbb{k}(z)]-|S|,
\end{aligned}
$$

where we have written $v \mid a$ for $v \mid \operatorname{ord}_{a}$. Using $[K: \mathbb{k}(z)]=H_{K}(z)$ (see (5.1.6), Theorem 5.2.2 follows.

Proof of Theorem 5.2.1. Let $x, y$ be a solution of equation (5.2.1). Then

$$
\begin{equation*}
H_{K}(x)=H_{K}\left(\alpha x \cdot \alpha^{-1}\right) \leq H_{K}(\alpha x)+H_{K}\left(\alpha^{-1}\right) \leq H_{K}(\alpha x)+H, \tag{5.2.4}
\end{equation*}
$$

and similarly $H_{K}(y) \leq H_{K}(\beta y)+H$. If $\alpha, \beta \in \mathbb{k}^{*}$, then Theorem 5.2.1 follows at once from Theorem 5.2.2. Suppose that $\alpha, \beta$ are not both in $\mathbb{k}^{*}$. Then $H \geq 1$.

If $\alpha x \in \mathbb{k}^{*}$ then $\beta y \in \mathbb{k}^{*}$, hence their heights are zero and (5.2.2) immediately follows.

Now suppose that $\alpha x, \beta y \notin \mathbb{k}^{*}$. Let $S_{\alpha}$ denote the set of valuations $v \in$ $\mathcal{M}_{K}$ with $v(\alpha) \neq 0$ and define $S_{\beta}$ similarly. In view of (5.1.9) we have $\left|S_{\alpha}\right| \leq$ $2 H_{K}(\alpha) \leq 2 H$ and similarly $\left|S_{\beta}\right| \leq 2 H$. Then it follows that $v(\alpha x)=$ $v(\beta y)=v(1)=0$ for $v \in \mathcal{M}_{K} \backslash\left(S \cup S_{\alpha} \cup S_{\beta}\right)$. Now using (5.2.4) and applying Theorem 5.2.2 to $\alpha x, \beta y, 1$ with $S \cup S_{\alpha} \cup S_{\beta}$ instead of $S$, we obtain

$$
\begin{aligned}
\max \left(H_{K}(x), H_{K}(y)\right) & \leq \max \left(H_{K}(\alpha x), H_{K}(\beta y)\right)+H \\
& \leq|S|+\left|S_{\alpha}\right|+\left|S_{\beta}\right|+2 g_{K / \mathbf{k}}-2+H \\
& \leq 5 H+|S|+2 g_{K / \mathbf{k}}-2
\end{aligned}
$$

Remark. In contrast with the number field case, for function fields there are effective results for $S$-unit equations in the more unknowns case as well; see Mason(1986,1988), Brownawell and Masser (1986) and, for further references, e.g. Evertse and Győry (2015).

### 5.3 The Catalan equation

As before, $K$ is a function field in one variable over an algebraically closed field $\mathbb{k}$ of characteristic 0 , and $S$ is a finite set of valuations on $K$ over $\mathbb{k}$. We deduce consequences for a slight generalization of the Catalan equation, i.e.,

$$
\begin{equation*}
x^{m}-y^{n}= \pm 1, \tag{5.3.1}
\end{equation*}
$$

both in the cases that $x, y$ assume their values in $\mathcal{O}_{S}$ and that $x, y$ assume their values in $K$.

Theorem 5.3.1. (i) Let $S$ be a finite set of valuations of $K$, let $m, n$ be integers with $m, n \geq 2, m n>4$ and let $x, y \in \mathcal{O}_{S} \backslash \mathbb{k}$ satisfy (5.3.1).

Then

$$
\begin{equation*}
m H_{K}(x), n H_{K}(y) \leq 6\left(|S|+2 g_{K / \mathbb{k}}-2\right) \tag{5.3.2}
\end{equation*}
$$

(ii) Let $m, n$ be integers with $n \geq m \geq 2, m n \geq 10$ and let $x, y \in K \backslash \mathbb{k}$ satisfy (5.3.1). Then

$$
\begin{equation*}
m H_{K}(x), n H_{K}(y) \leq 20\left(g_{K / \mathbf{k}}-1\right) \tag{5.3.3}
\end{equation*}
$$

Since $H_{K}(x), H_{K}(y)$ are $\geq 1$, 5.3.2) and (5.3.3) imply upper bounds for $m$ and $n$.

Proof. (i) Let $S_{1}$ be the set of $v \in \mathcal{M}_{K} \backslash S$ with $v(x)>0$ and $S_{2}$ the set of $v \in \mathcal{M}_{K} \backslash S$ with $v(y)>0$. Then

$$
\left|S_{1}\right| \leq H_{K}(x)=\frac{1}{m} H_{K}\left(x^{m}\right), \quad\left|S_{2}\right| \leq H_{K}(y)=\frac{1}{n} H_{K}\left(y^{n}\right) \leq \frac{1}{n} H_{K}\left(x^{m}\right)
$$

So by Theorem 5.2.2.

$$
\begin{aligned}
H_{K}\left(x^{m}\right) & \leq|S|+\left|S_{1}\right|+\left|S_{2}\right|+2 g_{K / \mathbb{k}}-2 \\
& \leq|S|+2 g_{K / \mathbb{k}}-2+\left(\frac{1}{m}+\frac{1}{n}\right) H_{K}\left(x^{m}\right) \\
& \leq|S|+2 g_{K / \mathbb{k}}-2+\frac{5}{6} H_{K}\left(x^{m}\right),
\end{aligned}
$$

implying (5.3.2).
(ii) Assume without loss of generality that $n \geq m$. Let $S_{1}$ be the set of $v \in \mathcal{M}_{K}$ with $v(x)>0$, let $S_{2}$ be the set of $v \in \mathcal{M}_{K}$ with $v(y)>0$, and let $S_{3}$ be the set of $v \in \mathcal{M}_{K}$ with $v(x)<0$; for these places we have also $v(y)<0$. Then

$$
\left|S_{1}\right| \leq \frac{1}{m} H_{K}\left(x^{m}\right), \quad\left|S_{2}\right|,\left|S_{3}\right| \leq H_{K}(y)=\frac{1}{n} H_{K}\left(x^{m}\right)
$$

and Theorem 5.2.2 gives

$$
\begin{aligned}
H_{K}\left(x^{m}\right) & \leq\left|S_{1}\right|+\left|S_{2}\right|+\left|S_{3}\right|+2 g_{K / \mathbb{k}}-2 \\
& \leq 2 g_{K / \mathbf{k}}-2+\left(\frac{1}{m}+\frac{2}{n}\right) H_{K}\left(x^{m}\right) \\
& \leq 2 g_{K / \mathbb{k}}-2+\frac{9}{10} H_{K}\left(x^{m}\right),
\end{aligned}
$$

implying (5.3.3).

### 5.4 Thue equations

We denote as before by $K$ a function field in one variable over an algebraically closed field $\mathbb{k}$ of characteristic 0 and by $S$ a finite set of valuations on $K$ over $\mathbb{k}$. In this section, we consider the Thue equation

$$
\begin{equation*}
F(x, y)=1 \quad \text { in } \quad x, y \in \mathcal{O}_{S} \tag{5.4.1}
\end{equation*}
$$

where $F$ is a binary form of degree $n \geq 3$ with coefficients in $K$ and with non-zero discriminant.

Using a method of Osgood $(1973,1975)$, Schmidt $(1976,1978)$ derived bounds for the heights of the solutions of (5.4.1). Later, by means of his Theorem 5.2.2 above Mason (1984) gave a better bound for the heights of the integral solutions over $\mathbb{k}[z]$ in the case when $F$ factorizes into linear factors over $K$. For a refinement, see Dvornicich and Zannier (1994).

We prove the following version of the theorems of Schmidt and Mason.
Theorem 5.4.1. Every solution $x, y$ of equation (5.4.1) satisfies

$$
\begin{align*}
\max \left(H_{K}(x), H_{K}(y)\right) & \leq H_{K}(x, y) \\
& \leq(8 n+62) H_{K}(F)+4|S|+8 g_{K / \mathbb{k}} . \tag{5.4.2}
\end{align*}
$$

If the splitting field of $F$ is $K$ we have the stronger estimate

$$
\begin{equation*}
H_{K}(x, y) \leq 62 H_{K}(F)+4|S|+8 g_{K / \mathbf{k}} . \tag{5.4.3}
\end{equation*}
$$

Our proof is different from that of Schmidt and Mason. It is based on Theorem 5.2.1 above. No special importance is attached to the constants in our upper bounds, which could be improved with some extra effort. However, such improvements would be irrelevant for our application in Chapter 9 .

Proof of Theorem 5.4.1. Write the binary form $F$ in the form

$$
F(X, Y)=a_{0} X^{n}+a_{1} X^{n-1} Y+\cdots+a_{n} Y^{n}
$$

We may suppose without loss of generality that $a_{0} \neq 0$. Indeed, this can be achieved by a linear transformation of the shape

$$
X=X^{\prime}, \quad Y=a X^{\prime}+Y^{\prime}
$$

with $a \in \mathbb{k}^{*}$ such that $F(1, a) \neq 0$. Then putting

$$
F^{\prime}\left(X^{\prime}, Y^{\prime}\right):=F\left(X^{\prime}, a X^{\prime}+Y^{\prime}\right)=a_{0}^{\prime} X^{\prime n}+a_{1}^{\prime} X^{\prime n-1} Y^{\prime}+\cdots+a_{n}^{\prime} Y^{\prime n}
$$

we have $a_{0}^{\prime}=F(1, a) \neq 0$. Then with $x^{\prime}=x, y^{\prime}=-a x+y$ we have $F^{\prime}\left(x^{\prime}, y^{\prime}\right)=1$. Observe that $\min \left(v\left(x^{\prime}\right), v\left(y^{\prime}\right)\right)=\min (v(x), v(y))$ for $v \in$ $\mathcal{M}_{K}$, so $H_{K}\left(x^{\prime}, y^{\prime}\right)=H_{K}(x, y)$. Further, each coefficient $a_{i}^{\prime}$ of $F^{\prime}$ is a $\mathbb{k}$-linear combination of $a_{0}, \ldots, a_{n}$, hence $v\left(a_{i}^{\prime}\right) \geq \min \left(v\left(a_{0}\right), \ldots, v\left(a_{n}\right)\right)=v(F)$ for $v \in \mathcal{M}_{K}, i=0, \ldots, n$, that is, $v\left(F^{\prime}\right) \geq v(F)$ for $v \in \mathcal{M}_{K}$. But by symmetry, the reverse inequality also holds so we have $v\left(F^{\prime}\right)=v(F)$ for $v \in \mathcal{M}_{K}$. Hence $H_{K}\left(F^{\prime}\right)=H_{K}(F)$.

We assume henceforth that $a_{0} \neq 0$. For the moment, we assume that $F$ has splitting field $K$. We can write (5.4.1) as

$$
\begin{equation*}
a_{0}\left(x-\alpha_{1} y\right) \cdots\left(x-\alpha_{n} y\right)=1 \quad \text { in } \quad x, y \in \mathcal{O}_{S} \tag{5.4.4}
\end{equation*}
$$

where, by our assumption, $\alpha_{1}, \ldots, \alpha_{n}$ are distinct elements of $K$. Denote by $S_{\alpha}$ the set of valuations $v \in \mathcal{M}_{S}$ for which $v\left(a_{0}\right) \neq 0$ and $v\left(\alpha_{i}\right) \neq 0$ for $i=1, \ldots, n$. Notice that in view of (5.1.16),

$$
\begin{equation*}
\left|S_{\alpha}\right| \leq 2\left(H_{K}\left(a_{0}\right)+\sum_{i=1}^{n} H_{K}\left(\alpha_{i}\right)\right) \leq 4 H_{K}(F) . \tag{5.4.5}
\end{equation*}
$$

Putting $S^{\prime}=S \cup S_{\alpha}$, the elements $x, y, a_{0}, \alpha_{1}, \ldots, \alpha_{n}$ and $\beta_{i}=x-\alpha_{i} y$ for $i=1, \ldots, n$ are all $S^{\prime}$-integers in $K$. Now it follows from (5.4.4) that $\beta_{i}$ is an $S^{\prime}$-unit in $K$ for $i=1, \ldots n$.

We have

$$
\begin{equation*}
-\gamma \frac{\beta_{i}}{\beta_{j}}+(\gamma+1) \frac{\beta_{l}}{\beta_{j}}=1 \tag{5.4.6}
\end{equation*}
$$

for any distinct $i, j, l$, where $\gamma=\left(\alpha_{j}-\alpha_{l}\right) /\left(\alpha_{l}-\alpha_{i}\right)$ and $\beta_{i} / \beta_{j}, \beta_{l} / \beta_{j}$ are $S^{\prime}$-units. If $\beta_{i} / \beta_{j} \in \mathbb{k}$ then $H_{K}\left(\beta_{i} / \beta_{j}\right)=0$. Otherwise, by Theorem 5.2.1 we have

$$
H_{K}\left(\beta_{i} / \beta_{j}\right) \leq 5 H+\left|S^{\prime}\right|+2 g_{K / \mathbf{k}}-2
$$

for each distinct $i, j$ with $1 \leq i, j \leq n$, where $H=\max \left(H_{K}(\gamma), H_{K}(\gamma+1)\right)$. In view of (5.1.16) we have $H \leq 2 H_{K}(F)$, while by (5.4.5) we have $\left|S^{\prime}\right| \leq$
$|S|+4 H_{K}(F)$. Thus we get

$$
\begin{equation*}
H_{K}\left(\beta_{i} / \beta_{j}\right) \leq 14 H_{K}(F)+|S|+2 g_{K / \mathbf{k}}=: C \tag{5.4.7}
\end{equation*}
$$

for each $i, j$ with $1 \leq i, j \leq n, i \neq j$. Here we have removed the -2 -term to incorporate the case $\beta_{i} / \beta_{j} \in \mathbb{k}$.

We infer from (5.4.4) that

$$
\beta_{j}^{-n}=a_{0} \prod_{i=1}^{n} \frac{\beta_{i}}{\beta_{j}}, \quad j=1, \ldots, n
$$

whence, using (5.1.16), we get

$$
\begin{equation*}
n H_{K}\left(\beta_{j}\right) \leq H_{K}(F)+n C . \tag{5.4.8}
\end{equation*}
$$

But we have

$$
x=\frac{\beta_{i} \alpha_{j}-\alpha_{i} \beta_{j}}{\alpha_{j}-\alpha_{i}}, \quad y=\frac{\beta_{i}-\beta_{j}}{\alpha_{j}-\alpha_{i}},
$$

and so, using again (5.1.16) and (5.4.8), it follows that

$$
H_{K}(x) \leq \frac{2 n+2}{n} H_{K}(F)+2 C .
$$

For $H_{K}(y)$ we get the same upper bound, so, in view of $H_{K}(x, y) \leq H_{K}(x)+$ $H_{K}(y)$,

$$
H_{K}(x, y) \leq \frac{4 n+4}{n} H_{K}(F)+4 C .
$$

Together with (5.4.7) this gives (5.4.3).
Now assume that $F$ has splitting field $L$ over $K$. Let $\Delta:=[L: K]$, and let $T$ be the set of valuations on $L$ lying above those in $S$. Then (5.4.3) holds with $L, T$ instead of $K$, $S$, i.e.,

$$
\begin{equation*}
H_{L}(x, y) \leq 62 H_{L}(F)+4|T|+8 g_{L / \mathbb{k}} . \tag{5.4.9}
\end{equation*}
$$

We have $H_{L}(x, y)=\Delta H_{K}(x, y), H_{L}(F)=\Delta H_{K}(F),|T| \leq \Delta|S|$ and lastly, by Lemma 5.1.1,

$$
g_{L / \mathbb{k}} \leq \Delta\left(g_{K / \mathbb{k}}+n H_{K}(F)\right) .
$$

By inserting this into (5.4.9) and dividing by $\Delta$ we obtain (5.4.2). This completes our proof.

### 5.5 Hyper- and superelliptic equations

Let again $\mathbb{k}$ be an algebraically closed field of characteristic $0, K$ a function field in one variable over $\mathbb{k}$, and $S$ a finite subset of $\mathcal{M}_{K}$.

Let $f \in K[X]$ be a polynomial of degree $n$ with non-zero discriminant. Consider the hyperelliptic equation

$$
\begin{equation*}
f(x)=y^{2} \quad \text { in } \quad x, y \in \mathcal{O}_{S} \tag{5.5.1}
\end{equation*}
$$

where $n \geq 3$, and the superelliptic equation

$$
\begin{equation*}
f(x)=y^{m} \quad \text { in } \quad x, y \in \mathcal{O}_{S} \tag{5.5.2}
\end{equation*}
$$

where $m \geq 3, n \geq 2$,
Schmidt (1978) gave an explicit upper bound for the heights of the solutions $x, y$ of (5.5.1) in the case when all the zeros of $f$ lie in $K$. Using his Theorem 5.2 .2 above, Mason $(1983,1984)$ derived explicit upper bounds for the heights of the solutions of equation (5.5.1), (5.5.2) but only under the assumptions that the zeros of $f$ belong to $K$ and $S$ consists of the infinite valuations of $K$ (i.e., those valuations $v$ with $v(z)<0$, where $z$ is an element of $K \backslash \mathbb{k}$ that is chosen and fixed in advance). Bérczes, Evertse and Győry (2014) needed results without these conditions, and so they extended Mason's results to the most general situation when the splitting field of $f$ and the set of valuations $S$ are arbitrary.

Below we present results similar to Proposition 4.7 and Proposition 4.6 of Bérczes, Evertse and Győry (2014), with different upper bounds. In our proofs we will follow Mason. Both proofs will be based on Theorem 5.2.2.

Theorem 5.5.1. Every solution $x$, y of equation (5.5.1) satisfies

$$
\begin{align*}
H_{K}(x) & \leq(8 n+42) H_{K}(f)+8|S|+8 g_{K / \mathbf{k}}  \tag{5.5.3}\\
H_{K}(y) & \leq\left(4 n^{2}+21 n+1\right) H_{K}(f)+4 n|S|+4 n g_{K / \mathbb{k}} . \tag{5.5.4}
\end{align*}
$$

In case of equation (5.5.2) we can even estimate $m$ from above, provided that $y \notin \mathbb{k}$.

Theorem 5.5.2. Every solution $x, y$ of equation (5.5.2) satisfies

$$
\begin{align*}
H_{K}(x) & \leq(6 n+18) H_{K}(f)+3|S|+6 g_{K / \mathbb{k}}  \tag{5.5.5}\\
m H_{K}(y) & \leq\left(6 n^{2}+18 n+1\right) H_{K}(f)+3 n|S|+6 n g_{K / \mathbf{k}} . \tag{5.5.6}
\end{align*}
$$

It will be more convenient to prove first Theorem 5.5.2 and then Theorem 5.5.1 Similarly as in the case of Thue equations, in both proofs we first consider the case when $f$ splits completely over $K$ and then deduce the general case.

Proof of Theorem 5.5.2 We fix a solution $(x, y)$ of (5.5.2). First we assume that $f(X)=a_{0} \prod_{i=1}^{n}\left(X-\alpha_{i}\right)$ with $a_{0} \in K^{*}$ and with distinct elements $\alpha_{1}, \ldots, \alpha_{n}$ of $K$. We shall apply Theorem 5.2 .2 to the identity

$$
\begin{equation*}
\left(x-\alpha_{1}\right)+\left(\alpha_{2}-x\right)+\left(\alpha_{1}-\alpha_{2}\right)=0 . \tag{5.5.7}
\end{equation*}
$$

We assume without loss of generality that $\alpha_{1}, \ldots, \alpha_{n}$ are arranged in order of increasing height, and so, in view of (5.1.16),

$$
\begin{equation*}
\sum_{i=1}^{s} H_{K}\left(\alpha_{i}\right) \leq s H_{K}(f) / n \text { for } s=1, \ldots, n \tag{5.5.8}
\end{equation*}
$$

Put $\beta_{i}=\prod_{j \neq i}\left(\alpha_{i}-\alpha_{j}\right), i=1, \ldots, n$. Then we have

$$
H_{K}\left(\beta_{i}\right) \leq H_{K}(f)+(n-2) H_{K}\left(\alpha_{i}\right) \quad \text { for } \quad i=1, \ldots, n .
$$

Denote by $W$ the set of valuations $v$ on $K$ at which one or more of the following occur:

$$
\begin{equation*}
v \in S, \quad v(f)<0, \quad v\left(a_{0}\right)>0, \quad v\left(\beta_{1} \beta_{2}\right)>0 . \tag{5.5.9}
\end{equation*}
$$

The number of $v$ with $v(f)<0$ is at most $H_{K}(f)$; the number of $v$ with $v\left(a_{0}\right)>0$ is at $\operatorname{most} H_{K}\left(a_{0}^{-1}\right)=H_{K}\left(a_{0}\right) \leq H_{K}(f)$; and lastly, by (5.5.8) the number of $v$ with $v\left(\beta_{1} \beta_{2}\right)>0$ is at most

$$
\begin{aligned}
H_{K}\left(\left(\beta_{1} \beta_{2}\right)^{-1}\right) & =H_{K}\left(\beta_{1} \beta_{2}\right) \leq H_{K}\left(\beta_{1}\right)+H_{K}\left(\beta_{2}\right) \\
& \leq 2 H_{K}(f)+(n-2)\left(H_{K}\left(\alpha_{1}\right)+H_{K}\left(\alpha_{2}\right)\right) \leq\left(4-\frac{4}{n}\right) H_{K}(f) .
\end{aligned}
$$

So altogether,

$$
\begin{equation*}
|W| \leq|S|+\left(6-\frac{4}{n}\right) H_{K}(f) . \tag{5.5.10}
\end{equation*}
$$

It is easy to check that for $v \in \mathcal{M}_{K} \backslash W$,

$$
\begin{align*}
& v\left(a_{0}\right)=0, \quad v(f)=0, \quad v\left(\alpha_{i}\right)=0 \text { for } i=1, \ldots, n,  \tag{5.5.11}\\
& v\left(\alpha_{i}-\alpha_{j}\right)=0 \text { for } i=1,2, j \neq i,
\end{align*}
$$

so that $\min \left(v\left(x-\alpha_{i}\right), v\left(x-\alpha_{j}\right)\right)=0$ for $i=1,2, j \neq i$. Hence

$$
\begin{equation*}
v\left(x-\alpha_{i}\right)=m v(y) \equiv 0(\bmod m) \text { for } i=1,2 . \tag{5.5.12}
\end{equation*}
$$

Denoting by $S_{i}, i=1,2$, the set of $v \in \mathcal{M}_{K} \backslash W$ such that $v\left(x-\alpha_{i}\right)>0$, and applying Theorem 5.2.2 to (5.5.7) we get

$$
\begin{equation*}
H_{K}\left(\frac{x-\alpha_{1}}{\alpha_{1}-\alpha_{2}}\right) \leq\left|W \cup S_{1} \cup S_{2}\right|+2 g_{K / \mathbb{k}} . \tag{5.5.13}
\end{equation*}
$$

By (5.5.12) we have $m\left|S_{i}\right| \leq H_{K}\left(x-\alpha_{i}\right) \leq H_{K}(x)+H_{K}\left(\alpha_{i}\right)$ for $i=1,2$. Therefore we can deduce from (5.5.13) that

$$
\begin{aligned}
& H_{K}(x)-2 H_{K}\left(\alpha_{1}\right)-H_{K}\left(\alpha_{2}\right) \\
& \quad \leq \frac{1}{m}\left(2 H_{K}(x)+H_{K}\left(\alpha_{1}\right)+H_{K}\left(\alpha_{2}\right)\right)+|W|+2 g_{K / \mathbf{k}} .
\end{aligned}
$$

Together with (5.5.10),(5.5.8) and $m \geq 3$ this gives

$$
\begin{aligned}
& \left(1-\frac{2}{m}\right) H_{K}(x) \\
& \left.\quad \leq \frac{1}{n} H_{K}(f)+\left(1+\frac{1}{m}\right) \cdot \frac{2}{n} H_{K}(f)+|S|+\left(6-\frac{4}{n}\right) H_{K}(f)\right)+2 g_{K / \mathbf{k}} \\
& \quad \leq 6 H_{K}(f)+|S|+2 g_{K / \mathbf{k}},
\end{aligned}
$$

and thus,

$$
\begin{equation*}
H_{K}(x) \leq 18 H_{K}(f)+3|S|+6 g_{K / \mathbf{k}} . \tag{5.5.14}
\end{equation*}
$$

Now assume that $f$ has splitting field $L$ over $K$. Put $\Delta:=[L: K]$ and let $T$ be the set of valuations of $L$ lying above the valuations in $S$. The inequality (5.5.14) holds with $L, T$ instead of $K, S$. Inserting

$$
\left.\begin{array}{l}
H_{L}(x)=\Delta H_{K}(x), \quad H_{L}(y)=\Delta H_{K}(y), \quad H_{L}(f)=\Delta H_{K}(f),  \tag{5.5.15}\\
|T| \leq \Delta|S|, \quad g_{L / k} \leq \Delta\left(g_{K / \mathbf{k}}+n H_{K}(f)\right) \text { (from Lemma 5.1.1) }
\end{array}\right\}
$$

and then dividing by $\Delta$, inequality (5.5.5) follows.
Writing $f(X)=a_{0} X^{n}+a_{1} X^{n-1}+\cdots+a_{n}$, we get for $v \in \mathcal{M}_{K}$,

$$
\begin{aligned}
m v(y) & \left.=v(f(x)) \geq \min _{0 \leq i \leq n}\left(v\left(a_{i}\right)+i v(x)\right)\right) \\
& \geq \min (0, v(f))+n \min (0, v(x)),
\end{aligned}
$$

and thus,

$$
\begin{equation*}
m H_{K}(y) \leq H_{K}(f)+n H_{K}(x), \tag{5.5.16}
\end{equation*}
$$

which combined with (5.5.5) gives (5.5.6). This completes the proof of our theorem.

Proof of Theorem 5.5.1. Let $(x, y)$ be a solution of (5.5.1) in $\mathcal{O}_{S}$. We start again with assuming that $f$ has splitting field $K$, so that $f(X)=a_{0}(X-$ $\left.\alpha_{1}\right) \cdots\left(X-\alpha_{n}\right)$ with $\alpha_{1}, \ldots, \alpha_{n}$ distinct elements of $K$. We assume again that $H_{K}\left(\alpha_{1}\right) \leq \cdots \leq H_{K}\left(\alpha_{n}\right)$, so that we have again (5.5.8). Let again

$$
\beta_{i}=\prod_{j \neq i}\left(\alpha_{i}-\alpha_{j}\right) \quad(i=1, \ldots, n) .
$$

We now take for $W$ the set of valuations $v$ of $K$ satisfying at least one of the conditions

$$
\begin{equation*}
v \in S, \quad v(f)<0, \quad v\left(a_{0}\right)>0, \quad v\left(\beta_{1} \beta_{2} \beta_{3}\right)>0 . \tag{5.5.17}
\end{equation*}
$$

Then by a similar computation as in (5.5.10,

$$
\begin{equation*}
|W| \leq|S|+\left(8-\frac{6}{n}\right) H_{K}(f) . \tag{5.5.18}
\end{equation*}
$$

Moreover, similarly to (5.5.11) we have for $v \in \mathcal{M}_{K} \backslash W$,

$$
\begin{align*}
& v\left(a_{0}\right)=0, \quad v(f)=0, \quad v\left(\alpha_{i}\right)=0 \text { for } i=1, \ldots, n, \\
& v\left(\alpha_{i}-\alpha_{j}\right)=0 \text { for } i=1,2,3, j \neq i, \tag{5.5.19}
\end{align*}
$$

and similarly to (5.5.12),

$$
\begin{equation*}
v\left(x-\alpha_{i}\right) \equiv 0(\bmod 2) \text { for } v \in \mathcal{M}_{K} \backslash W, i=1,2,3 . \tag{5.5.20}
\end{equation*}
$$

Consider the field $M$ generated over $K$ by the square roots of $x-\alpha_{1}, x-\alpha_{2}$ and $x-\alpha_{3}$. Let $\delta:=[M: K]$. We first compute an upper bound for the genus
of $M$, to be used in later arguments. By the Riemann-Hurwitz formula (see (5.1.5) we have

$$
2 g_{M / \mathbb{k}}-2=\delta\left(2 g_{K / \mathbb{k}}-2\right)+\sum_{v \in \mathcal{M}_{K}} \sum_{w \mid v}\left(e_{w \mid v}-1\right) .
$$

Let $v \in \mathcal{M}_{K} \backslash W$ and choose a local parameter $z_{v}$ of $v$. By (5.5.20) the square roots of $x-\alpha_{i}(i=1,2,3)$ can be expressed as Laurent series in $z_{v}$, so for the embeddings of $M$ in $\overline{K_{v}}$ as described in (5.1.1), all $e_{i}$ are equal to 1 . In other words, we have $e_{w \mid v}=1$ for all $v \in \mathcal{M}_{K} \backslash W$ and all valuations $w$ of $M$ lying above $v$. We lastly observe that by (5.1.2) we have $\sum_{w \mid v} e_{w \mid v}=\delta$ for $v \in W$. By inserting this into the Riemann-Hurwitz formula, we get

$$
\begin{equation*}
2 g_{M / \mathbf{k}}-2 \leq \delta\left(2 g_{K / \mathbf{k}}-2+|W|\right) \tag{5.5.21}
\end{equation*}
$$

Let $\mu:=\left(\alpha_{2}-\alpha_{1}\right)\left(\alpha_{3}-\alpha_{2}\right)\left(\alpha_{1}-\alpha_{3}\right)$, let $V$ be the set of valuations of $K$ satisfying at least one of the conditions

$$
v \in S, \quad v\left(a_{0}\right)>0, \quad v(f)<0, \quad v(\mu)>0
$$

By (5.5.8), we have

$$
\begin{align*}
|V| & \leq|S|+2 H_{K}(f)+2\left(H_{K}\left(\alpha_{1}\right)+H_{K}\left(\alpha_{2}\right)+H_{K}\left(\alpha_{3}\right)\right) \\
& \leq|S|+\left(2+\frac{6}{n}\right) H_{K}(f) \tag{5.5.22}
\end{align*}
$$

For $v \in \mathcal{M}_{K} \backslash V$ we have similarly to (5.5.19),

$$
\begin{align*}
& v\left(a_{0}\right)=0, \quad v(f)=0, v\left(\alpha_{i}\right)=0 \text { for } i=1, \ldots, n  \tag{5.5.23}\\
& v\left(\alpha_{i}-\alpha_{j}\right)=0 \text { for } i=1,2,3
\end{align*}
$$

Let $U$ be the set of valuations of $M$ lying above those in $V$, and denote by $\mathcal{O}_{U}$ the ring of $U$-integers, and by $\mathcal{O}_{U}^{*}$ the ring of $U$-units in $M$. Choose $\xi_{1}, \xi_{2}, \xi_{3} \in$ $M$ such that

$$
\xi_{i}^{2}=x-\alpha_{i}, \quad i=1,2,3
$$

Then by (5.5.23) we have $\xi_{i} \in \mathcal{O}_{U}$ for $i=1,2,3$. Further, let $\gamma_{i}, \widehat{\gamma}_{i}(i=$
$1,2,3$ ) be given by

$$
\begin{array}{ll}
\gamma_{1}=\xi_{2}-\xi_{3}, & \widehat{\gamma_{1}}=\xi_{2}+\xi_{3} \\
\gamma_{2}=\xi_{3}-\xi_{1}, & \widehat{\gamma_{2}}=\xi_{3}+\xi_{1} \\
\gamma_{3}=\xi_{1}-\xi_{2}, & \widehat{\gamma_{3}}=\xi_{1}+\xi_{2}
\end{array}
$$

Then

$$
\gamma_{1} \widehat{\gamma_{1}}=\alpha_{3}-\alpha_{2}, \quad \gamma_{2} \widehat{\gamma_{2}}=\alpha_{1}-\alpha_{3} \quad \gamma_{3} \widehat{\gamma_{3}}=\alpha_{2}-\alpha_{1},
$$

which together with (5.5.23) implies that

$$
\gamma_{i}, \widehat{\gamma}_{i} \in \mathcal{O}_{U}^{*} \text { for } i=1,2,3
$$

By applying Theorem 5.2.2 to the relations

$$
\begin{array}{ll}
\gamma_{1}+\gamma_{2}+\gamma_{3}=0, & \gamma_{1}+\widehat{\gamma_{2}}-\widehat{\gamma_{3}}=0 \\
\widehat{\gamma_{1}}-\gamma_{2}-\widehat{\gamma_{3}}=0, & \widehat{\gamma_{1}}-\widehat{\gamma_{2}}+\gamma_{3}=0
\end{array}
$$

and inserting $|U| \leq \delta|V|$, 5.5 .21 , (5.5.18), (5.5.22) we infer that the quantities

$$
H_{M}\left(\gamma_{2} / \gamma_{3}\right), \quad H_{M}\left(\widehat{\gamma_{2}} / \gamma_{3}\right), \quad H_{M}\left(\gamma_{2} / \widehat{\gamma_{3}}\right), \quad H_{M}\left(\widehat{\gamma_{2}} / \widehat{\gamma_{3}}\right)
$$

are all bounded above by

$$
\begin{aligned}
|U|+2 g_{M / \mathbb{k}} & \leq \delta\left(|V|+|W|+2 g_{K / \mathbb{k}}\right) \\
& \leq \delta\left(10 H_{K}(f)+2|S|+2 g_{K / \mathbb{k}}\right)=: \delta N .
\end{aligned}
$$

Since $x-\alpha_{1}=\xi_{1}^{2}=\frac{1}{4}\left(\widehat{\gamma_{2}}-\gamma_{2}\right)^{2}, x-\alpha_{3}=\xi_{3}^{2}=\frac{1}{4}\left(\widehat{\gamma_{2}}+\gamma_{2}\right)^{2}$ it follows that

$$
\frac{2 x-\alpha_{1}-\alpha_{3}}{\alpha_{2}-\alpha_{1}}=\frac{1}{2}\left(\left(\widehat{\gamma_{2}} / \gamma_{3}\right)\left(\widehat{\gamma_{2}} / \widehat{\gamma_{3}}\right)+\left(\gamma_{2} / \gamma_{3}\right)\left(\gamma_{2} / \widehat{\gamma_{3}}\right)\right),
$$

whence

$$
H_{K}\left(\frac{2 x-\alpha_{1}-\alpha_{3}}{\alpha_{2}-\alpha_{1}}\right)=\delta^{-1} H_{M}\left(\frac{2 x-\alpha_{1}-\alpha_{3}}{\alpha_{2}-\alpha_{1}}\right) \leq 4 N .
$$

Together with (5.5.8) this implies

$$
H_{K}(x) \leq 4 N+2 H_{K}\left(\alpha_{1}\right)+H_{K}\left(\alpha_{2}\right)+H_{K}\left(\alpha_{3}\right) \leq 4 N+\frac{4}{n} H_{K}(f)
$$

which in view of $n \geq 3$ leads to

$$
\begin{equation*}
H_{K}(x) \leq 42 H_{K}(f)+8|S|+8 g_{K / \mathbb{k}} . \tag{5.5.24}
\end{equation*}
$$

Now consider the general case that $f$ has arbitrary splitting field $L$ over $K$, and let $\Delta:=[L: K]$. Denote by $T$ the set of valuations of $L$ lying above those in $S$. Then (5.5.24) holds with $L, T$ instead of $K, S$. By inserting (5.5.15) and dividing by $\Delta$ we obtain (5.5.3). Together with (5.5.16) with $m=2$ this implies (5.5.4). This completes our proof.

## Chapter 6

## Tools from effective commutative algebra

In this chapter we have collected some algorithmic results for fields finitely generated over $\mathbb{Q}$ and for integral domains of characteristic 0 finitely generated over $\mathbb{Z}$. Our main references are Seidenberg (1974) and Aschenbrenner (2004).

By saying that given any input from a specified set we can determine effectively an output, we mean that there exists an algorithm (i.e., a deterministic Turing machine) which, for any choice of input from the specified set, computes the output in a finite number of steps. We say that an object is given effectively if it is given in such a form that it can serve as input for an algorithm.

We agree once more that upper case characters such as $X, Y$ denote variables whereas lower case characters denote elements of rings or fields. Given a ring $R$, we denote by $R^{m, n}$ the $R$-module of $m \times n$-matrices with elements in $R$, and by $R^{n}$ the $R$-module of $n$-dimensional column vectors with coordinates in $R$.

From matrices $A, B$ with the same number of rows, we form a matrix $[A, B]$ by placing the columns of $B$ after those of $A$. Likewise, from two matrices $A, B$ with the same number of columns we form $\left[{ }_{B}^{A}\right]$ by placing the rows of $B$ below those of $A$.

The logarithmic height $h(\mathcal{A})$ of a finite set $\mathcal{A}=\left\{a_{1}, \ldots, a_{t}\right\} \subset \mathbb{Z}$ is defined by $h(\mathcal{A}):=\log \max \left(\left|a_{1}\right|, \ldots,\left|a_{t}\right|\right)$. The logarithmic height $h(U)$ of a matrix $U$ with entries in $\mathbb{Z}$ is defined by the logarithmic height of the set of entries of $U$. The logarithmic height $h(P)$ of a polynomial $P$ with coefficients in $\mathbb{Z}$ is the logarithmic height of the set of coefficients of $P$. By the degree
of a polynomial we always mean its total degree, and the total degree of a polynomial $P$ is denoted by $\operatorname{deg} P$.

As in Section 4.1, we write

$$
\log ^{*} u:=\max (1, \log u) \text { for } u>0, \log ^{*} 0:=1
$$

We use notation $O(\cdot)$ as an abbreviation for $c \times$ the expression between the parentheses, where $c$ is an effectively computable positive absolute constant. At each occurrence of $O(\cdot)$, the value of $c$ may be different.

### 6.1 Effective linear algebra over polynomial rings

We have taken some material from Evertse and Győry $(2013,2015)$ on effective results for systems of linear equations over polynomial rings over a field or over $\mathbb{Z}$, with some small improvements here and there. For convenience of the reader we have repeated some details.

Lemma 6.1.1. Let $U \in \mathbb{Z}^{m, n}$ and $\mathbf{b} \in \mathbb{Z}^{m}$.
(i) The $\mathbb{Z}$-module of $\mathbf{y} \in \mathbb{Z}^{n}$ with $U \mathbf{y}=\mathbf{0}$ is generated by vectors in $\mathbb{Z}^{n}$ of logarithmic height at most $m h(U)+\frac{1}{2} m \log m$.
(ii) Assume that $U \mathbf{y}=\mathbf{b}$ is solvable in $\mathbb{Z}^{n}$. Then it has a solution $\mathbf{y} \in \mathbb{Z}^{n}$ with $h(\mathbf{y}) \leq m h([U, \mathbf{b}])+\frac{1}{2} m \log m$.

Proof. (i) We follow Aschenbrenner (2004), Lemma 4.2 and Section 5. Let $\mathcal{M}$ be the $\mathbb{Z}$-module of $\mathbf{y} \in \mathbb{Z}^{n}$ with $U \mathbf{y}=\mathbf{0}$. We may assume without loss of generality that $m \leq n$ and $U$ has rank $m$, so that $U$ has non-singular submatrices of order $m$. Let $U_{1}, \ldots, U_{k}$ be the non-singular submatrices of $U$ of order $m$, and put $\delta_{j}:=\operatorname{det} U_{j}$ for $j=1, \ldots, k$ and $\delta:=\operatorname{gcd}\left(\delta_{1}, \ldots, \delta_{k}\right)$.

We first prove that for $j=1, \ldots, k$ there are $\mathbf{y}_{1, j}, \ldots, \mathbf{y}_{n-m, j} \in \mathcal{M}$ such that

$$
\begin{align*}
& \text { for every } \mathbf{y} \in \mathcal{M} \text {, there are } b_{i, j} \in \mathbb{Z} \text { with } \mathbf{y}=\left(\delta / \delta_{j}\right) \sum_{i=1}^{n-m} b_{i, j} \mathbf{y}_{i, j}  \tag{6.1.1}\\
& h\left(\mathbf{y}_{i, j}\right) \leq m h(U)+\frac{1}{2} m \log m \text { for } i=1, \ldots, n-m \tag{6.1.2}
\end{align*}
$$

It suffices to prove this for $j=1$. After permuting the columns of $U$ we may assume that $U_{1}$ consists of the first $m$ columns of $U$. Let $V_{1}$ consist of the last $n-m$ columns of $U$. For $\mathbf{y} \in \mathcal{M}$, let $\mathbf{y}^{(1)}$ consist of the first $m$ coordinates
of $\mathbf{y}$, and $\mathbf{y}^{(2)}$ of the last $n-m$ coordinates of $\mathbf{y}$. Then $U_{1} \mathbf{y}^{(1)}=-V_{1} \mathbf{y}^{(2)}$, or equivalently

$$
\mathbf{y}^{(1)}=-U_{1}^{-1} V_{1} \mathbf{y}^{(2)} .
$$

For $i=1, \ldots, n-m$, let $\mathbf{y}_{i, 1}$ be the solution $\mathbf{y}$ of $U \mathbf{y}=\mathbf{0}$ for which $\mathbf{y}^{(2)}=$ $\left(\delta_{1} / \delta\right) \mathbf{e}_{i}$, where $\mathbf{e}_{i}$ is the $i$-th standard basis vector of $\mathbb{Z}^{n-m}$. The coordinates of $\mathbf{y}_{i, 1}$ are all of the shape $\pm\left(\delta_{1} / \delta\right) \operatorname{det} W / \operatorname{det} U_{1}= \pm \delta^{-1} \operatorname{det} W$, where $W$ is the determinant of some submatrix of $U$ of order $m$. Hence $\mathbf{y}_{i, 1} \in \mathbb{Z}^{n}$, implying $\mathbf{y}_{i, 1} \in \mathcal{M}$ for $i=1, \ldots, n-m$. Further, by Hadamard's inequality we have 6.1.2). If $\mathbf{y}=\left(b_{1}, \ldots, b_{n}\right)^{T} \in \mathcal{M}$ then $\mathbf{y}=\left(\delta / \delta_{1}^{-1}\right) \sum_{i=m+1}^{n} b_{i} \mathbf{y}_{i-m, 1}$. This proves (6.1.1).

There are integers $a_{1}, \ldots, a_{k}$ such that $a_{1} \delta_{1}+\cdots+a_{k} \delta_{k}=\delta$. Applying (6.1.2) we see that for $y \in \mathcal{M}$ we have

$$
\mathbf{y}=\sum_{j=1}^{k} a_{j}\left(\sum_{i=1}^{n-m} b_{i, j} \mathbf{y}_{i, j}\right) .
$$

This implies that the $\mathbf{y}_{i, j}$ generate $\mathcal{M}$.
(ii) Assume without loss of generality that $U$ and $[U, \mathbf{b}]$ have rank $m$. By a result of Borosh et al. (1989), $U \mathbf{y}=\mathbf{b}$ has a solution $\mathbf{y} \in \mathbb{Z}^{n}$ such that the absolute values of the entries of $y$ are bounded above by the maximum of the absolute values of the $m \times m$-subdeterminants of $[U, \mathbf{b}]$. The upper bound for $h(\mathbf{y})$ as in the lemma easily follows from Hadamard's inequality.

Theorem 6.1.2. Let $F$ be a field, $r \geq 1$, and $R:=F\left[X_{1}, \ldots, X_{r}\right]$. Further, let $V$ be an $m \times n$-matrix and $\mathbf{b}$ an m-dimensional column vector, both consisting of polynomials from $R$ of degree $\leq d$ where $d \geq 1$.
(i) The $R$-module of $\mathrm{x} \in R^{n}$ with $V \mathbf{x}=\mathbf{0}$ is generated by vectors $\mathbf{x}$ whose coordinates are polynomials of degree at most $(2 m d)^{2^{r}}$.
(ii) Suppose that $V \mathbf{x}=\mathbf{b}$ is solvable in $\mathbf{x} \in R^{n}$. Then it has a solution $\mathbf{x}$ whose coordinates are polynomials of degree at most $(2 m d)^{2^{r}}$.

Proof. See Aschenbrenner (2004, Thms. 3.2, 3.4). Results of this type were obtained earlier, but not with a completely correct proof, by Hermann (1926) and Seidenberg (1974).

Part (ii) of Theorem 6.1.2 gives an effective method to decide ideal membership in $F\left[X_{1}, \ldots, X_{r}\right]$, provided that the field $F$ is given effectively (a notion which we are not going to formalize):

Corollary 6.1.3. Given $b, f_{1}, \ldots, f_{M} \in F\left[X_{1}, \ldots, X_{r}\right]$, it can be decided effectively whether belongs to the ideal $\mathcal{I}:=\left(f_{1}, \ldots, f_{M}\right)$ of $F\left[X_{1}, \ldots, X_{r}\right]$.

Proof. Let $d:=\max \left(\operatorname{deg} b, \operatorname{deg} f_{1}, \ldots, \operatorname{deg} f_{M}\right)$. If $b \in \mathcal{I}$ then there are $x_{1}, \ldots, x_{M} \in F\left[X_{1}, \ldots, X_{r}\right]$ of degree at most $(2 d)^{2^{r}}$ such that $b=x_{1} f_{1}+$ $\cdots x_{M} f_{M}$. By comparing the coefficients of the polynomials on the left- and right hand side, we get an inhomogeneous system of linear equations over $F$ whose solvability can be checked by standard linear algebra.

Corollary 6.1.4. Let $R:=\mathbb{Q}\left[X_{1}, \ldots, X_{r}\right]$. Further, let $V$ be an $m \times n$-matrix consisting of polynomials in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ of degrees at most $d$ and logarithmic heights at most $h$ where $d \geq 1, h \geq 1$. Then the $R$-module of $\mathbf{x} \in R^{n}$ with $V \mathbf{x}=\mathbf{0}$ is generated by vectors $\mathbf{x}$, consisting of polynomials in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ of degree at most $(2 m d)^{2^{r}}$ and logarithmic height at most $(2 m d)^{6^{r}}$.

Proof. By Theorem 6.1.2 (i) we have to study $V \mathrm{x}=0$, restricted to vectors $\mathbf{x} \in R^{n}$ consisting of polynomials in $R$ of degree at most $(2 d)^{2^{r}}$. Let $\mathbf{y}$ be the tuple of coefficients of the polynomials in x . Then $\mathrm{y} \in \mathbb{Q}^{n^{*}}$, where $n^{*} \leq n(2 m d)^{r \cdot 2^{r}}$. Further, $V \mathbf{x}$ consists of $m$ polynomials in $\mathbb{Q}\left[X_{1}, \ldots, X_{r}\right]$ of degree at most $d+(2 m d)^{2^{r}}$ all whose coefficients have to be set to 0 . The total number of coefficients of $V \mathbf{x}$ is $m^{*} \leq m\left(d+(2 m d)^{2^{r}}\right)^{r}$. Thus, the system of equations $V \mathbf{x}=0$ in polynomials in $R$ of degree at most $(2 m d)^{2^{r}}$ reduces to a system of equations $U \mathbf{y}=\mathbf{0}$ in $\mathbf{y} \in \mathbb{Q}^{n^{*}}$, where $U \in \mathbb{Z}^{m^{*}, n^{*}}$. By Lemma 6.1.1(i), the solution space of this system is generated by vectors y in $\mathbb{Z}^{n^{*}}$ of logarithmic height at most $\frac{1}{2} m^{*} \log m^{*}+m^{*} h(U) \leq(2 m d)^{6^{r}} h=: T$. Hence the corresponding vectors $\mathbf{x}$ consist of polynomials in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ of logarithmic height at most $T$.

Theorem 6.1.5. Let $r \geq 1$ and let $V$ be an $m \times n$-matrix and $\mathbf{b}$ a non-zero m-dimensional column vector consisting of polynomials in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ of degree at most $d$ and logarithmic height at most $h$ where $d \geq 1, h \geq 1$.
(i) The solution set of $\mathbf{x} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]^{n}$ with $V \mathbf{x}=\mathbf{0}$ is generated by vectors $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]^{n}$ with

$$
\operatorname{deg} x_{i} \leq(2 m d)^{\exp \left((2 r)^{r}\right)}, \quad h\left(x_{i}\right) \leq(2 m d)^{\exp \left((6 r)^{r}\right)} h \text { for } i=1, \ldots, n .
$$

(ii) Assume that

$$
\begin{equation*}
V \mathbf{x}=\mathbf{b} \tag{6.1.3}
\end{equation*}
$$

is solvable in $\mathbf{x} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]^{n}$. Then (6.1.3) has a solution $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right)$
$\in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]^{n}$ with

$$
\left.\begin{array}{l}
\operatorname{deg} x_{i} \leq d_{0}:=(2 m d)^{\exp O\left(r \log ^{*} r\right)} h,  \tag{6.1.4}\\
h\left(x_{i}\right) \leq h_{0}:=(2 m d)^{\exp O\left(r \log ^{*} r\right)} h^{r+1}
\end{array}\right\} \text { for } i=1, \ldots, n
$$

Proof. (i) This follows from Aschenbrenner (2004, Thm. 4.1) except for the height bound. The height bound can be derived from Lemma 6.1.1(i), with similar computations as in the proof of Corollary 6.1.4.
(ii) This follows from Aschenbrenner (2004, Thm. 6.5) except for the height bound. To derive such a height bound, let us restrict to solutions $\mathbf{x}=$ $\left(x_{1}, \ldots, x_{n}\right)$ of (6.1.3) with $\operatorname{deg} x_{i} \leq d_{0}$ for $i=1, \ldots, n$, and denote by $\mathbf{y}$ the vector of coefficients of the polynomials $x_{1}, \ldots, x_{n}$. Then (6.1.3) translates into a system of linear equations $U \mathbf{y}=\mathbf{b}^{*}$ which is solvable over $\mathbb{Z}$. Here, the number $m^{*}$ of equations, i.e., number of rows of $U$, is $\leq\left(d_{0}+d\right)^{r}$. Further, $h\left(U, \mathbf{b}^{*}\right) \leq h$. By Lemma 6.1.1(ii), $U \mathbf{y}=\mathbf{b}^{*}$ has a solution $\mathbf{y}$ with coordinates in $\mathbb{Z}$ of logarithmic height at most

$$
m^{*} h+\frac{1}{2} m^{*} \log m^{*} \leq(2 d)^{\exp O\left(r \log ^{*} r\right)} h^{r+1}=: h_{0}
$$

It follows that (6.1.3) has a solution $\mathrm{x} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]^{n}$ satisfying (6.1.4).

Aschenbrenner (2004) gives an example which shows that the upper bound for the degrees of the $x_{i}$ cannot depend on $d$ and $r$ only.

Part (ii) of Theorem 6.1.5 gives an effective criterion for ideal membership in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ :

Corollary 6.1.6. Given $b, f_{1}, \ldots, f_{M} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$, it can be decided effectively whether b belongs to the ideal $\mathcal{I}:=\left(f_{1}, \ldots, f_{M}\right)$ of $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$.

Proof. By Theorem6.1.5, if $b \in \mathcal{I}$ then there are $x_{1}, \ldots, x_{M} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ with upper bounds for the degrees and heights as in 6.1.4) with $m=1, n=$ $M$, such that $b=\sum_{i=1}^{M} x_{i} f_{i}$. It requires only a finite computation to check whether such $x_{i}$ exist.

Theorem 6.1.7. Let $f_{1}, \ldots, f_{M}$ be polynomials in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ of total degrees at most $d$ and logarithmic heights at most h. Let $\overline{\mathcal{I}}$ be the ideal of $\mathbb{Q}\left[X_{1}, \ldots, X_{r}\right]$ generated by $f_{1}, \ldots, f_{M}$. Then $\mathcal{I}:=\overline{\mathcal{I}} \cap \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ is an ideal generated by polynomials of total degrees at most $d+(2 d)^{(2 r)^{r}}$ and logarithmic heights at most $(6 r)^{r} \log (2 d)+h$.

Proof. The upper bound for the degrees follows from Aschenbrenner (2004), Theorem 4.7. But in his proof he uses Corollary 3.5 of his paper, some details of the proof of which he has left to the reader and which were not fully obvious to us. So we provide an argument avoiding Aschenbrenner's Corollary 3.5. First consider for a fixed positive integer $a$ the ideal $\mathcal{I}_{a}$ of polynomials $x \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ such that $a x$ is in the ideal of $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ generated by $f_{1}, \ldots, f_{M}$. We can find these $x$ by solving the equation

$$
x_{1} f_{1}+\cdots+x_{M} f_{M}-a x=0 \text { in }\left(x, x_{1}, \ldots, x_{M}\right) \in \mathbb{Z}\left[X_{1}, \ldots, X_{M}\right]^{M+1}
$$

By Theorem 6.1.5, the solutions $\left(x, x_{1}, \ldots, x_{M}\right)$ of this equation form a module over $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$, generated by tuples of polynomials of total degree at most $C:=(2 d)^{(2 r)^{r}}$. Hence for every positive integer $a, \mathcal{I}_{a}$ is generated by polynomials $g_{1} f_{1}+\cdots+g_{M} f_{M}$, where $g_{1}, \ldots, g_{M} \in a^{-1} \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ and $\operatorname{deg} g_{i} \leq C$ for $i=1, \ldots, M$. It follows that $\mathcal{I}=\cup_{a} \mathcal{I}_{a}$ is generated by polynomials $g_{1} f_{1}+\cdots+g_{M} f_{M}$, where

$$
\left\{\begin{array}{l}
g_{1}, \ldots, g_{M} \in \mathbb{Q}\left[X_{1}, \ldots, X_{r}\right],  \tag{6.1.5}\\
g_{1} f_{1}+\cdots+g_{M} f_{M} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right], \\
\operatorname{deg} g_{1}, \ldots, \operatorname{deg} g_{M} \leq C
\end{array}\right.
$$

The $\mathbb{Q}$-vector space $\mathcal{V}$ of $g_{1} f_{1}+\cdots+g_{M} f_{M}$ with $g_{1}, \ldots, g_{M}$ satisfying (6.1.5) but without the condition $g_{1} f_{1}+\cdots+g_{M} f_{M} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ is contained in the vector space of polynomials in $\mathbb{Q}\left[X_{1}, \ldots, X_{r}\right]$ of degree $\leq C+d$, whence has dimension $N \leq\binom{ C+d+r}{r}$. Further, $\mathcal{V}$ is generated by the polynomials $X_{1}^{j_{1}} \cdots X_{r}^{j_{r}} f_{i}\left(i=1, \ldots, M, j_{1}+\cdots+j_{r} \leq C\right)$, hence we can select a basis $b_{1}, \ldots, b_{N}$ of $\mathcal{V}$ from this set. Notice that $b_{1}, \ldots, b_{N}$ belong to $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ and have logarithmic heights $\leq h$. By Cassels (1959), Chap. V, Lemma 8, the $\mathbb{Z}$-module of polynomials $g_{1} f_{1}+\cdots+g_{M} f_{M}$ with (6.1.5) has a basis $c_{1}, \ldots, c_{N}$ with $c_{i}=\sum_{j=1}^{i} \xi_{i, j} b_{j}$ for $j=1, \ldots, N$, where $\xi_{i, j} \in \mathbb{Q}$ and $\left|\xi_{i, j}\right| \leq 1$ for all $i, j$. These polynomials $c_{1}, \ldots, c_{N}$ generate $\mathcal{I}$, have total degrees at most $C+d$, and logarithmic heights at most

$$
h+\log N \leq h+\log \binom{C+d+r}{r} \leq h+(6 r)^{r} \log 2 d
$$

This proves our theorem.

### 6.2 Finitely generated fields over $\mathbb{Q}$

To a field $K=\mathbb{Q}\left(z_{1}, \ldots, z_{r}\right)$ that is finitely generated over $\mathbb{Q}$ we may associate the polynomial ideal

$$
\mathcal{I}:=\left\{f \in \mathbb{Q}\left[X_{1}, \ldots, X_{r}\right]: f\left(z_{1}, \ldots, z_{r}\right)=0\right\} .
$$

By Hilbert's Basis Theorem, the ideal $\mathcal{I}$ is finitely generated, that is, $\mathcal{I}=$ $\left(f_{1}, \ldots, f_{M}\right)$ with $f_{1}, \ldots, f_{M} \in \mathbb{Q}\left[X_{1}, \ldots, X_{r}\right]$. Thus, $K$ is isomorphic to the quotient field of

$$
\begin{equation*}
\mathbb{Q}\left[X_{1}, \ldots, X_{r}\right] /\left(f_{1}, \ldots, f_{M}\right), \tag{6.2.1}
\end{equation*}
$$

and $z_{1}, \ldots, z_{r}$ may be identified with the residue classes of $X_{1}, \ldots, X_{r}$ modulo $\left(f_{1}, \ldots, f_{M}\right)$. We call $\left(f_{1}, \ldots, f_{M}\right)$ an ideal representation for $K$. We say that $K$ is given effectively if an ideal representation for it is given.

Notice that for polynomials $f_{1}, \ldots, f_{M} \in \mathbb{Q}\left[X_{1}, \ldots, X_{r}\right]$ to form an ideal representation of a field, it is necessary and sufficient that $\left(f_{1}, \ldots, f_{M}\right)$ be a prime ideal of $\mathbb{Q}\left[X_{1}, \ldots, X_{r}\right]$. This can be verified effectively, see for instance Seidenberg (1974, Sect. 46, p. 293) (there in fact Seidenberg gives a method to determine the prime ideals associated to a given ideal $\mathcal{I}$, which certainly enables one to decide whether $\mathcal{I}$ is a prime ideal).

Let $K=\mathbb{Q}\left(z_{1}, \ldots, z_{r}\right)$ be a field with given ideal representation $\mathcal{I}=$ $\left(f_{1}, \ldots, f_{M}\right)$. We say that $y \in K$ is given/can be computed (in terms of $z_{1}, \ldots, z_{r}$ ), if polynomials $g, h \in \mathbb{Q}\left[X_{1}, \ldots, X_{r}\right]$ are given/can be computed such that $y=g\left(z_{1}, \ldots, z_{r}\right) / h\left(z_{1}, \ldots, z_{r}\right)$. Thanks to Theorem 6.1.2 we can verify whether an expression $g\left(z_{1}, \ldots, z_{r}\right) / h\left(z_{1}, \ldots, z_{r}\right)$ is well-defined (i.e., $h\left(z_{1}, \ldots, z_{r}\right) \neq 0$ or equivalently, $h \notin \mathcal{I}$ ) and whether two expressions $g_{i}\left(z_{1}, \ldots, z_{r}\right) / h_{i}\left(z_{1}, \ldots, z_{r}\right)(i=1,2)$ are equal (i.e., $\left.g_{1} h_{2}-g_{2} h_{1} \in \mathcal{I}\right)$.

We note that if $y_{1}, \ldots, y_{m}$ are given in terms of $z_{1}, \ldots, z_{r}$, then for any given polynomial $h \in \mathbb{Q}\left[Y_{1}, \ldots, Y_{m}\right]$ it can be decided whether $h\left(y_{1}, \ldots, y_{m}\right) \neq$ 0 . Moreover, for any two given $g, h \in \mathbb{Q}\left[Y_{1}, \ldots, Y_{m}\right]$ with $h\left(y_{1}, \ldots, y_{m}\right) \neq 0$ one can compute $g\left(y_{1}, \ldots, y_{m}\right) / h\left(y_{1}, \ldots, y_{m}\right)$ in terms of $z_{1}, \ldots, z_{r}$.

Finally, if $y_{1}, \ldots, y_{m}$ are elements of $K$ given in terms of $z_{1}, \ldots, z_{r}$ then we say that $y$ is given/can be computed in terms of $y_{1}, \ldots, y_{m}$, if $g, h \in$ $\mathbb{Q}\left[Y_{1}, \ldots, Y_{m}\right]$ are given/can be computed, such that $h\left(y_{1}, \ldots, y_{m}\right) \neq 0$ and $y=g\left(y_{1}, \ldots, y_{m}\right) / h\left(y_{1}, \ldots, y_{m}\right)$.

Theorem 6.2.1. (i) For any $r \geq 1$ and any effectively given field $K=$ $\mathbb{Q}\left(z_{1}, \ldots, z_{r}\right)$ we can:
(i) determine a permutation $x_{1}, \ldots, x_{q}, y_{1}, \ldots, y_{t}$ of $z_{1}, \ldots, z_{r}$ such that
$x_{1}, \ldots, x_{q}$ are algebraically independent over $\mathbb{Q}$ and $y_{1}, \ldots, y_{t}$ are algebraic over $\mathbb{Q}\left(x_{1}, \ldots, x_{q}\right)$;
(ii) determine the monic minimal polynomial of $y_{1}$ over $\mathbb{Q}\left(x_{1}, \ldots, x_{q}\right)$ with coefficients given in terms of $x_{1}, \ldots, x_{q}$, and for $i=2, \ldots, t$, determine the monic minimal polynomial of $y_{i}$ over $\mathbb{Q}\left(x_{1}, \ldots, x_{q}, y_{1}, \ldots, y_{i-1}\right)$ with coefficients given in terms of $x_{1}, \ldots, x_{q}, y_{1}, \ldots, y_{i-1}$.

Proof. Repeated application of Seidenberg (1974, §23 on p. 284 and $\S 25$ on p. 285).

Theorem 6.2.2. For any effectively given field $K=\mathbb{Q}\left(z_{1}, \ldots, z_{r}\right)$ and any $y_{1}, \ldots, y_{s}, y \in K$ given in terms of $z_{1}, \ldots, z_{r}$ we can:
(i) determine a finite set of generators for the ideal

$$
\left\{f \in \mathbb{Q}\left[X_{1}, \ldots, X_{s}\right]: f\left(y_{1}, \ldots, y_{s}\right)=0\right\}
$$

(ii) decide whether $y \in \mathbb{Q}\left(y_{1}, \ldots, y_{s}\right)$ and if so, determine $g$, $h \in \mathbb{Z}\left[X_{1}, \ldots, X_{s}\right]$ such that $y=g\left(y_{1}, \ldots, y_{s}\right) / h\left(y_{1}, \ldots, y_{s}\right)$.

Proof. (i) Seidenberg (1974, §27, p. 287).
(ii) By (i) one can compute a finite set of generators for the ideal of $f \in$ $\mathbb{Q}\left[X_{1}, \ldots, X_{s+1}\right]$ such that $f\left(y_{1}, \ldots, y_{s}, y\right)=0$. Using Theorem 6.2.1 one can decide whether $y$ is algebraic over $\mathbb{Q}\left(y_{1}, \ldots, y_{s}\right)$, if so compute its monic minimal polynomial over $\mathbb{Q}\left(y_{1}, \ldots, y_{s}\right)$, and check if it has degree 1 .

Theorem 6.2.3. For any effectively given field $K=\mathbb{Q}\left(z_{1}, \ldots, z_{r}\right)$ and any polynomial $\mathcal{F} \in K\left[X_{1}, \ldots, X_{m}\right]$ with coefficients given in terms of $z_{1}, \ldots, z_{r}$, we can determine a factorization of $\mathcal{F}$ into irreducible polynomials of $K\left[X_{1}, \ldots, X_{m}\right]$, whose coefficients are all given in terms of $z_{1}, \ldots, z_{r}$. In particular we can decide whether $\mathcal{F}$ is irreducible.

Proof. This follows from Seidenberg (1974), sections 33-35 (p. 289). For $m=1$, a more precise quantitative version can be deduced from Proposition 8.2 .3 in Chapter 8 below.

Theorem 6.2.4. For any effectively given field $K=\mathbb{Q}\left(z_{1}, \ldots, z_{r}\right)$ and any monic irreducible polynomial $\mathcal{F} \in K[X]$ with coefficients given in terms of $z_{1}, \ldots, z_{r}$, we can:
(i) determine a finite set of generators for the ideal

$$
\left\{f \in \mathbb{Q}\left[X_{1}, \ldots, X_{r}, Y\right]: f\left(z_{1}, \ldots, z_{r}, y\right)=0\right\}
$$

where $y$ is a root of $\mathcal{F}$;
(ii) for any $a \in K(y)$ given in terms of $z_{1}, \ldots, z_{r}, y$, determine $b_{0}, \ldots, b_{\mathcal{F}-1} \in$ $K$, given in terms of $z_{1}, \ldots, z_{r}$, such that $a=\sum_{i=0}^{\operatorname{deg} \mathcal{F}-1} b_{i} y^{i}$.

Proof. Put $L:=K(y), d:=[L: K]$. Let $\left(f_{1}, \ldots, f_{M}\right)$ be an ideal representation for $K$. We may express $\mathcal{F}$ as $X^{d}+\left(a_{1} / b\right) X^{d-1}+\cdots+\left(a_{d} / b\right)$ where $a_{1}, \ldots, a_{d}, b$ are given as polynomials with integer coefficients in $z_{1}, \ldots, z_{r}$.

Let $y^{\prime}:=b y$. Then $K\left(y^{\prime}\right)=L$, and $y^{\prime}$ has minimal polynomial $X^{d}+$ $a_{1} X^{d-1}+\cdots+b^{d-1} a_{d}$ over $K$. We can write $b^{i-1} a_{i}=h_{i}\left(z_{1}, \ldots, z_{r}\right)$ with $h_{i} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ for $i=1, \ldots, d$. Then the ideal of polynomials $Q \in$ $\mathbb{Q}\left[X_{1}, \ldots, X_{r}, Y\right]$ with $Q\left(z_{1}, \ldots, z_{r}, y^{\prime}\right)=0$ is generated by $f_{1}, \ldots, f_{M}$ and $Y^{d}+\sum_{i=1}^{d} h_{i} Y^{d-i}$ and so these polynomials provide an ideal representation for $L$. Using Theorem 6.2.2, we can compute a finite set of generators for the ideal of $f \in \mathbb{Q}\left[X_{1}, \ldots, X_{r}, Y\right]$ with $f\left(z_{1}, \ldots, z_{r}, y\right)=0$.

Using division by $\mathcal{F}$ with remainder, from an expression of $a \in L$ in terms of $z_{1}, \ldots, z_{r}, y$ we can compute an expression $\sum_{i=0}^{d-1} b_{i} y^{i}$, with $b_{i} \in K$ given in terms of $z_{1}, \ldots, z_{r}$.

Corollary 6.2.5. For any effectively given field $K=\mathbb{Q}\left(z_{1}, \ldots, z_{r}\right)$ and any polynomial $\mathcal{F} \in K[X]$ with coefficients given in terms of $z_{1}, \ldots, z_{r}$ we can determine effectively an ideal representation for the splitting field of $\mathcal{F}$ over $K$.

Proof. We proceed by induction on $n:=\operatorname{deg} \mathcal{F}$. For $n=1$ our assertion is clear. Let $n \geq 2$. By Theorem 6.2 .3 we can compute an irreducible factor $\mathcal{F}_{1} \in K[X]$ of $\mathcal{F}$ in terms of $z_{1}, \ldots, z_{r}$ and then adjoin a zero $y_{1}$ of $\mathcal{F}_{1}$ to $K$. By Theorem 6.2.4 we can compute an ideal representation for $K_{1}:=K\left(y_{1}\right)$, and then by the induction hypothesis an ideal representation for the splitting field of $\mathcal{F}(X) /\left(X-y_{1}\right)$ over $K_{1}$. This is then the splitting field of $\mathcal{F}$ over $K$.

Corollary 6.2.6. For any effectively given ideal representations for $K=$ $\mathbb{Q}\left(z_{1}, \ldots, z_{r}\right)$ and a finite extension $L=\mathbb{Q}\left(z_{1}, \ldots, z_{r}, y_{1}, \ldots, y_{n}\right)$ of $K$, we can:
(i) determine effectively an element $y$ of $L$ in terms of $z_{1}, \ldots, z_{r}, y_{1}, \ldots, y_{n}$ such that $L=K(y)$, together with the monic minimal polynomial of $y$ over $K$, with coefficients given in terms of $z_{1}, \ldots, z_{r}$;
(ii) for any $a \in L$ given in terms of $z_{1}, \ldots, z_{r}, y_{1}, \ldots, y_{n}$, determine effectively $b_{0}, \ldots, b_{[L: K]-1} \in K$ in terms of $z_{1}, \ldots, z_{r}$ such that $a=\sum_{i=0}^{[L: K]-1} b_{i} y^{i}$.

Proof. Let $K$ be the effectively given field, put $K_{0}:=K$ and for $i=1, \ldots, n$, define $K_{i}:=K\left(y_{1}, \ldots, y_{i}\right)$, put $d_{i}:=\left[K_{i}: K_{i-1}\right]$, and denote by $G_{i}$ the monic minimal polynomial of $y_{i}$ over $K_{i-1}$. The coefficients of $G_{i}$ can be computed in terms of $z_{1}, \ldots, z_{r}, y_{1}, \ldots, y_{i-1}$ by means of Theorem6.2.4. Put $d:=[L: K]$. Then

$$
\left\{\omega_{1}, \ldots, \omega_{d}\right\}:=\left\{y_{1}^{k_{1}} \cdots y_{n}^{k_{n}}: 0 \leq k_{j}<d_{j}, j=1, \ldots, n\right\}
$$

is a $K$-basis of $L=K_{n}$. Using Theorem 6.2.4 we can compute, for any element of $L$ given in terms of $z_{1}, \ldots, z_{r}, y_{1}, \ldots, y_{n}$, an expression of this element as a $K$-linear combination of $\omega_{1}, \ldots, \omega_{d}$, with coefficients given in terms of $z_{1}, \ldots, z_{r}$. As is well-known, there are integers $c_{1}, \ldots, c_{d}$ of absolute values at most $d^{2}$ such that $y:=c_{1} \omega_{1}+\cdots+c_{d} \omega_{d}$ is a primitive element of $L$ over $K$. For each choice of the $c_{i}$ we can check whether $y$ is primitive by expressing $1, y, \ldots, y^{d-1}$ as $K$-linear combinations of $\omega_{1}, \ldots, \omega_{d}$ and check if they are linearly independent over $K$. Having found such an $y$, we can express $\omega_{1}, \ldots, \omega_{d}$, and thus every element of $L$, as $K$-linear combinations of $1, y, \ldots, y^{d-1}$ with coefficients given in terms of $z_{1}, \ldots, z_{r}$. In particular, we can express $y^{d}$ as such a linear combination, and thus find the monic minimal polynomial of $y$.

Remark. From Corollary 8.3 .4 one can deduce quantitative versions of Corollaries 6.2.5 and 6.2.6

### 6.3 Finitely generated integral domains over $\mathbb{Z}$

We need some analogues of the results mentioned above for finitely generated integral domains $\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]$ of characteristic 0 . First we recall some basic concepts introduced in Section 2.1 .

To an integral domain $A=\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]$ of characteristic 0 we may associate the polynomial ideal

$$
\mathcal{I}:=\left\{f \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]: f\left(z_{1}, \ldots, z_{r}\right)=0\right\} .
$$

By Hilbert's Basis Theorem, there are finitely many polynomials $f_{1}, \ldots, f_{M} \in$ $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ such that $\mathcal{I}=\left(f_{1}, \ldots, f_{M}\right)$. Thus, $A$ is isomorphic to

$$
\begin{equation*}
\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right] /\left(f_{1}, \ldots, f_{M}\right) \tag{6.3.1}
\end{equation*}
$$

and $z_{1}, \ldots, z_{r}$ may be identified with the residue classes of $X_{1}, \ldots, X_{r}$ modulo $\left(f_{1}, \ldots, f_{M}\right)$. We call $\left(f_{1}, \ldots, f_{M}\right)$ an ideal representation for $A$. We say that $A$ is effectively given if an ideal representation for it is given.

Notice that for polynomials $f_{1}, \ldots, f_{M} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ to form an ideal representation of an integral domain, it is necessary and sufficient that $\mathcal{I}:=$ $\left(f_{1}, \ldots, f_{M}\right)$ be a prime ideal of $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ and $\mathcal{I} \cap \mathbb{Z}=(0)$. This is equivalent to $\overline{\mathcal{I}}:=\mathcal{I} \cdot \mathbb{Q}\left[X_{1}, \ldots, X_{r}\right]$ being a prime ideal of $\mathbb{Q}\left[X_{1}, \ldots, X_{r}\right]$, $\overline{\mathcal{I}} \cap \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]=\mathcal{I}$ and $1 \notin \overline{\mathcal{I}}$. For instance by Seidenberg (1974), Sect. 46 , p. 293, one can check that $\overline{\mathcal{I}}$ is a prime ideal in $\mathbb{Q}\left[X_{1}, \ldots, X_{r}\right]$ and by Theorem 6.1.2 (ii) one can check that $1 \notin \overline{\mathcal{I}}$. To verify that $\overline{\mathcal{I}} \cap \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]=\mathcal{I}$, one can compute a set of generators for $\overline{\mathcal{I}} \cap \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ using Theorem 6.1.7, and then check, using Theorem 6.1.5, whether these generators belong to $\mathcal{I}$.

Let $A=\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]$ be an integral domain with given ideal representation $\mathcal{I}=\left(f_{1}, \ldots, f_{M}\right)$. We say that $y \in A$ is given/can be computed (as a polynomial in $z_{1}, \ldots, z_{r}$ ), if a polynomial $g \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ is given/can be computed such that $y=g\left(z_{1}, \ldots, z_{r}\right)$. Thanks to Corollary 6.1.6 we can verify whether two expressions $g_{i}\left(z_{1}, \ldots, z_{r}\right)(i=1,2)$ are equal (i.e., $g_{1}-g_{2} \in$ $\mathcal{I})$.

Finally, if $y_{1}, \ldots, y_{m}$ are given elements of $A$, then we say that $y$ is given/can be computed as a polynomial in terms of $y_{1}, \ldots, y_{m}$, if $g \in \mathbb{Z}\left[Y_{1}, \ldots, Y_{m}\right]$ are given/can be computed, such that $y=g\left(y_{1}, \ldots, y_{m}\right)$.

Theorem 6.3.1. For any effectively given integral domain $A=\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]$ of characteristic 0 and any given monic irreducible polynomial $\mathcal{F} \in A[X]$ with coefficients given as polynomials in $z_{1}, \ldots, z_{r}$, we can:
(i) determine effectively a finite set of generators for the ideal

$$
\left\{f \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}, Y\right]: f\left(z_{1}, \ldots, z_{r}, y\right)=0\right\}
$$

where $y$ is a root of $\mathcal{F}$;
(ii) for any $a \in A[y]$ given as polynomial in $z_{1}, \ldots, z_{r}, y$, determine effectively $b_{0}, \ldots, b_{\operatorname{deg} \mathcal{F}-1} \in A$, given as polynomials in $z_{1}, \ldots, z_{r}$, such that $a=\sum_{i=0}^{\operatorname{deg} \mathcal{F}-1} b_{i} y^{i}$.

Proof. Similar to Theorem 6.2.4
Theorem 6.3.2. For any effectively given integral domain $A=\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]$ of characteristic 0 finitely generated over $\mathbb{Z}$, any $m \times n$-matrix $V$ with entries
in the quotient field $K$ of $A$, and any column vector $\mathbf{b} \in K^{n}$, all with entries given in terms of $z_{1}, \ldots, z_{r}$ we can:
(i) determine effectively a finite set of generators, with coordinates given as polynomials in $z_{1}, \ldots, z_{r}$, for the $A$-module $\left\{\mathbf{x} \in A^{n}: V \mathbf{x}=\mathbf{0}\right\}$;
(ii) decide whether $V \mathbf{x}=\mathbf{b}$ is solvable in $\mathbf{x} \in A^{n}$ and if so, determine effectively a solution with coordinates given as polynomials in $z_{1}, \ldots, z_{r}$.

Proof. After multiplication with a suitable non-zero element of $A$, we may assume that $V$ and $\mathbf{b}$ have their entries in $A$, and are given as polynomials with integer coefficients in $z_{1}, \ldots, z_{r}$. Suppose that $A$ is given by an ideal representation $\mathcal{I}=\left(f_{1}, \ldots, f_{M}\right)$. By choosing representatives for the entries of $V$ and $\mathbf{b}$ in $R:=\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ we can rewrite the systems of linear equations in (i), (ii) as systems of linear congruence equations modulo $\mathcal{I}$ in unknowns from $R$. By writing the elements of $\mathcal{I}$ as $R$-linear combinations of $f_{1}, \ldots, f_{M}$, we can rewrite these congruence systems as systems of linear equations as considered in Theorem 6.1.5 and apply the latter.

Theorem 6.3.3. For any effectively given field $K=\mathbb{Q}\left(z_{1}, \ldots, z_{r}\right)$ and any $y_{1}, \ldots, y_{s}, y \in K$ given in terms of $z_{1}, \ldots, z_{r}$ we can:
(i) determine effectively a finite set of generators for the ideal

$$
\mathcal{J}:=\left\{f \in \mathbb{Z}\left[X_{1}, \ldots, X_{s}\right]: f\left(y_{1}, \ldots, y_{s}\right)=0\right\}
$$

(ii) decide whether $y \in \mathbb{Z}\left[y_{1}, \ldots, y_{s}\right]$ and if so, determine effectively $g \in$ $\mathbb{Z}\left[X_{1}, \ldots, X_{s}\right]$ such that $y=g\left(y_{1}, \ldots, y_{s}\right)$.
Proof. (i) Theorem 6.2.2 (i) provides an algorithm to compute a finite set of generators for the ideal

$$
\overline{\mathcal{J}}:=\left\{f \in \mathbb{Q}\left[X_{1}, \ldots, X_{s}\right]: f\left(y_{1}, \ldots, y_{s}\right)=0\right\}
$$

and subsequently, by means of Theorem 6.1.7 one can determine a finite set of generators for $\overline{\mathcal{J}} \cap \mathbb{Z}\left[X_{1}, \ldots, X_{s}\right]=\mathcal{J}$.
(ii) By Theorem 6.2 .2 (ii) it can be decided whether $y \in \mathbb{Q}\left(y_{1}, \ldots, y_{s}\right)$ and if so, elements $a, b$ of $\mathbb{Z}\left[y_{1}, \ldots, y_{s}\right]$ can be computed, both represented as polynomials with integer coefficients in $y_{1}, \ldots, y_{s}$, such that $y=a / b$. By Theorem 6.3.2, it can be decided whether $a / b \in \mathbb{Z}\left[y_{1}, \ldots, y_{s}\right]$ and if so, a polynomial $g \in \mathbb{Z}\left[X_{1}, \ldots, X_{s}\right]$ can be computed such that $a / b=g\left(y_{1}, \ldots, y_{s}\right)$.

Let $A=\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]$ be an effectively given integral domain finitely generated over $\mathbb{Z}$, and $K$ its quotient field. We consider finitely generated
$A$-modules contained in $K$ (so in other words, fractional ideals of $A$ ). The $A$-module generated by elements $y_{1}, \ldots, y_{m}$ is denoted by $\left(y_{1}, \ldots, y_{m}\right)$. We say that such a module is given/can be determined in terms of $z_{1}, \ldots, z_{r}$, if a finite set of generators for it is given/can be determined in terms of $z_{1}, \ldots, z_{r}$.

We say that a finitely generated $A$-module $\mathcal{M} \subset K$ is given if a finite set of $A$-module generators for $\mathcal{M}$ is given.

Theorem 6.3.4. For any two given $A$-submodules $\mathcal{M}_{1}, \mathcal{M}_{2}$ of $K$, one can
(i) decide whether $\mathcal{M}_{1} \subseteq \mathcal{M}_{2}$;
(ii) compute a finite set of $A$-module generators for $\mathcal{M}_{1} \cap \mathcal{M}_{2}$.

Proof. Let $\mathcal{M}_{1}=\left(a_{1}, \ldots, a_{u}\right), \mathcal{M}_{2}=\left(b_{1}, \ldots, b_{v}\right)$ with the $a_{i}, b_{j} \in K$ given in terms of $z_{1}, \ldots, z_{r}$. Then (i) comes down to checking whether $a_{1}, \ldots, a_{u} \in$ $\mathcal{M}_{2}$, which is a special case of part (ii) of Theorem 6.3.2. To determine a finite set of $A$-module generators for $\mathcal{M}_{1} \cap \mathcal{M}_{2}$, using part (i) of Theorem 6.3.2 one first determines a finite set of $A$-module generators for the solution set $\left(x_{1}, \ldots, x_{u}, y_{1}, \ldots, y_{v}\right) \in A^{u+v}$ of $\sum_{i=1}^{u} x_{i} a_{i}=\sum_{j=1}^{v} y_{j} b_{j}$ and then for each generator one takes the coordinates $x_{1}, \ldots, x_{u}$, and subsequently $\sum_{i=1}^{u} x_{i} a_{i}$.

The quotient module of two $A$-modules $\mathcal{M}_{1}, \mathcal{M}_{2}$ with $\mathcal{M}_{1} \subseteq \mathcal{M}_{2}$ is given by $\mathcal{M}_{2} / \mathcal{M}_{1}:=\left\{a+\mathcal{M}_{1}: a \in \mathcal{M}_{2}\right\}$, with the usual addition and scalar multiplication of cosets. By a full system of representatives for $\mathcal{M}_{2} / \mathcal{M}_{1}$ we mean a subset of $\mathcal{M}_{2}$ consisting of precisely one element from each of the cosets $a+\mathcal{M}_{1}\left(a \in \mathcal{M}_{2}\right)$.

Theorem 6.3.5. For any effectively given integral domain $A=\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]$ finitely generated over $\mathbb{Z}$ and any two given finitely generated $A$-modules $\mathcal{M}_{1}, \mathcal{M}_{2}$ with $\mathcal{M}_{1} \subseteq \mathcal{M}_{2}$ contained in the quotient field of $A$, it can be decided whether $\mathcal{M}_{2} / \mathcal{M}_{1}$ is finite. If this is the case, a full system of representatives for $\mathcal{M}_{2} / \mathcal{M}_{1}$ can be determined in terms of $z_{1}, \ldots, z_{r}$.

Proof. The proof is too lengthy to be inserted here. See for instance Evertse and Győry (2017b, Prop. 3.6).

Let $A$ be an integral domain, $K$ its quotient field, and $G$ a finite extension of $K$. Then we denote by $A_{G}$ the integral closure of $A$ in $G$. In particular, $A_{K}$ is the integral closure of $A$ in its quotient field $K$. Recall that $G$ is effectively given if an irreducible polynomial $P \in K[X]$ is given such that $G \cong K[X] /(P)$. The irreducibility of $P$ can be checked for instance by means of Theorem 6.2.3.

Theorem 6.3.6. Assume that $A$ and a finite extension $G$ of its quotient field $K$ are effectively given. Then one can compute a finite set of $A$-module generators for $A_{G}$. Moreover, one can compute an ideal representation for $A_{G}$.

Proof. A method to compute a finite set of $A$-module generators for $A_{G}$ can be derived by combining results of Nagata (1956), de Jong (1998), Matsumura (1986) and Matsumoto (2000), see for more details Evertse and Győry (2017a, Cor. 10.7.18). Then an ideal representation for $A_{G}$ can be computed using Theorem 6.3.3.

We finish with two consequences, related to Theorems 1.6.1 and 1.6.3
Corollary 6.3.7. Assume that $A$ is effectively given. Let $n$ be an integer $\geq 2$. Then one can effectively decide whether the quotient $A$-module $\left(\frac{1}{n} A \cap A_{K}\right) / A$ is finite and if so, compute a full system of representatives for $\left(\frac{1}{n} A \cap A_{K}\right) / A$. Proof. Immediate consequence of Theorems 6.3.4 6.3.6.

Let again $A$ be effectively given and denote by $K$ its quotient field. Recall that a finite étale $K$-algebra $\Omega$ is effectively given if a separable polynomial $P \in K[X]$ is given such that $\Omega \cong K[X] /(P)$. The separability of $P$ can be checked for instance by means of Theorem 6.2.3.

Let $n:=\operatorname{deg} P$ and denote by $\theta$ the residue class of $X$ modulo $P$. Then $\left\{1, \theta, \ldots, \theta^{n-1}\right\}$ is a $K$-basis of $\Omega$ and every element of $\Omega$ can be expressed uniquely as $\sum_{i=0}^{n-1} a_{i} \theta^{i}$ with all $a_{i} \in K$. We say that such an element is given if the $a_{i}$ are given in terms of $z_{1}, \ldots, z_{r}$.

We say that a finitely generated $A$-module $\mathcal{M} \subset \Omega$ is effectively given, if $\omega_{1}, \ldots, \omega_{u}$ are given such that $\mathcal{M}=\left\{\sum_{i=1}^{u} x_{i} \omega_{i}: x_{1}, \ldots, x_{m} \in A\right\}$.

Corollary 6.3.8. Assume that $A$, a finite étale $K$-algebra $\Omega$, and a finitely generated $A$-module $\mathcal{M} \subset \Omega$ are effectively given.
(i) For any given $\alpha \in \Omega$ it can be decided whether $\alpha \in \mathcal{M}$.
(ii) A set of $A$-module generators for $\mathcal{M} \cap K$ can be determined effectively in terms of $z_{1}, \ldots, z_{r}$.

Proof. Let $P, n, \theta$ be as above. Further, let $\left\{\omega_{1}, \ldots, \omega_{u}\right\}$ be an effectively given set of $A$-module generators for $\mathcal{M}$. Then $\omega_{1}, \ldots, \omega_{u}$ can be expressed as $K$-linear combinations of $1, \theta, \ldots, \theta^{n-1}$, with coefficients given in terms of $z_{1}, \ldots, z_{r}$. Then we may express elements of $\mathcal{M}$ as $\sum_{k=0}^{n-1} \ell_{k}(\mathbf{x}) \theta^{k}$ with $\mathrm{x} \in A^{u}$, where $\ell_{0}, \ldots, \ell_{n-1}$ are linear forms from $K\left[X_{1}, \ldots, X_{u}\right]$ with coefficients given in terms of $z_{1}, \ldots, z_{r}$.
(i) Let $\alpha=\sum_{k=0}^{n-1} a_{k} \theta^{k}$, where $a_{0}, \ldots, a_{n-1} \in K$ are effectively given in terms of $z_{1}, \ldots, z_{r}$. Clearly, $\alpha \in \mathcal{M}$ if and only if there is $\mathbf{x} \in A^{u}$ with $\ell_{k}(\mathbf{x})=a_{k}$ for $k=0, \ldots, n-1$, and this can be ckecked by means of Theorem 6.3.2 (ii).
(ii) By Theorem 6.3 .2 (i), we can compute a set of $A$-module generators, say $\left\{\mathbf{x}_{1}, \ldots, \mathbf{x}_{v}\right\}$, for

$$
\left\{\mathbf{x} \in A^{u}: \ell_{1}(\mathbf{x})=\cdots=\ell_{n-1}(\mathbf{x})=0\right\} .
$$

Then $\left\{\ell_{0}\left(\mathbf{x}_{1}\right), \ldots, \ell_{0}\left(\mathbf{x}_{v}\right)\right\}$ is a set of $A$-module generators for $\mathcal{M} \cap K$.
Corollary 6.3.9. Assume that $A$, a finite étale $K$-algebra $\Omega$, and a finitely generated $A$-module $\mathcal{O} \subset \Omega$ are effectively given.
(i) It can be decided whether $\mathcal{O}$ is an $A$-order of $\Omega$.
(ii) If $\mathcal{O}$ is an $A$-order of $\Omega$, one can decide whether the quotient $A$-module $(\mathcal{O} \cap K) / A$ is finite, and if so, compute a full system of representatives for $(\mathcal{O} \cap K) / A$.

Proof. (i) Let $\left\{\omega_{1}, \ldots, \omega_{u}\right\}$ be a set of $A$-module generators for $\mathcal{O}$, and let $\ell_{0}, \ldots, \ell_{n-1}$ be the linear forms from the proof of Corollary 6.3.8.

We first have to verify that the linear forms $\ell_{0}, \ldots, \ell_{n-1}$ have rank $n$ over $K$, to make sure that $\mathcal{O}$ contains a $K$-basis of $\Omega$; this is simply a matter of computing a determinant. The next thing to verify is whether $1 \in \mathcal{O}$ and $\omega_{i} \omega_{j} \in \mathcal{O}$ for $i, j=1, \ldots, u$; this can be done using Corollary 6.3.8(i).
(ii) Using Corollary 6.3 .8 (ii) we can compute a finite set of $A$-module generators for $\mathcal{O} \cap K$. With these generators for $\mathcal{O} \cap K$ and Theorem 6.3.5, we can check whether $(\mathcal{O} \cap K) / A$ is finite, and if so, compute a full system of representatives.

## Chapter 7

## The effective specialization method

In this chapter we present our general effective specialization method and make it ready for application to our Diophantine equations under consideration.

The general idea of our method is to reduce our given Diophantine equations over $A$ to Diophantine equations of the same type over function fields and over number fields by means of an effective specialization method. In the first step we extend our equations to equations of the same form over a finitely generated overring $B$ of $A$ of a special type which is more convenient to deal with.

As was mentioned in the Introduction and Chapter 3 such an effective specialization argument was elaborated by Győry $(1983,1984 b)$ for decomposable form equations and discriminant equations over a restricted class of finitely generated integral domains $A$ containing both algebraic and transcendental elements, of which the elements have some "good" effective representations. That time, for any finitely generated domains $A$, no algorithm was known to select those solutions from the overring $B$ which are contained in $A$. A later effective result by Aschenbrenner (2004) on systems of linear equations over polynomial rings over $\mathbb{Z}$ enabled us in our paper Evertse and Győry (2013) to surmount this difficulty and extend the method to the case of arbitrary finitely generated domains $A$.

Below we follow closely our paper Evertse and Győry (2013). Save for some small modifications, Lemmas 7.2.3, 7.2.4, 7.2.6, 7.3.1, 7.3.3 and 7.4.27.4.7, as well as Propositions 7.2.5, 7.2.7 below are taken from that paper. For convenience of the reader, we reproduce here their proofs.

### 7.1 Notation

As in the previous chapters, for a polynomial $f$ with coefficients in $\mathbb{Z}$ we denote by $\operatorname{deg} f, h(f)$ its total degree and its logarithmic height, i.e., the logarithm of the maximum of the absolute values of its coefficients. Further, we define $\log ^{*} u:=\max (1, \log u)$ for $u>0$.

Let $A=\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]$ be an integral domain of characteristic 0 finitely generated over $\mathbb{Z}$, and denote by $K$ the quotient field of $A$. We assume that $r>0$. We have

$$
\begin{equation*}
A \cong \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right] / \mathcal{I} \tag{7.1.1}
\end{equation*}
$$

where $\mathcal{I}$ is the ideal of polynomials $f \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ such that $f\left(z_{1}, \ldots, z_{r}\right)=$ 0 . The ideal $\mathcal{I}$ is finitely generated. We assume that

$$
\begin{array}{ll}
\mathcal{I}=\left(f_{1}, \ldots, f_{M}\right) & \text { with } \operatorname{deg} f_{i} \leq d, h\left(f_{i}\right) \leq h \text { for } i=1, \ldots, M, \\
& \text { where } d \geq 1, h \geq 1 \tag{7.1.2}
\end{array}
$$

A representative for $\alpha \in A$ is a polynomial $\widetilde{\alpha} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ such that $\alpha=\widetilde{\alpha}\left(z_{1}, \ldots, z_{r}\right)$, or, with the representation (7.1.2 for $A, \alpha=\widetilde{\alpha}(\bmod \mathcal{I})$.

We assume that $K$ has transcendence degree $q \geq 0$ over $\mathbb{Q}$. For $q>0$, we assume without loss of generality that $z_{1}, \ldots, z_{q}$ are algebraically independent over $\mathbb{Q}$, and that $z_{1}=X_{1}, \ldots, z_{q}=X_{q}$. Write $t:=r-q$ and rename $z_{q+1}, \ldots, z_{r}$ as $y_{1}, \ldots, y_{t}$. Put

$$
\begin{align*}
& A_{0}:=\mathbb{Z}\left[X_{1}, \ldots, X_{q}\right], K_{0}:=\mathbb{Q}\left(X_{1}, \ldots, X_{q}\right) \text { if } q>0,  \tag{7.1.3}\\
& A_{0}:=\mathbb{Z}, K_{0}:=\mathbb{Q} \text { if } q=0
\end{align*}
$$

so that

$$
A=A_{0}\left[y_{1}, \ldots, y_{t}\right], \quad K=K_{0}\left(y_{1}, \ldots, y_{t}\right), \quad\left[K: K_{0}\right]<\infty .
$$

For $a \in A_{0}$ we denote by $\operatorname{deg} a, h(a)$ the total degree and logarithmic height of $a$ if $q>0$, while we put $\operatorname{deg} a:=0$ and $h(a):=\log |a|$ if $q=0$.

Recall that $A_{0}$ is a unique factorization domain with unit group $A_{0}^{*}=$ $\{ \pm 1\}$. This implies that any finite set $a_{1}, \ldots, a_{r}$ of non-zero elements of $A_{0}$ has an up to sign unique greatest common divisor $b:=\operatorname{gcd}\left(a_{1}, \ldots, a_{r}\right)$ such that $c \in A_{0}$ divides $a_{1}, \ldots, a_{r}$ if and only if $c$ divides $b$.

### 7.2 Construction of a more convenient ground domain $B$

In this section we prove in a more general form that there are $w \in A, g \in$ $A_{0} \backslash\{0\}$ such that

$$
A \subseteq B:=A_{0}\left[w, g^{-1}\right]
$$

and $w$ has minimal polynomial $\mathcal{F}(X)=X^{D}+\mathcal{F}_{1} X^{D-1}+\cdots+\mathcal{F}_{D}$ over $K_{0}$ with $\mathcal{F}_{i} \in A_{0}$ for $i=1, \ldots, D$. Further, we give explicit upper bounds in terms of $r, q, d, h$ for $D$ and the degrees and logarithmic heights of $g, \mathcal{F}_{1}, \ldots, \mathcal{F}_{D}$. Moreover, we require that $\mathcal{A} \subset B^{*}$ for some prescribed finite set $\mathcal{A}$.

We shall need several lemmas.
Lemma 7.2.1. Let $b_{1}, \ldots, b_{n} \in A_{0}$ and $b=b_{1} \cdots b_{n}$. Then

$$
\left|h(b)-\sum_{i=1}^{n} h\left(b_{i}\right)\right| \leq q \operatorname{deg} b .
$$

Proof. Consequence of Corollary 4.1.6.
Write $\mathbf{Y}:=\left(X_{q+1}, \ldots, X_{r}\right)$ and $K_{0}(\mathbf{Y}):=K_{0}\left(X_{q+1}, \ldots, X_{r}\right)$. Given $f \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$, we write $f^{*}$ for $f$ but viewed as a polynomial in the variables $\mathbf{Y}=\left(X_{q+1}, \ldots, X_{r}\right)$, with coefficients in $A_{0}$. Given $f \in K_{0}(\mathbf{Y})$, we denote by $\operatorname{deg}_{\mathbf{Y}} f$ its total degree with respect to $\mathbf{Y}$; recall that the total degree deg $b$ of $b \in A_{0}$ is taken with respect to $X_{1}, \ldots, X_{q}$. With this notation (7.1.1) and (7.1.2) can be rewritten as

$$
\begin{align*}
& A \cong A_{0}[\mathbf{Y}] /\left(f_{1}^{*}, \ldots, f_{M}^{*}\right) \\
& \operatorname{deg}_{\mathbf{Y}} f_{i}^{*} \leq d \text { for } i=1, \ldots, M,  \tag{7.2.1}\\
& \text { the coefficients of } f_{1}^{*}, \ldots, f_{M}^{*} \text { in } A_{0} \text { have total degrees } \\
& \text { at most } d \text { and logarithmic heights at most } h .
\end{align*}
$$

Lemma 7.2.2. Let $\mathbb{k}$ be an algebraically closed field of characteristic 0 , let $s$ be a positive integer and let $\mathcal{X}$ be an algebraic subset of $\mathbb{k}^{s}$ given by polynomials of total degree at most $d$. Further, let $\mathcal{Y}$ be an algebraic subset of $\mathcal{X}$ such that $\mathcal{X} \backslash \mathcal{Y}$ is finite. Then $\mathcal{X} \backslash \mathcal{Y}$ has cardinality at most $d^{s}$.

Proof. See Corollary 7.5.3 of Evertse and Győry (2015). It is proved by re-
peated application of the version of Bezout's theorem from algebraic geometry as stated in Hartshorne (1977) Ch. 1, Theorem 7.7.

Let $D:=\left[K: K_{0}\right]$ and let $\sigma_{1}, \ldots, \sigma_{D}$ denote the $K_{0}$-isomorphic embeddings of $K$ in an algebraic closure $\overline{K_{0}}$ of $K_{0}$.
Lemma 7.2.3. (i) We have $D \leq d^{t}$.
(ii) There exist rational integers $a_{1}, \ldots, a_{t}$ with $\left|a_{i}\right| \leq D^{2}$ for $i=1, \ldots, t$, such that for $v:=a_{1} y_{1}+\cdots+a_{t} y_{t}$ we have $K=K_{0}(v)$.
Proof. (i) The images of $\left(y_{1}, \ldots, y_{t}\right)$ under $\sigma_{1}, \ldots, \sigma_{D}$ belong to

$$
\mathcal{W}:=\left\{\mathbf{y} \in \bar{K}_{0}^{t}: f_{1}^{*}(\mathbf{y})=\cdots=f_{M}^{*}(\mathbf{y})=0\right\}
$$

Conversely, each assignment $\mathbf{Y}=\left(X_{q+1}, \ldots, X_{r}\right) \mapsto \mathbf{y}$ with $\mathbf{y} \in \mathcal{W}$ yields a $K_{0}$-isomorphic embedding of $K$ in $\bar{K}_{0}$ since $K \cong K_{0}[\mathbf{Y}] /\left(f_{1}^{*}, \ldots, f_{M}^{*}\right)$. Thus $|\mathcal{W}|=D<\infty$. Now Lemma 7.2.2 with $\mathbb{k}=\bar{K}_{0}, \mathcal{X}=\mathcal{W}, \mathcal{Y}=\emptyset$ gives $|\mathcal{W}| \leq d^{t}$. Hence $D \leq d^{t}$.
(ii) For integers $a_{1}, \ldots, a_{t}$, the quantity $v:=a_{1} y_{1}+\cdots+a_{t} y_{t}$ generates $K$ over $K_{0}$ if and only if $a_{1} \sigma_{1}\left(y_{1}\right)+\cdots+a_{t} \sigma_{t}\left(y_{t}\right)$ are distinct for $i=1, \ldots, D$. There are integers $a_{j}$ with $\left|a_{j}\right| \leq D^{2}, j=1, \ldots, t$, for which this holds.
Lemma 7.2.4. There are $\mathcal{G}_{0}, \ldots, \mathcal{G}_{D} \in A_{0}$ such that

$$
\begin{equation*}
\sum_{i=0}^{D} \mathcal{G}_{i} v^{D-i}=0, \quad \mathcal{G}_{0} \cdot \mathcal{G}_{D} \neq 0 \tag{7.2.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{deg} \mathcal{G}_{i} \leq(2 d)^{\exp O(r)}, \quad h\left(\mathcal{G}_{i}\right) \leq(2 d)^{\exp O(r)} h \tag{7.2.3}
\end{equation*}
$$

for $i=0, \ldots, D$.
Proof. We write $\mathbf{Y}:=\left(X_{q+1}, \ldots, X_{r}\right)$ and $\mathbf{Y}^{\mathbf{u}}:=X_{q+1}^{u_{1}} \cdot \ldots \cdot X_{q+t}^{u_{t}},|\mathbf{u}|:=$ $u_{1}+\cdots+u_{t}$ for tuples of non-negative integers $\mathbf{u}=\left(u_{1}, \ldots, u_{t}\right)$. Further, we define $W:=\sum_{j=1}^{t} a_{j} X_{q+j}$.

Since $v$ has degree $D$ over $K_{0}$, elements $\mathcal{G}_{0}, \ldots, \mathcal{G}_{D}$ of $A_{0}$ as in (7.2.2) exist. By (7.2.1) there are $g_{1}^{*}, \ldots, g_{M}^{*} \in A_{0}[\mathbf{Y}]$ with the property

$$
\begin{equation*}
\sum_{i=0}^{D} \mathcal{G}_{i} W^{D-i}=\sum_{j=1}^{M} g_{j}^{*} f_{j}^{*} \tag{7.2.4}
\end{equation*}
$$

By Theorem 6.1.2 (ii), applied with the field $F=K_{0}$, there are polynomials $g_{j}^{*} \in K_{0}[\mathbf{Y}]$ satisfying (7.2.4) of degrees at most $(2 \max (d, D))^{2^{t}} \leq$ $\left(2^{d^{t}}\right)^{2^{t}}=: d^{\prime}$ in Y. Multiplying $\mathcal{G}_{0}, \ldots, \mathcal{G}_{D}$ with an appropriate non-zero factor from $A_{0}$, we may assume that $g_{j}^{*}$ are polynomials in $A_{0}[\mathbf{Y}]$ of degree at most $d^{\prime}$ in Y. Considering (7.2.4) with such polynomials $g_{j}^{*}$, we obtain

$$
\begin{equation*}
\sum_{i=0}^{D} \mathcal{G}_{i} W^{D-i}=\sum_{j=1}^{M}\left(\sum_{|\mathbf{u}| \leq d^{\prime}} g_{j, \mathbf{u}} \mathbf{Y}^{\mathbf{u}}\right)\left(\sum_{|\mathbf{v}| \leq d} f_{j, \mathbf{v}} \mathbf{Y}^{\mathbf{v}}\right) \tag{7.2.5}
\end{equation*}
$$

where $g_{j, \mathbf{u}} \in A_{0}$ and $f_{j}^{*}=\sum_{|\mathbf{v}| \leq d} f_{j, \mathbf{v}} \mathbf{Y}^{\mathbf{v}}$ with $f_{j, \mathbf{v}} \in A_{0}$. Here $\mathcal{G}_{0}, \ldots, \mathcal{G}_{D}$ and the polynomials $g_{j, \mathbf{u}}$ are viewed as the unknowns of (7.2.5). Thus (7.2.5) has solutions with $\mathcal{G}_{0} \cdot \mathcal{G}_{D} \neq 0$.

Consider (7.2.5) as a system of linear equations $V \mathbf{x}=\mathbf{0}$ over $K_{0}$, where $\mathbf{x}$ consists of $\mathcal{G}_{i}, i=0, \ldots, D$, and $g_{j . \mathbf{u}}, j=1, \ldots, M,|\mathbf{u}| \leq d^{\prime}$. Using Lemma 7.2.3. (4.1.7), 4.1.8), we get that the polynomial $W^{D-i}=\left(\sum_{k=1}^{t} a_{k} X_{q+k}\right)^{D-i}$ has logarithmic height at most $O\left(D \log \left(2 D^{2} t\right)\right) \leq(2 d)^{O(t)}$. Together with (7.2.1) this gives that the entries of the matrix $V$ are elements of $A_{0}$ of degrees at most $d$ and logarithmic heights at most $h^{\prime}:=\max \left((2 d)^{O(t)}, h\right)$. Further, the number of rows of $V$ is at most the number of monomials in $\mathbf{Y}$ of degree at most $d+d^{\prime}$ which is bounded above by

$$
m_{0}:=\binom{d+d^{\prime}+t}{t} \leq(2 d)^{\exp O(r)}
$$

In view of Corollary 6.1 .4 the $A_{0}$-module of solutions of (7.2.5) is generated by vectors $\mathbf{x}=\left(\mathcal{G}_{0}, \ldots, \mathcal{G}_{D},\left\{g_{i, \mathbf{u}}\right\}\right)$, whose coordinates are elements from $A_{0}$ of degrees and logarithmic heights at most

$$
\left(2 m_{0} d\right)^{2^{q}},\left(2 m_{0} d\right)^{6^{q}} h^{\prime},
$$

respectively. Among these vectors x there is one with $\mathcal{G}_{0} \neq 0$ and also one with $\mathcal{G}_{D} \neq 0$ since otherwise (7.2.5) would have no solution with $\mathcal{G}_{0} \cdot \mathcal{G}_{D} \neq$ 0 , contradicting what we already observed about (7.2.2) and (7.2.3). Either, among these vectors $\mathbf{x}$ there is one with $\mathcal{G}_{0} \mathcal{G}_{D} \neq 0$; or there is no such vector but then among these vectors there are $\mathbf{x}_{1}$ with $\mathcal{G}_{0}=0, \mathcal{G}_{D} \neq 0$ and $\mathbf{x}_{2}$ with $\mathcal{G}_{0} \neq 0, \mathcal{G}_{D}=0$, so that $\mathbf{x}:=\mathbf{x}_{1}+\mathbf{x}_{2}$ has $\mathcal{G}_{0} \mathcal{G}_{D} \neq 0$. Using the above established upper bound for $m_{0}$, we infer that in both cases, the coordinates
of $\mathbf{x}$ have degrees and logarithmic heights at most

$$
(2 d)^{\exp O(r)},(2 d)^{\exp O(r)} h,
$$

respectively. This completes the proof.
It will be more convenient to work with

$$
w:=\mathcal{G}_{0} v=\mathcal{G}_{0}\left(a_{1} y_{1}+\cdots+a_{t} y_{t}\right) \text { if } D \geq 2, \quad w:=1 \text { if } D=1 .
$$

Notice that by (7.2.3) and the estimates $\left|a_{i}\right| \leq D^{2} \leq d^{2 r}$ from Lemma 7.2.3, this $w$ belongs to $A$ and has a representative $\widetilde{w} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ with

$$
\begin{equation*}
\operatorname{deg} \widetilde{w} \leq(2 d)^{\exp O(r)}, \quad h(\widetilde{w}) \leq(2 d)^{\exp O(r)} h \tag{7.2.6}
\end{equation*}
$$

The following proposition follows at once from Lemmas 7.2.1, 7.2.3 and 7.2.4

Proposition 7.2.5. We have $K=K_{0}(w)$, where $w \in A$, $w$ is integral over $A_{0}$ and $w$ has minimal polynomial $\mathcal{F}(X)=X^{D}+\mathcal{F}_{1} X^{D-1}+\cdots+\mathcal{F}_{D}$ over $K_{0}$ such that

$$
\mathcal{F}_{i} \in A_{0}, \operatorname{deg} \mathcal{F}_{i} \leq(2 d)^{\exp O(r)}, h\left(\mathcal{F}_{i}\right) \leq(2 d)^{\exp O(r)} h
$$

for $i=1, \ldots, D$.
In what follows, we fix such a $w \in A$. Since $A_{0}=\mathbb{Z}\left[X_{1}, \ldots, X_{q}\right]$ is a unique factorization domain, the greatest common divisor of a finite set of elements of $A_{0}$ is well-defined and uniquely determined up to sign. With every $\alpha \in K$ we associate an up to sign unique tuple $P_{\alpha, 0}, \ldots, P_{\alpha, D-1}, Q_{\alpha}$ from $A_{0}$ such that

$$
\begin{equation*}
\alpha=Q_{\alpha}^{-1} \sum_{j=0}^{D-1} P_{\alpha, j} w^{j} \text { with } Q_{\alpha} \neq 0, \operatorname{gcd}\left(P_{\alpha, 0}, \ldots, P_{\alpha, D-1}, Q_{\alpha}\right)=1 \tag{7.2.7}
\end{equation*}
$$

We keep the notation from (7.1.2). Set

$$
\begin{aligned}
\overline{\operatorname{deg}} \alpha & :=\max \left(\operatorname{deg} P_{\alpha, 0}, \ldots, \operatorname{deg} P_{\alpha, D-1}, \operatorname{deg} Q_{\alpha}\right), \\
\bar{h}(\alpha) & :=\max \left(h\left(P_{\alpha, 0}\right), \ldots, h\left(P_{\alpha, D-1}\right), h\left(Q_{\alpha}\right)\right) .
\end{aligned}
$$

Lemma 7.2.6. Let $\alpha \in K^{*}$ and let $(a, b)$ be a pair of representatives for $\alpha$ with $a, b \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right], b \neq \mathcal{I}$. Put

$$
d_{0}:=\max (d, \operatorname{deg} a, \operatorname{deg} b), h_{0}:=\max (h, h(a), h(b)) .
$$

Then

$$
\overline{\operatorname{deg}} \alpha \leq\left(2 d_{0}\right)^{\exp O(r)}, \bar{h}(\alpha) \leq\left(2 d_{0}\right)^{\exp O(r)} h_{0} .
$$

Proof. Consider the linear equation

$$
\begin{equation*}
Q=\sum_{j=0}^{D-1} P_{j} w^{j} \tag{7.2.8}
\end{equation*}
$$

in unknowns $P_{0}, \ldots, P_{D-1}, Q \in A_{0}$. Since $\alpha \in K=K_{0}(w)$ and $w$ has degree $D$ over $K_{0}$, the equation (7.2.8) has a solution with $Q \neq 0$. Put again $\mathbf{Y}:=\left(X_{q+1}, \ldots, X_{r}\right)$ and set $Y:=\mathcal{G}_{0}\left(\sum_{j=1}^{t} a_{j} X_{q+j}\right)$. According to our general convention, we write $a^{*}, b^{*}$ for $a, b$, viewed as polynomials in $\mathbf{Y}$ with coefficients in $A_{0}$. By (7.2.1), there exist $g_{j}^{*} \in A_{0}[\mathbf{Y}]$ such that

$$
\begin{equation*}
Q a^{*}-b^{*} \sum_{j=0}^{D-1} P_{j} Y^{j}=\sum_{j=1}^{M} g_{j}^{*} f_{j}^{*} . \tag{7.2.9}
\end{equation*}
$$

By Theorem 6.1.2 (ii) this identity holds with polynomials $g_{j}^{*} \in K_{0}[\mathbf{Y}]$ of degree at most $\left(2 \max \left(d_{0}, D\right)\right)^{2^{t}} \leq\left(2 d_{0}\right)^{t \cdot 2^{t}}$ in $\mathbf{Y}$; by multiplying the tuple $\left(P_{0}, \ldots, P_{D-1}, Q\right)$ with a suitable non-zero element of $A_{0}$ we can make it so that the $g_{j}^{*}$ belong to $A_{0}[\mathbf{Y}]$. Now, as in the proof of Lemma 7.2.4, we can rewrite (7.2.9) as a system of linear equations over $K_{0}$ and then Corollary 6.1.4 can be applied. It follows that (7.2.8) is satisfied by $P_{0}, \ldots, P_{D-1}, Q \in$ $A_{0}$ with $Q \neq 0$ and

$$
\begin{aligned}
\operatorname{deg} P_{0}, \ldots, \operatorname{deg} P_{D-1}, \operatorname{deg} Q & \leq\left(2 d_{0}\right)^{\exp O(r)} \\
h\left(P_{0}\right), \ldots, h\left(P_{D-1}\right), h(Q) & \leq\left(2 d_{0}\right)^{\exp O(r)} h_{0}
\end{aligned}
$$

Dividing $P_{0}, \ldots, P_{D-1}, Q$ by their greatest common divisor and using Lemma 7.2.1 we get $P_{\alpha, 0}, \ldots, P_{\alpha, D-1}, Q_{\alpha} \in A_{0}$ satisfying (7.2.7) and

$$
\begin{aligned}
\operatorname{deg} P_{\alpha, 0}, \ldots, \operatorname{deg} P_{\alpha, D-1}, \operatorname{deg} Q_{\alpha} & \leq\left(2 d_{0}\right)^{\exp O(r)} \\
h\left(P_{\alpha, 0}\right), \ldots, h\left(P_{\alpha, D-1}\right), h\left(Q_{\alpha}\right) & \leq\left(2 d_{0}\right)^{\exp O(r)} h_{0}
\end{aligned}
$$

This proves our lemma.
Proposition 7.2.7. Let $w$ be as in Proposition 7.2.5, and let $\mathcal{A}$ be a finite (possibly empty) subset of $K^{*}$ of cardinality $k \geq 0$. For $\alpha \in \mathcal{A}$, let $\left(a_{\alpha}, b_{\alpha}\right)$ be a pair of representatives of $\alpha$ with $a_{\alpha}, b_{\alpha} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right], b_{\alpha} \notin \mathcal{I}$. Put

$$
d_{1}:=\max \left(d, \max _{\alpha \in \mathcal{A}}\left(\operatorname{deg} a_{\alpha}, \operatorname{deg} b_{\alpha}\right)\right)
$$

and

$$
h_{1}:=\max \left(h, \max _{\alpha \in \mathcal{A}}\left(h\left(a_{\alpha}\right), h\left(b_{\alpha}\right)\right)\right) .
$$

Then there is a non-zero $g \in A_{0}$ such that

$$
\begin{equation*}
A \subseteq B:=A_{0}\left[w, g^{-1}\right], \mathcal{A} \subset B^{*} \tag{7.2.10}
\end{equation*}
$$

and

$$
\left.\begin{array}{rl}
\operatorname{deg} g & \leq(k+1)\left(2 d_{1}\right)^{\exp O(r)}  \tag{7.2.11}\\
h(g) & \leq(k+1)\left(2 d_{1}\right)^{\exp O(r)} h_{1}
\end{array}\right\}
$$

Proof. Take

$$
g:=\prod_{i=1}^{t} Q_{y_{i}} \cdot \prod_{\alpha \in \mathcal{A}}\left(Q_{\alpha} \cdot Q_{\alpha^{-1}}\right)
$$

where, as above, $A=A_{0}\left[y_{1}, \ldots, y_{t}\right]$. In general, we have $Q_{\beta} \cdot \beta \in A_{0}[w]$ for $\beta \in K^{*}$. Hence we have $g \beta \in A_{0}[w]$ for $\beta=y_{1}, \ldots, y_{t}$ and for each $\alpha, \alpha^{-1}$ with $\alpha \in \mathcal{A}$. This implies (7.2.10). The inequalities (7.2.11) follow at once from Lemmas 7.2.6 and 7.2.1.

We shall use Proposition 7.2 .7 in various special cases. Before stating the first, we introduce some further notation and prove a lemma.

We recall that $a_{0}, a_{1}, \ldots, a_{n} \in A$ are the coefficients of the binary form $F(X, Y)$ in Section 2.3, resp. of the polynomial $F(X)$ in Section 2.4, while $\delta \in A \backslash\{0\}$ is the term occurring in the Thue equation (2.3.1) and the superelliptic equation (2.4.3). Further, $\widetilde{a_{0}}, \widetilde{a_{1}}, \ldots, \widetilde{a_{n}}, \widetilde{\delta}$ denote their representatives in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ with degrees at most $d$ and logarithmic heights at most $h$, where $d \geq 1, h \geq 1$. Denote by $\widetilde{F}$ the binary form $F(X, Y)$, resp. the polynomial $F(X)$ with coefficients $a_{0}, a_{1}, \ldots, a_{n}$ replaced by $\widetilde{a_{0}}, \widetilde{a_{1}}, \ldots, \widetilde{a_{n}}$, and by $D_{\widetilde{F}}$ the discriminant of $\widetilde{F}$. Then the assumption $D_{F} \neq 0$ implies $D_{\widetilde{F}} \notin \mathcal{I}$.

With the above notation and assumptions from Sections 2.3 and 2.4 , the following lemma holds.

Lemma 7.2.8. For the discriminant $D_{\widetilde{F}}$ we have the following inequalities:

$$
\begin{align*}
& \operatorname{deg} D_{\widetilde{F}} \leq(2 n-2) d  \tag{7.2.12}\\
& h\left(D_{\widetilde{F}}\right) \leq(2 n-2)\left[\log \left(2 n^{2}\binom{d+r}{r}\right)+h\right] \tag{7.2.13}
\end{align*}
$$

This is Lemma 3.2 of Bérczes, Evertse and Győry (2014).
Proof. Recall that $D_{\widetilde{F}}$ can be expressed as

$$
D_{\widetilde{F}}= \pm\left|\begin{array}{ccccccc}
\widetilde{a}_{0} & \widetilde{a}_{1} & \cdots & \cdots & \widetilde{a}_{n} & &  \tag{7.2.14}\\
& \ddots & & & & \ddots & \\
& & \widetilde{a}_{0} & \widetilde{a}_{1} & \cdots & \cdots & \widetilde{a}_{n} \\
\widetilde{a}_{1} & 2 \widetilde{a}_{2} & \cdots & n \widetilde{a}_{n} & & & \\
n \widetilde{a}_{0} & (n-1) \widetilde{a}_{1} & \cdots & \widetilde{a}_{n-1} & & & \\
& \ddots & & & \ddots & & \\
& & & n \widetilde{a}_{0} & (n-1) \widetilde{a}_{1} & \cdots & \widetilde{a}_{n-1}
\end{array}\right|
$$

with on the first $n-2$ rows of the determinant $\widetilde{a}_{0}, \ldots, \widetilde{a}_{n}$, on the $(n-1)$ st row $\widetilde{a}_{1}, 2 \widetilde{a}_{2}, \ldots, n \widetilde{a}_{n}$ and on the last $n-1$ rows $n \widetilde{a}_{0}, \ldots, \widetilde{a}_{n-1}$; see e.g. Section 1.4 in Evertse and Győry (2017b). Now Lemma 7.2 .8 follows at once from Lemma 4.1.7, using that the determinant has $(2 n-2)!\leq(2 n-2)^{2 n}$ terms and that each of the $\widetilde{a_{i}}$ has at most $\binom{d+r}{r}$ non-zero coefficients.

We can now apply Proposition 7.2.7 to the numbers $\alpha_{1}=\delta, \alpha_{2}=\delta^{-1}$, $\alpha_{3}=D_{F}, \alpha_{4}=D_{F}^{-1}$. Then the pairs $(\widetilde{\delta}, 1),(1, \widetilde{\delta}),\left(D_{\widetilde{F}}, 1\right),\left(1, D_{\widetilde{F}}\right)$ represent these numbers. Using the upper bounds for $\operatorname{deg} D_{\widetilde{F}}, h\left(D_{\widetilde{F}}\right)$ provided by Lemma 7.2.8 as well as $\operatorname{deg} \widetilde{\delta} \leq d, h(\widetilde{\delta}) \leq h$ that we assumed in Sections 2.3 and 2.4, we obtain immediately from Proposition 7.2.7 the following.
Proposition 7.2.9. There is a non-zero $g \in A_{0}$ such that

$$
\begin{equation*}
A \subseteq B:=A_{0}\left[w, g^{-1}\right], \quad \delta, D_{F} \in B^{*} \tag{7.2.15}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{deg} g \leq(n d)^{\exp O(r)}, h(g) \leq(n d)^{\exp O(r)} h \tag{7.2.16}
\end{equation*}
$$

This is Proposition 3.3 of Bérczes, Evertse and Győry (2014).

### 7.3 Comparison of different degrees and heights

We keep the above notation. Namely, $A$ is finitely generated over $\mathbb{Z}$, i.e. $A \cong \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right] /\left(f_{1}, \ldots, f_{M}\right)$, where $f_{1}, \ldots, f_{M} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$. In this section we compare, for $\alpha \in A \backslash\{0\}$, certain degrees and heights related to $\alpha$ and an appropriate representative $\widetilde{\alpha} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ of $\alpha$. Lemma 7.2.6 provided upper bounds for $\overline{\operatorname{deg}} \alpha$ and $\bar{h}(\alpha)$ in terms of the degrees and heights of $a, b$, where $(a, b)$ is a pair of representative for $\alpha$. Conversely, we have the following.

Lemma 7.3.1. Let $\lambda \in K^{*}$ and let $\alpha$ be a non-zero element of $A$. Let $(a, b)$ with $a, b \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ be a pair of representatives for $\lambda$. Put

$$
\begin{aligned}
d_{2} & :=\max \left(1, \operatorname{deg} f_{1}, \ldots, \operatorname{deg} f_{M}, \operatorname{deg} a, \operatorname{deg} b, \overline{\operatorname{deg}} \lambda \alpha\right), \\
h_{2} & :=\max \left(1, h\left(f_{1}\right), \ldots, h\left(f_{M}\right), h(a), h(b), \bar{h}(\lambda \alpha)\right) .
\end{aligned}
$$

Then $\alpha$ has a representative $\widetilde{\alpha} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ such that

$$
\begin{aligned}
\operatorname{deg} \widetilde{\alpha} & \leq\left(2 d_{2}\right)^{\exp O\left(r \log ^{*} r\right)} h_{2}, \\
h(\widetilde{\alpha}) & \leq\left(2 d_{2}\right)^{\exp O\left(r \log ^{*} r\right)} h_{2}^{r+1} .
\end{aligned}
$$

If moreover $\alpha \in A^{*}$, then $\alpha^{-1}$ has a representative $\widetilde{\alpha}^{\prime} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ with

$$
\begin{aligned}
\operatorname{deg} \widetilde{\alpha}^{\prime} & \leq\left(2 d_{2}\right)^{\exp O\left(r \log ^{*} r\right)} h_{2}, \\
h\left(\widetilde{\alpha}^{\prime}\right) & \leq\left(2 d_{2}\right)^{\exp O\left(r \log ^{*} r\right)} h_{2}^{r+1} .
\end{aligned}
$$

In the special case with $\lambda=1$ and $a=b=1$ we get the following corollary which will be useful in some applications.

Corollary 7.3.2. Let $\alpha \in A \backslash\{0\}$, and let

$$
\begin{aligned}
d_{2}^{\prime} & :=\max \left(1, \operatorname{deg} f_{1}, \ldots, \operatorname{deg} f_{M}, \overline{\operatorname{deg}} \alpha\right), \\
h_{2}^{\prime} & :=\max \left(1, h\left(f_{1}\right), \ldots, h\left(f_{M}\right), \bar{h}(\alpha)\right)
\end{aligned}
$$

Then $\alpha$ has a representative $\widetilde{\alpha} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ such that

$$
\begin{aligned}
\operatorname{deg} \widetilde{\alpha} & \leq\left(2 d_{2}^{\prime}\right)^{\exp O\left(r \log ^{*} r\right)} h_{2}^{\prime}, \\
h(\widetilde{\alpha}) & \leq\left(2 d_{2}^{\prime}\right)^{\exp O\left(r \log ^{*} r\right)} h_{2}^{\prime++1} .
\end{aligned}
$$

Proof of Lemma 7.3.1. With the identification of $z_{i}$ with $X_{i}$ for $i=1, \ldots, q$
we may view $A_{0}$ as a subring of $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$. Let $Y:=\mathcal{G}_{0}\left(\sum_{i=1}^{t} a_{i} X_{q+i}\right)$. We have

$$
\begin{equation*}
\lambda \alpha=Q^{-1} \sum_{i=0}^{D-1} P_{i} w^{i} \tag{7.3.1}
\end{equation*}
$$

with $P_{0}, \ldots, P_{D-1}, Q \in A_{0}$ and $\operatorname{gcd}\left(P_{0}, \ldots, P_{D-1}, Q\right)=1$. In view of (7.3.1), $\widetilde{\alpha} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ is a representative for $\alpha$ if and only if there exist $g_{1}, \ldots, g_{m} \in$ $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ such that

$$
\begin{equation*}
\widetilde{\alpha} \cdot(Q \cdot a)+\sum_{i=1}^{m} g_{i} f_{i}=b \sum_{i=0}^{D-1} P_{i} Y^{i} \tag{7.3.2}
\end{equation*}
$$

We may consider (7.3.2) as an inhomogeneous linear equation over $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ in the unknowns $\widetilde{\alpha}, g_{1}, \ldots, g_{m}$. By Lemmas 7.2.3, 7.2.4, 7.2.5 and 7.2.6 the degrees and logarithmic heights of $Q a$ and $b \sum_{i=0}^{D=1} P_{i} Y^{i}$ are bounded above by

$$
\left(2 d_{2}\right)^{\exp O(r)}, \quad\left(2 d_{2}\right)^{\exp O(r)} h_{2},
$$

respectively. Theorem 6.1.5 implies that (7.3.2) has a solution with upper bounds for $\operatorname{deg} \widetilde{\alpha}, h(\widetilde{\alpha})$, as stated in the lemma.

Now suppose that $\alpha \in A^{*}$. Then (7.3.1) gives as above that $\widetilde{\alpha}^{\prime} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ is a representative for $\alpha^{-1}$ if and only if there are $g_{1}^{\prime}, \ldots, g_{m}^{\prime} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ such that

$$
\widetilde{\alpha}^{\prime} b \sum_{i=0}^{D-1} P_{i} Y^{i}+\sum_{i=1}^{m} g_{i}^{\prime} f_{i}=Q a .
$$

Similarly as above, this equation has a solution with upper bounds for $\operatorname{deg} \widetilde{\alpha}^{\prime}$, $h\left(\widetilde{\alpha}^{\prime}\right)$ as stated in the lemma.

We next deduce some estimates for the $\overline{\mathrm{deg}}$ of elements from $K$, by applying the results from Chapter 5. Let as above $K_{0}=\mathbb{Q}\left(X_{1}, \ldots, X_{q}\right)$, $K=K_{0}(y), A_{0}=\mathbb{Z}\left[X_{1}, \ldots, X_{q}\right], B=\mathbb{Z}\left[X_{1}, \ldots, X_{q}, w, g^{-1}\right]$. Choose an algebraic closure $\bar{K}_{0}$ of $K_{0}$. Then there are precisely $D K_{0}$-isomorphic embeddings of $K$ into $\bar{K}_{0}$ which we denote by $\alpha \mapsto \alpha^{(j)}, j=1, \ldots, D$.

For $i=1, \ldots, q$, let $\mathbb{k}_{i}$ be the algebraic closure of $\mathbb{Q}\left(X_{1}, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{q}\right)$ in $\overline{K_{0}}$. Then $A_{0}$ is contained in $\mathbb{K}_{i}\left(X_{i}\right)$. Consider the function field

$$
L_{i}:=\mathbb{k}_{i}\left(X_{i}, w^{(1)}, \ldots, w^{(D)}\right)
$$

This is the splitting field of the polynomial $\mathcal{F}(X)=X^{D}+\mathcal{F}_{1} X^{D-1}+\cdots+\mathcal{F}_{D}$ over $\mathbb{k}_{i}\left(X_{i}\right)$. The subring

$$
B_{i}:=\mathbb{k}_{i}\left[X_{i}, w^{(1)}, \ldots, w^{(D)}, g^{-1}\right]
$$

of $L_{i}$ contains $B=\mathbb{Z}\left[X_{1}, \ldots, X_{q}, w, g^{-1}\right]$ as a subring. Define

$$
\Delta_{i}:=\left[L_{i}: \mathbb{K}_{i}\left(X_{i}\right)\right] .
$$

We shall apply some estimates from Section 5.1 with $X_{i}, \mathbb{k}_{i}, L_{i}$ instead of $z, \mathbb{k}, K$. The height $H_{L_{i}}$ is taken with respect to $L_{i} / \mathbb{k}_{i}$. For $P \in A_{0}$ we denote by $\operatorname{deg}_{X_{i}} P$ the degree of $P$ in the variable $X_{i}$. We recall that $g$ from Proposition 7.2.7, and the coefficients $\mathcal{F}_{1}, \ldots, \mathcal{F}_{D}$ of the polynomial $\mathcal{F}(X)$ from Proposition 7.2.7 are contained in $A_{0}$.
Lemma 7.3.3. For $\alpha \in K$ we have

$$
\overline{\operatorname{deg}} \alpha \leq \sum_{i=1}^{q} \Delta_{i}^{-1} \sum_{j=1}^{D} H_{L_{i}}\left(\alpha^{(j)}\right)+q D \max \left(\operatorname{deg} \mathcal{F}_{1}, \ldots, \operatorname{deg} \mathcal{F}_{D}\right) .
$$

Remark. It will be convenient to have estimates in which only $d$ and $r$ occur. Inserting the bounds for $\operatorname{deg} \mathcal{F}_{i}$ from Proposition 7.2.5 and the estimate $D \leq$ $d^{t}$ from Lemma 7.2.3 we obtain

$$
\begin{equation*}
\overline{\operatorname{deg}} \alpha \leq(2 d)^{\exp O(r)}+r d^{r} \max _{i, j} \Delta_{i}^{-1} H_{L_{i}}\left(\alpha^{(j)}\right) \tag{7.3.3}
\end{equation*}
$$

where the maximum is taken over $i=1, \ldots, u, j=1, \ldots, D$.
Proof of Lemma 7.3.3 Put

$$
d^{*}:=\max \left(\operatorname{deg} \mathcal{F}_{1}, \ldots, \operatorname{deg} \mathcal{F}_{D}\right)
$$

We have

$$
\alpha=Q^{-1} \sum_{j=0}^{D-1} P_{j} w^{j}
$$

for certain $P_{0}, \ldots, P_{D-1}, Q \in A_{0}$ with $\operatorname{gcd}\left(Q, P_{0}, \ldots, P_{D-1}\right)=1$. It is clear that

$$
\begin{align*}
& \overline{\operatorname{deg}} \alpha \leq \sum_{i=1}^{q} \mu_{i},  \tag{7.3.4}\\
& \text { where } \mu_{i}:=\max \left(\operatorname{deg}_{X_{i}} Q, \operatorname{deg}_{X_{i}} P_{0}, \ldots, \operatorname{deg}_{X_{i}} P_{D-1}\right)
\end{align*}
$$

Using the height properties listed in Section 5.1, we now estimate $\mu_{1}, \ldots, \mu_{q}$ from above. Fix $i \in\{1, \ldots, q\}$. By taking conjugates over $K_{0}$ we infer

$$
\alpha^{(k)}=Q^{-1} \sum_{j=0}^{D-1} P_{j} \cdot\left(w^{(k)}\right)^{j}, \text { for } k=1, \ldots, D
$$

Let $\Omega$ be the $D \times D$ matrix with rows

$$
(1, \ldots, 1),\left(w^{(1)}, \ldots, w^{(D)}\right), \ldots,\left(\left(w^{(1)}\right)^{D-1}, \ldots,\left(w^{(D)}\right)^{D-1}\right) .
$$

By Cramer's rule we get $P_{j} / Q=\delta_{j} / \delta$, where $\delta=\operatorname{det} \Omega$ and $\delta_{j}$ is the determinant of the matrix obtained by replacing the $j$-th row of $\Omega$ by $\left(\alpha^{(1)}, \ldots, \alpha^{(D)}\right)$.

Gauss' Lemma implies that $P_{0}, \ldots, P_{D-1}, Q$ are relatively prime in the ring $\mathbb{k}_{i}\left[X_{i}\right]$. Hence by (5.1.10) (with $X_{i}$ in place of $z$ ) we obtain

$$
\mu_{i}=H_{\mathbb{k}_{i}\left(X_{i}\right)}^{\mathrm{hom}}\left(Q, P_{0}, \ldots, P_{D-1}\right)
$$

$\operatorname{But}\left(\delta, \delta_{1}, \ldots, \delta_{D}\right)$ is a scalar multiple of $\left(Q, P_{0}, \ldots, P_{D-1}\right)$. Combining (5.1.9), (5.1.11) and inserting $\left[L_{i}: \mathbb{k}_{i}\left(X_{i}\right)\right]=\Delta_{i}$, we deduce that

$$
\begin{equation*}
\mu_{i}=\Delta_{i}^{-1} H_{L_{i}}^{\mathrm{hom}}\left(Q, P_{0}, \ldots, P_{D-1}\right)=\Delta_{i}^{-1} H_{L_{i}}^{\mathrm{hom}}\left(\delta, \delta_{1}, \ldots, \delta_{D}\right) \tag{7.3.5}
\end{equation*}
$$

We now estimate from above the right-hand side. It follows that for every valuation $v$ of $L_{i} / \mathbb{k}_{i}$

$$
\begin{aligned}
& -\min \left(v(\delta), v\left(\delta_{1}\right), \ldots, v\left(\delta_{D}\right)\right) \\
& \quad \leq-D \sum_{j=1}^{D} \min \left(0, v\left(w^{(j)}\right)\right)-\sum_{j=1}^{D} \min \left(0, v\left(\alpha^{(j)}\right)\right),
\end{aligned}
$$

and then summation over $v$ gives

$$
\begin{equation*}
H_{L_{i}}^{\mathrm{hom}}\left(\delta, \delta_{1}, \ldots, \delta_{D}\right) \leq D \sum_{j=1}^{D} H_{L_{i}}\left(w^{(j)}\right)+\sum_{j=1}^{D} H_{L_{i}}\left(\alpha^{(j)}\right) \tag{7.3.6}
\end{equation*}
$$

A combination of (5.1.14), (5.1.11), (5.1.10) yields

$$
\begin{align*}
\Delta_{i}^{-1} \sum_{j=1}^{D} H_{L_{i}}\left(w^{(j)}\right) & =\Delta_{i}^{-1} H_{L_{i}}^{\mathrm{hom}}(\mathcal{F})=H_{\mathbb{k}_{i}\left(X_{i}\right)}^{\mathrm{hom}}(\mathcal{F}) \\
& =\max \left(\operatorname{deg}_{X_{i}} \mathcal{F}_{1}, \ldots, \operatorname{deg}_{X_{i}} \mathcal{F}_{D}\right) \leq d^{*} \tag{7.3.7}
\end{align*}
$$

Together with (7.3.5), (7.3.6) this gives

$$
\mu_{i} \leq D d^{*}+\Delta_{i}^{-1} \sum_{j=1}^{D} H_{L_{i}}\left(\alpha^{(j)}\right) .
$$

Now these bounds for $i=1, \ldots, q$ together with (7.3.4) imply our lemma.

We have the following converse of Lemma 7.3.3.

Lemma 7.3.4. Let $\alpha \in K^{*}$ and $\alpha^{(1)}, \ldots, \alpha^{(D)}$ be as in Lemma 7.3.3. Then

$$
\begin{equation*}
\max _{i, j} \Delta_{i}^{-1} H_{L_{i}}\left(\alpha^{(j)}\right) \leq 2 D \overline{\operatorname{deg}} \alpha+D \max \left(\operatorname{deg} \mathcal{F}_{1}, \ldots, \operatorname{deg} \mathcal{F}_{D}\right) \tag{7.3.8}
\end{equation*}
$$

This is a slight refinement of Lemma 4.4 in Bérczes, Evertse and Győry (2014).

Remark. Inserting the bounds for $\operatorname{deg} \mathcal{F}_{i}$ from Proposition 7.2 .5 and the estimate $D \leq d^{t}$ from Lemma 7.2.3 we obtain

$$
\begin{equation*}
\max _{i, j} \Delta_{i}^{-1} H_{L_{i}}\left(\alpha^{(j)}\right) \leq(2 d)^{\exp O(r)}+2 d^{r} \overline{\operatorname{deg}} \alpha . \tag{7.3.9}
\end{equation*}
$$

Proof of Lemma 7.3.4 Define again

$$
d^{*}:=\max \left(\operatorname{deg} \mathcal{F}_{1}, \ldots, \operatorname{deg} \mathcal{F}_{D}\right)
$$

Consider the representation of $\alpha$ of the form (7.2.7). Then we have

$$
\alpha^{(j)}=Q_{\alpha}^{-1} \sum_{k=0}^{D-1} P_{\alpha, k}\left(w^{(j)}\right)^{k}, \text { for } j=1, \ldots, D,
$$

since $P_{\alpha, k}$ and $Q_{\alpha}$ are in $K_{0}$. Using (5.1.7) and (5.1.8), we get

$$
\begin{equation*}
H_{L_{i}}\left(\alpha^{(j)}\right) \leq \sum_{k=0}^{D-1} H_{L_{i}}\left(P_{\alpha, k} / Q_{\alpha}\right)+\sum_{k=0}^{D-1} k H_{L_{i}}\left(w^{(j)}\right) \tag{7.3.10}
\end{equation*}
$$

However, we have

$$
\begin{align*}
H_{L_{i}}\left(P_{\alpha, k} / Q_{\alpha}\right) & \leq \Delta_{i} H_{\mathbb{k}_{i}\left(X_{i}\right)}\left(P_{\alpha, k} / Q_{\alpha}\right) \leq \Delta_{i}\left(\operatorname{deg}_{X_{i}} P_{\alpha, k}+\operatorname{deg}_{X_{i}} Q_{\alpha}\right) \\
& \leq \Delta_{i}\left(\operatorname{deg} P_{\alpha, k}+\operatorname{deg} Q_{\alpha}\right) \\
& \leq 2 \Delta_{i} \overline{\operatorname{deg}} \alpha . \tag{7.3.11}
\end{align*}
$$

Further, it follows from the proof of (7.3.7) and from Lemma 7.2.3(i) that

$$
\begin{equation*}
\sum_{k=0}^{D-1} k H_{L_{i}}\left(w^{(j)}\right) \leq D \Delta_{i} \max _{1 \leq k \leq D} \operatorname{deg}_{X_{i}} \mathcal{F}_{k} \leq D \Delta_{i} d^{*} \tag{7.3.12}
\end{equation*}
$$

Now (7.3.10), (7.3.11) and 7.3.12) give (7.3.8).

### 7.4 Specializations

In this section we first prove some results about our specialization homomorphisms from $B$ to $\overline{\mathbb{Q}}$, where $B$ denotes the overring of $A$ from Proposition 7.2.7

If $q=0$, no specialization argument is needed. Hence, in this section we assume that $q>0$. We start with some auxiliary results that are used in the construction of our specializations.

Lemma 7.4.1. Let $\alpha_{1}, \ldots, \alpha_{m} \in \overline{\mathbb{Q}}$ with $G(X):=\left(X-\alpha_{1}\right) \cdots\left(X-\alpha_{m}\right) \in$ $\mathbb{Z}[X]$. Then

$$
\left|h(G)-\sum_{i=1}^{m} h\left(\alpha_{i}\right)\right| \leq m \log 2 .
$$

Proof. This is a special case of Corollary 4.1.5.
Lemma 7.4.2. Let $\alpha_{1}, \ldots, \alpha_{m} \in \overline{\mathbb{Q}}$ be distinct and suppose that $G(X):=$ $\left(X-\alpha_{1}\right) \cdots\left(X-\alpha_{m}\right) \in \mathbb{Z}[X]$. Let $q, p_{0}, \ldots, p_{m-1}$ be integers with
$\operatorname{gcd}\left(q, p_{0}, \ldots, p_{m-1}\right)=1$ and put

$$
\beta_{i}:=\sum_{j=0}^{m-1}\left(p_{j} / q\right) \alpha_{i}^{j} \text { for } i=1, \ldots, m .
$$

Then

$$
\log \max \left(|q|,\left|p_{0}\right|, \ldots,\left|p_{m-1}\right|\right) \leq 2 m^{2}+(m-1) h(G)+\sum_{i=1}^{m} h\left(\beta_{i}\right)
$$

Proof. For $m=1$ the assertion is obvious, hence we assume $m \geq 2$. Let $L=\mathbb{Q}\left(\alpha_{1}, \ldots, \alpha_{m}\right)$. Denote by $\Omega$ the $m \times m$ matrix with rows $\left(\alpha_{1}^{i}, \ldots, \alpha_{m}^{i}\right)$ for $i=0, \ldots, m-1$. By Cramer's rule we get $p_{i} / q=\delta_{i} / \delta, i=0, \ldots, m-1$, where $\delta=\operatorname{det} \Omega$ and $\delta_{i}$ is the determinant of the matrix obtained by replacing the $i$-the row of $\Omega$ by $\left(\beta_{i}, \ldots, \beta_{m}\right)$. Put

$$
\mu:=\log \max \left(|q|,\left|p_{0}\right|, \ldots,\left|p_{m-1}\right|\right)
$$

Since $\left(\delta, \delta_{0}, \ldots, \delta_{m-1}\right)$ is a scalar multiple of $\left(q, p_{0}, \ldots, p_{m-1}\right)$, we have by (4.1.4) and (4.1.6)

$$
\begin{align*}
\mu & =h^{\mathrm{hom}}\left(q, p_{0}, \ldots, p_{m-1}\right)=h^{\mathrm{hom}}\left(\delta, \delta_{0}, \ldots, \delta_{m-1}\right) \\
& =\frac{1}{d} \sum_{v \in \mathcal{M}_{L}} \log \max \left(|\delta|_{v},\left|\delta_{0}\right|_{v}, \ldots,\left|\delta_{m-1}\right|_{v}\right) . \tag{7.4.1}
\end{align*}
$$

Estimating the determinants using Hadamard's inequality for the infinite places and the ultrametric inequality for the finite places, we get

$$
\begin{aligned}
& \max \left(|\delta|_{v},\left|\delta_{0}\right|_{v}, \ldots,\left|\delta_{m-1}\right|_{v}\right) \\
& \quad \leq m^{m s(v) / 2} \cdot \prod_{i=1}^{m} \max \left(1,\left|\alpha_{i}\right|_{v}\right)^{m-1} \cdot \max \left(1,\left|\beta_{i}\right|_{v}\right)
\end{aligned}
$$

for $v \in \mathcal{M}_{L}$, where $s(v)=1$ if $v$ is real, $s(v)=2$ if $v$ is complex, and $s(v)=0$ if $v$ is finite. Together with 7.4.1) this gives

$$
\mu \leq \frac{1}{2} m \log m+\sum_{i=1}^{m}\left((m-1) h\left(\alpha_{i}\right)+h\left(\beta_{i}\right)\right) .
$$

Combining this with Corollary 4.1.5 we get Lemma 7.4.2.
Let again $A_{0}=\mathbb{Z}\left[X_{1}, \ldots, X_{q}\right]$. Given $b \in A_{0}, \mathbf{u}=\left(u_{1}, \ldots, u_{q}\right) \in \mathbb{Z}^{q}$ we denote by $b(\mathbf{u})$ the image of $b$ under $X_{i} \mapsto u_{i}(i=1, \ldots, q)$.

Lemma 7.4.3. Let $b \in A_{0}$ have degree $\mathcal{D}$. Let $\mathcal{N}$ be a finite subset of $\mathbb{Z}$ of cardinality $>\mathcal{D}$. Then

$$
\left|\left\{\mathbf{u} \in \mathcal{N}^{q}: b(\mathbf{u})=0\right\}\right| \leq \mathcal{D}|\mathcal{N}|^{q-1}
$$

Proof. We proceed by induction on $q$. For $q=1$ the assertion is obvious. Let $q \geq 2$. Write

$$
b=\sum_{i=0}^{\mathcal{D}_{0}} b_{i} X_{q}^{i},
$$

where $b_{i} \in \mathbb{Z}\left[X_{1}, \ldots, X_{q-1}\right]$ and $b_{\mathcal{D}_{0}} \neq 0$. Then $\operatorname{deg} b_{\mathcal{D}_{0}} \leq \mathcal{D}-\mathcal{D}_{0}$. By the induction hypothesis, there are at most $\left(\mathcal{D}-\mathcal{D}_{0}\right)|\mathcal{N}|^{q-2} \cdot|\mathcal{N}|$ tuples $\left(u_{1}, \ldots, u_{q-1}, u_{q}\right) \in \mathcal{N}^{q}$ with $b_{\mathcal{D}_{0}}\left(u_{1}, \ldots, u_{q-1}\right)=0$ and $u_{q}$ arbitrary. Further, there are at most $|\mathcal{N}|^{q-1} \cdot \mathcal{D}_{0}$ tuples $\mathbf{u} \in \mathcal{N}^{q}$ with $b_{\mathcal{D}_{0}}\left(u_{1}, \ldots, u_{q-1}\right) \neq 0$ and $b\left(u_{1}, \ldots, u_{q}\right)=0$. Summing these two quantities implies that $b$ has at most $\mathcal{D}|\mathcal{N}|^{q-1}$ zeros in $\mathcal{N}^{q}$.

For $f \in A_{0}=\mathbb{Z}\left[X_{1}, \ldots, X_{q}\right]$ and $p \in \mathcal{M}_{\mathbb{Q}}:=\{\infty\} \cup\{$ primes $\}$, we define $|f|_{p}$ to be the maximum of the $|\cdot|_{p}$-values of the coefficients of $f$.

Lemma 7.4.4. Let $b_{1}, b_{2} \in A_{0}$ have degrees $\mathcal{D}_{1}, \mathcal{D}_{2}$, respectively, and let $N$ be an integer $\geq \max \left(1, \mathcal{D}_{1}, \mathcal{D}_{2}\right)$. Define

$$
\mathcal{S}:=\left\{\mathbf{u} \in \mathbb{Z}^{q}:|\mathbf{u}| \leq N, b_{2}(\mathbf{u}) \neq 0\right\} .
$$

Then $\mathcal{S}$ is non-empty, and

$$
\begin{equation*}
\left|b_{1}\right|_{p} \leq U_{p}^{q} \max \left\{\left|b_{1}(\mathbf{u})\right|_{p}: \mathbf{u} \in \mathcal{S}\right\} \text { for } p \in \mathcal{M}_{\mathbb{Q}} \tag{7.4.2}
\end{equation*}
$$

where $U_{\infty}=(4 N)^{\mathcal{D}_{1}}, U_{p}=(2 N)^{\mathcal{D}_{1}}$ if $p$ is a prime with $p \leq 2 N$, and $U_{p}=1$ if $p$ is a prime $>2 N$.

Proof. We proceed by induction on $q$, starting with $q=0$. In the case $q=0$ we interpret $b_{1}, b_{2}$ as non-zero constants. Then the lemma is trivial. Let $q \geq 1$. The lemma is obviously true if $\mathcal{D}_{1}=0$ so we assume $\mathcal{D}_{1} \geq 1$. Put

$$
\mathcal{C}_{p}:=\max \left\{\left|b_{1}(\mathbf{u})\right|_{p}: \mathbf{u} \in \mathcal{S}\right\} \text { for } p \in \mathcal{M}_{\mathbb{Q}} .
$$

Write

$$
b_{1}=\sum_{j=0}^{\mathcal{D}_{1}^{\prime}} b_{1, j} X_{q}^{j}, \quad b_{2}=\sum_{j=0}^{\mathcal{D}_{2}^{\prime}} b_{2, j} X_{q}^{j},
$$

where the $b_{1, j}, b_{2, j}$ belong to $\mathbb{Z}\left[X_{1}, \ldots, X_{q-1}\right]$ and $b_{1, \mathcal{D}_{1}^{\prime}}, b_{2, \mathcal{D}_{2}^{\prime}} \neq 0$. By the induction hypothesis, the set

$$
\mathcal{S}^{\prime}:=\left\{\mathbf{u}^{\prime} \in \mathbb{Z}^{q-1}:\left|\mathbf{u}^{\prime}\right| \leq N, b_{2, \mathcal{D}_{2}^{\prime}}\left(\mathbf{u}^{\prime}\right) \neq 0\right\}
$$

is non-empty and moreover,

$$
\begin{align*}
& \max _{0 \leq j \leq \mathcal{D}_{1}^{\prime}}\left|b_{1, j}\right|_{p} \leq U_{p}^{q-1} \mathcal{C}_{p}^{\prime} \text { for } p \in \mathcal{M}_{\mathbb{Q}}  \tag{7.4.3}\\
& \quad \text { where } \mathcal{C}_{p}^{\prime}:=\max \left\{\left|b_{1, j}\left(\mathbf{u}^{\prime}\right)\right|_{p}: \mathbf{u}^{\prime} \in \mathcal{S}^{\prime}, j=0, \ldots, \mathcal{D}_{1}^{\prime}\right\}
\end{align*}
$$

We fix $p \in \mathcal{M}_{\mathbb{Q}}$ and estimate $\mathcal{C}_{p}^{\prime}$ from above in terms of $\mathcal{C}_{p}$. Take $\mathbf{u}^{\prime} \in \mathcal{S}^{\prime}$ such that $\mathcal{C}_{p}^{\prime}=\max _{0 \leq j \leq \mathcal{D}_{1}^{\prime}}\left|b_{1, j}\left(\mathbf{u}^{\prime}\right)\right|_{p}$. There exist at least $2 N+1-\mathcal{D}_{2}^{\prime} \geq$ $\mathcal{D}_{1}^{\prime}+1$ integers $u_{q}$ with $\left|u_{q}\right| \leq N$ such that $b_{2}\left(\mathbf{u}^{\prime}, u_{q}\right) \neq 0$. Let $a_{0}, \ldots, a_{\mathcal{D}_{1}^{\prime}}$ be distinct integers from this set. Using Lagrange's interpolation formula we obtain

$$
b_{1}\left(\mathbf{u}^{\prime}, X_{q}\right)=\sum_{j=0}^{\mathcal{D}_{1}^{\prime}} b_{1 j}\left(\mathbf{u}^{\prime}\right) X_{q}^{j}=\sum_{j=0}^{\mathcal{D}_{1}^{\prime}} b_{1}\left(\mathbf{u}^{\prime}, a_{j}\right)\left(\prod_{\substack{i=0 \\ i \neq j}}^{\mathcal{D}_{1}^{\prime}} \frac{X_{q}-a_{i}}{a_{j}-a_{i}}\right) .
$$

First, consider $p=\infty$. The coefficients of a polynomial $\prod_{k=1}^{m}\left(X-c_{k}\right)$ with $c_{1}, \ldots, c_{m} \in \mathbb{C}$ have absolute values at most $\prod_{k=1}^{m}\left(1+\left|c_{k}\right|\right)$. Hence

$$
\begin{aligned}
\mathcal{C}_{\infty}^{\prime} & =\max _{0 \leq j \leq \mathcal{D}_{1}^{\prime}}\left|b_{1 j}\left(\mathbf{u}^{\prime}\right)\right| \leq \mathcal{C}_{\infty} \sum_{j=0}^{\mathcal{D}_{1}^{\prime}} \prod_{\substack{i=0 \\
i \neq j}}^{\mathcal{D}_{1}^{\prime}}\left(1+\left|a_{i}\right|\right) \\
& \leq \mathcal{C}_{\infty}\left(\mathcal{D}_{1}^{\prime}+1\right)(N+1)^{\mathcal{D}_{1}^{\prime}} \leq U_{\infty} \mathcal{C}_{\infty}
\end{aligned}
$$

Now let $p$ be a prime and let $k$ be the largest integer such that $p^{k} \leq 2 N$. Then for all $i, j$ with $0 \leq i<j \leq \mathcal{D}_{1}^{\prime}$ we have

$$
\left|a_{i}-a_{j}\right|_{p} \geq p^{-k} \geq \begin{cases}(2 N)^{-1} & \text { if } p \leq 2 N \\ 1 & \text { if } p>2 N\end{cases}
$$

and thus,

$$
\mathcal{C}_{p}^{\prime}=\max _{0 \leq j \leq \mathcal{D}_{1}^{\prime}}\left|b_{1 j}\left(\mathbf{u}^{\prime}\right)\right|_{p} \leq \mathcal{C}_{p} \max _{0 \leq j \leq \mathcal{D}_{1}^{\prime}} \prod_{\substack{i=0 \\ i \neq j}}^{\mathcal{D}_{1}^{\prime}}\left|a_{j}-a_{i}\right|_{p}^{-1} \leq U_{p} \mathcal{C}_{p} .
$$

So $\mathcal{C}_{p}^{\prime} \leq U_{p} \mathcal{C}_{p}$ for all $p \in \mathcal{M}_{\mathbb{Q}}$. Combining this with (7.4.3) we obtain (7.4.2).

We now define our specializations $B \rightarrow \overline{\mathbb{Q}}$ and prove some properties. These specializations were introduced by Győry $(1983,1984)$ and, in a refined form, by Evertse and Győry (2013); see Chapter 3 .

We recall that in this section $q>0$ is assumed. Apart from that we keep the notation and assumptions from Section 7.2. In particular,

$$
\begin{aligned}
& A_{0}=\mathbb{Z}\left[X_{1}, \ldots, X_{q}\right], \quad K_{0}=\mathbb{Q}\left(X_{1}, \ldots, X_{q}\right), \\
& K=\mathbb{Q}\left(X_{1}, \ldots, X_{q}, w\right), \quad B=\mathbb{Z}\left[X_{1}, \ldots, X_{q}, w, g^{-1}\right],
\end{aligned}
$$

where $g \in A_{0}$ is the polynomial from Proposition 7.2.7, $w$ is integral over $A_{0}$ and $w$ has minimal polynomial

$$
\mathcal{F}(X):=X^{D}+\mathcal{F}_{1} X^{D-1}+\cdots+\mathcal{F}_{D} \in A_{0}[X]
$$

over $K_{0}$. By construction, $\mathcal{A} \subset B^{*}$, where $\mathcal{A}$ is the finite set from Proposition 7.2.7. In the case $D=1$, we take $w=1, \mathcal{F}=X-1$.

Let $d_{1}, h_{1}$ be the quantities from Proposition 7.2.7 and $k$ the cardinality of $\mathcal{A}$. Further, define

$$
\begin{cases}d_{3}:=\max \left(d, \operatorname{deg} \mathcal{F}_{1}, \ldots, \operatorname{deg} \mathcal{F}_{D}\right), & d_{4}:=\max \left(d_{3}, \operatorname{deg} g\right)  \tag{7.4.4}\\ h_{3}:=\max \left(h, h\left(\mathcal{F}_{1}\right), \ldots, h\left(\mathcal{F}_{D}\right)\right), & h_{4}:=\max \left(h_{3}, h(g)\right) .\end{cases}
$$

By Propositions 7.2.5 and 7.2.7 we have

$$
\begin{cases}d_{3} \leq(2 d)^{\exp O(r)}, & d_{4} \leq(k+1)\left(2 d_{1}\right)^{\exp O(r)}  \tag{7.4.5}\\ h_{3} \leq(2 d)^{\exp O(r)} h, & h_{4} \leq(k+1)\left(2 d_{1}\right)^{\exp O(r)} h_{1}\end{cases}
$$

Further, we will frequently use the consequence of Lemma 7.2.3(i),

$$
\begin{equation*}
D \leq d^{r} . \tag{7.4.6}
\end{equation*}
$$

Let $\mathbf{u}=\left(u_{1}, \ldots, u_{q}\right) \in \mathbb{Z}^{q}$. Then the substitution $X_{1} \mapsto u_{1}, \ldots, X_{q} \mapsto u_{q}$ defines a ring homomorphism (specialization) from a subring of $K_{0}$ to $\mathbb{Q}$

$$
\varphi_{\mathbf{u}}: \alpha \mapsto \alpha(\mathbf{u}):\left\{b_{1} / b_{2}: b_{1}, b_{2} \in A_{0}, b_{2}(\mathbf{u}) \neq 0\right\} \rightarrow \mathbb{Q}
$$

We want to define ring homomorphisms from $B$ to $\overline{\mathbb{Q}}$ and for this, we have to impose some restrictions on $\mathbf{u}$. Let $\Delta_{\mathcal{F}}$ denote the discriminant of $\mathcal{F}$ (with $\Delta_{\mathcal{F}}:=1$ if $D=\operatorname{deg} \mathcal{F}=1$ ), and let

$$
\begin{equation*}
\mathcal{T}:=\Delta_{\mathcal{F}} \mathcal{F}_{D} \cdot g \tag{7.4.7}
\end{equation*}
$$

Then $\mathcal{T} \in A_{0}$. Since $\Delta_{\mathcal{F}}$ is a polynomial of degree $2 D-2$ with integer coefficients in $\mathcal{F}_{1}, \ldots, \mathcal{F}_{D}$, we deduce easily that

$$
\begin{equation*}
\operatorname{deg} \mathcal{T} \leq(2 D-1) d_{3}+d_{4} \leq 2 D d_{4} \tag{7.4.8}
\end{equation*}
$$

Assume that

$$
\mathcal{T}(\mathbf{u}) \neq 0
$$

Then $g(\mathbf{u}) \neq 0$ and the polynomial

$$
\mathcal{F}_{\mathbf{u}}:=X^{D}+\mathcal{F}_{1}(\mathbf{u}) X^{D-1}+\cdots+\mathcal{F}_{D}(\mathbf{u})
$$

has $D$ distinct zeros which are all different from 0 , say $w_{1}(\mathbf{u}), \ldots, w_{D}(\mathbf{u})$. Thus, for $j=1, \ldots, D$ the assignment

$$
X_{1} \mapsto u_{1}, \ldots, X_{q} \mapsto u_{q}, w \mapsto w_{j}(\mathbf{u})
$$

defines a ring homomorphism $\varphi_{\mathbf{u}, j}$ from $B$ to $\overline{\mathbb{Q}}$; in the case $D=1$ it is just $\varphi_{\mathbf{u}}$. The image of $\alpha \in B$ under $\varphi_{\mathbf{u}, j}$ is denoted by $\alpha_{j}(\mathbf{u})$. We recall that the elements $\alpha$ of $B$ can be expressed as

$$
\begin{equation*}
\alpha=\sum_{i=0}^{D-1}\left(P_{i} / Q\right) w^{i} \text { with relatively prime } P_{0}, \ldots, P_{D-1}, Q \in A_{0} \tag{7.4.9}
\end{equation*}
$$

Since $\alpha \in B$, the denominator $Q$ must divide a power of $g$. Hence $Q(\mathbf{u}) \neq 0$. Thus we have

$$
\begin{equation*}
\alpha_{j}(\mathbf{u})=\sum_{i=0}^{D-1}\left(P_{i}(\mathbf{u}) / Q(\mathbf{u})\right) w_{j}(\mathbf{u})^{i}, j=1, \ldots, D \tag{7.4.10}
\end{equation*}
$$

Clearly, $\varphi_{\mathbf{u}, j}$ is the identity on $B \cap \mathbb{Q}$. Hence, if $\alpha \in B \cap \overline{\mathbb{Q}}$, then $\varphi_{\mathbf{u}, j}(\alpha)$ has the same minimal polynomial as $\alpha$ and so it is conjugate to $\alpha$.

For $\mathbf{u}=\left(u_{1}, \ldots, u_{q}\right) \in \mathbb{Z}^{q}$, we put $|\mathbf{u}|:=\max \left(\left|u_{1}\right|, \ldots,\left|u_{q}\right|\right)$. It is easy to show that for any $b \in A_{0}, \mathbf{u} \in \mathbb{Z}^{q}$,

$$
\begin{equation*}
\log |b(\mathbf{u})| \leq q \log \operatorname{deg} b+h(b)+\operatorname{deg} b \log \max (1,|\mathbf{u}|) . \tag{7.4.11}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
h\left(\mathcal{F}_{\mathbf{u}}\right) \leq q \log d_{3}+h_{3}+d_{3} \log \max (1,|\mathbf{u}|) \tag{7.4.12}
\end{equation*}
$$

and so by Corollary 4.1.5.

$$
\begin{equation*}
\sum_{j=1}^{D} h\left(w_{j}(\mathbf{u})\right) \leq D+1+q \log d_{3}+h_{3}+d_{3} \log \max (1,|\mathbf{u}|) \tag{7.4.13}
\end{equation*}
$$

Define the algebraic number fields $K_{\mathbf{u}, j}:=\mathbb{Q}\left(w_{j}(\mathbf{u})\right), j=1, \ldots, D$. We derive an upper bound for the discriminant $D_{K_{\mathbf{u}, j}}$ of $K_{\mathbf{u}, j}$.

Lemma 7.4.5. Let $\mathbf{u} \in \mathbb{Z}^{q}$ with $\mathcal{T}(\mathbf{u}) \neq 0$. Then for $j=1, \ldots, D$ the field $K_{\mathbf{u}, j}$ has degree $\left[K_{\mathbf{u}, j}: \mathbb{Q}\right] \leq D$ and absolute discriminant

$$
\left|D_{K_{\mathbf{u}, j}}\right| \leq D^{2 D-1}\left(d_{3}^{q} e^{h_{3}} \max (1,|\mathbf{u}|)^{d_{3}}\right)^{2 D-2} .
$$

Remark. Inserting (7.4.5), (7.4.6) we obtain

$$
\begin{equation*}
\log \left|D_{K_{\mathbf{u}, j}}\right| \leq(2 d)^{\exp O(r)}(h+\log \max (1,|\mathbf{u}|)) . \tag{7.4.14}
\end{equation*}
$$

Proof. Let $j \in\{1, \ldots, D\}$. As observed above, $w_{j}(\mathbf{u})$ is a zero of $\mathcal{F}_{\mathbf{u}}$, which is a monic polynomial in $\mathbb{Z}[X]$ of degree $D$. Hence $\left[K_{\mathbf{u}, j}: \mathbb{Q}\right] \leq D$. To estimate the discriminant of $K_{\mathbf{u}, j}$, let $\mathcal{P}_{j}$ denote the monic minimal polynomial of $w_{j}(\mathbf{u})$ over $\mathbb{Q}$, which necessarily has its coefficients from $\mathbb{Z}$. Then $D_{K_{\mathbf{u}, j}}$ divides the discriminant of $\mathcal{P}_{j}$, which is the discriminant of the order $\mathbb{Z}\left[w_{j}(\mathbf{u})\right]$. Using the expression of the discriminant of a monic polynomial as the product of the squares of the differences of its zeros, it is easy to see that the discriminant of $\mathcal{P}_{j}$ divides that of $\mathcal{F}_{\mathbf{u}}$ in the ring of algebraic integers and so also in $\mathbb{Z}$. Denoting the latter discriminant by $\Delta$, we infer that $D_{K_{\mathbf{u}, j}}$ divides $\Delta$ in $\mathbb{Z}$.

It remains to estimate from above $|\Delta|$. We can express this as a determinant similar to (7.2.14), replacing $n$ by $D$ and $\widetilde{a_{0}}, \ldots, \widetilde{a_{n}}$ by the coefficients of $\mathcal{F}_{\mathbf{u}}$. Hadamard's inequality gives that the absolute value of this determi-
nant can be estimated from above by the product of the Euclidean norms of its rows. Letting $H$ denote the maximum of the absolute values of the coefficients of $\mathcal{F}_{\mathbf{u}}$, this leads to

$$
\begin{aligned}
|\Delta| & \leq(D+1)^{(D-2) / 2}\left(1^{2}+\cdots+D^{2}\right)^{D / 2} H^{2 D-2} \\
& =(D+1)^{(D-2) / 2}\left(\frac{1}{6} D(D+1)(2 D+1)\right)^{D / 2} H^{2 D-2} \\
& \leq D^{2 D-1} H^{2 D-2},
\end{aligned}
$$

provided that $D \geq 3$. For $D=1,2$, the inequality $|\Delta| \leq D^{2 D-1} H^{2 D-2}$ can be verified by direct computation. Inserting (7.4.12), i.e., $H \leq d_{3}^{q} e^{h_{3}} \max (1,|\mathbf{u}|)^{d_{3}}$, we arrive at

$$
|\Delta| \leq D^{2 D-1}\left(d_{3}^{q} e^{h_{3}} \max (1,|\mathbf{u}|)^{d_{3}}\right)^{2 D-2}
$$

This implies our lemma.

Finally, we state and prove two lemmas which relate $\bar{h}(\alpha)$ to the heights of $\alpha_{j}(\mathbf{u})$ for $\alpha \in B, \mathbf{u} \in \mathbb{Z}^{q}$.

Lemma 7.4.6. Let $\mathbf{u} \in \mathbb{Z}^{q}$ with $\mathcal{T}(\mathbf{u}) \neq 0$. Further, let $\alpha \in B$. Then for $j=1, \ldots, D$

$$
\begin{aligned}
h\left(\alpha_{j}(\mathbf{u})\right) \leq & D^{2} \\
& +q\left(D \log d_{3}+\log \overline{\operatorname{deg}} \alpha\right)+ \\
& +D h_{3}+\bar{h}(\alpha)+\left(D d_{3}+\overline{\operatorname{deg}} \alpha\right) \log \max (1,|\mathbf{u}|)
\end{aligned}
$$

Remark. Inserting (7.4.5), (7.4.6) we derive the estimate

$$
\begin{equation*}
h\left(\alpha_{j}(\mathbf{u})\right) \leq \bar{h}(\alpha)+(2 d)^{\exp O(r)}(h+(\overline{\operatorname{deg}} \alpha+1) \log \max (1,|\mathbf{u}|)) . \tag{7.4.15}
\end{equation*}
$$

Proof. Let $P_{0}, \ldots, P_{D-1}, Q$ be as in 7.4.9. Let $L=\mathbb{Q}\left(w_{j}(\mathbf{u})\right)$. We denote by $\mathcal{M}_{L}$ the set of places of $L$, and $|\cdot|_{v}\left(v \in \mathcal{M}_{L}\right)$ the corresponding absolute values normalized as in Section 4.1. Then for $v \in \mathcal{M}_{L}$ we have

$$
\left|\alpha_{j}(\mathbf{u})\right|_{v} \leq D^{s(v)} T_{v} \max \left(1,\left|w_{j}(\mathbf{u})\right|_{v}\right)^{D-1}
$$

where $s(v)=1$ if $v$ is real, $s(v)=2$ if $v$ is complex, $s(v)=0$ if $v$ is finite, and

$$
T_{v}=\max \left(1,\left|P_{0}(\mathbf{u}) / Q(\mathbf{u})\right|_{v}, \ldots,\left|P_{D-1}(\mathbf{u}) / Q(\mathbf{u})\right|_{v}\right) .
$$

Hence

$$
\begin{equation*}
h\left(\alpha_{j}(\mathbf{u})\right) \leq \log D+\frac{1}{[L: \mathbb{Q}]} \sum_{v \in \mathcal{M}_{L}} \log T_{v}+(D-1) h\left(w_{j}(\mathbf{u})\right) . \tag{7.4.16}
\end{equation*}
$$

We infer that

$$
\begin{aligned}
\frac{1}{[L: \mathbb{Q}]} \sum_{v \in \mathcal{M}_{L}} \log T_{v} & =h\left(P_{0}(\mathbf{u}) / Q(\mathbf{u}), \ldots, P_{D-1}(\mathbf{u}) / Q(\mathbf{u})\right) \\
& =h^{\mathrm{hom}}\left(Q(\mathbf{u}), P_{0}(\mathbf{u}), \ldots, P_{D-1}(\mathbf{u})\right) \\
& \leq \log \max \left(|Q(\mathbf{u})|,\left|P_{0}(\mathbf{u})\right|, \ldots,\left|P_{D-1}(\mathbf{u})\right|\right) \\
& \leq q \log \overline{\operatorname{deg}} \alpha+\bar{h}(\alpha)+\overline{\operatorname{deg}} \alpha \cdot \log \max (1,|\mathbf{u}|) .
\end{aligned}
$$

Combining this with (7.4.13) and (7.4.16), the lemma follows.
Lemma 7.4.7. Let $\alpha \in B, \alpha \neq 0$, and let $N$ be an integer such that

$$
N \geq \max \left(\overline{\operatorname{deg}} \alpha, 2 D d_{3}+2(q+1)\left(d_{4}+1\right)\right)
$$

Then the set

$$
\mathcal{S}:=\left\{\mathbf{u} \in \mathbb{Z}^{q}:|\mathbf{u}| \leq N, \mathcal{T}(\mathbf{u}) \neq 0\right\}
$$

is non-empty and

$$
\bar{h}(\alpha) \leq(6 N)^{q+4}\left(h_{4}+H\right)
$$

where $H:=\max \left\{h\left(\alpha_{j}(\mathbf{u})\right): \mathbf{u} \in \mathcal{S}, j=1, \ldots, D\right\}$.
Remark. In view of (7.4.5), (7.4.6 we may take here

$$
\begin{equation*}
N=\max \left(\overline{\operatorname{deg}} \alpha,(k+1)\left(2 d_{1}\right)^{\exp O(r)}\right), \tag{7.4.17}
\end{equation*}
$$

and get an upper bound

$$
\begin{equation*}
\bar{h}(\alpha) \leq\left(2 d_{1}\right)^{\exp O(r)}((k+1)+\overline{\operatorname{deg}} \alpha)^{q+5}\left(h_{1}+H\right), \tag{7.4.18}
\end{equation*}
$$

where $k=|\mathcal{A}|$, with $\mathcal{A}$ the set from Proposition 7.2.7.
Proof. Lemmas 7.4.4, 7.4.6 and our assumption on $N$ imply that $\mathcal{S}$ is nonempty. We proceed with estimating $\bar{h}(\alpha)$. Let $P_{0}, \ldots, P_{D-1}, Q \in A_{0}$ be as in (7.4.9). We analyze $Q$ more closely. Let

$$
g= \pm p_{1}^{k_{1}} \cdot \ldots \cdot p_{m}^{k_{m}} g_{1}^{\ell_{1}} \cdot \ldots \cdot g_{n}^{\ell_{n}}
$$

be the up to the sign of the irreducible factors unique factorization of $g$ in $A_{0}$, where $p_{1}, \ldots, p_{m}$ are distinct prime numbers, and $g_{1}, \ldots, g_{n}$ are irreducible elements of $A_{0}$ of positive degree with $g_{i} \neq \pm g_{j}$ for all $i, j$ with $1 \leq i<j \leq$ $n$. By Corollary 4.1.5 we have

$$
\begin{equation*}
\sum_{i=1}^{n} \ell_{i} h\left(g_{i}\right) \leq q d_{4}+h_{4} \tag{7.4.19}
\end{equation*}
$$

Since $\alpha \in B$, the polynomial $Q$ is also composed of $p_{1}, \ldots, p_{m}, g_{1}, \ldots, g_{n}$. Thus

$$
\begin{equation*}
Q=a Q^{\prime} \text { with } a= \pm p_{1}^{k_{1}^{\prime}} \cdot \ldots \cdot p_{m}^{k_{m}^{\prime}}, Q^{\prime}=g_{1}^{\ell_{1}^{\prime}} \cdot \ldots \cdot g_{n}^{\ell_{n}^{\prime}} \tag{7.4.20}
\end{equation*}
$$

for certain non-negative integers $k_{1}^{\prime}, \ldots, k_{m}^{\prime}, \ell_{1}^{\prime}, \ldots, \ell_{n}^{\prime}$. Clearly

$$
\ell_{1}^{\prime}+\cdots+\ell_{n}^{\prime} \leq \operatorname{deg} Q \leq \overline{\operatorname{deg}} \alpha \leq N
$$

and by Lemma 7.2.1 and 7.4.19)

$$
\begin{align*}
h\left(Q^{\prime}\right) & \leq q \operatorname{deg} Q+\sum_{i=1}^{n} \ell_{i}^{\prime} h\left(g_{i}\right) \leq N\left(q+q d_{4}+h_{4}\right) \\
& \leq N^{2}\left(h_{4}+1\right) \tag{7.4.21}
\end{align*}
$$

By virtue of (7.4.11) we have for $\mathbf{u} \in \mathcal{S}$

$$
\begin{aligned}
\log \left|Q^{\prime}(\mathbf{u})\right| & \leq q \log d_{4}+h\left(Q^{\prime}\right)+\operatorname{deg} Q \log N \\
& \leq \frac{3}{2} N \log N+N^{2}\left(h_{4}+1\right) \leq N^{2}\left(h_{4}+2\right)
\end{aligned}
$$

Hence

$$
h\left(Q^{\prime}(\mathbf{u}) \alpha_{j}(\mathbf{u})\right) \leq N^{2}\left(h_{4}+2\right)+H
$$

for $\mathbf{u} \in \mathcal{S}, j=1, \ldots, D$. Further, by (7.4.10) and (7.4.20) we have

$$
Q^{\prime}(\mathbf{u}) \alpha_{j}(\mathbf{u})=\sum_{i=0}^{D-1}\left(P_{i}(\mathbf{u}) / a\right) w_{j}(\mathbf{u})^{i} .
$$

Set

$$
\delta(\mathbf{u}):=\operatorname{gcd}\left(a, P_{0}(\mathbf{u}), \ldots, P_{D-1}(\mathbf{u})\right)
$$

Then by applying Lemma 7.4.2 together with (7.4.12) we infer that

$$
\begin{align*}
& \log \left(\frac{\max \left(|a|,\left|P_{0}(\mathbf{u})\right|, \ldots,\left|P_{D-1}(\mathbf{u})\right|\right)}{\delta(\mathbf{u})}\right) \\
& \leq 2 D^{2}+(D-1) h\left(\mathcal{F}_{\mathbf{u}}\right)+D\left(N^{2}\left(h_{4}+2\right)+H\right) \\
& \leq 2 D^{2}+(D-1)\left(q \log d_{4}+h_{4}+d_{4} \log N\right)+D\left(N^{2}\left(h_{4}+2\right)+H\right) \\
& \leq N^{3}\left(h_{4}+2\right)+D H . \tag{7.4.22}
\end{align*}
$$

Our assumption that $Q, P_{0}, \ldots, P_{D-1}$ are relatively prime in $A_{0}$ implies that the greatest common divisor of $a$ and the coefficients of $P_{0}, \ldots, P_{D-1}$ is 1 . Let $p \in\left\{p_{1}, \ldots, p_{m}\right\}$ be one of the prime factors of $a$. There is $j \in$ $\{0, \ldots, D-1\}$ such that $\left|P_{j}\right|_{p}=1$. Our assumption on $N$ and (7.4.8) implies that $N \geq \max \left(\operatorname{deg} \mathcal{T}, \operatorname{deg} P_{j}\right)$. This means that Lemma 7.4.4 can be applied with $g_{1}=P_{j}$ and $g_{2}=\mathcal{T}$. It follows that

$$
\max \left\{\left|P_{j}(\mathbf{u})\right|_{p}: \mathbf{u} \in \mathcal{S}\right\} \geq \begin{cases}(2 N)^{-q N} & \text { if } p \leq 2 N \\ 1 & \text { if } p>2 N\end{cases}
$$

that is, there is $\mathbf{u}_{p} \in \mathcal{S}$ with

$$
\left|P_{j}\left(\mathbf{u}_{p}\right)\right|_{p} \geq \begin{cases}(2 N)^{-q N} & \text { if } p \leq 2 N \\ 1 & \text { if } p>2 N\end{cases}
$$

Thus,

$$
\left|\delta\left(\mathbf{u}_{p}\right)\right|_{p} \geq \begin{cases}(2 N)^{-q N} & \text { if } p \leq 2 N  \tag{7.4.23}\\ 1 & \text { if } p>2 N\end{cases}
$$

For $\mathbf{u} \in \mathcal{S}$, let $\mathcal{P}_{\mathbf{u}}$ be the set of primes $p$ dividing $a$ with $\mathbf{u}_{p}=\mathbf{u}$, and $\mathcal{P}_{\mathbf{u}}^{\prime}$ the set of primes $p \in \mathcal{P}_{\mathbf{u}}$ with $p \leq 2 N$. Then by (7.4.22), (7.4.23) we have for $\mathbf{u} \in \mathcal{S}$,

$$
\begin{aligned}
\sum_{p \in \mathcal{P}_{\mathbf{u}}} \log |a|_{p}^{-1} & \leq \log |a / \delta(\mathbf{u})|+\sum_{p \in \mathcal{P}_{\mathbf{u}}^{\prime}} \log |\delta(\mathbf{u})|_{p}^{-1} \\
& \leq N^{3}\left(h_{4}+2\right)+D H+\left|\mathcal{P}_{\mathbf{u}}^{\prime}\right| \cdot q N \log 2 N .
\end{aligned}
$$

Summing over $\mathbf{u} \in \mathcal{S}$, using that $|\mathcal{S}| \leq(3 N)^{q}$, we get

$$
\begin{align*}
\log |a| & \leq(3 N)^{q}\left(N^{3}\left(h_{4}+2\right)+D H\right)+2 q N^{2} \log 2 N \\
& \leq(3 N)^{q+4}\left(h_{4}+H\right) . \tag{7.4.24}
\end{align*}
$$

Together with (7.4.20) and (7.4.21) this gives

$$
\begin{equation*}
h(Q) \leq(4 N)^{q+4}\left(h_{4}+H\right) . \tag{7.4.25}
\end{equation*}
$$

Further, the right-hand side of (7.4.24 provides also an upper bound for $\log \delta(\mathbf{u})$ for $\mathbf{u} \in \mathcal{S}$. A combination of this with (7.4.22) gives

$$
\log \max \left\{\left|P_{j}(\mathbf{u})\right|: \mathbf{u} \in \mathcal{S}, j=0, \ldots, D-1\right\} \leq(5 N)^{q+4}\left(h_{4}+H\right) .
$$

Another application of Lemma 7.4.4 yields

$$
h\left(P_{j}\right) \leq q N \log 4 N+(5 N)^{q+4}\left(h_{4}+H\right) \leq(6 N)^{q+4}\left(h_{4}+H\right)
$$

for $j=0, \ldots, D-1$. Together with (7.4.25) this gives the upper bound for $\bar{h}(\alpha)$ as claimed in our lemma.

### 7.5 Multiplicative independence

We prove a general effective multiplicative independence result for elements of a finitely generated field.

Recall that non-zero elements $\gamma_{1}, \ldots, \gamma_{s}$ of a field are called multiplicatively independent if there is no tuple $\left(b_{1}, \ldots, b_{s}\right) \in \mathbb{Z}^{s}$ with at least one of the $b_{i}$ not equal to 0 , such that $\gamma_{1}^{b_{1}} \cdots \gamma_{s}^{b_{s}}=1$.

We start with a result over number fields, and with the help of the specialization theory worked out above, we extend this to arbitrary finitely generated fields.

We state and prove a result on multiplicative dependence over number fields due to Loxton and van der Poorten (1983), which is not the strongest one available at present, but which amply suffices for our purposes. By $d_{L}$ we denote the degree of a number field $L$ and by $w_{L}$ the number of its roots of unity. Further, $m\left(d_{L}\right)$ denotes the height lower bound from Lemma 4.1.2. with $d$ replaced by $d_{L}$.
Lemma 7.5.1. Let $L$ be an algebraic number field, and let $\gamma_{0}, \ldots, \gamma_{s}$ be nonzero elements of $L$ such that $\gamma_{0}, \ldots, \gamma_{s}$ are multiplicatively dependent, but any
s elements among $\gamma_{0}, \ldots, \gamma_{s}$ are multiplicatively independent. Then there are non-zero integers $k_{0}, \ldots, k_{s}$ such that

$$
\begin{aligned}
& \gamma_{0}^{k_{0}} \cdots \gamma_{s}^{k_{s}}=1 \\
& \quad\left|k_{i}\right| \leq s!\cdot w_{L} m\left(d_{L}\right)^{-s} h\left(\gamma_{0}\right) \cdots h\left(\gamma_{s}\right) / h\left(\gamma_{i}\right) \text { for } i=0, \ldots, s
\end{aligned}
$$

Remark. Loher and Masser (2004, Cor. 2.3) obtained the asymptotically sharper upper bound, based on an idea of Kunrui Yu,

$$
\left|k_{i}\right| \leq 58\left(s!e^{s} / s^{s}\right) d_{L}^{s+1}\left(\log d_{L}\right) h\left(\gamma_{0}\right) \cdots h\left(\gamma_{s}\right) / h\left(\gamma_{i}\right) \text { for } i=0, \ldots, s
$$

Proof. We follow Loxton and van der Poorten. The result is trivially true if $s=0$ so we assume that $s \geq 1$. By assumption, there are non-zero integers $b_{0}, \ldots, b_{s}$ such that

$$
\begin{equation*}
\gamma_{0}^{b_{0}} \cdots \gamma_{s}^{b_{s}}=1 \tag{7.5.1}
\end{equation*}
$$

Without loss of generality,

$$
\begin{equation*}
\left|b_{0}\right| \cdot h\left(\gamma_{0}\right) \geq\left|b_{i}\right| \cdot h\left(\gamma_{i}\right) \quad \text { for } i=1, \ldots, s \tag{7.5.2}
\end{equation*}
$$

The tuple $\left(b_{0}, \ldots, b_{s}\right)$ is uniquely determined up to a scalar factor, because if $\left(b_{0}^{\prime}, \ldots, b_{s}^{\prime}\right)$ is any other tuple of non-zero integers with $\gamma_{0}^{b_{0}^{\prime}} \cdots \gamma_{s}^{b_{s}^{\prime}}=1$, then

$$
\gamma_{1}^{b_{0}^{\prime} b_{1}-b_{0} b_{1}^{\prime}} \cdots \gamma_{s}^{b_{0}^{\prime} b_{s}-b_{0} b_{s}^{\prime}}=1
$$

and thus, $b_{0}^{\prime} b_{i}-b_{0} b_{i}^{\prime}=0$ for $i=1, \ldots, s$ by the multiplicative independence of $\gamma_{1}, \ldots, \gamma_{s}$.

Let $\left(\theta_{k}\right)_{k \geq 0}$ be a sequence of positive reals increasing to $m\left(d_{L}\right)$. For every $k$, consider the $(s+1)$-dimensional symmetric convex body, consisting of the points $\left(x_{0}, \ldots, x_{s}\right) \in \mathbb{R}^{s+1}$ with

$$
\sum_{i=1}^{s} h\left(\gamma_{i}\right)\left|x_{i}-\frac{b_{i}}{b_{0}} x_{0}\right| \leq \theta_{k}, \quad\left|x_{0}\right| \leq s!\theta_{k}^{-s} h\left(\gamma_{1}\right) \cdots h\left(\gamma_{s}\right) .
$$

This body has volume $2^{s+1}$, so by Minkowski's convex body theorem it contains a non-zero point $\mathbf{l}_{k} \in \mathbb{Z}^{s+1}$. But among the points $\mathbf{l}_{k}(k \geq 0)$ there are only finitely many distinct ones, since they all lie in a bounded set independent of $k$. Hence there is a non-zero $\mathbf{l}=\left(l_{0}, \ldots, l_{s}\right) \in \mathbb{Z}^{s+1}$ belonging to the
above defined convex bodies for infinitely many $k$. But then this point satisfies

$$
\begin{equation*}
\sum_{i=1}^{s} h\left(\gamma_{i}\right)\left|l_{i}-\frac{b_{i}}{b_{0}} l_{0}\right|<m\left(d_{L}\right), \quad\left|l_{0}\right| \leq s!\cdot m\left(d_{L}\right)^{-s} h\left(\gamma_{1}\right) \cdots h\left(\gamma_{s}\right) . \tag{7.5.3}
\end{equation*}
$$

For $i=0, \ldots, s$, choose $\beta_{i}$ such that $\beta_{i}^{b_{0}}=\gamma_{i}$. By (7.5.1), $\zeta:=\beta_{0}^{b_{0}} \cdots \beta_{s}^{b_{s}}$ is a root of unity. From the height properties (4.1.3) we infer

$$
\begin{aligned}
h\left(\gamma_{0}^{l_{0}} \cdots \gamma_{s}^{l_{s}}\right) & =h\left(\gamma_{0}^{l_{0}} \cdots \gamma_{s}^{l_{s}} \zeta^{-l_{0}}\right)=h\left(\beta_{1}^{b_{0} l_{1}-b_{1} l_{0}} \cdots \beta_{s}^{b_{0} l_{s}-b_{s} l_{0}}\right) \\
& \leq \sum_{i=1}^{s} h\left(\beta_{i}\right)\left|b_{0} l_{i}-b_{i} l_{0}\right|=\sum_{i=1}^{s} h\left(\gamma_{i}\right)\left|l_{i}-\frac{b_{i}}{b_{0}} l_{0}\right|<m\left(d_{L}\right) .
\end{aligned}
$$

So by Lemma 4.1.2, $\gamma_{0}^{l_{0}} \cdots \gamma_{s}^{l_{s}}$ is a root of unity. It follows that $\gamma_{0}^{k_{0}} \cdots \gamma_{s}^{k_{s}}=1$, where $k_{i}:=w_{L} l_{i}$ for $i=0, \ldots, s$. Since we assumed that any $s$ elements among $\gamma_{0}, \ldots, \gamma_{s}$ are multiplicatively independent, the integers $k_{0}, \ldots, k_{s}$ are all non-zero.

It remains to estimate $k_{0}, \ldots, k_{s}$. By (7.5.3) we have

$$
\left|k_{0}\right| \leq s!w_{L} m\left(d_{L}\right)^{-s} h\left(\gamma_{1}\right) \cdots h\left(\gamma_{s}\right)
$$

Further, $\left(k_{0}, \ldots, k_{s}\right)$ is up to a scalar multiple equal to $\left(b_{0}, \ldots, b_{s}\right)$, and so, in view of (7.5.2), we have for $i=1, \ldots, s$,

$$
\left|k_{i}\right|=\left|\frac{b_{i}}{b_{0}} k_{0}\right| \leq \frac{h\left(\gamma_{0}\right)}{h\left(\gamma_{i}\right)} \cdot\left|k_{0}\right| \leq s!w_{L} m\left(d_{L}\right)^{-s} h\left(\gamma_{0}\right) \cdots h\left(\gamma_{s}\right) / h\left(\gamma_{i}\right) .
$$

This proves our lemma.
We prove a generalization for arbitrary finitely generated integral domains. As before, let $A=\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right] \supseteq \mathbb{Z}$ be an integral domain finitely generated over $\mathbb{Z}$ with quotient field $K$, and suppose that the ideal $\mathcal{I}$ of polynomials $f \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ with $f\left(z_{1}, \ldots, z_{r}\right)=0$ is generated by $f_{1}, \ldots, f_{M}$. Let $\gamma_{0}, \ldots, \gamma_{s}$ be non-zero elements of $K$, and for $i=0, \ldots, s$, let $\left(g_{i, 1}, g_{i, 2}\right)$ be a pair of representative for $\gamma_{i}$, i.e. elements of $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ such that

$$
\gamma_{i}=\frac{g_{i, 1}\left(z_{1}, \ldots, z_{r}\right)}{g_{i, 2}\left(z_{1}, \ldots, z_{r}\right)}
$$

Proposition 7.5.2. Assume that $\gamma_{0}, \ldots, \gamma_{s}$ are multiplicatively dependent. Further, assume that $f_{1}, \ldots, f_{M}$ and $g_{i, 1}, g_{i, 2}(i=0, \ldots, s)$ have degrees at most $d$ and logarithmic heights at most $h$, where $d \geq 1, h \geq 1$. Then there are integers $k_{0}, \ldots, k_{s}$, not all zero, such that

$$
\begin{align*}
& \gamma_{0}^{k_{0}} \cdots \gamma_{s}^{k_{s}}=1  \tag{7.5.4}\\
& \left|k_{i}\right| \leq(2 d)^{\exp O(r+s)} h^{s} \text { for } i=0, \ldots, s \tag{7.5.5}
\end{align*}
$$

This is Lemma 7.2 of Evertse and Győry (2013).

Proof. We may assume without loss of generality that any $s$ elements among $\gamma_{0}, \ldots, \gamma_{s}$ are multiplicatively independent (if this is not the case, take a minimal multiplicatively independent subset of $\left\{\gamma_{0}, \ldots, \gamma_{s}\right\}$ and proceed further with this subset). We first assume that $q>0$. We use an argument of van der Poorten and Schlickewei (1991). Keeping the above notation and assumptions from Chapter 7 , we assume that $z_{1}=X_{1}, \ldots, z_{q}=X_{q}$ is a transcendence basis of $K$, and rename $z_{q+1}, \ldots, z_{r}$ as $y_{1}, \ldots, y_{t}$, respectively. For brevity, we include the case $t=0$ as well in our proof. But it should be possible to prove in this case a sharper result by means of a more elementary method. We keep the notation and assumptions from Section 7.2, in particular,

$$
A_{0}=\mathbb{Z}\left[X_{1}, \ldots, X_{q}\right], \quad K_{0}=\mathbb{Q}\left(X_{1}, \ldots, X_{q}\right), \quad K=\mathbb{Q}\left(X_{1}, \ldots, X_{q}, w\right),
$$

where $w$ is integral over $A_{0}$ and $w$ has minimal polynomial

$$
\mathcal{F}(X):=X^{D}+\mathcal{F}_{1} X^{D-1}+\cdots+\mathcal{F}_{D} \in A_{0}[X]
$$

over $K_{0}$. In the case $D=1$, we take $w=1, \mathcal{F}=X-1$. We construct a specialization such that among the images of $\gamma_{0}, \ldots, \gamma_{s}$ no $s$ elements are multiplicatively dependent, and then apply Lemma 7.5.1.

Let $V \geq 2 d$ be a positive integer. Later we shall make our choice of $V$ more precise. Define the set

$$
\begin{align*}
\mathcal{V}:=\{\mathbf{v}= & \left(v_{0}, \ldots, v_{s}\right) \in \mathbb{Z}^{s+1} \backslash\{0\}: \\
& \left.\left|v_{i}\right| \leq V \text { for } i=0, \ldots, s, \text { and with } v_{i}=0 \text { for some } i\right\} . \tag{7.5.6}
\end{align*}
$$

Then

$$
\gamma_{\mathbf{v}}:=\left(\prod_{i=0}^{s} \gamma_{i}^{v_{i}}\right)-1(\mathbf{v} \in \mathcal{V})
$$

are non-zero elements of $K$. It is easy to show that for $\mathbf{v} \in \mathcal{V}, \gamma_{\mathbf{v}}$ has a pair of representatives $\left(g_{1, \mathbf{v}}, g_{2, \mathbf{v}}\right)$ such that

$$
\operatorname{deg} g_{1, \mathbf{v}}, \operatorname{deg} g_{2, \mathbf{v}} \leq s d V
$$

In the case $t>0$, there exists by Proposition 7.2.7 a non-zero $g \in A_{0}$ such that

$$
A \subseteq B:=A_{0}\left[w, g^{-1}\right], \gamma_{\mathbf{v}} \in B^{*} \text { for } \mathbf{v} \in \mathcal{V}
$$

and

$$
\operatorname{deg} g \leq V^{s+1}(2 s d V)^{\exp O(r)} \leq V^{\exp O(r+s)}
$$

In the case $t=0$ this holds true as well, with $w=1$ and $g=\prod_{\mathbf{v} \in \mathcal{V}}\left(g_{1, \mathbf{v}} \cdot g_{2, \mathbf{v}}\right)$. We apply the theory of specializations explained in Section 7.4 above with this $g$. We put $\mathcal{T}:=\Delta_{\mathcal{F}} \mathcal{F}_{D} \cdot g$, where $\Delta_{\mathcal{F}}$ denotes the discriminant of $\mathcal{F}$. Using Proposition 7.2.5 and inserting the bound $D \leq d^{t}$ from Lemma 7.2.3 we get for $t>0$ :

$$
\left\{\begin{array}{l}
d_{3}:=\max \left(d, \operatorname{deg} \mathcal{F}_{1}, \ldots, \operatorname{deg} \mathcal{F}_{D}\right) \leq(2 d)^{\exp O(r)}  \tag{7.5.7}\\
h_{3}:=\max \left(h, h\left(\mathcal{F}_{1}\right), \ldots, h\left(\mathcal{F}_{D}\right)\right) \leq(2 d)^{\exp O(r)} h,
\end{array}\right.
$$

with the provision $\operatorname{deg} 0=h(0)=-\infty$; this is true also if $t=0$. Combining this with Lemma 7.2.8, we obtain

$$
\operatorname{deg} \mathcal{T} \leq(2 D-1) d_{3}+\operatorname{deg} g \leq V^{\exp O(r+s)}
$$

By Lemma 7.4 .3 there exists $\mathbf{u} \in \mathbb{Z}^{q}$ with

$$
\begin{equation*}
\mathcal{T}(\mathbf{u}) \neq 0,|\mathbf{u}| \leq V^{\exp O(r+s)} \tag{7.5.8}
\end{equation*}
$$

We proceed further with this $\mathbf{u}$.
As was seen above, $\gamma_{\mathbf{v}} \in B^{*}$ for $\mathbf{v} \in \mathcal{V}$. By our choice of $\mathbf{u}$, there are $D$ distinct specialization maps $\varphi_{\mathbf{u}, j}(j=1, \ldots, D)$ from $B$ to $\overline{\mathbb{Q}}$. We fix one of these specializations, which we denote by $\varphi_{\mathbf{u}}$. Given $\alpha \in B$, we write $\alpha(\mathbf{u})$ for $\varphi_{\mathbf{u}}(\alpha)$. As the elements $\gamma_{\mathbf{v}}$ are all units in $B$, their images under $\varphi_{\mathbf{u}}$ are non-zero. Thus we have

$$
\begin{equation*}
\prod_{i=0}^{s} \gamma_{i}(\mathbf{u})^{v_{i}} \neq 1 \text { for } \mathbf{v} \in \mathcal{V} \tag{7.5.9}
\end{equation*}
$$

where $\mathcal{V}$ is defined by 7.5.6.

We use Lemma 7.4.6 to estimate the heights $h\left(\gamma_{i}(\mathbf{u})\right)$ for $i=0, \ldots, s$. Recall that by Lemma 7.2.6 we have

$$
\overline{\operatorname{deg}} \gamma_{i} \leq(2 d)^{\exp O(r)}, \bar{h}\left(\gamma_{i}\right) \leq(2 d)^{\exp O(r)} h
$$

for $i=0, \ldots, s$. By inserting these bounds and that for $|\mathbf{u}|$ from (7.5.8) into (7.4.15), we obtain for $i=0, \ldots, s$,

$$
\begin{align*}
h\left(\gamma_{i}(\mathbf{u})\right) & \leq(2 d)^{\exp O(r)}(1+h+\log \max (1,|\mathbf{u}|)) \\
& \leq(2 d)^{\exp O(r+s)}(1+h+\log V) . \tag{7.5.10}
\end{align*}
$$

We show that any $s$ numbers among $\gamma_{0}(\mathbf{u}), \ldots, \gamma_{s}(\mathbf{u})$ are multiplicatively independent, provided $V$ is chosen appropriately. Assume the contrary. By Lemma 7.5 .1 there are integers $k_{0}, \ldots, k_{s}$, at least one of which is non-zero and at least one of which is 0 , such that

$$
\begin{align*}
& \prod_{i=0}^{s} \gamma_{i}(\mathbf{u})^{k_{i}}=1 \\
& \quad\left|k_{i}\right| \leq(2 d)^{\exp O(r+s)}(1+h+\log V)^{s-1} \text { for } i=0, \ldots, s . \tag{7.5.11}
\end{align*}
$$

We now choose $V$ large enough such that this upper bound for the numbers $\left|k_{i}\right|$ is smaller than $V$. This is satisfied with

$$
\begin{equation*}
V=(2 d)^{\exp O(r+s)} h^{s-1} \tag{7.5.12}
\end{equation*}
$$

where the constant in the $O$-symbol in (7.5.12) is sufficiently large compared with that of (7.5.11). But then we have $\prod_{i=0}^{s} \gamma_{i}(\mathbf{u})^{v_{i}}=1$ for some $\mathbf{v} \in \mathcal{V}$, contrary to $(7.5 .9)$. Hence we conclude that with the choice $(7.5 .12)$ for $V$, there exists $\mathbf{u} \in \mathbb{Z}^{q}$ with (7.5.8), such that any $s$ numbers among $\gamma_{0}(\mathbf{u}), \ldots, \gamma_{s}(\mathbf{u})$ are multiplicatively independent. Of course, the numbers $\gamma_{0}(\mathbf{u}), \ldots, \gamma_{s}(\mathbf{u})$ are multiplicatively dependent, since they are the images under $\varphi_{\mathbf{u}}$ of $\gamma_{0}, \ldots, \gamma_{s}$ which are multiplicatively dependent. Substituting (7.5.12) into (7.5.10) we obtain

$$
\begin{equation*}
h\left(\gamma_{i}(\mathbf{u})\right) \leq(2 d)^{\exp O(r+s)} h \text { for } i=0, \ldots, s \tag{7.5.13}
\end{equation*}
$$

Now Lemma 7.5 .1 implies that there are non-zero integers $k_{0}, \ldots, k_{s}$ such
that

$$
\begin{align*}
& \prod_{i=0}^{s} \gamma_{i}(\mathbf{u})^{k_{i}}=1  \tag{7.5.14}\\
& \quad\left|k_{i}\right| \leq(2 d)^{\exp O(r+s)} h^{s} \text { for } i=0, \ldots, s \tag{7.5.15}
\end{align*}
$$

Our assumption concerning $\gamma_{0}, \ldots, \gamma_{s}$ implies that there are non-zero integers $\ell_{0}, \ldots, \ell_{s}$ such that $\prod_{i=0}^{s} \gamma_{i}^{\ell_{i}}=1$. Hence $\prod_{i=0}^{s} \gamma_{i}(\mathbf{u})^{\ell_{i}}=1$. Together with (7.5.14) this yields

$$
\prod_{i=1}^{s} \gamma_{i}(\mathbf{u})^{\ell_{0} k_{i}-\ell_{i} k_{0}}=1
$$

However, we have $\ell_{0} k_{i}-\ell_{i} k_{0}=0$ for $i=1, \ldots, s$ since $\gamma_{1}(\mathbf{u}), \ldots, \gamma_{s}(\mathbf{u})$ are multiplicatively independent, that is,

$$
\ell_{0} \cdot\left(k_{0}, \ldots, k_{s}\right)=k_{0} \cdot\left(\ell_{0}, \ldots, \ell_{s}\right)
$$

It follows that

$$
\prod_{i=0}^{s} \gamma_{i}^{k_{i}}=\zeta
$$

for some root of unity $\zeta$. But $\varphi_{\mathbf{u}}(\zeta)=1$ and it is conjugate to $\zeta$. Hence $\zeta=1$. So in fact we have $\prod_{i=0}^{s} \gamma_{i}^{k_{i}}=1$ with non-zero integers $k_{i}$ satisfying (7.5.15). This proves our Proposition, but under the assumption $q>0$. If $q=0$ then a much simpler argument, without specializations, gives $h\left(\gamma_{i}\right) \leq$ $(2 d)^{\exp O(r+s)} h$ for $i=0, \ldots, s$ in place of (7.5.13). Then the proof is finished in the same way as in the case $q>0$.

Corollary 7.5.3. Let $\gamma_{0}, \gamma_{1}, \ldots, \gamma_{s} \in K^{*}$, and suppose that $\gamma_{1}, \ldots, \gamma_{s}$ are multiplicatively independent and

$$
\gamma_{0}=\gamma_{1}^{k_{1}} \cdots \gamma_{s}^{k_{s}}
$$

for certain integers $k_{1}, \ldots, k_{s}$. Let $d, h$ be as in Proposition 7.5.2. Then

$$
\left|k_{i}\right| \leq(2 d)^{\exp O(r+s)} h^{s} \text { for } i=1, \ldots, s .
$$

Proof. By Proposition 7.5.2, and by the multiplicative independence of $\gamma_{1}, \ldots, \gamma_{s}$,
there are integers $\ell_{0}, \ldots, \ell_{s}$ such that

$$
\begin{aligned}
& \prod_{i=0}^{s} \gamma_{i}^{\ell_{i}}=1 \\
& \ell_{0} \neq 0,\left|\ell_{i}\right| \leq(2 d)^{\exp O(r+s)} h^{s} \text { for } i=0, \ldots, s
\end{aligned}
$$

But then we have also

$$
\prod_{i=1}^{s} \gamma_{i}^{\ell_{0} k_{i}-\ell_{i}}=1
$$

whence $\ell_{0} k_{i}-\ell_{i}=0$ for $i=1, \ldots, s$. It follows that

$$
\left|k_{i}\right|=\left|\ell_{i} / \ell_{0}\right| \leq(2 d)^{\exp O(r+s)} h^{s} \text { for } i=1, \ldots, s,
$$

which is what we wanted to prove.

## Chapter 8

## Degree-height estimates

Let as before $A$ be an integral domain of characteristic 0 that is finitely generated over $\mathbb{Z}, K$ its quotient field, and $\bar{K}$ an algebraic closure of $K$. We introduce so-called degree-height estimates for elements of $\bar{K}$, which may be seen as an analogue for the naive height (height of the minimal polynomial over $\mathbb{Z}$ ) of an algebraic number. Our goal is to give a degree-height estimate for $\beta \in \bar{K}$ in terms of degree-height estimates for $\alpha_{1}, \ldots, \alpha_{m} \in \bar{K}$, if $\beta$ is related to the $\alpha_{i}$ by $P\left(\beta, \alpha_{1}, \ldots, \alpha_{m}\right)=0$ for some given $P \in \mathbb{Z}\left[X, X_{1}, \ldots, X_{m}\right]$ that is monic in $X$. Estimates of this type will be crucial in Chapter 10 .

### 8.1 Definitions

We keep using the following notation: $A=\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]$ with $r>0$ is an integral domain of characteristic $0, K$ is its quotient field, and $\bar{K}$ is an algebraic closure of $K$. Further, $\mathcal{I}$ is the ideal of $f \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ with $f\left(z_{1}, \ldots, z_{r}\right)=0$, so that

$$
\begin{equation*}
A \cong \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right] / \mathcal{I} \tag{8.1.1}
\end{equation*}
$$

We assume again that

$$
\begin{align*}
\mathcal{I}=\left(f_{1}, \ldots, f_{M}\right) & \text { with } \operatorname{deg} f_{i} \leq d, h\left(f_{i}\right) \leq h \text { for } i=1, \ldots, M, \\
& \text { where } d \geq 1, h \geq 1 . \tag{8.1.2}
\end{align*}
$$

We now introduce the notion of degree-height estimate. Given a monic polynomial $G \in K[X]$, we call $\left(g_{0}, \ldots, g_{n}\right)$ a tuple of representatives for $G$
if $g_{0}, \ldots, g_{n} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right], g_{0} \notin \mathcal{I}$ and

$$
G=X^{n}+\frac{g_{1}\left(z_{1}, \ldots, z_{r}\right)}{g_{0}\left(z_{1}, \ldots, z_{r}\right)} X^{n-1}+\cdots+\frac{g_{n}\left(z_{1}, \ldots, z_{r}\right)}{g_{0}\left(z_{1}, \ldots, z_{r}\right)} .
$$

We write

$$
G \prec\left(d^{*}, h^{*}\right)
$$

if $G$ has a tuple of representatives $\left(g_{0}, \ldots, g_{n}\right)$ with total degree $\operatorname{deg} g_{i} \leq d^{*}$ and logarithmic height $h\left(g_{i}\right) \leq h^{*}$ for $i=0, \ldots, n$, and call $\left(d^{*}, h^{*}\right)$ a degreeheight estimate for $G$.

In case that $G$ is a monic polynomial in $A[X]$, we call $\left(g_{1}, \ldots, g_{n}\right)$ an integral tuple of representatives for $G$ if $g_{1}, \ldots, g_{n} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ and

$$
G=X^{n}+g_{1}\left(z_{1}, \ldots, z_{r}\right) X^{n-1}+\cdots+g_{n}\left(z_{1}, \ldots, z_{r}\right) .
$$

We write

$$
G \stackrel{\text { int }}{\prec}\left(d^{*}, h^{*}\right)
$$

if $G$ has an integral tuple of representatives $\left(g_{1}, \ldots, g_{n}\right)$ with $\operatorname{deg} g_{i} \leq d^{*}$, $h\left(g_{i}\right) \leq h^{*}$ for $i=1, \ldots, n$.

Let $\alpha \in \bar{K}$. We denote the monic minimal polynomial of $\alpha$ over $K$ by $F_{\alpha}$. We denote by $d_{K}(\alpha)$ the degree of $\alpha$ over $K$, i.e., the degree of $F_{\alpha}$. We define a tuple of representatives for $\alpha$ to be a tuple of representatives for $F_{\alpha}$. We write

$$
\alpha \prec\left(d^{*}, h^{*}\right) \text { if } F_{\alpha} \prec\left(d^{*}, h^{*}\right)
$$

and call $\left(d^{*}, h^{*}\right)$ a degree-height estimate for $\alpha$. In case that $F_{\alpha} \in A[X]$, an integral tuple of representatives for $F_{\alpha}$ is also called an integral tuple of representatives for $\alpha$, and we write

$$
\alpha \stackrel{\mathrm{int}}{\prec}\left(d^{*}, h^{*}\right) \text { if } F_{\alpha} \stackrel{\mathrm{int}}{\prec}\left(d^{*}, h^{*}\right) .
$$

In particular, if $\alpha \in K$ then $\alpha \prec\left(d^{*}, h^{*}\right)$ if $\alpha$ has a pair of representatives each of which has total degree at most $d^{*}$ and logarithmic height at most $h^{*}$, while if $\alpha \in A$, then $\alpha \stackrel{\text { int }}{\prec}\left(d^{*}, h^{*}\right)$ if $\alpha$ has a representative of total degree at most $d^{*}$ and logarithmic height at most $h^{*}$.

We should mention here that there Moriwaki (2000) developed a sophisticated height theory for points in projective space $\mathbb{P}^{n}(\bar{K})$, based on Arakelov intersection theory, which may be seen as an analogue of the theory of abso-
lute Weil heights over $\mathbb{P}^{n}(\overline{\mathbb{Q}})$. We preferred to keep our presentation down to earth and to use the naive degree-height estimates introduced above. It would be of interest to figure out how our degree-height estimates relate to Moriwaki's height.

As mentioned above, our aim is to give a degree-height estimate for $\beta \in \bar{K}$ in terms of degree-height estimates for $\alpha_{1}, \ldots, \alpha_{m} \in \bar{K}$, if $\beta$ is related to the $\alpha_{i}$ by $P\left(\beta, \alpha_{1}, \ldots, \alpha_{m}\right)=0$ for some given $P \in \mathbb{Z}\left[X, X_{1}, \ldots, X_{m}\right]$ that is monic in $X$. We outline our procedure. Consider the polynomial

$$
G(X):=\prod_{i_{1}=1}^{n_{1}} \cdots \prod_{i_{m}=1}^{n_{m}} P\left(X, \alpha_{1}^{\left(i_{1}\right)}, \ldots, \alpha_{m}^{\left(i_{m}\right)}\right)
$$

where $\alpha_{i}^{\left(i_{j}\right)}\left(j=1, \ldots, n_{i}\right)$ are the conjugates of $\alpha_{i}$ over $K$, for $i=1, \ldots, m$. The polynomial $G$ is monic and by the theory of symmetric functions, its coefficients belong to $K$ and can be expressed in terms of the coefficients of the monic minimal polynomials of $\alpha_{1}, \ldots, \alpha_{m}$ over $K$. This enables us to derive a degree-height estimate for $G$. The polynomial $G$ has $\beta$ as a zero and thus, is a multiple of the monic minimal polynomial of $\beta$, but in general it is not equal to this minimal polynomial. To get a degree-height estimate for the minimal polynomial of $\beta$, hence of $\beta$ itself, we use estimates for degree-height estimates of the factors in $K[X]$ of a given polynomial in $K[X]$. We will derive such estimates in Section 8.2. In Section 8.3 we derive the degree-height estimate for $\beta$ in the way explained above, and give some further applications.

### 8.2 Estimates for factors of polynomials

We obtain explicit degree-height estimates for the monic divisors in $K[X]$ of a given monic polynomial in $K[X]$. Probably this would have been possible by making explicit arguments from Seidenberg (1974). We have chosen to use instead the specialization theory developed in Sections 7.2-7.4. We keep our assumptions that $A=\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]$ and that the ideal $\mathcal{I}$ of $f \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ with $f\left(z_{1}, \ldots, z_{r}\right)=0$ is generated by polynomials $f_{1}, \ldots, f_{M}$ with (8.1.2). We assume again that $z_{1}=X_{1}, \ldots, z_{q}=X_{q}$ is a transcendence basis of $K$, and write

$$
A_{0}=\mathbb{Z}\left[X_{1}, \ldots, X_{q}\right], \quad K_{0}=\mathbb{Q}\left(X_{1}, \ldots, X_{q}\right) .
$$

We will work with a domain $B=A_{0}\left[w, g^{-1}\right]$ as in Proposition 7.2.7 where we take $\mathcal{A}=\left\{\Delta_{\mathcal{F}}\right\}$, with $\Delta_{\mathcal{F}}$ the discriminant of the polynomial $\mathcal{F}$ from

Proposition 7.2.5, so that $\Delta_{\mathcal{F}} \in B^{*}$. Since $g \in A_{0}$ and since $w$ is integral over $A_{0}$, we have in fact

$$
\begin{equation*}
\Delta_{\mathcal{F}} \in K_{0} \cap B^{*}=A_{0}\left[g^{-1}\right]^{*} . \tag{8.2.1}
\end{equation*}
$$

We take the quantities $d_{1}, h_{1}$ defined in Proposition 7.2.7. In our situation we have

$$
d_{1}=\max \left(d, \operatorname{deg} \Delta_{\mathcal{F}}\right), \quad h_{1}=\max \left(h, h\left(\Delta_{\mathcal{F}}\right)\right) .
$$

By estimates completely similar to those in Lemma 7.2.8 we have

$$
\begin{aligned}
& \operatorname{deg} \Delta_{\mathcal{F}} \leq(2 D-2) d^{*}, \\
& h\left(\Delta_{\mathcal{F}}\right) \leq(2 D-2)\left(\log \left(2 D^{2}\left(\binom{d_{*}+r}{r}\right)+h^{*}\right),\right.
\end{aligned}
$$

where
$D=\left[K: K_{0}\right], d^{*}=\max \left(\operatorname{deg} \mathcal{F}_{1}, \ldots, \operatorname{deg} \mathcal{F}_{D}\right), h^{*}=\max \left(h\left(\mathcal{F}_{1}\right), \ldots, h\left(\mathcal{F}_{D}\right)\right)$.
Invoking the estimates $D \leq d^{r}$ implied by Lemma 7.2 .3 and those for $\operatorname{deg} \mathcal{F}_{i}$, $h\left(\mathcal{F}_{i}\right)$ from Lemma 7.2.5, we obtain

$$
\begin{equation*}
d_{1} \leq(2 d)^{\exp O(r)}, \quad h_{1} \leq(2 d)^{\exp O(r)} h \tag{8.2.2}
\end{equation*}
$$

We start with some preparatory lemmas.
Lemma 8.2.1. The above domain $B$ is integrally closed.
Proof. Denote by $x \mapsto x^{(i)}\left(i=1, \ldots, D=\left[K: K_{0}\right]\right)$ the $K_{0}$-isomorphic embeddings of $K$ in an algebraic closure $\overline{K_{0}}$ of $K_{0}$. Let $\beta \in K$ be integral over $B$. Then $\beta$ is integral over $A_{0}\left[g^{-1}\right]$. We have

$$
\beta=\sum_{j=0}^{D-1} b_{j} w^{j} \text { with } b_{0}, \ldots, b_{D-1} \in K_{0}
$$

and thus,

$$
\beta^{(i)}=\sum_{j=0}^{D-1} b_{j}\left(w^{(i)}\right)^{j} \text { for } i=1, \ldots, D .
$$

Viewing this as a system of linear equations in $b_{0}, \ldots, b_{D-1}$, we get by Cramer's rule,

$$
b_{j}=\Delta_{j} / \Delta \text { for } j=0, \ldots, D-1,
$$

where $\Delta=\operatorname{det}\left(\left(w^{(i)}\right)^{j-1}\right)_{i=1, \ldots, D, j=0, \ldots, D-1}$ and where $\Delta_{j}$ is the determinant obtained by replacing $\left(w^{(i)}\right)^{j-1}$ by $\beta^{(i)}$ for $i=1, \ldots, D$. Using Vandermonde's identity $\Delta^{2}=\prod_{1 \leq j<k \leq D}\left(w^{(j)}-w^{(k)}\right)^{2}=\Delta_{\mathcal{F}}$, we obtain

$$
b_{j}=\Delta_{j} \cdot \Delta / \Delta_{\mathcal{F}} \text { for } j=0, \ldots, D-1
$$

Clearly, $\Delta_{j} \Delta \in K$. Recall that the polynomial $\mathcal{F}$ is monic in $A_{0}[X]$, so $w$ is integral over $A_{0}$. Hence the $w^{(i)}(i=1, \ldots, D)$ are integral over $A_{0}$. Further, $\beta$ is integral over $B$, hence over $A_{0}\left[g^{-1}\right]$, and so the $\beta^{(i)}(i=1, \ldots, D)$ are integral over $A_{0}\left[g^{-1}\right]$. It follows that $\Delta_{j} \Delta$ is integral over $A_{0}\left[g^{-1}\right]$, and so it belongs to $A_{0}\left[g^{-1}\right]$ since the latter is a localization of a unique factorization domain, hence integrally closed. Together with (8.2.1) this implies $b_{j} \in A_{0}\left[g^{-1}\right]$ for $j=0, \ldots, D-1$. We conclude that $\beta \in \bar{B}$, as required.

Lemma 8.2.2. Let $F \in B[X]$ be a monic polynomial, and $G \in K[X]$ a monic polynomial that divides $F$ in $K[X]$. Then $G \in B[X]$.

Proof. For certain $\alpha_{1}, \ldots, \alpha_{n} \in \bar{K}$ we have $F=\left(X-\alpha_{1}\right) \cdots\left(X-\alpha_{n}\right), G=$ $\left(X-\alpha_{1}\right) \cdots\left(X-\alpha_{m}\right)$. Since $\alpha_{1}, \ldots, \alpha_{m}$ are integral over $B$, the coefficients of $G$ are also integral over $B$. These must belong to $B$, since $B$ is integrally closed.

We are now ready to prove our result concerning the degree-height estimates of the factors of a given polynomial.

Proposition 8.2.3. Let $d_{5} \geq d, h_{5} \geq h$ and let $F \in K[X]$ be a monic polynomial of degree $n \geq 2$ with $F \prec\left(d_{5}, h_{5}\right)$. Then for each monic polynomial $G \in K[X]$ dividing $F$ we have

$$
G \prec\left(\left(n d_{5}\right)^{\exp O(r)},\left(n d_{5}\right)^{\exp O(r)} h_{5}\right) .
$$

Proof. We can write $F=X^{n}+\left(a_{1} / a_{0}\right) X^{n-1}+\cdots+\left(a_{n} / a_{0}\right)$ where $a_{0}, \ldots, a_{n} \in$ $A, a_{0} \neq 0$, and $a_{i}$ has a representative $\widetilde{a_{i}}$ with $\operatorname{deg} \widetilde{a_{i}} \leq d_{5}, h\left(\widetilde{a_{i}}\right) \leq h_{5}$, for $i=0, \ldots, n$. Define $F^{*}(X):=a_{0}^{n} F\left(X / a_{0}\right)$. Then

$$
F^{*}(X)=X^{n}+a_{1}^{*} X^{n-1}+\cdots+a_{n}^{*} \text { where } a_{i}^{*}:=a_{i} a_{0}^{i-1} \text { for } i=1, \ldots, n .
$$

Clearly, $F^{*} \in \underset{\sim}{A}[X] \subseteq B[X]$ and by Corollary 4.1.6, $a_{i}^{*}(i=1, \ldots, n)$ has a representative $a_{i}^{*}$ with

$$
\begin{equation*}
\operatorname{deg} \widetilde{a_{i}^{*}} \leq n d_{5}, \quad h\left(\widetilde{a_{i}^{*}}\right) \leq n\left(r d_{5}+h_{5}\right) \text { for } i=1, \ldots, n . \tag{8.2.3}
\end{equation*}
$$

Together with Lemma 7.2.6 this implies

$$
\begin{align*}
\overline{\operatorname{deg}} a_{i}^{*} & \leq\left(n d_{5}\right)^{\exp O(r)} \text { for } i=1, \ldots, n,  \tag{8.2.4}\\
\bar{h}\left(a_{i}^{*}\right) & \leq\left(n d_{5}\right)^{\exp O(r)} h_{5} \text { for } i=1, \ldots, n . \tag{8.2.5}
\end{align*}
$$

Let $G \in K[X]$ be a monic divisor of $F$ of degree $m$, say, and set $G^{*}(X):=$ $a_{0}^{m} G\left(X / a_{0}\right)$. Then $G^{*}$ is a monic divisor of $F^{*}$ in $K[X]$, so by Lemma 8.2.2.

$$
\begin{equation*}
G^{*} \in B[X], \quad G^{* *}:=F^{*} / G^{*} \in B[X] . \tag{8.2.6}
\end{equation*}
$$

Write

$$
G^{*}(X)=X^{m}+b_{1}^{*} X^{m-1}+\cdots+b_{m}^{*}
$$

Then

$$
\begin{align*}
G(X) & =a_{0}^{-m} G^{*}\left(a_{0} X\right) \\
& =X^{m}+\left(b_{1}^{*} a_{0}^{-1}\right) X^{m-1}+b_{2}^{*} a_{0}^{-2} X^{m-2}+\cdots+b_{m}^{*} a_{0}^{-m} \tag{8.2.7}
\end{align*}
$$

We first estimate $\overline{\operatorname{deg}} b_{i}^{*}$ for $i=1, \ldots, m$ by making a reduction to function field heights. For $k=1, \ldots, q$, let $\mathbb{k}_{k}$ be the algebraic closure in $\overline{K_{0}}$ of $\mathbb{Q}\left(X_{1}, \ldots, X_{k-1}, X_{k+1}, \ldots, X_{q}\right)$, and

$$
L_{k}:=\mathbb{k}_{k}\left(X_{k}, w^{(1)}, \ldots, w^{(D)}\right), \quad \Delta_{k}:=\left[L_{k}: \mathbb{k}_{k}\left(X_{k}\right)\right],
$$

where $w^{(1)}, \ldots, w^{(D)}$ are the conjugates of $w$ over $K_{0}$. From (8.2.4), (7.3.9), with the notation as in Lemma 7.3.4, we deduce

$$
\Delta_{k}^{-1} H_{L_{k}}\left(\left(a_{i}^{*}\right)^{(j)}\right) \leq\left(n d_{5}\right)^{\exp O(r)}
$$

for $k=1, \ldots, q, j=1, \ldots, D, i=1, \ldots, m$. The polynomial $G^{*}$ divides $F^{*}$ in $L_{k}[X]$, so by (5.1.15) and (5.1.14),

$$
\Delta_{k}^{-1} H_{L_{k}}\left(\left(b_{i}^{*}\right)^{(j)}\right) \leq\left(n d_{5}\right)^{\exp O(r)}
$$

which together with (7.3.3) yields

$$
\begin{equation*}
\overline{\operatorname{deg}} b_{i}^{*} \leq\left(n d_{5}\right)^{\exp O(r)} \text { for } i=1, \ldots, m \tag{8.2.8}
\end{equation*}
$$

The next step is to estimate $\bar{h}\left(b_{i}^{*}\right)$ for $i=1, \ldots, m$. Inequalities (8.2.5), (7.4.15) imply that for $i=1, \ldots, n, j=1, \ldots, D$ and for each $\mathbf{u} \in \mathbb{Z}^{q}$ with

$$
\begin{aligned}
|\mathbf{u}| \leq\left(n d_{5}\right)^{\exp O(r)}, \mathcal{T}(\mathbf{u}) & \neq 0 \\
& h\left(\left(a_{i}^{*}\right)_{j}(\mathbf{u})\right) \leq\left(n d_{5}\right)^{\exp O(r)} h_{5},
\end{aligned}
$$

where $\left(a_{i}^{*}\right)_{j}(\mathbf{u})$ is the image of $a_{i}^{*}$ under the specialization homomorphism $\varphi_{\mathbf{u}, j}$. By 8.2.6), this homomorphism is also defined on the coefficients of $G^{*}, G^{* *}$. By applying $\varphi_{\mathbf{u}, j}$ to the coefficients of $G^{*}$ and $G^{* *}$, we see that the image of $G^{*}$ under $\varphi_{\mathbf{u}, j}$ divides the image of $F^{*}$ in $K_{\mathbf{u}, j}[X]$. Now we infer from Corollary 4.1.4 and inequality (4.1.5),

$$
h\left(\left(b_{i}^{*}\right)_{j}(\mathbf{u})\right) \leq\left(n d_{5}\right)^{\exp O(r)} h_{5}
$$

for $i=1, \ldots, m, j=1, \ldots, D$ and $\mathbf{u} \in \mathbb{Z}^{q}$ with $|\mathbf{u}| \leq\left(n d_{5}\right)^{\exp O(r)}, \mathcal{T}(\mathbf{u}) \neq$ 0 , where $\left(b_{i}^{*}\right)_{j}(\mathbf{u})$ is the image of $b_{i}^{*}$ under $\varphi_{\mathbf{u}, j}$. An application of inequality (7.4.18) with $k=1$, using (8.2.2), 8.2.8, then gives

$$
\begin{equation*}
\bar{h}\left(b_{i}^{*}\right) \leq\left(n d_{5}\right)^{\exp O(r)} h_{5} \text { for } i=1, \ldots, m . \tag{8.2.9}
\end{equation*}
$$

Inequalities 8.2.8 and 8.2.9 mean that there are $P_{i, 0}, \ldots, P_{i, D-1}, Q_{i} \in A_{0}$ such that

$$
\begin{align*}
& b_{i}^{*}=Q_{i}^{-1} \sum_{j=0}^{D-1} P_{i, j} w^{j} \text { with } \\
& \operatorname{deg} P_{i, j}, \operatorname{deg} Q_{i} \leq\left(n d_{5}\right)^{\exp O(r)}, h\left(P_{i, j}\right), h\left(Q_{i}\right) \leq\left(n d_{5}\right)^{\exp O(r)} h_{5} \tag{8.2.10}
\end{align*}
$$

for $i=1, \ldots, n, j=0, \ldots, D-1$. The $P_{i, j}$ and $Q_{i}$ belong to $A_{0}=\mathbb{Z}\left[X_{1}, \ldots, X_{q}\right] \subset$ $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$. Let $\widetilde{w}$ be the representative for $w$ from (7.2.6). From (8.2.10), 8.2.7) and some computations, using Lemma 4.1.7, it follows that ( $\left.\widetilde{b_{0}}, \ldots, \widehat{b}_{m}\right)$, given by

$$
\begin{aligned}
\widetilde{b_{0}} & ={\widetilde{a_{0}}}^{m} Q_{1} \cdots Q_{m}, \\
\widetilde{b_{i}} & ={\widetilde{a_{0}}}^{m-i} \prod_{\substack{k=1 \\
k \neq i}}^{m} Q_{k} \sum_{j=0}^{D-1} P_{i, j} \widetilde{w}^{j} \text { for } i=1, \ldots, m
\end{aligned}
$$

is a tuple of representatives for $G$ with

$$
\operatorname{deg} \widetilde{b_{i}} \leq\left(n d_{5}\right)^{\exp O(r)}, h\left(\widetilde{b_{i}}\right) \leq\left(n d_{5}\right)^{\exp O(r)} h_{5} \text { for } i=0, \ldots, m
$$

This proves Proposition 8.2.3

### 8.3 Consequences

Given degree-height estimates for certain elements $\alpha_{1}, \ldots, \alpha_{m}$ of $\bar{K}$, and given $P \in \mathbb{Z}\left[X, X_{1}, \ldots, X_{m}\right]$, we derive a degree-height estimate for $\beta$ satisfying $P\left(\beta, \alpha_{1}, \ldots, \alpha_{m}\right)=0$. Further, we give a degree-height estimate for a primitive element of a given finite extension $K\left(\alpha_{1}, \ldots, \alpha_{m}\right)$ of $K$. Lastly, we give degree-height estimates for solutions of systems of linear equations with coefficients from $\bar{K}$. These results are all consequences of the work from the previous section, together with a simple estimate for symmetric polynomials that we deduce below. The quantities $d, h$ satisfy (8.1.2).

Let $\mathbf{X}_{i}=\left(X_{i, 1}, \ldots, X_{i, n_{i}}\right)(i=1, \ldots, m)$ be blocks of variables. The block $\mathbf{Y}_{i}=\left(Y_{i, 1}, \ldots, Y_{i, n_{i}}\right)$ of elementary symmetric polynomials in $\mathbf{X}_{i}$ is given by

$$
X^{n_{i}}-Y_{i, 1} X^{n_{i}-1}+\cdots+(-1)^{n_{i}} Y_{i, n_{i}}=\left(X-X_{i, 1}\right) \cdots\left(X-X_{i, n_{i}}\right) .
$$

Let $R$ be any commutative ring with 1 . A polynomial $F \in R\left[\mathbf{X}_{1}, \ldots, \mathbf{X}_{m}\right]$ is called symmetric in $\mathbf{X}_{1}, \ldots, \mathbf{X}_{m}$ if

$$
F\left(\sigma_{1}\left(\mathbf{X}_{1}\right), \ldots, \sigma_{m}\left(\mathbf{X}_{m}\right)\right)=F\left(\mathbf{X}_{1}, \ldots, \mathbf{X}_{m}\right)
$$

for each tuple $\left(\sigma_{1}, \ldots, \sigma_{m}\right)$, with $\sigma_{i}$ a permutation of the variables in $\mathbf{X}_{i}$, for $i=1, \ldots, m$. By the theory of symmetric polynomials, such $F$ can be expressed as

$$
F\left(\mathbf{X}_{1}, \ldots, \mathbf{X}_{m}\right)=F^{\text {sym }}\left(\mathbf{Y}_{1}, \ldots, \mathbf{Y}_{m}\right)
$$

where $F^{\text {sym }}$ is a polynomial with coefficients in $R$. Further, if $F$ has total degree $\mathcal{D}$, then $F^{\text {sym }}$ is an $R$-linear combination of monomials $\prod_{i=1}^{m} \prod_{h=1}^{n_{i}} Y_{i, h}^{k_{i, h}}$, with

$$
\begin{equation*}
\sum_{i=1}^{m} \sum_{h=1}^{n_{h}} h \cdot k_{i, h} \leq \mathcal{D} \tag{8.3.1}
\end{equation*}
$$

We define the scalar product of any two tuples $\mathbf{a}=\left(a_{i}: i \in I\right), \mathbf{b}=\left(b_{i}: i \in\right.$ $I)$ with entries in some commutative ring by $\langle\mathbf{a}, \mathbf{b}\rangle=\sum_{i \in I} a_{i} b_{i}$.
Lemma 8.3.1. Let $F \in R\left[\mathbf{X}_{1}, \ldots, \mathbf{X}_{m}\right]$ be of total degree $\mathcal{D}$ and symmetric in the blocks $\mathbf{X}_{1}, \ldots, \mathbf{X}_{m}$. Denote by $\mathbf{f}$ the tuple of non-zero coefficients of $F$.

Then

$$
F^{\mathrm{sym}}\left(\mathbf{Y}_{1}, \ldots, \mathbf{Y}_{m}\right)=\sum_{\mathbf{k}}\left\langle\mathbf{s}_{\mathbf{k}}, \mathbf{f}\right\rangle \prod_{i=1}^{m} \prod_{h=1}^{n_{i}} Y_{i, h}^{k_{i, h}}
$$

where the sum is taken over all tuples $\mathbf{k}=\left(k_{1,1}, \ldots, k_{m, n_{m}}\right)$ of non-negative integers with (8.3.1), and where $\mathrm{s}_{\mathrm{k}}$ is a tuple with entries in $\mathbb{Z}$ of absolute value at most $3^{\mathcal{D}+n_{1}+\cdots+n_{m}}$.

Proof. For a tuple of non-negative integers $\mathbf{j}=\left(j_{1,1}, \ldots, j_{m, n_{m}}\right)$ we write $X^{\mathbf{j}}:=\prod_{i=1}^{m} \prod_{h=1}^{n_{m}} X_{i, h}^{j_{i, h}}, Y^{\mathbf{j}}:=\prod_{i=1}^{m} \prod_{h=1}^{n_{m}} Y_{i, h}^{j_{i, h}}$. For a tuple of non-negative integers $\mathbf{k}=\left(k_{1,1}, \ldots, k_{m, n_{m}}\right)$ with $k_{i, 1} \geq \cdots \geq k_{i, n_{i}} \geq 0$ for $i=1, \ldots, m$, let $\mathcal{J}_{\mathbf{k}}$ be the minimal set of tuples of non-negative integers $\mathbf{j}=\left(j_{1,1}, \ldots, j_{m, n_{m}}\right)$ such that $\mathbf{k} \in \mathcal{J}_{\mathbf{k}}$ and

$$
F_{\mathbf{k}}:=\sum_{\mathbf{j} \in \mathcal{J}_{\mathbf{k}}} \mathbf{X}^{\mathbf{j}}
$$

is symmetric in $\mathbf{X}_{1}, \ldots, \mathbf{X}_{m}$. Then

$$
F=\sum_{\mathbf{k}} f_{\mathbf{k}} F_{\mathbf{k}}
$$

where $f_{\mathbf{k}} \in R$ and the sum is taken over those tuples $\mathbf{k}$ with

$$
k_{i, 1} \geq \cdots \geq k_{i, n_{i}} \geq 0 \text { for } i=1, \ldots, m, \sum_{i=1}^{m} \sum_{h=1}^{n_{i}} k_{i, h} \leq \mathcal{D}
$$

By the theory of symmetric polynomials, $F_{\mathbf{k}}^{\text {sym }}$ has its coefficients in $\mathbb{Z}$. It suffices to show that these coefficients have absolute values at most $3^{\mathcal{E}+n}$, where $n:=n_{1}+\cdots+n_{m}$ and $\mathcal{E}:=\sum_{i=1}^{m} \sum_{h=1}^{n_{i}} k_{i, h}$. We have

$$
F_{\mathbf{k}}^{\mathrm{sym}}\left(\mathbf{Y}_{1}, \ldots, \mathbf{Y}_{m}\right)=\sum_{\mathbf{j}} f_{\mathbf{k}, \mathbf{j}}^{\mathrm{sym}} \mathbf{Y}^{\mathbf{j}}
$$

where the sum is over the tuples $\mathbf{j}=\left(j_{1,1}, \ldots, j_{m, n_{m}}\right)$ of non-negative integers with $\sum_{i=1}^{m} \sum_{h=1}^{n_{i}} h \cdot j_{i, h} \leq \mathcal{E}$, and the $f_{\mathbf{k}, \mathbf{j}}^{\text {sym }}$ are integers. Let $\mathbf{D}_{r}$ denote the set of vectors $\mathbf{z}=\left(z_{1,1}, \ldots, z_{m, n_{m}}\right) \in \mathbb{C}^{n}$ with $\left|z_{i, j}\right| \leq r$ for $i=1, \ldots, m, h=$ $1, \ldots, n_{i}$. Let $\gamma$ be the circle with center 0 and radius 1 in the complex plane, traversed counterclockwise. Recall that if $x_{i, 1}, \ldots, x_{i, n_{i}}, y_{i, 1}, \ldots, y_{i, n_{i}} \in \mathbb{C}$ are such that $X^{n}-y_{i, 1} X^{n-1}+\cdots+(-1)^{n} y_{i, n_{i}}=\left(X-x_{i, 1}\right) \cdots\left(X-x_{i, n_{i}}\right)$, then
$\max _{h}\left|x_{i, h}\right| \leq 1+\max _{h}\left|y_{i, h}\right|$. This leads to

$$
\begin{aligned}
\left|f_{\mathbf{k}, \mathbf{j}}^{\mathrm{sym}}\right| & =(2 \pi)^{-n}\left|\oint_{\gamma} \cdots \oint_{\gamma} F_{\mathbf{k}}^{\mathrm{sym}}(\mathbf{z}) \prod_{i=1}^{m} \prod_{h=1}^{n_{h}}\left(z_{i, h}^{-j_{i, h}-1} d z_{i, h}\right)\right| \\
& \leq \sup _{\mathbf{y} \in \mathbf{D}_{1}}\left|F_{\mathbf{k}}^{\operatorname{sym}}(\mathbf{y})\right| \leq \sup _{\mathbf{x} \in \mathbf{D}_{2}}\left|F_{\mathbf{k}}(\mathbf{x})\right| \\
& \leq\left|\mathcal{J}_{\mathbf{k}}\right| \cdot 2^{\mathcal{E}} \leq\binom{ n+\mathcal{E}-1}{\mathcal{E}} 2^{\mathcal{E}} \leq \sum_{k=0}^{n+\mathcal{E}-1}\binom{n+\mathcal{E}-1}{k} 2^{k} \leq 3^{\mathcal{E}+n},
\end{aligned}
$$

as required.
Before proving the result mentioned in the beginning of this section, we make a simple observation. Let $\alpha \in \bar{K}^{*}$. If $\left(\widetilde{g_{0}}, \ldots, \widetilde{g_{n}}\right)$ is a tuple of representatives for $\alpha$, i.e., for its minimal polynomial $F_{\alpha}$ over $K$, then clearly its reverse $\left(\widetilde{g_{n}}, \ldots, \widetilde{g_{0}}\right)$ is a tuple of representatives for $\alpha^{-1}$. This shows that

$$
\begin{equation*}
\alpha \prec\left(d^{*}, h^{*}\right) \Leftrightarrow \alpha^{-1} \prec\left(d^{*}, h^{*}\right) . \tag{8.3.2}
\end{equation*}
$$

We now state and prove the main result of this section. In the somewhat elaborate computations, we use the properties of heights and lengths of polynomials, stated in (4.1.7), 4.1.8) and Lemma 4.1.7.
Proposition 8.3.2. Let $P \in \mathbb{Z}\left[X, X_{1}, \ldots, X_{m}\right]$ be such that

$$
\operatorname{deg}_{X} P \geq 1, \quad P \text { is monic in } X
$$

and let $\alpha_{1}, \ldots, \alpha_{m} \in \bar{K}$ be such that

$$
\operatorname{deg}_{K} \alpha_{i}=n_{i}, \quad \alpha_{i} \prec\left(d_{6}, h_{6}\right) \text { for } i=1, \ldots, m,
$$

where $d_{6} \geq d, h_{6} \geq h$. Lastly, let $\beta \in \bar{K}$ satisfy

$$
P\left(\beta, \alpha_{1}, \ldots, \alpha_{m}\right)=0
$$

Then

$$
\begin{aligned}
& \operatorname{deg}_{K} \beta \leq\left(\operatorname{deg}_{X} P\right) \cdot n_{1} \cdots n_{m}, \\
& \beta \prec\left(\mathcal{R}_{1}^{\exp O(r)}, \mathcal{R}_{1}^{\exp O(r)}\left(h(P)+h_{6}\right)\right),
\end{aligned}
$$

where $\mathcal{R}_{1}:=2 m \cdot n_{1} \cdots n_{m} \cdot \operatorname{deg} P \cdot d_{6}$.

Proof. The estimate for $\operatorname{deg}_{K} \beta$ being clear, we proceed with computing a degree-height estimate for $\beta$.

For $i=1, \ldots, m$, let $\mathbf{X}_{i}=\left(X_{i, 1}, \ldots, X_{i, n_{i}}\right)$ and let $\mathbf{Y}_{i}=\left(Y_{i, 1}, \ldots, Y_{i, n_{i}}\right)$ be the elementary symmetric polynomials in $\mathbf{X}_{i}$. Consider the polynomial

$$
Q\left(X, \mathbf{X}_{1}, \ldots, \mathbf{X}_{m}\right)=\prod_{h_{1}=1}^{n_{1}} \cdots \prod_{h_{m}=1}^{n_{m}} P\left(X, X_{1, h_{1}}, \ldots, X_{m, h_{m}}\right) .
$$

This is symmetric in $\mathbf{X}_{1}, \ldots, \mathbf{X}_{m}$, hence

$$
Q\left(X, \mathbf{X}_{1}, \ldots, \mathbf{X}_{m}\right)=Q^{\mathrm{sym}}\left(X, \mathbf{Y}_{1}, \ldots, \mathbf{Y}_{m}\right),
$$

for some polynomial $Q^{\text {sym }}$ with integer coefficients. For $i=1, \ldots, m$, let $\left(g_{i, 0}, \ldots, g_{i, n_{i}}\right)$ be a tuple of representatives for $F_{\alpha_{i}}$ in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$, with

$$
\begin{equation*}
\operatorname{deg} g_{i, h} \leq d_{6}, h\left(g_{i, h}\right) \leq h_{6} \text { for } h=1, \ldots, n_{i} . \tag{8.3.3}
\end{equation*}
$$

Then the monic minimal polynomial of $\alpha_{i}$ over $K$ is given by

$$
F_{\alpha_{i}}:=X^{n_{i}}+\sum_{h=1}^{n_{i}} \frac{g_{i, h}\left(z_{1}, \ldots, z_{r}\right)}{g_{i, 0}\left(z_{1}, \ldots, z_{r}\right)} X^{n_{i}-h} .
$$

Let $\alpha_{i}^{(h)}\left(h=1, \ldots, n_{i}\right)$ be the conjugates of $\alpha_{i}$ over $K$. Further, let $\mathbf{b}_{i}:=$ $\left(b_{i, 1}, \ldots, b_{i, n_{i}}\right)$ be the tuple of elementary symmetric functions of $\alpha_{i}^{(1)}, \ldots, \alpha_{i}^{\left(n_{i}\right)}$. Then $b_{i, h}=(-1)^{h} g_{i, j}\left(z_{1}, \ldots, z_{r}\right) / g_{i, 0}\left(z_{1}, \ldots, z_{r}\right)$ for $h=1, \ldots, n_{i}$. Clearly,

$$
\prod_{h_{1}=1}^{n_{1}} \cdots \prod_{h_{m}=1}^{n_{m}} P\left(X, \alpha_{1}^{\left(h_{1}\right)}, \ldots, \alpha_{m}^{\left(h_{m}\right)}\right)=Q^{\mathrm{sym}}\left(X, \mathbf{b}_{1}, \ldots, \mathbf{b}_{m}\right)=: G(X)
$$

is a monic polynomial in $K[X]$ with $G(\beta)=0$. By replacing $Y_{i, h}$ by $(-1)^{h} g_{i, h} / g_{i, 0}$ for $i=1, \ldots, m, h=1, \ldots, n_{i}$ in $Q^{\text {sym }}$ we obtain a polynomial $\widetilde{G}$ in $X$ with coefficients in $\mathbb{Q}\left(X_{1}, \ldots, X_{r}\right)$. By substituting $z_{i}$ for $X_{i}$ in $\widetilde{G}$ for $i=1, \ldots, r$ we obtain again $G$. Clearing the denominators of $\widetilde{G}$ we get a polynomial

$$
\begin{align*}
F & :=\left(g_{1,0} \cdots g_{m, 0}\right)^{\operatorname{deg} Q^{\text {sym }}} \widetilde{G} \\
& =\sum_{k=0}^{\operatorname{deg} Q^{\text {sym }}}\left(\sum_{\mathbf{u}} a_{k}(\mathbf{u}) \prod_{i=1}^{m} \prod_{h=0}^{n_{i}} g_{i, h}^{u_{i, h}}\right) X^{k} \in \mathbb{Z}\left[X, X_{1}, \ldots, X_{r}\right], \tag{8.3.4}
\end{align*}
$$

where the inner sum is taken over tuples $\mathbf{u}=\left(u_{1,0}, \ldots, u_{m, n_{m}}\right)$ of nonnegative integers with $\sum_{i, h} u_{i, h}=m \operatorname{deg} Q^{\text {sym }}$ and the $a_{k}(\mathbf{u})$ are up to sign coefficients of $Q^{\text {sym }}$. According to the definition we have

$$
G \prec(\operatorname{deg} F, h(F)) .
$$

Since $G \in K[X]$ is monic and $G(\beta)=0$, the monic minimal polynomial $F_{\beta}$ of $\beta$ over $K$ divides $G$. So by Proposition 8.2.3,

$$
\begin{align*}
& \beta \prec\left(\left(2 d^{*}\right)^{\exp O(r)},\left(2 d^{*}\right)^{\exp O(r)} h^{*}\right), \\
& \quad \text { where } d^{*}:=\operatorname{deg} G \cdot \max (d, \operatorname{deg} F), \quad h^{*}:=\max (h, h(F)) . \tag{8.3.5}
\end{align*}
$$

We estimate the right-hand side of (8.3.5) from above. Let $n:=n_{1}+\cdots+$ $n_{m}+1$ and $\mathcal{D}:=\operatorname{deg} P, H:=H(P)$. We have

$$
\begin{aligned}
& \operatorname{deg} Q=\mathcal{D} \cdot n_{1} \cdots n_{m}, \\
& L(Q) \leq L(P)^{n_{1} \cdots n_{m}} \leq\left(\binom{\mathcal{D}+n}{\mathcal{D}} H\right)^{n_{1} \cdots n_{m}},
\end{aligned}
$$

where in the second estimate we used (4.1.7) and 4.1.8). From Lemma 8.3.1 we infer

$$
\begin{align*}
& \operatorname{deg} Q^{\text {sym }} \leq \mathcal{D} \cdot n_{1} \cdots n_{m}  \tag{8.3.6}\\
& H\left(Q^{\text {sym }}\right) \leq 3^{\mathcal{D} n_{1} \cdots n_{m}+n}\left(\binom{\mathcal{D}+n}{\mathcal{D}} H\right)^{n_{1} \cdots n_{m}} \tag{8.3.7}
\end{align*}
$$

Using (8.3.6) we get

$$
\operatorname{deg} G \leq \mathcal{D} \cdot n_{1} \cdots n_{m}
$$

and, using (8.3.3), (8.3.4), (8.3.6),

$$
\operatorname{deg} F \leq m \operatorname{deg} Q^{\operatorname{sym}}\left(d_{6}+1\right) \leq(m+1) \mathcal{D} n_{1} \cdots n_{m} \cdot d_{6}
$$

leading to

$$
\begin{equation*}
d^{*} \leq(m+1)\left(\mathcal{D} n_{1} \cdots n_{m}\right)^{2} d_{6} . \tag{8.3.8}
\end{equation*}
$$

For the height of $F$ we get, using (4.1.7), (4.1.8), (8.3.7), (8.3.3), (8.3.4),

$$
\begin{aligned}
& H(F) \leq L(F) \leq L\left(Q^{\text {sym }}\right)\left(\max _{i, h} L\left(g_{i, h}\right)\right)^{m \mathcal{D} n_{1} \cdots n_{m}} \\
& \leq 3^{\mathcal{D} n_{1} \cdots n_{m}+n}\binom{\mathcal{D} n_{1} \cdots n_{m}+n}{n}\binom{\mathcal{D}+n}{\mathcal{D}}^{n_{1} \cdots n_{m}} H^{n_{1} \cdots n_{m}}\left(\binom{d_{6}+r}{r} \exp h_{6}\right)^{m \mathcal{D} n_{1} \cdots n_{m}} .
\end{aligned}
$$

Using $n \leq 2 m \cdot n_{1} \cdots n_{m}$, this leads to

$$
h(F) \leq 5 m \mathcal{D} \cdot n_{1} \cdots n_{m}\left(\log (\mathcal{D}+n)+r+d_{6}+h(P)+h_{6}\right),
$$

which is clearly an upper bound for $h^{*}$. By substituting this bound and the upper bound for $d^{*}$ from (8.3.8) into (8.3.5) we arrive at

$$
F_{\beta} \prec\left(\mathcal{R}_{1}^{\exp O(r)}, \mathcal{R}_{1}^{\exp O(r)}\left(h(P)+h_{6}\right)\right) .
$$

This proves Proposition 8.3.2.
Corollary 8.3.3. Let $Q \in \mathbb{Z}\left[X_{1}, \ldots, X_{m}\right]$ be a non-constant polynomial, and let $\alpha_{1}, \ldots, \alpha_{m} \in \bar{K}$ be such that

$$
\operatorname{deg}_{K} \alpha_{i}=n_{i}, \quad \alpha_{i} \prec\left(d_{6}, h_{6}\right) \text { for } i=1, \ldots, m,
$$

where $d_{6} \geq d, h_{6} \geq h$. Then $\operatorname{deg}_{K} Q\left(\alpha_{1}, \ldots, \alpha_{m}\right) \leq n_{1} \cdots n_{m}$ and

$$
Q\left(\alpha_{1}, \ldots, \alpha_{m}\right) \prec\left(\mathcal{R}_{2}^{\exp O(r)}, \mathcal{R}_{2}^{\exp O(r)}\left(h(Q)+h_{6}\right)\right),
$$

where $\mathcal{R}_{2}:=2 m \cdot n_{1} \cdots n_{m} \cdot \operatorname{deg} Q \cdot d_{6}$.
Proof. Apply Proposition 8.3.2 with $P=X-Q\left(X_{1}, \ldots, X_{m}\right)$.
Corollary 8.3.4. Let $\alpha_{0}, \alpha_{1}, \ldots, \alpha_{m} \in \bar{K}$ be such that

$$
\begin{aligned}
\alpha_{0} \in L & :=K\left(\alpha_{1}, \ldots, \alpha_{m}\right) \\
\operatorname{deg}_{K} \alpha_{i} & =n_{i}, \quad \alpha_{i} \prec\left(d_{6}, h_{6}\right) \text { for } i=0, \ldots, m,
\end{aligned}
$$

where $d_{6} \geq d, h_{6} \geq h$. Put $\mathcal{E}:=[L: K]$ and

$$
\mathcal{R}_{3}:=2 m \cdot n_{1} \cdots n_{m} \cdot d_{6}, \quad \mathcal{R}_{4}:=2 m \cdot n_{0}^{n_{0}} \cdots n_{m}^{n_{m}} \cdot d_{6} .
$$

(i) There is $\theta \in L$ such that $L=K(\theta)$, $\theta$ has monic minimal polynomial
$F_{\theta} \in A[X]$ over $K$, and

$$
\theta \stackrel{\text { int }}{\prec}\left(\mathcal{R}_{3}^{\exp O(r)}, \mathcal{R}_{3}^{\exp O(r)} h_{6}\right) .
$$

(ii) We have $\alpha_{0}=\sum_{j=0}^{\mathcal{E}-1} p_{j} \theta^{j}$, where

$$
p_{j} \in K, p_{j} \prec\left(\mathcal{R}_{4}^{\exp O(r)}, \mathcal{R}_{4}^{\exp O(r)} h_{6}\right) \text { for } j=0, \ldots, \mathcal{E}-1 .
$$

Proof. We start with some preliminaries, before proving (i) and (ii).
Let $\sigma_{1}, \ldots, \sigma_{\mathcal{E}}: L \hookrightarrow \bar{K}$ denote the $K$-isomorphic embeddings of $L$. Denote by $\alpha_{k}^{(1)}, \ldots, \alpha_{k}^{\left(n_{k}\right)}$ the conjugates of $\alpha_{k}$ over $K$, for $k=0, \ldots, m$.

There are rational integers $a_{1}, \ldots, a_{m}$ with $\left|a_{i}\right| \leq \mathcal{E}^{2}$ for $i=1, \ldots, m$, such that the quantities $\sum_{j=1}^{m} a_{j} \sigma_{i}\left(\alpha_{j}\right)(i=1, \ldots, \mathcal{E})$ are pairwise distinct. Let $\gamma:=\sum_{j=1}^{m} a_{j} \alpha_{j}$. Then $\sigma_{i}(\gamma)(i=1, \ldots, \mathcal{E})$ are pairwise distinct, and thus, $L=K(\gamma)$. By applying Corollary 8.3.3 with $Q:=a_{1} X_{1}+\cdots+a_{m} X_{m}$, we get

$$
\gamma \prec\left(\mathcal{R}_{3}^{\exp O(r)}, \mathcal{R}_{3}^{\exp O(r)} h_{6}\right) .
$$

That is, there are $g_{i} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right](i=0, \ldots, \mathcal{E})$ such that for the monic minimal polynomial $F_{\gamma}$ of $\gamma$ over $K$ we have

$$
\begin{align*}
F_{\gamma}= & X^{\mathcal{E}}+\left(b_{1} / b_{0}\right) X^{\mathcal{E}-1}+\cdots+\left(b_{\mathcal{E}} / b_{0}\right), \text { with } \\
& b_{i}:=g_{i}\left(z_{1}, \ldots, z_{r}\right), \\
& \operatorname{deg} g_{i} \leq \mathcal{R}_{3}^{\exp O(r)}, h\left(g_{i}\right) \leq \mathcal{R}_{3}^{\exp O(r)} h_{6} \text { for } i=0, \ldots, \mathcal{E} . \tag{8.3.9}
\end{align*}
$$

(i) Let $\theta:=b_{0} \gamma$. Then $L=K(\theta), \theta$ has monic minimal polynomial

$$
F_{\theta}=b_{0}^{\mathcal{E}} F_{\theta}\left(X / b_{0}\right)=X^{\mathcal{E}}+b_{1} X^{\mathcal{E}-1}+b_{0} b_{2} X^{\mathcal{E}-2}+\cdots+b_{0}^{\mathcal{E}-1} b_{\mathcal{E}} \in A[X]
$$

over $K$, and by Lemma 4.1.7, applied with the polynomials $g_{i}$,

$$
\theta \stackrel{\text { int }}{\prec}\left(\mathcal{R}_{3}^{\exp O(r)}, \mathcal{R}_{3}^{\exp O(r)} h_{6}\right) .
$$

This proves (i).
(ii) There are $q_{j} \in K(j=0, \ldots, \mathcal{E}-1)$ such that

$$
\alpha_{0}=\sum_{j=0}^{\mathcal{E}-1} q_{j} \gamma^{j} .
$$

Hence

$$
\sigma_{i}\left(\alpha_{0}\right)=\sum_{j=0}^{\mathcal{E}-1} q_{j} \cdot \sigma_{i}(\gamma)^{j} \text { for } k=1, \ldots, m, i=1, \ldots, \mathcal{E}
$$

By Cramer's rule, we have

$$
q_{j}=\Delta_{j} / \Delta \text { for } j=0, \ldots, \mathcal{E}-1
$$

where $\Delta:=\operatorname{det}\left(\sigma_{i}(\gamma)^{j}\right)_{i=1, \ldots, \mathcal{E}, j=0, \ldots, \mathcal{E}-1}$, and $\Delta_{j}$ is the determinant obtained by replacing $\sigma_{i}(\gamma)^{j}$ by $\sigma_{i}\left(\alpha_{0}\right)$, for $i=1, \ldots, \mathcal{E}$. We clearly have

$$
\begin{equation*}
\alpha_{0}=\sum_{j=0}^{\mathcal{E}-1} p_{j} \beta^{j} \text { with } p_{j}=\Delta_{j} /\left(\Delta g_{0}^{j}\right) \text { for } j=0, \ldots, \mathcal{E}-1 \tag{8.3.10}
\end{equation*}
$$

Using $\sigma_{i}(\gamma)=\sum_{\ell=1}^{m} a_{\ell} \sigma_{i}\left(\alpha_{\ell}\right)$, we see that both $\Delta_{j}$ and $\Delta$ are polynomials with integer coefficients in $\alpha_{0}^{(1)}, \ldots, \alpha_{0}^{\left(n_{0}\right)}, \ldots, \alpha_{m}^{(1)}, \ldots, \alpha_{m}^{\left(n_{m}\right)}$. These polynomials have degrees at most $\mathcal{E}^{2}$. Further, a similar computation as carried out in the proof of Proposition 8.3.2, using (4.1.7), (4.1.8), shows that these polynomials have logarithmic heights at most $O\left(\mathcal{E}^{2} \log \left(2 m \mathcal{E}^{2}\right)\right)$ with the implied constant being absolute. Using Corollary 8.3 .3 and $\mathcal{E} \leq n_{1} \cdots n_{m}$, this shows that

$$
\Delta_{0}, \ldots, \Delta_{\mathcal{E}-1}, \Delta \prec\left(\mathcal{R}_{4}^{\exp O(r)}, \mathcal{R}_{4}^{\exp O(r)} h_{6}\right)
$$

while (8.3.9) gives $b_{0} \stackrel{\text { int }}{\prec}\left(\mathcal{R}_{3}^{\exp O(r)}, \mathcal{R}_{3}^{\exp O(r)} h_{6}\right)$. Now applying (8.3.2) and Corollary 8.3 .3 to 8.3.10), using the estimates for $\Delta_{j}, \Delta, g_{0}$ just established, we arrive at

$$
p_{j} \prec\left(\mathcal{R}_{4}^{\exp O(r)}, \mathcal{R}_{4}^{\exp O(r)} h_{6}\right) \text { for } j=0, \ldots, \mathcal{E}-1
$$

This proves (ii).
Corollary 8.3.5. Let $k, l$ be positive integers, $\mathcal{A}=\left(\alpha_{i, j}\right)_{i=1, \ldots, k j=1, \ldots, l}$ a $k \times$ $l$-matrix and $\mathbf{b}=\left(\beta_{1}, \ldots, \beta_{k}\right)^{T}$ a $k$-dimensional column vector, both with
entries in $\bar{K}$, satisfying

$$
\begin{aligned}
& \operatorname{deg}_{K} \alpha_{i, j} \leq n_{i, j}, \alpha_{i, j} \prec\left(d_{7}, h_{7}\right), \quad \operatorname{deg}_{K} \beta_{i} \leq n_{i}, \beta_{i} \prec\left(d_{7}, h_{7}\right) \\
& \text { for } i=1, \ldots, k, j=1, \ldots, l,
\end{aligned}
$$

where $d_{7} \geq d, h_{7} \geq h$. Suppose that

$$
\begin{equation*}
\mathcal{A} \mathrm{x}=\mathrm{b} \tag{8.3.11}
\end{equation*}
$$

has a solution $\mathbf{x}=\left(x_{1}, \ldots, x_{l}\right)^{T} \in A^{l}$. Then (8.3.11) has such a solution with

$$
\begin{equation*}
x_{j} \stackrel{\text { int }}{\prec}\left(\mathcal{R}_{5}^{\exp O\left(r \log ^{*} r\right)} h_{7}, \mathcal{R}_{5}^{\exp O\left(r \log ^{*} r\right)} h_{7}^{r+1}\right) \text { for } j=1, \ldots, l \text {, } \tag{8.3.12}
\end{equation*}
$$

where $\mathcal{R}_{5}:=\left(\prod_{i=1}^{k} \prod_{j=1}^{l} n_{i, j}^{n_{i, j}}\right) \cdot\left(\prod_{i=1}^{k} n_{i}^{n_{i}}\right) 2 k l d_{7}$.

Proof. Let $L$ be the extension of $K$ generated by the $\alpha_{i, j}$ and the $\beta_{i}$, for $i=$ $1, \ldots, k, j=1, \ldots, l$, and put $\mathcal{E}:=[L: K]$. By Corollary 8.3 .4 there exists $\theta \in L$ such that $L=K(\theta), \theta$ has monic minimal polynomial $F_{\theta} \in A[X]$, and moreover, there exist $a_{i, j, h}, b_{i, h} \in K$, for $i=1, \ldots, k, j=1, \ldots, l$, $h=0, \ldots, \mathcal{E}-1$ such that

$$
\begin{aligned}
& \alpha_{i, j}=\sum_{h=0}^{\mathcal{E}-1} a_{i, j, h} \theta^{h}, \quad \beta_{i}=\sum_{h=0}^{\mathcal{E}-1} b_{i, h} \theta^{h}, \\
& \text { for } i=1, \ldots, k, j=1, \ldots, l,
\end{aligned}
$$

and

$$
\begin{aligned}
& a_{i, j, h}, b_{i, h} \prec\left(\mathcal{R}_{5}^{\exp O(r)}, \mathcal{R}_{5}^{\exp O(r)} h_{7}\right) \\
& \text { for } i=1, \ldots, k, j=1 \ldots l, h=0, \ldots, \mathcal{E}-1 .
\end{aligned}
$$

This means that

$$
a_{i, j, h}=\frac{g_{i, j, h}^{\prime}\left(z_{1}, \ldots, z_{r}\right)}{g_{i, j, h}^{\prime \prime}\left(z_{1}, \ldots, z_{r}\right)}, \quad b_{i, h}=\frac{g_{i, j}^{\prime}\left(z_{1}, \ldots, z_{r}\right)}{g_{i, j}^{\prime \prime}\left(z_{1}, \ldots, z_{r}\right)},
$$

where $g_{i, j, h}^{\prime}, g_{i, j, h}^{\prime \prime}, g_{i, j}^{\prime}, g_{i, j}^{\prime \prime}$ are polynomials from $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ of total degree at most $\mathcal{R}_{5}^{\exp O(r)}$ and logarithmic height at most $\mathcal{R}_{5}^{\exp O(r)} h_{7}$, for $i=1, \ldots, k$, $j=1, \ldots, l, h=0, \ldots, \mathcal{E}-1$. Take the product of the denominators of the
$a_{i, j, h}, b_{i, j}$,

$$
c_{0}:=\left(\prod_{i=1}^{k} \prod_{j=1}^{l} \prod_{h=0}^{\mathcal{E}-1} g_{i, j, h}^{\prime \prime}\left(z_{1}, \ldots, z_{r}\right)\right) \cdot\left(\prod_{i=1}^{k} \prod_{j=1}^{l} g_{i, j}^{\prime \prime}\left(z_{1}, \ldots, z_{r}\right)\right)
$$

and put

$$
a_{i, j, h}^{\prime}:=c_{0} a_{i, j, h}, b_{i, h}^{\prime}:=c_{0} b_{i, h} \text { for all } i, j, h .
$$

Then the $a_{i, j, h}^{\prime}, b_{i, h}^{\prime}$ all belong to $A$ and by Lemma 4.1.7 we have

$$
\begin{align*}
& a_{i, j, h}^{\prime}, b_{i, h}^{\prime} \stackrel{\text { int }}{\prec}\left(\mathcal{R}_{5}^{\exp O(r)}, \mathcal{R}_{5}^{\exp O(r)} h_{7}\right) \\
& \text { for } i=1, \ldots, k, j=1, \ldots, l, h=0, \ldots, \mathcal{E}-1 . \tag{8.3.13}
\end{align*}
$$

Writing

$$
\mathcal{A}_{h}=\left(a_{i, j, h}^{\prime}\right)_{i=1, \ldots, k, j=1, \ldots, l}, \quad \mathbf{b}_{h}=\left(b_{1, h}^{\prime}, \ldots, b_{k, h}^{\prime}\right)^{T}
$$

we get

$$
c_{0} \mathcal{A}=\sum_{h=0}^{\mathcal{E}-1} \theta^{h} \mathcal{A}_{h}, \quad c_{0} \mathbf{b}=\sum_{h=0}^{\mathcal{E}-1} \theta^{h} \mathbf{b}_{h} .
$$

Therefore, the solution set in $\mathrm{x} \in A^{l}$ of 8.3.11) is equal to that of

$$
\sum_{h=0}^{\mathcal{E}-1} \theta^{h} \mathcal{A}_{h} \mathbf{x}=\sum_{h=0}^{\mathcal{E}-1} \theta^{h} \mathbf{b}_{h}
$$

whence to that of

$$
\begin{equation*}
\mathcal{A}_{h} \mathbf{x}=\mathbf{b}_{h} \text { for } h=0, \ldots, \mathcal{E}-1, \tag{8.3.14}
\end{equation*}
$$

since $1, \theta, \ldots, \theta^{\mathcal{E}-1}$ are linearly independent over $K$. Let $\widetilde{\mathcal{A}_{h}}, \widetilde{\mathbf{b}_{h}}$ consist of representatives in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ of the entries of $\mathcal{A}_{h}, \mathbf{b}_{h}$, which by (8.3.13) we may choose of total degrees at most $\mathcal{R}_{5}^{\exp O(r)}$ and logarithmic heights at most $\mathcal{R}_{5}^{\exp O(r)} h_{7}$. Then system (8.3.14) is equivalent to

$$
\widetilde{\mathcal{A}_{h}} \widetilde{\mathbf{x}} \equiv \widetilde{\mathbf{b}_{h}}(\bmod \mathcal{I}) \text { for } h=0, \ldots, \mathcal{E}-1,
$$

to be solved in $\widetilde{\mathbf{x}} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]^{l}$, where the components of $\widetilde{\mathbf{x}}$ are represen-
tatives for the components of $\mathbf{x}$, and we can rewrite the latter system as

$$
\widetilde{\mathcal{A}_{h}} \widetilde{\mathbf{x}}=\widetilde{\mathbf{b}_{h}}+\sum_{i=1}^{M} f_{i} \widetilde{\mathbf{y}_{i, h}}(h=0, \ldots, \mathcal{E}-1)
$$

with solutions

$$
\left(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}_{1,0}}, \ldots, \widetilde{\mathbf{y}_{M, \mathcal{E}-1}}\right) \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]^{l(M \mathcal{E}+1)}
$$

where $\left\{f_{1}, \ldots, f_{M}\right\}$ is the set of generators for $\mathcal{I}$ that we have chosen in the very beginning. Note that this system contains altogether

$$
k \mathcal{E} \leq k\left(\prod_{i=1}^{k} \prod_{j=1}^{l} n_{i, j}\right) \cdot\left(\prod_{i=1}^{k} n_{i}\right)
$$

linear equations, and that each coefficient of this system is a polynomial in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ of total degree at most $\mathcal{R}_{5}^{\exp O(r)}$ and logarithmic height at most $\mathcal{R}_{5}^{\exp O(r)} h_{7}$. So by Theorem 6.1.5, if this system has a solution with coordinates in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$, then it has such a solution, of which each coordinate has total degree and logarithmic height at most

$$
\mathcal{R}_{5}^{\exp O\left(r \log ^{*} r\right)} h_{7}, \quad \mathcal{R}_{5}^{\exp O\left(r \log ^{*} r\right)} h_{7}^{r+1},
$$

respectively. This implies that system 8.3.14, hence the original system 8.3.11) we started with, has a solution $\mathbf{x}=\left(x_{1}, \ldots, x_{l}\right)^{T} \in A^{l}$ with 8.3.12). This completes the proof of Corollary 8.3.5.

## Chapter 9

## Proofs of the results from Sections 2.2-2.5; use of specializations

In this chapter we prove the general effective theorems from Chapter 2 on unit equations, Thue equations, hyper- and superelliptic equations and the Catalan equation over finitely generated domains of the form $A=\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]$.

We consider our equations over a more convenient finitely generated overring $B$ of $A$, constructed in Chapter 7 . Then, to prove our theorems, we reduce our equations first to the function field case and then to the number field case by means of the effective specializations described in Chapter 7 .

Sections $9.1,9.2$ and 9.3 are devoted to unit equations, Thue equations and hyper- and superelliptic equations, while Section 9.4 deals with the Catalan equation. Using some results from Chapter 7, in Section 9.1 we first reduce the estimates of the sizes of appropriate representatives of the solutions $x, y$ in $B$ resp. in $B^{*}$ to bounding the degrees $\overline{\operatorname{deg}} x, \overline{\operatorname{deg}} y$ and heights $\bar{h}(x), \bar{h}(y)$, introduced in Chapter 7 . Then, by means of the effective results from Chapter 5 concerning the corresponding equations over function fields we derive in Section 9.2 bounds for $\overline{\operatorname{deg}} x, \overline{\operatorname{deg}} y$. Finally, combining the effective specializations presented in Chapter 7 with the corresponding effective results from Chapter 4 over number fields we give in Section 9.3 effective upper bounds for $\bar{h}(x), \bar{h}(y)$ which completes the proof of our general effective results over finitely generated domains.

As was pointed out in Section 2.5, in case of the Catalan equation it is enough to derive an effective upper bound for the unknown exponents. In Section 9.4 we combine the corresponding effective results from Chapters 4 and 5 over number fields resp. function fields with a simplified version, used by Brindza (1993) and Koymans (2017), of our general method to bound the
exponents under consideration.

### 9.1 A reduction

For convenience, we repeat some notation and definitions. As before, let $A=$ $\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]$ with $r>0$ be an integral domain of characteristic 0 which is finitely generated over $\mathbb{Z}$, and let $K$ denote its quotient field. Then we have

$$
\begin{equation*}
A \cong \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right] / \mathcal{I} \tag{9.1.1}
\end{equation*}
$$

where $\mathcal{I}$ is the ideal of polynomials $f \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ such that $f\left(z_{1}, \ldots, z_{r}\right)=$ 0 . The ideal $\mathcal{I}$ is finitely generated. Assume as before that

$$
\begin{gather*}
\mathcal{I}=\left(f_{1}, \ldots, f_{M}\right) \text { with } \operatorname{deg} f_{i} \leq d, h\left(f_{i}\right) \leq h \text { for } i=1, \ldots, M, \\
\text { where } d \geq 1, h \geq 1 . \tag{9.1.2}
\end{gather*}
$$

Suppose that $K$ has transcendence degree $q \geq 0$ over $\mathbb{Q}$. If $q>0$, we assume without loss of generality that $z_{1}=X_{1}, \ldots, z_{q}=X_{q}$ is a transcendence basis for $K / \mathbb{Q}$. We define as before,

$$
\begin{array}{lll}
A_{0}:=\mathbb{Z}\left[X_{1}, \ldots, X_{q}\right], & K_{0}:=\mathbb{Q}\left(X_{1}, \ldots, X_{q}\right) & \text { if } q>0, \\
A_{0}:=\mathbb{Z}, & K_{0}:=\mathbb{Q} & \text { if } q=0 .
\end{array}
$$

For $q \geq 0, A_{0}$ is a unique factorization domain. For $f \in A_{0} \backslash\{0\}$ we denote by $\operatorname{deg} f$ and $h(f)$ the (total) degree and logarithmic height of $f$, where in the case $q=0$ we put $\operatorname{deg} f:=0$ and $h(f):=\log |f|$ if $f \in \mathbb{Z} \backslash\{0\}$.

By Proposition 7.2 .5 there is a $w \in A$, integral over $A_{0}$ such that $K=$ $K_{0}(w)$ and $w$ has minimal polynomial $\mathcal{F}(X)=X^{D}+\mathcal{F}_{1} X^{D-1}+\cdots+\mathcal{F}_{D}$ over $K_{0}$ such that

$$
\begin{equation*}
\mathcal{F}_{j} \in A_{0}, \operatorname{deg} \mathcal{F}_{j} \leq(2 d)^{\exp O(r)}, h\left(\mathcal{F}_{j}\right) \leq(2 d)^{\exp O(r)} h \tag{9.1.3}
\end{equation*}
$$

for $j=1, \ldots, D$ and, by Lemma 7.2.3,

$$
\begin{equation*}
D \leq d^{t} \text { where } t=r-q . \tag{9.1.4}
\end{equation*}
$$

In what follows, we fix such an element $w$. With every $\alpha \in K$ we associate an up to sign unique tuple $P_{\alpha, 0}, \ldots, P_{\alpha, D-1}, Q_{\alpha}$ from $A_{0}$ such that (7.2.7) holds,
i.e.

$$
\alpha=Q_{\alpha}^{-1} \sum_{j=0}^{D-1} P_{\alpha, j} w^{j} \text { with } Q_{\alpha} \neq 0, \operatorname{gcd}\left(P_{\alpha, 0}, \ldots, P_{\alpha, D-1}, Q_{\alpha}\right)=1 .
$$

Then, as in Section 7.2 we define

$$
\begin{aligned}
& \overline{\operatorname{deg}} \alpha:=\max \left(\operatorname{deg} P_{\alpha, 0}, \ldots, \operatorname{deg} P_{\alpha, D-1}, \operatorname{deg} Q_{\alpha}\right), \\
& \bar{h}(\alpha):=\max \left(h\left(P_{\alpha, 0}\right), \ldots, h\left(P_{\alpha, D-1}\right), h\left(Q_{\alpha}\right)\right) .
\end{aligned}
$$

We shall deal separately with unit equations, Thue equations and hyperand superelliptic equations.

### 9.1.1 Unit equations

We shall deduce our general Theorem 2.2.1 on unit equations from the following.
Proposition 9.1.1. Let $g \in A_{0} \backslash\{0\}$ and put

$$
\begin{align*}
d_{4} & :=\max \left(d, \operatorname{deg} g, \operatorname{deg} \mathcal{F}_{1}, \ldots, \operatorname{deg} \mathcal{F}_{D}\right), \\
h_{4} & :=\max \left(h, h(g), h\left(\mathcal{F}_{1}\right), \ldots, h\left(\mathcal{F}_{D}\right)\right)
\end{align*}
$$

Define the domain $B:=A_{0}\left[w, g^{-1}\right]$. Then for every pair $(\varepsilon, \eta)$ with

$$
\begin{equation*}
\varepsilon+\eta=1, \varepsilon, \eta \in B^{*} \tag{9.1.6}
\end{equation*}
$$

we have

$$
\begin{align*}
\overline{\operatorname{deg}} \varepsilon, \overline{\operatorname{deg}} \eta & \leq 4 q D^{2} d_{4},  \tag{9.1.7}\\
\bar{h}(\varepsilon), \bar{h}(\eta) & \leq \exp O\left(2 D\left(q+d_{4}\right)\left(\log ^{*}\left(2 D\left(q+d_{4}\right)\right)\right)^{2} \cdot h_{4}\right) . \tag{9.1.8}
\end{align*}
$$

The proof of Proposition 9.1 .1 is given in the Subsections 9.2.1, 9.3.1 In Subsection 9.2.1 we deduce the degree bound (9.1.7). Here, our main tool is Mason's effective result on $S$-unit equations in function fields; see Mason (1983) or Theorem 5.2.1 in Section 5.2. In subsection 9.3.1 we prove (9.1.8) by combining 9.1 .7 with our general specialization method from Evertse and Győry (2013), as presented in Chapter 7 , and with an effective result of Győry and Yu (2006) on $S$-unit equations over number fields; see also Theorem4.3.1 in Chapter 4 .

We now deduce Theorem 2.2.1 from Proposition 9.1.1.
Proof of Theorem 2.2.1. Let $a, b, c$ be the coefficients in the unit equation (2.2.1), and $\tilde{a}, \tilde{b}, \tilde{c}$ the representatives for $a, b, c$ from the statement of Theorem 2.2.1. Then by assumption

$$
\max \left(\operatorname{deg} f_{1}, \ldots, \operatorname{deg} f_{M}, \operatorname{deg} \tilde{a}, \operatorname{deg} \tilde{b}, \operatorname{deg} \tilde{c}\right) \leq d
$$

and

$$
\max \left(h\left(f_{1}\right), \ldots, h\left(f_{M}\right), h(\tilde{a}), h(\tilde{b}), h(\tilde{c})\right) \leq h,
$$

where $d \geq 1, h \geq 1$. Further, as was mentioned above, by Proposition 7.2.5 and Lemma 7.2 .3 we have $K=K_{0}(w)$ with $w \in A$, integral over $A_{0}$, and $w$ has minimal polynomial $\mathcal{F}(X)=X^{D}+\mathcal{F}_{1} X^{D-1}+\cdots+\mathcal{F}_{D}$ over $K_{0}$ such that 9.1.3 and 9.1.4 hold.

In view of Proposition 7.2.7 there is a non-zero $g \in A_{0}$ such that

$$
A \subseteq B:=A_{0}\left[w, g^{-1}\right], a, b, c \in B^{*}
$$

and

$$
\begin{equation*}
\operatorname{deg} g \leq(2 d)^{\exp O(r)}, \quad h(g) \leq(2 d)^{\exp O(r)} h . \tag{9.1.9}
\end{equation*}
$$

Let $(x, y)$ be a solution of equation (2.2.1) and put $x_{1}:=a x / c, y_{1}:=b y / c$. Then $x_{1}+y_{1}=1$ and $x_{1}, y_{1} \in B^{*}$. By (7.4.5) we have $d_{4} \leq(2 d)^{\exp O(r)}, h_{4} \leq$ $(2 d)^{\exp O(r)} h$. We apply now Proposition 9.1.1 with $\varepsilon=x_{1}, \eta=y_{1}$. It follows from Proposition 9.1.1 that

$$
\begin{align*}
\overline{\operatorname{deg}} x_{1} & \leq 4 q d^{2 t}(2 d)^{\exp O(r)} \leq(2 d)^{\exp O(r)},  \tag{9.1.10}\\
\bar{h}\left(x_{1}\right) & \leq \exp \left((2 d)^{\exp O(r)} h\right) \tag{9.1.11}
\end{align*}
$$

We use Lemma 7.3 .1 with $\lambda=a / c$, which is represented by $(\tilde{a}, \tilde{c})$. Choosing $a=\tilde{a}, b=\tilde{c}$ and $\alpha=x$ in Lemma 7.3.1, we have $\lambda \alpha=x_{1}$, and in view of 9.1.10) and (9.1.11) we have

$$
d_{2} \leq(2 d)^{\exp O(r)}, h_{2} \leq \exp \left((2 d)^{\exp O(r)} h\right)
$$

We infer that $x, x^{-1}$ have representatives $\tilde{x}, \tilde{x}^{\prime}$ in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ such that

$$
\operatorname{deg} \tilde{x}, \operatorname{deg} \tilde{x}^{\prime}, h(\tilde{x}), h\left(\tilde{x}^{\prime}\right) \leq \exp \left((2 d)^{\exp O(r)} h\right) .
$$

In the same way one can derive similar upper bounds for the degrees and logarithmic heights of representatives for $y$ and $y^{-1}$. This completes the proof of Theorem 2.2.1.

In our proof of Theorem 2.2.3, given below, we need the following lemma.
Lemma 9.1.2. Let $A \cong \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right] /\left(f_{1}, \ldots, f_{M}\right)$ be an integral domain. Let $\beta \in A \backslash\{0\}$, and let $\widetilde{\beta} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ be a representative for $\beta$. Then

$$
A\left[\beta^{-1}\right] \cong \mathbb{Z}\left[X_{1}, \ldots, X_{r}, Y\right] /\left(f_{1}, \ldots, f_{M}, 1-\widetilde{\beta} Y\right)
$$

Proof. For $f \in A[Y]$, define $f^{*}:=Y^{\operatorname{deg} f} f\left(Y^{-1}\right)$. Then for $f \in A[Y]$ we have

$$
\begin{aligned}
f\left(\beta^{-1}\right)=0 & \Leftrightarrow f^{*}(\beta)=0 \Leftrightarrow f^{*}=(Y-\beta) h^{*} \text { for some } h^{*} \in A[Y] \\
& \Leftrightarrow f=(1-\beta Y) h \text { for some } h \in A[Y] .
\end{aligned}
$$

Now via the ring homomorphism $A[Y] \mapsto A\left[\beta^{-1}\right]: f \mapsto f\left(\beta^{-1}\right)$ we obtain $A[Y] /(1-\beta Y) \cong A\left[\beta^{-1}\right]$. This implies

$$
\mathbb{Z}\left[X_{1}, \ldots, X_{r}, Y\right] /\left(f_{1}, \ldots, f_{M}, 1-\widetilde{\beta} Y\right) \cong A[Y] /(1-\beta Y) \cong A\left[\beta^{-1}\right]
$$

Proof of Theorem 2.2.3. We keep the notation and assumptions from the statement of Theorem 2.2.3. For $i=1, \ldots, s$, let

$$
\alpha_{i}:=g_{i, 1}\left(z_{1}, \ldots, z_{r}\right), \beta_{i}:=g_{i, 2}\left(z_{1}, \ldots, z_{r}\right),
$$

with $\alpha_{i}, \beta_{i} \in A$ so that $\gamma_{i}=\alpha_{i} / \beta_{i}$ and define the ring

$$
\widetilde{A}:=A\left[\alpha_{1}^{-1}, \beta_{1}^{-1}, \ldots, \alpha_{s}^{-1}, \beta_{s}^{-1}\right] .
$$

Then by repeatedly applying Lemma 9.1 .2 we obtain

$$
\widetilde{A} \cong \mathbb{Z}\left[X_{1}, \ldots, X_{r}, X_{r+1}, \ldots, X_{r+2 s}\right] / \widetilde{\mathcal{I}}
$$

with

$$
\begin{gathered}
\tilde{\mathcal{I}}=\left(f_{1}, \ldots, f_{M}, g_{1,1} X_{r+1}-1, g_{1,2} X_{r+2}-1, g_{2,1} X_{r+3}-1, g_{2,2} X_{r+4}-1,\right. \\
\left.\ldots, g_{s, 1} X_{r+2 s-1}-1, g_{s, 2} X_{r+2 s}-1\right)
\end{gathered}
$$

Let $\left(u_{1}, \ldots, v_{s}\right)$ be a solution of (2.2.2) and put

$$
\varepsilon:=\prod_{i=1}^{s} \gamma_{i}^{u_{i}}, \eta:=\prod_{i=1}^{s} \gamma_{i}^{v_{i}} .
$$

Then

$$
a \varepsilon+b \eta=c, \varepsilon, \eta \in \tilde{A}^{*}
$$

By Theorem 2.2.1, $\varepsilon$ has a representative $\widetilde{\varepsilon} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r+2 s}\right]$ of degree and logarithmic height both bounded above by

$$
\exp \left((2 d)^{\exp O(r+s)} h\right)
$$

Now Corollary 7.5.3 implies

$$
\left|u_{i}\right| \leq \exp \left((2 d)^{\exp O(r+s)} h\right) \text { for } i=1, \ldots, s
$$

For $\left|v_{i}\right|(i=1, \ldots, s)$ we derive a similar upper bound. This completes the proof of Theorem 2.2.3.

### 9.1.2 Thue equations

As in Section 2.3, let

$$
F(X, Y)=a_{0} X^{n}+a_{1} X^{n-1} Y+\cdots+a_{n} Y^{n} \in A[X, Y]
$$

be a binary form of degree $n \geq 3$ with discriminant $D_{F} \neq 0$ and let $\delta \in$ $A \backslash\{0\}$. Recall that for $a_{0}, \ldots, a_{n}, \delta$ we have chosen representatives $\tilde{a}_{0}, \ldots, \tilde{a}_{n}, \tilde{\delta}$ such that $\tilde{\delta}$ and the discriminant $D_{\tilde{F}}$ of $\tilde{F}:=\sum_{j=0}^{n} \tilde{a}_{j} X^{n-j}$ are not contained in $\mathcal{I}$, and that $f_{1}, \ldots, f_{M}, \tilde{a}_{0}, \ldots, \tilde{a}_{n}, \tilde{\delta}$ have degrees at most $d$ and logarithmic heights at most $h$ where $d \geq 1, h \geq 1$.

Theorem 2.3.1 will be deduced from the following.
Proposition 9.1.3. Let $g \in A_{0} \backslash\{0\}$ with the properties specified in Proposi-
tion 7.2.9 and consider the integral domain

$$
\begin{equation*}
A \subseteq B:=A_{0}\left[w, g^{-1}\right] \text {, where } \delta, D_{F} \in B^{*} \tag{9.1.12}
\end{equation*}
$$

Then for the solutions $x, y$ of the equation

$$
\begin{equation*}
F(x, y)=\delta \quad \text { in } x, y \in B \tag{9.1.13}
\end{equation*}
$$

we have

$$
\begin{align*}
\overline{\operatorname{deg}} x, \overline{\operatorname{deg}} y & \leq(n d)^{\exp O(r)}  \tag{9.1.14}\\
\bar{h}(x), \bar{h}(y) & \leq \exp \left(n!(n d)^{\exp O(r)} h\right) \tag{9.1.15}
\end{align*}
$$

Proposition 9.1.3 will be proved in Subsections 9.2.2, 9.3.2.
We now deduce Theorem 2.3.1 from Proposition 9.1.3.
Proof of Theorem 2.3.1. Let $x, y$ be a solution of equation 2.3.1) in $A$. In view of $9.1 .12 x, y$ are also contained in $B:=A_{0}\left[w, g^{-1}\right]$, where $w, g$ have the properties specified in Proposition 7.2.5 resp. in Proposition 7.2.9. Then, by Proposition 9.1 .3 the inequalities 9.1 .14 and 9.1 .15 hold. Applying now Corollary 7.3 .2 to $x$ and $y$, we infer that $x, y$ have representatives $\tilde{x}, \tilde{y}$ in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ which satisfy (2.3.2).

### 9.1.3 Hyper- and superelliptic equations

Recall that as in Section 2.4, $F(X)=a_{0} X^{n}+a_{1} X^{n-1}+\cdots+a_{n} \in A[X]$ is a polynomial with $a_{0} \neq 0$ and with discriminant $D_{F} \neq 0$, that $\delta \in A \backslash\{0\}$ and that for $a_{0}, \ldots, a_{n}, \delta$ we have chosen representatives $\tilde{a}_{0}, \ldots, \tilde{a}_{n}, \tilde{\delta}$ from $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ with degrees at most $d$ and logarithmic heights at most $h$ such that $\tilde{\delta}$ and the discriminant of $\tilde{F}:=\sum_{j=0}^{n} \tilde{a}_{j} X^{n-j}$ are not in $\mathcal{I}$.

We shall deduce Theorem 2.4.1 from the following

Proposition 9.1.4. Let $g \in A_{0} \backslash\{0\}$ with the properties specified in Proposition 7.2.9 such that for the overring $B:=A_{0}\left[w, g^{-1}\right]$ of $A$, we have $\delta, D_{F} \in$ $B^{*}$. Further, let $m$ be an integer $\geq 2$, and assume that $n \geq 3$ if $m=2$ and $n \geq 2$ if $m \geq 3$. Then for all solutions $x, y$ of the equation

$$
\begin{equation*}
F(x)=\delta y^{m} \text { in } x, y \in B \tag{9.1.16}
\end{equation*}
$$

we have

$$
\begin{align*}
& \overline{\operatorname{deg}} x, \overline{\operatorname{deg}} y \leq(n d)^{\exp O(r)}  \tag{9.1.17}\\
& m \leq(n d)^{\exp O(r)} \text { if } y \notin \overline{\mathbb{Q}}  \tag{9.1.18}\\
& \bar{h}(x), \bar{h}(y) \leq \exp \left(m^{3}(n d)^{\exp O(r)} h\right) . \tag{9.1.19}
\end{align*}
$$

We prove (9.1.17) and 9.1.18) in Subsection 9.2.3 and 9.1.19) in Subsection 9.3.3.

We now deduce Theorem 2.4.1 from Proposition 9.1.4
Proof of Theorem 2.4.1. Let $x, y$ be a solution of equation (2.4.1). In view of $A \subseteq B$, the pair $x, y$ is a solution also in $B$. Then, by Proposition 9.1 .4 the inequalities 9.1.17) and 9.1.19) hold. Applying Corollary 7.3 .2 to $x$ and $y$, we infer that $x, y$ have representatives $\tilde{x}, \tilde{y}$ in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ satisfying (2.4.2).

Together with Proposition 9.1.4, the below proposition implies Theorem 2.4.2.

Proposition 9.1.5. Suppose that equation 9.1.16 has a solution $x, y$ with $y \in \overline{\mathbb{Q}}$ such that $y \neq 0$ and $y$ is not a root of unity. Then

$$
m \leq \exp \left((n d)^{\exp O(r)} h\right)
$$

Proposition 9.1 .5 will be proved at the end of Section 9.3 .
Proof of Theorem 2.4.2 Immediate from 9.1.18) and 9.1.19).

### 9.2 Bounding the degrees

In this section we prove separately the inequalities 9.1.7) from Proposition 9.1.1, 9.1.14) from Proposition 9.1 .3 and 9.1.17) from Proposition 9.1.4. The main tools will be Theorem 5.2.1 on unit equations, Theorem 5.4.1 on Thue equations and Theorems $5.5 .1,5.5 .2$ on hyper- and superelliptic equations over function fields.

We recall some notation and introduce further notation. The case $q=$ 0 being trivial, in this section we assume that $q>0$. Let as above $K_{0}=$ $\mathbb{Q}\left(X_{1}, \ldots, X_{q}\right), K=K_{0}(w), A_{0}=\mathbb{Z}\left[X_{1}, \ldots, X_{q}\right], B=A_{0}\left[w, g^{-1}\right]$. Choose an algebraic closure $\bar{K}_{0}$ of $K_{0}$. Then there are $D K_{0}$-isomorphic embeddings of $K$ into $\bar{K}_{0}$ which we denote by $\alpha \mapsto \alpha^{(j)}, j=1, \ldots, D$.

As in Section 7.3, let $\mathbb{k}_{i}$ be an algebraic closure of $\mathbb{Q}\left(X_{1}, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{q}\right)$ for $i=1, \ldots, q$. Then $A_{0}$ is contained in $\mathbb{k}_{i}\left[X_{i}\right]$. Consider the function field

$$
L_{i}:=\mathbb{k}_{i}\left(X_{i}, w^{(1)}, \ldots, w^{(D)}\right), \quad i=1, \ldots, q,
$$

where $w^{(1)}=w, \ldots, w^{(D)}$ denote the conjugates of $w$ over $K_{0}$ in $\bar{K}_{0}$. Then $L_{i}$ is the splitting field of the polynomial $\mathcal{F}(X)=X^{D}+\mathcal{F}_{1} X^{D-1}+\cdots+\mathcal{F}_{D}$ over $\mathbb{k}_{i}\left(X_{i}\right)$ with $\mathcal{F}_{j} \in \mathbb{k}_{i}\left[X_{i}\right], j=1, \ldots, D$. The subring

$$
B_{i}:=\mathbb{k}_{i}\left[X_{i}, w^{(1)}, \ldots, w^{(D)}, g^{-1}\right]
$$

of $L_{i}$ contains $B=\mathbb{Z}\left[X_{1}, \ldots, X_{q}, w, g^{-1}\right]$ as a subring. Put $\Delta_{i}:=\left[L_{i}:\right.$ $\mathbb{k}_{i}\left(X_{i}\right)$.

Let $g_{L_{i} / \mathbb{k}_{i}}$ denote the genus of $L_{i} / \mathbb{k}_{i}$ and $H_{L_{i}}$ the height taken with respect to $L_{i} / \mathbb{k}_{i}$. By Lemma 5.1.1, applied with $\mathbb{k}_{i}, X_{i}, \mathbb{k}_{i}\left(X_{i}\right), L_{i}$ instead of $\mathbb{k}, z, K, L$ and with $F=\mathcal{F}=X^{D}+\mathcal{F}_{1} X^{D-1}+\cdots+\mathcal{F}_{D}$, and using (9.1.3) we obtain

$$
\begin{equation*}
g_{L_{i} / \mathbb{k}_{i}} \leq \Delta_{i} D \max _{j} \operatorname{deg}_{X_{i}} \mathcal{F}_{j} \leq \Delta_{i} D \max _{j} \operatorname{deg} \mathcal{F}_{j} . \tag{9.2.1}
\end{equation*}
$$

Let $S_{i}$ denote the subset of valuations $v$ of $L_{i} / \mathbb{k}_{i}$ such that $v\left(X_{i}\right)<0$ or $v(g)>0$. Every valuation of $\mathbb{k}_{i}\left(X_{i}\right)$ can be extended to at most $\Delta_{i}$ valuations of $L_{i}$. Thus $L_{i}$ has at most $\Delta_{i}$ valuations $v$ with $v\left(X_{i}\right)<0$ and at most $\Delta_{i} \operatorname{deg} g$ valuations with $v(g)>0$. Hence, using also 9.1.9, we infer

$$
\begin{equation*}
\left|S_{i}\right| \leq \Delta_{i}+\Delta_{i} \operatorname{deg}_{X_{i}} g \leq \Delta_{i}(1+\operatorname{deg} g) \leq \Delta_{i}(2 d)^{\exp O(r)} \tag{9.2.2}
\end{equation*}
$$

Since $w^{(1)}, \ldots, w^{(D)}$ lie in $L_{i}$ and are all integral over $\mathbb{K}_{i}\left[X_{i}\right]$, they belong to $\mathcal{O}_{S_{i}}$, i.e., the ring of $S_{i}$-integers in $L_{i}$. Further, $g^{-1} \in \mathcal{O}_{S_{i}}$. Consequently, if $\alpha \in B=A_{0}\left[w, g^{-1}\right]$, then $\alpha^{(j)} \in \mathcal{O}_{S_{i}}$ for $j=1, \ldots, D, i=1, \ldots, q$.

### 9.2.1 Unit equations

We now prove the upper bound 9.1.7) in Proposition 9.1.1.
Proof of (9.1.7). Keeping the above notation, let $(\varepsilon, \eta)$ be a solution of equation (9.1.6) $\varepsilon+\eta=1$ in $\varepsilon, \eta \in B^{*}$. Then we have

$$
\varepsilon^{(j)}+\eta^{(j)}=1, \varepsilon^{(j)}, \eta^{(j)} \in \mathcal{O}_{S_{i}}^{*} \text { for } j=1, \ldots, D
$$

and $i=1, \ldots, q$. We apply Theorem 5.2.1, insert the upper bounds 9.2.1), (9.2.2) and use $\operatorname{deg} g, \operatorname{deg} \mathcal{F}_{j} \leq d_{4}$ from 9.1.5) for $j=1, \ldots, D$. It follows that for $j=1, \ldots, D$ we have either $\varepsilon^{(j)} \in \mathbb{k}_{i}$ or

$$
H_{L_{i}}\left(\varepsilon^{(j)}\right) \leq\left|S_{i}\right|+2 g_{L_{i} / \mathbb{k}_{i}}-2 \leq 3 \Delta_{i} D d_{4} .
$$

Of course, the last upper bound is valid also if $\varepsilon^{(j)} \in \mathbb{k}_{i}$. Together with Lemma 7.3.3 this implies

$$
\overline{\operatorname{deg}} \varepsilon \leq q D d_{4}+q \cdot 3 D d_{4} \leq 4 q D^{2} d_{4} .
$$

For $\overline{\operatorname{deg}} \eta$ we obtain the same upper bound. This proves 9.1.7.

### 9.2.2 Thue equations

Keeping the notation introduced at the beginning of Section 9.2 , we prove the upper bound 9.1.14) from Proposition 9.1.3.
Proof of (9.1.14). Let $x, y$ be a solution of equation (9.1.13). Put $F^{\prime}:=\delta^{-1} F$, and let $F^{\prime(j)}$ denote the binary form obtained by taking the $j$-th conjugates of the coefficients of $F^{\prime}$. Let $i \in\{1, \ldots, q\}, j \in\{1, \ldots, D\}$. Then $F^{\prime(j)} \in$ $L_{i}[X, Y]$ and we get

$$
F^{\prime(j)}\left(x^{(j)}, y^{(j)}\right)=1 \text { with } x^{(j)}, y^{(j)} \in \mathcal{O}_{S_{i}} .
$$

By Theorem 5.4.1 we obtain

$$
\begin{equation*}
\max \left(H_{L_{i}}\left(x^{(j)}\right), H_{L_{i}}\left(y^{(j)}\right)\right) \leq(8 n+62) H_{L_{i}}\left(F^{\prime(j)}\right)+8 g_{L_{i} / \mathbb{k}_{i}}+4\left|S_{i}\right| . \tag{9.2.3}
\end{equation*}
$$

We estimate from above the parameters occurring in this bound. We start with $H_{L_{i}}\left(F^{\prime(j)}\right)$. Recall that $F^{\prime}(X, Y)=\delta^{-1}\left(a_{0} X^{n}+a_{1} X^{n-1} Y+\cdots+a_{n} Y^{n}\right)$. Using the properties of heights from Section 5.1, and inequality (7.3.9), we infer that

$$
\begin{aligned}
H_{L_{i}}\left(F^{\prime(j)}\right) & =H_{L_{i}}\left(a_{0}^{(j)}, \ldots, a_{n}^{(j)}\right) \leq H_{L_{i}}\left(a_{0}^{(j)}\right)+\cdots+H_{L_{i}}\left(a_{n}^{(j)}\right) \\
& \leq \Delta_{i}\left(2 d^{r}\left(\overline{\operatorname{deg}} a_{0}+\cdots+\overline{\operatorname{deg}} a_{n}\right)+n(2 d)^{\exp O(r)}\right)
\end{aligned}
$$

By Lemma 7.2 .6

$$
\overline{\operatorname{deg}} a_{i} \leq(2 d)^{\exp O(r)} \text { for } i=1, \ldots, n
$$

Thus

$$
\begin{align*}
H_{L_{i}}\left(F^{\prime(j)}\right) & \leq \Delta_{i}\left((n+1)(2 d)^{\exp O(r)}+n(2 d)^{\exp O(r)}\right) \\
& \leq \Delta_{i}(n d)^{\exp O(r)} \tag{9.2.4}
\end{align*}
$$

Next we estimate the genus $g_{L_{i} / \mathbb{k}_{i}}$. Using (9.2.1), Proposition 7.2.5 and Lemma 7.2.3, we deduce that

$$
\begin{equation*}
g_{L_{i} / \mathbb{k}_{i}} \leq \Delta_{i} D \max _{j} \operatorname{deg} \mathcal{F}_{j} \leq \Delta_{i}(2 d)^{\exp O(r)} \tag{9.2.5}
\end{equation*}
$$

Lastly, we estimate $\left|S_{i}\right|$. Combining Proposition 7.2 .9 with 9.2 .2 we obtain

$$
\begin{equation*}
\left|S_{i}\right| \leq \Delta_{i}(n d)^{\exp O(r)} \tag{9.2.6}
\end{equation*}
$$

Inserting the bound 9.2.4, (9.2.5), (9.2.6 into 9.2.3), we infer

$$
\begin{equation*}
\max \left(H_{L_{i}}\left(x^{(j)}\right), H_{L_{i}}\left(y^{(j)}\right)\right) \leq \Delta_{i}(n d)^{\exp O(r)} \tag{9.2.7}
\end{equation*}
$$

In view of Lemma 7.3.3, the inequalities 9.2.7), $D \leq d^{r}, q \leq r$, and Propositions 7.2.5 and 7.2.9 we obtain

$$
\overline{\operatorname{deg}} x \leq q D(n d)^{\exp O(r)}+\sum_{i=1}^{q} \Delta_{i}^{-1} \sum_{j=1}^{D} H_{L_{i}}\left(x^{(j)}\right) \leq(n d)^{\exp O(r)}
$$

and similarly for $\overline{\operatorname{deg}} y$. This proves (9.1.14).

### 9.2.3 Hyper- and superelliptic equations

Using the notation of Subsection 9.1 .3 we prove the bounds in 9.1.17) and 9.1.18) from Proposition 9.1.4.

Proof of (9.1.17). We follow the proof of (9.1.14) in Proposition 9.1.3, and use the same notation. In particular, $\mathbb{k}_{i}, L_{i}, S_{i}, g_{L_{i} / \mathbb{k}_{i}}, \Delta_{i}$ have the same meaning and for $\alpha \in B$ and $j=1, \ldots, D$ the $j$ th conjugate $\alpha^{(j)}$ is the one corresponding to $w^{(j)}$. Put $F^{\prime}:=\delta^{-1} F$, and let $F^{\prime(j)}$ denote the polynomial obtained by taking the $j$ th conjugates of the coefficients of $F^{\prime}$.

We keep the argument together for both hyper- and superelliptic equations by using the worse bounds everywhere. Let $x, y \in B$ be a solution of 9.1.16,
where $m, n \geq 2$ and $n \geq 3$ if $m=2$. Then we get

$$
F^{\prime(j)}\left(x^{(j)}\right)=\left(y^{(j)}\right)^{m}, x^{(j)}, y^{(j)} \in \mathcal{O}_{S_{i}} .
$$

Combining Theorems 5.5.1 and 5.5.2 we obtain the generous bound

$$
H_{L_{i}}\left(x^{(j)}\right), m H_{L_{i}}\left(y^{(j)}\right) \leq 16 n^{2}\left(H_{L_{i}}\left(F^{\prime(j)}\right)+g_{L_{i} / \mathbb{k}_{i}}+\left|S_{i}\right|\right) .
$$

For $H_{L_{i}}\left(F^{\prime(j)}\right), g_{L_{i} / \mathbb{k}_{i}},\left|S_{i}\right|$ we have precisely the same estimates as (9.2.4), 9.2.5) and 9.2.6. Then a similar computation as in the proof of 9.1.14) leads to

$$
\begin{equation*}
H_{L_{i}}\left(x^{(j)}\right), m H_{L_{i}}\left(y^{(j)}\right) \leq \Delta_{i}(n d)^{\exp O(r)} . \tag{9.2.8}
\end{equation*}
$$

Now applying Lemma 7.3.3 and ignoring $m$ for the moment we get, similarly to the proof of 9.1 .14

$$
\overline{\operatorname{deg}} x, \overline{\operatorname{deg}} y \leq(n d)^{\exp O(r)}
$$

which was to be proved.

We proceed to deduce the upper bound 9.1.18) for $m$ in the SchinzelTijdeman equation. We need the following lemma, originally proved by Brindza (1993). We have included another proof.

Lemma 9.2.1. We have

$$
\bigcap_{i=1}^{q} \mathbb{k}_{i}=\overline{\mathbb{Q}} .
$$

Proof. Clearly, $\overline{\mathbb{Q}} \subseteq \cap_{i=1}^{q} \mathbb{k}_{i}$. To prove the reverse inclusion, let $\alpha \in \cap_{i=1}^{q} \mathbb{k}_{i}$. Define the fields $\mathbb{F}_{i}:=\mathbb{Q}\left(X_{1}, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{q}\right)(i=1, \ldots, q)$ and as before $K_{0}:=\mathbb{Q}\left(X_{1}, \ldots, X_{q}\right)$. For $i=1, \ldots, q$, let $P_{i} \in \mathbb{F}_{i}(X)$ be the monic minimal polynomial of $\alpha$ over $\mathbb{F}_{i}$, and let $P \in K_{0}(X)$ be the monic minimal polynomial of $\alpha$ over $K_{0}$. Then for $i=1, \ldots, q, P$ divides $P_{i}$ in $K_{0}(X)$. But this is possible only if the coefficients of $P$ lie in $\mathbb{F}_{i}$ for $i=1, \ldots, q$. So the coefficients of $P$ lie in $\cap_{i=1}^{q} \mathbb{F}_{i}$, implying $\alpha \in \overline{\mathbb{Q}}$.

Proof of (9.1.18). Assume that in 9.1.16) y $\notin \overline{\mathbb{Q}}$. Then $y \notin \mathbb{k}_{i}$ for at least one index $i$ by Lemma 9.2.1. Since $y \in B \subset \mathbb{k}_{i}\left(X_{i}, w\right)$ and $\left[\mathbb{k}_{i}\left(X_{i}, w\right)\right.$ :
$\left.\mathbb{k}_{i}\left(X_{i}\right)\right] \leq D$, it follows that

$$
\begin{aligned}
H_{L_{i}}(y) & =\left[L_{i}: \mathbb{k}_{i}\left(X_{i}, w\right)\right] H_{\mathbb{k}_{i}\left(X_{i}, w\right)}(y) \geq\left[L_{i}: \mathbb{k}_{i}\left(X_{i}, w\right)\right] \\
& \geq \Delta_{i} / D .
\end{aligned}
$$

Together with 9.2.8) and $D \leq d^{r}$ (see Lemma 7.2 .3 (i)), this gives

$$
m \leq(n d)^{\exp O(r)}
$$

which is 9.1.18.

### 9.3 Bounding the heights, specializations

Combining our degree bounds established in Section 9.2 with the effective specialization method from Chapter 7 and the corresponding effective results from Chapter 4 over number fields, we derive effective bounds for the heights $\bar{h}$ of the solutions of unit equations, Thue-equations and hyper- and superelliptic equations. As was seen in Section 9.1, this will complete our effective proofs for the general version of our equations considered over finitely generated domains.

Before proving the height bounds, we recall again some notation and collect some preparatory results from Chapters 7 and 4. Let as above $A=$ $\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]$ be an integral domain of characteristic 0 finitely generated over $\mathbb{Z}, K$ its quotient field, $q$ the transcendence degree of $K, z_{1}=X_{1}, \ldots, z_{q}=$ $X_{q}$ algebraically independent over $\mathbb{Q}$, and $A_{0}=\mathbb{Z}\left[X_{1}, \ldots, X_{q}\right]$ if $q>0$, $A_{0}=\mathbb{Z}$ otherwise. Let $w \in A$ with minimal polynomial $\mathcal{F}(X)=X^{D}+$ $\mathcal{F}_{1} X^{D-1}+\cdots+\mathcal{F}_{D} \in A_{0}[X]$ be as in Proposition 7.2.5, $\Delta_{\mathcal{F}}$ the discriminant of $\mathcal{F}, g \in A_{0} \backslash\{0\}, A \subseteq B:=A_{0}\left[w, g^{-1}\right]$ as in Proposition 7.2.7, and $\mathcal{T}=\Delta_{\mathcal{F}} \mathcal{F}_{D} g$ as in (7.4.7). Moreover, in case of the Thue equation (9.1.13) $F(x, y)=\delta$ and the superelliptic equation 9.1.16) $F(x)=\delta y^{m}$, we apply Proposition 7.2 .9 with $\delta$ and the discriminant $D_{F}$ of $F$ belonging to $B^{*}$.

For $\mathbf{u} \in \mathbb{Z}^{q}$ with $\mathcal{T}(\mathbf{u}) \neq 0$, the polynomial $\mathcal{F}_{\mathbf{u}}(X)=X^{D}+\mathcal{F}_{1}(\mathbf{u}) X^{D-1}+$ $\cdots+\mathcal{F}_{D}(\mathbf{u})$ has distinct zeros $w_{1}(\mathbf{u}), \ldots, w_{D}(\mathbf{u})$ in $\overline{\mathbb{Q}}$ which are all nonzero. Consequently, for $j=1, \ldots, D$ the substitution $X_{1} \mapsto u_{1}, \ldots, X_{q} \mapsto$ $u_{q}, w \mapsto w_{j}(\mathbf{u})$ defines a ring homomorphism $\varphi_{\mathbf{u}, j}$ from $B$ to $\overline{\mathbb{Q}}$. The image of $\alpha \in B$ under $\varphi_{\mathbf{u}, j}$ is denoted by $\alpha_{j}(\mathbf{u})$. Then $\varphi_{\mathbf{u}, j}(B)$ is contained in the algebraic number field $K_{\mathbf{u}, j}:=\mathbb{Q}\left(w_{j}(\mathbf{u})\right)$.

For a fixed $j \in\{1, \ldots, D\}$ and a suitably chosen finite extension $L$ of
$K_{\mathbf{u}, j}$, we let $S$ denote the set of places of $L$ which consists of all infinite places and all finite places lying above the rational prime divisors of $g(\mathbf{u})$. Note that $w_{j}(\mathbf{u})$ is an algebraic integer and $g(\mathbf{u}) \in \mathcal{O}_{S}^{*}$. Thus $\varphi_{\mathbf{u}, j}(B) \subseteq \mathcal{O}_{S}$ and $\varphi_{\mathbf{u}, j}\left(B^{*}\right) \subseteq \mathcal{O}_{S}^{*}$. Further, since $\delta, D_{F} \in B^{*}$, we have $\delta_{j}(\mathbf{u}) \neq 0$ and $D_{F, j}(\mathbf{u}) \neq 0$.

As in Chapter 4, $d_{L}, \mathcal{O}_{L}, \mathcal{M}_{L}, D_{L}, h_{L}, r_{L}$ and $R_{L}$ denote the degree, ring of integers, set of places, discriminant, class number, unit rank and regulator of $L$. The absolute norm of an ideal $\mathfrak{a}$ of $\mathcal{O}_{L}$ is denoted by $N(\mathfrak{a})$.

If $S$ consists only of the infinite places of $L$, we put $P_{S}:=2, Q_{S}:=$ 2. If $S$ contains also finite places, we let $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{w}$ denote the prime ideals corresponding to the finite places of $S$ and put

$$
\begin{equation*}
P_{S}:=\max \left(N\left(\mathfrak{p}_{1}\right), \ldots N\left(\mathfrak{p}_{w}\right)\right), Q_{S}:=N\left(\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{w}\right) \tag{9.3.1}
\end{equation*}
$$

The $S$-regulator is denoted by $R_{S}$. In case that $S$ consists only of the infinite places of $L$ it is just $R_{L}$, while otherwise

$$
R_{S}=h_{S} R_{L} \prod_{i=1}^{w} \log N\left(\mathfrak{p}_{i}\right),
$$

where $h_{S}$ is a positive divisor of $h_{L}$, see 4.1.10). Further,

$$
\begin{equation*}
R_{S} \leq\left|D_{L}\right|^{1 / 2}\left(\log ^{*}\left|D_{L}\right|\right)^{d_{L}-1}\left(\log ^{*} Q_{S}\right)^{w} \tag{9.3.2}
\end{equation*}
$$

see (4.1.13).
Finally, in Subsections 9.3 .2 and 9.3 .3 we shall need the discriminant estimates from Lemmas 4.1.10 and 4.1.11.

In the proofs in Subsections 9.3.1, 9.3.2 and 9.3 .3 most of the above notation and results will be used without any further mention.

### 9.3.1 Unit equations

We prove the height bound 9.1.8).
Proof of 9.1.8. Let $\varepsilon, \eta$ be a solution of equation 9.1.6. We first consider the case $q>0$. Pick $\mathbf{u} \in \mathbb{Z}^{q}$ with $\mathcal{T}(\mathbf{u}) \neq 0$, let $j \in\{1, \ldots, D\}$ and $L:=$ $K_{\mathbf{u}, j}$. Putting $S$ as above, we have $\varphi_{\mathbf{u}, j}(B) \subseteq \mathcal{O}_{S}$ and $\varphi_{\mathbf{u}, j}\left(B^{*}\right) \subseteq \mathcal{O}_{S}^{*}$. Hence it follows from 9.1.6) that

$$
\begin{equation*}
\varepsilon_{j}(\mathbf{u})+\eta_{j}(\mathbf{u})=1, \varepsilon_{j}(\mathbf{u}), \eta_{j}(\mathbf{u}) \in \mathcal{O}_{S}^{*} \tag{9.3.3}
\end{equation*}
$$

where $\varepsilon_{j}(\mathbf{u}), \eta(\mathbf{u})$ are the images of $\varepsilon, \eta$ under $\varphi_{\mathbf{u}, j}$. Applying Theorem4.3.1 with $\alpha=\beta=1, H=1$ to equation 9.3.3), we get

$$
\begin{equation*}
\max \left(h\left(\varepsilon_{j}(\mathbf{u})\right), h\left(\eta_{j}(\mathbf{u})\right)\right) \leq c_{1} P_{S} R_{S}\left(1+\log ^{*} R_{S} / \log P_{S}\right) \tag{9.3.4}
\end{equation*}
$$

with

$$
c_{1}=s^{2 s+3.5} 2^{7 s+27}(\log (2 s)) d_{L}^{2(s+1)}\left(\log ^{*}\left(2 d_{L}\right)\right)^{3},
$$

where $s$ is the cardinality of $S$.

We estimate from above the upper bound (9.3.4). By (9.1.5), $g \in A_{0} \backslash\{0\}$ has degree at most $d_{4}$ and height at most $h_{4}$. Hence

$$
\begin{equation*}
|g(\mathbf{u})| \leq d_{4}^{q} e^{h_{4}} \max (1,|\mathbf{u}|)^{d_{4}}=: R(\mathbf{u}) . \tag{9.3.5}
\end{equation*}
$$

Since $d_{L}:=[L: \mathbb{Q}] \leq D$, the cardinality $s$ of $S$ is at most $D(1+w)$ where $w$ denotes the number of prime divisors of $g(\mathbf{u})$. In view of the inequality from prime number theory $w \leq O\left(\log ^{*}|g(\mathbf{u})| / \log ^{*} \log ^{*}|g(\mathbf{u})|\right)$, we obtain

$$
\begin{equation*}
s \leq O\left(\frac{D \log ^{*} R(\mathbf{u})}{\log ^{*} \log ^{*} R(\mathbf{u})}\right) . \tag{9.3.6}
\end{equation*}
$$

From this it is easy to deduce that

$$
\begin{equation*}
c_{1} \leq \exp O\left(D \log ^{*} D \log ^{*} R(\mathbf{u})\right) . \tag{9.3.7}
\end{equation*}
$$

We now estimate $P_{S}$ and $R_{S}$. By (9.3.5) we have

$$
\begin{equation*}
P_{S} \leq Q_{S} \leq|g(\mathbf{u})|^{D} \leq \exp O\left(D \log ^{*} R(\mathbf{u})\right) \tag{9.3.8}
\end{equation*}
$$

To estimate $R_{S}$, we use (9.3.2). Using Lemma 7.4.5 and $d_{3} \leq d_{4}$, we infer that

$$
\left|D_{L}\right| \leq D^{2 D-1}\left(d_{4}^{q} e^{h_{4}} \max (1,|\mathbf{u}|)^{d_{4}}\right)^{2 D-2} \leq \exp O\left(D \log ^{*} D \log ^{*} R(\mathbf{u})\right)
$$

and this gives

$$
\left|D_{L}\right|^{1 / 2}\left(\log ^{*}\left|D_{L}\right|\right)^{D-1} \leq \exp O\left(D \log ^{*} D \log ^{*} R(\mathbf{u})\right) .
$$

Together with the estimates (9.3.6, 9.3.8) for $s$ and $Q_{S}$, this yields

$$
\begin{align*}
R_{S} & \leq \exp O\left(D \log ^{*} D \log ^{*} R(\mathbf{u})+s \log ^{*} \log ^{*} Q\right) \\
& \leq \exp O\left(D \log ^{*} D \log ^{*} R(\mathbf{u})\right) \tag{9.3.9}
\end{align*}
$$

Collecting now 9.3.7)-9.3.9), we infer that the right hand side of 9.3.4) is bounded above by $\exp O\left(D \log ^{*} D \log ^{*} R(\mathbf{u})\right.$ ). So we obtain from 9.3.4 that

$$
\begin{equation*}
h\left(\varepsilon_{j}(\mathbf{u})\right), h\left(\eta_{j}(\mathbf{u})\right) \leq \exp O\left(D \log ^{*} D \log ^{*} R(\mathbf{u})\right) . \tag{9.3.10}
\end{equation*}
$$

We apply Lemma 7.4.7 with $N:=4 D^{2}\left(q+d_{4}+1\right)^{2}$. From the already established (9.1.7) it follows that $\overline{\operatorname{deg}} \varepsilon, \overline{\operatorname{deg}} \eta \leq N$. Further, in view of $d_{4} \geq$ $d_{3}$ we have $N \geq 2 D d_{3}+2\left(d_{4}+1\right)(q+1)$. Hence indeed, we can apply Lemma 7.4.7 with this value of $N$. It follows that the set

$$
\mathcal{S}:=\left\{\mathbf{u} \in \mathbb{Z}^{q}:|\mathbf{u}| \leq N, \mathcal{T}(\mathbf{u}) \neq 0\right\}
$$

is not empty. For $\mathbf{u} \in \mathcal{S}, j=1, \ldots, D$, we deduce from (9.3.5) and 9.3.10) that

$$
\begin{aligned}
h\left(\varepsilon_{j}(\mathbf{u})\right) & \leq \exp O\left(D \log ^{*} D\left(q \log d_{4}+h_{4}+d_{4} \log ^{*} N\right)\right) \\
& \leq \exp O\left(N^{1 / 2}\left(\log ^{*} N\right)^{2}+\left(D \log ^{*} D\right) h_{4}\right),
\end{aligned}
$$

and so by Lemma 7.4.7

$$
\bar{h}(\varepsilon) \leq \exp O\left(N^{1 / 2}\left(\log ^{*} N\right)^{2} h_{4}\right)
$$

For $\bar{h}(\eta)$ we obtain the same upper bound. This gives (9.1.8) for $q>0$.
Next assume that $q=0$. In this case $A_{0}=\mathbb{Z}, K=\mathbb{Q}(w)$ is a number field containing $B=\mathbb{Z}\left[w, g^{-1}\right]$, where $w$ is an algebraic integer with minimal polynomial $\mathcal{F}(X)=X^{D}+\mathcal{F}_{1} X^{D-1}+\cdots+\mathcal{F}_{D} \in \mathbb{Z}[X]$ over $\mathbb{Q}$, and $g$ is a non-zero rational integer. By assumption, $\log |g| \leq h_{4}, \log ^{*}\left|\mathcal{F}_{j}\right| \leq h_{4}$ for $j=1, \ldots, D$. Denote by $w^{(1)}, \ldots, w^{(D)}$ the conjugates of $w$ over $\mathbb{Q}$ and let $L:=\mathbb{Q}\left(w^{(j)}\right)$ for some $j$. By a similar argument as in the proof of Lemma 7.4.5, we obtain $\left|D_{L}\right| \leq D^{2 D-1} e^{(2 D-2) h_{4}}$. The isomorphism defined by $w \mapsto$ $w^{(j)}$ maps $\mathbb{Q}(w)$ to $L$ and $B$ to $\mathcal{O}_{S}$, where $S$ consists of the infinite places of $L$ and of the prime ideals of $\mathcal{O}_{L}$ that divide $g$. The estimates 9.3.5)-9.3.9 remain valid if we replace $R(\mathbf{u})$ by $e^{h_{4}}$. Hence for any solution $\varepsilon, \eta$ of 9.1.6

$$
h\left(\varepsilon^{(j)}\right), h\left(\eta^{(j)}\right) \leq \exp O\left(\left(D \log ^{*} D\right) h_{4}\right),
$$

where $\varepsilon^{(j)}, \eta^{(j)}$ are the $j$ th conjugates of $\varepsilon, \eta$, respectively. Now an application of Lemma 7.4.2 with $G=\mathcal{F}, m=D, \beta_{i}=\varepsilon$ gives

$$
\bar{h}(\varepsilon) \leq \exp O\left(\left(D \log ^{*} D\right) h_{4}\right) .
$$

We obtain the same upper bound for $\bar{h}(\eta)$, whence 9.1.8) follows. The proof of Proposition 9.1.1 has been completed.

### 9.3.2 Thue equations

Concluding the proof of Theorem 2.3.1, it remains to prove (9.1.15) from Proposition 9.1.3.

Proof of 9.1.15). Let $x, y$ be a solution of equation 9.1.13) in $B$. Consider first the case $q>0$. We keep the notation of Chapter 7 and that of the introduction of Section 9.3. Recall that $\mathcal{T}=\Delta_{\mathcal{F}} \mathcal{F}_{D} g$ and, by (7.4.5) and (7.4.8),

$$
\operatorname{deg} \mathcal{T} \leq(n d)^{\exp O(r)}
$$

Choose $\mathbf{u} \in \mathbb{Z}^{q}$ with $\mathcal{T}(\mathbf{u}) \neq 0$, choose $j \in\{1, \ldots, D\}$ and denote by $F_{\mathbf{u}, j}, \delta_{j}(\mathbf{u}), x_{j}(\mathbf{u}), y_{j}(\mathbf{u})$ the images of $F, \delta, x, y$ under $\varphi_{\mathbf{u}, j}$. The coefficients of $F_{\mathbf{u}, j}$ belong to $K_{\mathbf{u}, j}$. Let $L$ denote the splitting field of $F_{\mathbf{u}, j}$ over $K_{\mathbf{u}, j}$, and $S$ the set of places of $L$ which consists of all infinite places and all finite places lying above the rational prime divisors of $g(\mathbf{u})$. Note that $w_{j}(\mathbf{u})$ is an algebraic integer and $g(\mathbf{u}) \in \mathcal{O}_{S}^{*}$. Hence $\varphi_{\mathbf{u}, j}(B) \subseteq \mathcal{O}_{S}, \varphi_{\mathbf{u}, j}\left(B^{*}\right) \subseteq \mathcal{O}_{S}^{*}$ and it follows from (9.1.13) that

$$
\begin{equation*}
F_{\mathbf{u}, j}\left(x_{j}(\mathbf{u}), y_{j}(\mathbf{u})\right)=\delta_{j}(\mathbf{u}), x_{j}(\mathbf{u}), y_{j}(\mathbf{u}) \in \mathcal{O}_{S} . \tag{9.3.11}
\end{equation*}
$$

Since by assumption $\delta, D_{F} \in B^{*}$, we have $\delta_{j}(\mathbf{u}) \neq 0$ and $D_{F, j}(\mathbf{u}) \neq 0$. Consequently, $F_{\mathbf{u}, j}$ is without multiple zeros. Then we can apply Theorem 4.4.1 to equation (9.3.11) and we get

$$
\begin{align*}
& \max \left(h\left(x_{j}(\mathbf{u})\right), h\left(y_{j}(\mathbf{u})\right)\right) \\
& \quad \leq c_{2} P_{S} R_{S}\left(1+\log ^{*} R_{S} / \log ^{*} P_{S}\right) \cdot\left(c_{3} R_{L}+\frac{h_{L}}{d_{L}} \log Q_{S}+2 n d_{L} H_{1}+H_{2}\right) \tag{9.3.12}
\end{align*}
$$

where $H_{1}=\max \left(1, h\left(F_{\mathbf{u}, j}\right)\right), H_{2}=\max \left(1, h\left(\delta_{\mathbf{u}, j}\right)\right)$,

$$
c_{2}=250 n^{6} s^{2 s+3.5} \cdot 2^{7 s+29}(\log 2 s) d_{L}^{2 s+4}\left(\log ^{*}\left(2 d_{L}\right)\right)^{3}
$$

and $c_{3}=0$ if $r_{L}=0,1 / d_{L}$ if $r_{L}=1$ and $29 e r_{L}!r_{L} \sqrt{r_{L}-1} \log d_{L}$ if $r_{L} \geq 2$.
We already proved in Subsection 9.2 .2 that 9.1 .14 in Proposition 9.1 .3 holds, i.e.

$$
\begin{equation*}
\overline{\operatorname{deg}} x, \overline{\operatorname{deg}} y \leq(n d)^{\exp O(r)} . \tag{9.3.13}
\end{equation*}
$$

Thus we can apply Lemma 7.4.7 with $\alpha=x$ resp. $y$ and

$$
N=\max \left((n d)^{\exp O(r)}, 2 D d_{3}+2(q+1)\left(d_{4}+1\right)\right)
$$

to get an upper bound for $\bar{h}(x), \bar{h}(y)$ which still depends on $h\left(x_{j}(\mathbf{u})\right), h\left(y_{j}(\mathbf{u})\right)$. Then, to prove 9.1.15), we have to bound from above the parameters occurring in (9.3.12).

In view of (7.4.5), $D \leq d^{r}$ (see Lemma (7.2.3) (i)) and $q \leq r$ we obtain

$$
\begin{equation*}
N \leq(n d)^{\exp O(r)} . \tag{9.3.14}
\end{equation*}
$$

Applying Lemma 7.4.7, inserting $D \leq d^{r}$ and the upper bound $h_{4} \leq(n d)^{\exp O(r)} h$ from (7.4.5), it follows that there are $\mathbf{u} \in \mathbb{Z}^{q}, j \in\{1, \ldots, D\}$ with

$$
\begin{equation*}
|\mathbf{u}| \leq(n d)^{\exp O(r)}, \mathcal{T}(\mathbf{u}) \neq 0 \tag{9.3.15}
\end{equation*}
$$

and

$$
\begin{align*}
& \max (\bar{h}(x), \bar{h}(y)) \\
& \quad \leq(n d)^{\exp O(r)} \max \left(h\left(x_{j}(\mathbf{u})\right), h\left(y_{j}(\mathbf{u})\right)\right) . \tag{9.3.16}
\end{align*}
$$

We proceed further with these $\mathbf{u}, j$ to derive upper bounds for the parameters corresponding to those which occur in 9.3.12).

Write $F(X, Y)=a_{0} X^{n}+a_{1} X^{n-1} Y+\cdots+a_{n} Y^{n}$ and put

$$
\overline{\operatorname{deg}} F:=\max _{0 \leq k \leq n} \overline{\operatorname{deg}} a_{k}, \bar{h}(F):=\max _{0 \leq k \leq n} \bar{h}\left(a_{k}\right) .
$$

Notice that by Lemma 7.2 .6 applied to $\delta$ and the coefficients of $F$ with the
choice $d_{0}=d, h_{0}=h$, we have

$$
\begin{align*}
\overline{\operatorname{deg}} F, \overline{\operatorname{deg}} \delta & \leq(2 d)^{\exp O(r)}  \tag{9.3.17}\\
\bar{h}(F), \bar{h}(\delta) & \leq(2 d)^{\exp O(r)} h . \tag{9.3.18}
\end{align*}
$$

It follows from Lemma 7.4.6, $q \leq r, D \leq d^{r}$, (7.4.5 (9.3.17), 9.3.18) and 9.3.15) that

$$
\begin{align*}
& h\left(F_{\mathbf{u}, j}\right) \leq D^{2}+q\left(D \log d_{3}\right.+\log \overline{\operatorname{deg}} F)+D h_{3}+\bar{h}(F)+ \\
&+\left(D d_{3}+\overline{\operatorname{deg}} F\right) \log \max (1,|\mathbf{u}|) \\
& \leq(n d)^{\exp O(r)} h . \tag{9.3.19}
\end{align*}
$$

Similarly, replacing $F$ by $\delta$, we obtain

$$
\begin{equation*}
h\left(\delta_{j}(\mathbf{u})\right) \leq(n d)^{\exp O(r)} h . \tag{9.3.20}
\end{equation*}
$$

We recall that $d_{L}$ and $D_{L}$ denote the degree and discriminant of $L$ over $\mathbb{Q}$. Since $\left[K_{\mathbf{u}, j}: \mathbb{Q}\right] \leq D$, we have $d_{L} \leq D n!$. Let $G(X):=F(X, 1)$ and let $\vartheta_{1}, \ldots, \vartheta_{n}$ be the zeros of $G$. We have $n^{\prime}=n$ if $a_{0} \neq 0$ and $n^{\prime}=n-1$ otherwise. Then $L=K_{\mathbf{u}, j}\left(\vartheta_{1}, \ldots, \vartheta_{n^{\prime}}\right)$. Let $d_{K_{u, j}}, d_{L_{i}}$ denote the degree and $D_{K_{u, j}}, D_{L_{i}}$ the discriminant of the number field $K_{u, j}$ resp. $L_{i}:=K_{\mathbf{u}, j}\left(\vartheta_{i}\right)$, $i=1, \ldots, n^{\prime}$. Then by Lemma 4.1.10 we have

$$
\begin{equation*}
\left|D_{L}\right| \leq \prod_{i=1}^{n^{\prime}}\left|D_{L_{i}}\right|^{d_{L} / d_{L_{i}}} . \tag{9.3.21}
\end{equation*}
$$

We now estimate $\left|D_{L}\right|$. First notice that by Lemma 7.4.5, using the estimates $q \leq r, D \leq d^{r}$, 7.4.4, (7.4.5), 9.3.15), we obtain

$$
\begin{align*}
\left|D_{K_{\mathbf{u}, j}}\right| & \leq D^{2 D-1}\left(d_{3}^{q} e^{h_{3}} \max (1,|\mathbf{u}|)^{d_{3}}\right)^{2 D-2} \\
& \leq \exp \left((n d)^{\exp O(r)} h\right) . \tag{9.3.22}
\end{align*}
$$

Further, by Lemma 4.1.11 and the estimates $D \leq d^{r}$, 9.3.19, 9.3.22 we get

$$
\begin{aligned}
\left|D_{L_{i}}\right| & \leq n^{(2 n-1) D} e^{\left(2 n^{2}-2\right) h\left(F_{\mathbf{u}, j}\right)}\left|D_{K_{\mathbf{u}, j}}\right|\left[L_{i}: K_{\mathbf{u}, j}\right] \\
& \leq \exp \left\{\left[L_{i}: K_{\mathbf{u}, j}\right](n d)^{\exp O(r)} h\right\} .
\end{aligned}
$$

Inserting this into 9.3.21) and using $d_{L_{i}}=\left[L_{i}: K_{u, j}\right] \cdot d_{K_{u, j}}$ we have

$$
\begin{align*}
\left|D_{L}\right| & \leq \exp \left\{(n d)^{\exp O(r)} h n d_{L} / d_{K_{\mathbf{u}, j}}\right\} \\
& \leq \exp \left\{n!(n d)^{\exp O(r)} h\right\} . \tag{9.3.23}
\end{align*}
$$

We follow now similar arguments as in the above proof of (9.1.8) concerning unit equations. The polynomial $g$ has degree at most $d_{4}$ and logarithmic height at most $h_{4}$ which satisfy (7.4.5). Further, $g(\mathbf{u}) \neq 0$ and by $q \leq r$ and 9.3.15

$$
\begin{equation*}
|g(\mathbf{u})| \leq d_{4}^{q} e^{h_{4}} \max (1,|\mathbf{u}|)^{d_{4}} \leq \exp \left\{(n d)^{\exp O(r)} h\right\} \tag{9.3.24}
\end{equation*}
$$

The cardinality $s$ of $S$ is at most $d_{L}(1+w)$, where $w$ denotes the number of distinct prime divisors of $g(\mathbf{u})$. By prime number theory

$$
\begin{equation*}
s=O\left(d_{L} \log ^{*}|g(\mathbf{u})| / \log ^{*} \log ^{*}|g(\mathbf{u})|\right) \tag{9.3.25}
\end{equation*}
$$

Together with 9.3.24), $D \leq d^{r}$ and $d_{L} \leq n!d^{r}$, this implies that $c_{2}$ can be estimated as

$$
\begin{equation*}
c_{2} \leq \exp \left\{n!(n d)^{\exp O(r)} h\right\} . \tag{9.3.26}
\end{equation*}
$$

Next, we estimate $P_{S}, Q_{S}$ and $R_{S}$. In view of (9.3.24), $d_{L} \leq n!d^{r}$ we have

$$
\begin{equation*}
P_{S} \leq Q_{S} \leq|g(u)|^{d_{L}} \leq \exp \left\{n!(n d)^{\exp O(r)} h\right\} \tag{9.3.27}
\end{equation*}
$$

To estimate $R_{S}$, we use 9.3.2). Then, by 9.3 .23 ) and $d_{L} \leq n!d^{r}$ we obtain

$$
\begin{equation*}
\left|D_{L}\right|^{1 / 2}\left(\log ^{*}\left|D_{L}\right|\right)^{d_{L}-1} \leq \exp \left\{n!(n d)^{\exp O(r)} h\right\} . \tag{9.3.28}
\end{equation*}
$$

Further, 9.3.25) and 9.3.27) imply

$$
\left(\log Q_{S}\right)^{s} \leq \exp \left\{O\left(d_{L} \frac{\log ^{*}|g(\mathbf{u})|}{\log ^{*} \log ^{*}|g(\mathbf{u})|}\left(\log d_{L}+\log ^{*} \log ^{*}|g(\mathbf{u})|\right)\right)\right\} .
$$

Together with 9.3 .24 this yields

$$
\begin{equation*}
R_{S} \leq \exp \left\{n!(n d)^{\exp O(r)} h\right\} \tag{9.3.29}
\end{equation*}
$$

Combining 9.3.2) with $R_{S}$ replaced by $R_{L}$ (when $\log Q_{S}<1$ ) with 9.3.28)
and $R_{L}>0.2052$ (see Friedman (1989) or Section 4.1), we obtain

$$
\begin{equation*}
\max \left(h_{L}, R_{L}\right) \leq \exp \left\{n!(n d)^{\exp O(r)} h\right\} \tag{9.3.30}
\end{equation*}
$$

Finally, using $r_{L}<d_{L} \leq n!d^{r}$, we deduce that

$$
\begin{equation*}
c_{3} \leq \exp O\left(d_{L} \log ^{*} d_{L}\right) \leq \exp \left\{n!(n d)^{\exp O(r)}\right\} \tag{9.3.31}
\end{equation*}
$$

From the estimates (9.3.21), (9.3.22), (9.3.26), (9.3.28), 9.3.29), (9.3.30) it follows that the upper bound in 9.3.12 is a sum and product of terms which are all bounded above by $\exp \left\{n!(n d)^{\exp O(r)} h\right\}$. Consequently,

$$
h\left(x_{j}(\mathbf{u})\right), h\left(y_{j}(\mathbf{u})\right) \leq \exp \left\{n!(n d)^{\exp O(r)} h\right\} .
$$

Inserting this into 9.3.16, we get the upper bound 9.1.15 for $q>0$.
Now assume that $q=0$. Then $A_{0}=\mathbb{Z}, K=\mathbb{Q}(w), B=\mathbb{Z}\left[w, g^{-1}\right]$, where $w$ is an algebraic integer with minimal polynomial $\mathcal{F}(X)=X^{D}+$ $\mathcal{F}_{1} X^{D-1}+\cdots+\mathcal{F}_{D} \in \mathbb{Z}[X]$ over $\mathbb{Q}$, and $g$ is a non-zero rational integer. By assumption (7.4.4), (7.4.5) we may assume that $\log |g| \leq h_{4}, \log ^{*} \mathcal{F}_{j} \leq h_{4}$ for $j=1, \ldots, D$. Denote by $w^{(1)}, \ldots, w^{(D)}$ the conjugates of $w$ over $\mathbb{Q}$, and let $K_{j}:=\mathbb{Q}\left(w^{(j)}\right)$ for $j=1, \ldots, D$. One can prove by a similar argument as in the proof of Lemma 7.4.5 that $\left|D_{K_{j}}\right| \leq D^{2 D-1} e^{(2 D-2) h_{4}}$. For $\alpha \in K$, we denote by $\alpha^{(j)}$ the conjugates of $\alpha$ over $\mathbb{Q}$, corresponding to $w^{(j)}$.

Instead of Lemma 7.4.7 we use Lemma 7.4.2, applied with $G=\mathcal{F}, m=$ $D$ and $\beta^{(j)}=x^{(j)}$ or $y^{(j)}$. Inserting (7.4.4), (7.4.5), this yields the estimate

$$
\begin{equation*}
\max (\bar{h}(x), \bar{h}(y)) \leq(n d)^{\exp O(r)}\left(h+\max _{1 \leq j \leq D} \max \left(h\left(x^{(j)}\right), h\left(y^{(j)}\right)\right)\right) \tag{9.3.32}
\end{equation*}
$$

We proceed further with the index $j$ for which the maximum is attained.
Now we can follow the argument above for the case $q>0$, except that in all estimates we have to take $q=0$, and replace $\max (1,|\mathbf{u}|)$ by $1, K_{\mathbf{u}, j}$ by $K_{j}$, $g(\mathbf{u})$ by $g, F_{\mathbf{u}, j}$ by $F_{j}$, where $F_{j}$ is the binary form obtained by taking the $j$-th conjugates of the coefficients of $F$, and $g(\mathbf{u})$ by $g$. This leads to an estimate

$$
h\left(x^{(j)}\right), h\left(y^{(j)}\right) \leq \exp \left\{n!(n d)^{\exp O(r)} h\right\}
$$

and combined with 9.3 .32 , this implies again 9.1 .15 which completes the proof of Proposition 9.1.3.

### 9.3.3 Hyper- and superelliptic equations

It remains to prove 9.1.19p from Proposition 9.1.4. The computations will be similar to those as in the above proof of 9.1 .15 but with some simplifications.

Proof of 9.1.19) of Proposition 9.1.4 Take a solution $x, y$ of equation 9.1.16 in $B$. Consider again first the case $q>0$. We use once more the polynomial $\mathcal{T}:=\Delta_{\mathcal{F}} \mathcal{F}_{D} g$ as in (7.4.7). Take again $\mathbf{u} \in \mathbb{Z}^{q}$ with $\mathcal{T}(\mathbf{u}) \neq 0$, choose $j \in\{1, \ldots, D\}$, and denote by $F_{\mathbf{u}, j}, \delta_{j}(\mathbf{u}), x_{j}(\mathbf{u}), y_{j}(\mathbf{u})$ the images of $F, \delta, x, y$ under the specialization $\varphi_{\mathbf{u}, j}$. In contrast to our arguments for Thue equations, now we do not have to deal with the splitting field of $F$. Put $L:=K_{\mathbf{u}, j}$ and choose for $S$ the set of places of $L$ which consists of all infinite places and the finite places lying above the rational prime divisors of $g(\mathbf{u})$. Then $\varphi_{\mathbf{u}, j}(B) \subseteq \mathcal{O}_{S}$, and

$$
\begin{equation*}
F_{\mathbf{u}, j}\left(x_{j}(\mathbf{u})\right)=\delta_{j}(\mathbf{u}) y_{j}(\mathbf{u})^{m}, \text { where } x_{j}(\mathbf{u}), y_{j}(\mathbf{u}) \in \mathcal{O}_{S} . \tag{9.3.33}
\end{equation*}
$$

We note that by assumption $\delta, D_{F} \in B^{*}$, hence $\delta_{j}(\mathbf{u}) \neq 0$ and $F_{\mathbf{u}, j}$ has nonzero discriminant. Since $F_{\mathbf{u}, j}$ has the same number of zeros and the same degree as $F$, the degree of $F_{\mathbf{u}, j}$ is $n \geq 3$ if $m=2$ and $n \geq 2$ if $m \geq 3$. Thus we can apply Theorems 4.5.1 and 4.5.2 to equation (9.3.33) according as $m=2$ or $m \geq 3$. Then we obtain

$$
h\left(x_{j}(\mathbf{u})\right), h\left(y_{j}(\mathbf{u})\right) \leq\left\{\begin{array}{l}
c_{4}\left|D_{L}\right| 0^{8 n^{2}} Q_{S}^{20 n^{3}} e^{50 n^{4} d_{L} \widehat{h}} \text { if } n \geq 3  \tag{9.3.34}\\
c_{5}^{m}{ }^{3}\left|D_{L}\right|^{2 m^{2} n^{2}} Q_{S}^{3 m^{2} n^{2}} e^{8 m^{2} n^{3} d_{L} \widehat{h}} \text { if } n \geq 2, m \geq 3
\end{array}\right.
$$

where $c_{4}=(4 n s)^{212 n^{4} s}, c_{5}=(6 n s)^{14 m^{3} n^{3} s}$ and $\widehat{h}$ is defined by 4.5.3).
It follows by precisely the same argument as in the case of Thue equations that there are $\mathbf{u} \in \mathbb{Z}^{q}$ and $j \in\{1, \ldots, D\}$ which satisfy (9.3.15) and 9.3.16). We proceed further with these $\mathbf{u}, j$.

We estimate from above the parameters occurring in the bounds in 9.3.34. First, we obtain the same estimates as in 9.3.19) and 9.3.20). These imply

$$
\begin{equation*}
\widehat{h} \leq(n+1) h\left(F_{\mathbf{u}, j}\right)+h\left(\delta_{j}(\mathbf{u})\right) \leq(n d)^{\exp O(r)} h . \tag{9.3.35}
\end{equation*}
$$

Further, we have similarly to (9.3.23)

$$
\begin{equation*}
\left|D_{L}\right| \leq \exp \left\{(n d)^{\exp O(r)} h\right\} \tag{9.3.36}
\end{equation*}
$$

and similarly to 9.3.24

$$
\begin{equation*}
|g(\mathbf{u})| \leq \exp \left\{(n d)^{\exp O(r)} h\right\} \tag{9.3.37}
\end{equation*}
$$

Now the set $S$ consists of places of $L$ instead of the splitting field of $F_{\mathbf{u}, j}$ over $K$. Because of $[L: \mathbb{Q}] \leq D$, we have $s \leq D(1+w)$, where $w$ is the number of distinct prime divisors of $g(\mathbf{u})$. This implies, instead of 9.3.25),

$$
\begin{equation*}
s=O\left(D \log ^{*}|g(\mathbf{u})| / \log ^{*} \log ^{*}|g(\mathbf{u})|\right) . \tag{9.3.38}
\end{equation*}
$$

Inserting 9.3.37) and $D \leq d^{r}$, we obtain for the quantities $c_{4}, c_{5}$ in 9.3.34

$$
\begin{equation*}
c_{4}, c_{5} \leq \exp \left\{(n d)^{\exp O(r)} h\right\} \tag{9.3.39}
\end{equation*}
$$

Lastly, by $D \leq d^{r}$ and 9.3.37) we have

$$
\begin{equation*}
P_{S} \leq Q_{S} \leq|g(\mathbf{u})|^{D} \leq \exp \left\{(n d)^{\exp O(r)} h\right\} . \tag{9.3.40}
\end{equation*}
$$

We now use (9.3.34). By inserting (9.3.35), (9.3.36), (9.3.39) and $d_{L} \leq$ $D \leq d^{r}$ into 9.3.34, we get

$$
\begin{equation*}
h\left(x_{j}(\mathbf{u})\right), h\left(y_{j}(\mathbf{u})\right) \leq \exp \left\{m^{3}(n d)^{\exp O(r)} h\right\} . \tag{9.3.41}
\end{equation*}
$$

Finally, inserting this into 9.3.16, we obtain 9.1.19) in the case $q>0$.
Now let $q=0$. For $\alpha \in K$, write $\alpha^{(j)}$ for the conjugate of $\alpha$ corresponding to $w^{(j)}$ and let $F_{j}$ be the polynomial obtained by taking the $j$ th conjugates of the coefficients of $F$. We simply follow the above arguments, replacing everywhere $q$ by $0, \max (1, \mathbf{u})$ by $1, K_{\mathbf{u}, j}$ by $K_{j}=\mathbb{Q}\left(w^{(j)}\right), F_{\mathbf{u}, j}$ by $F_{j}, x_{j}(\mathbf{u}), y_{j}(\mathbf{u})$ by $x^{(j)}, y^{(j)}$, and $g(\mathbf{u})$ by $g \in \mathbb{Z}$. Instead of (9.3.16) we have to use 9.3.32). Thus we obtain the same estimate as 9.3.41), but with $x^{(j)}, y^{(j)}$ instead of $x_{j}(\mathbf{u}), y_{j}(\mathbf{u})$. Via 9.3.32) we obtain 9.1.19). This completes our proof for Proposition 9.1.4

Proof of Proposition 9.1.5. Assume first that $q>0$. Let $x \in B, y \in B \cap \overline{\mathbb{Q}}$, $m \in \mathbb{Z}_{\geq 2}$ be a solution of equation (9.1.16) such that $y \neq 0$ and $y$ is not a root of unity. Choose again $\mathbf{u}, j$ such that they satisfy (9.3.15), (9.3.16). Note that $y_{j}(\mathbf{u})$ is a conjugate of $y$ since $y \in \overline{\mathbb{Q}}$. Hence it is not 0 or a root of unity.

By applying Theorem 4.5.3 to equation 9.3.33) we get

$$
\begin{equation*}
m \leq c_{6}\left|D_{L}\right|^{6 n} P_{S}^{n^{2}} e^{11 n d_{L} \widehat{h}} \tag{9.3.42}
\end{equation*}
$$

where $c_{6}=\left(10 n^{2} s\right)^{40 n s}$. By 9.3.37) and 9.3.38) the constant $c_{6}$ satisfies

$$
c_{6} \leq \exp \left\{(n d)^{\exp O(r)} h\right\}
$$

Further, we have the upper bounds (9.3.35) for $\widehat{h}$, 9.3.36) for $\left|D_{L}\right|$ and 9.3.40 for $P_{S}$. Inserting these estimates into the upper bound in 9.3.42), we get $m \leq \exp \left\{(n d)^{\exp O(r)} h\right\}$. In the case $q=0$, we obtain the same estimates, by making the same modifications as in the proof of Proposition 9.1.4. Our proof is complete.

### 9.4 The Catalan equation

In this section we complete the proof of Theorem 2.5.1 on the Catalan equation (2.5.1) $x^{m}-y^{n}=1$ in $x, y \in A$ and integers $m, n$ with $m, n>1$ and $m n>4$. We follow mostly the proofs of Brindza (1993) and Koymans (2017).

As before, $A=\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]$ denotes an integral domain finitely generated over $\mathbb{Z}$, with quotient field $K$. Then $A \cong \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right] / \mathcal{I}$, where $\mathcal{I}$ is the ideal of polynomials $f \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ such that $f\left(z_{1}, \ldots, z_{r}\right)=0$. Let $d \geq 1, h \geq 1$ and assume again that $\mathcal{I}=\left(f_{1}, \ldots, f_{M}\right)$ with $\operatorname{deg} f_{i} \leq d$, $h\left(f_{i}\right) \leq h$ for $i=1, \ldots, M$.

Proof of Theorem 2.5.1] Let $x, y, m, n$ be an arbitrary solution of equation (2.5.1) with non-zero $x, y \in A$, not roots of unity, and with integers $m, n$ such that $m, n>1$ and $m n>4$. We keep the notation and assumptions from the beginning of Sections 9.3 and 9.4 . Further, by Proposition 7.2 .7 there exists a non-zero $g \in A_{0}$ such that

$$
A \subseteq B=A_{0}\left[w, g^{-1}\right]
$$

and

$$
\begin{equation*}
\operatorname{deg} g \leq(2 d)^{\exp O(r)}, h(g) \leq(2 d)^{\exp O(r)} h . \tag{9.4.1}
\end{equation*}
$$

We shall work in this larger ring $B$ to bound $m$ and $n$.
First consider the case $q=0$. Then we have $A_{0}=\mathbb{Z}, K_{0}=\mathbb{Q}, K$ is a number field of degree $D \leq d^{r}$ and $D_{K}$ is the discriminant of $K$. Since $\left|D_{K}\right| \leq|D(\mathcal{F})|$ and, as was seen in the proof of Lemma 7.4.5, $|D(\mathcal{F})| \leq$ $D^{2 D-1} H(\mathcal{F})^{2 D-2}$, where $H(\mathcal{F})$ denotes the maximum of the absolute values
of the coefficients of $\mathcal{F}$, we infer that

$$
\left|D_{K}\right| \leq \exp \left((2 d)^{\exp O(r)} h\right) .
$$

Let $S$ denote the set of infinite places and of the finite places of $K$ corresponding to the prime ideal divisors, say $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{w}$, of $g$. Write $s=|S|$ and let $P_{S}, Q_{S}$ be as in (9.3.1). In view of (9.4.1) it follows that $s \leq(2 d)^{\exp O(r)} h$, $P_{S} \leq \exp \left((2 d)^{\exp O(r)} h\right)$ and $Q_{S} \leq|g|^{d} \leq \exp \left((2 d)^{\exp O(r)} h\right)$. Using the estimates for $s, P_{S}, Q_{S}$ and combining them with Theorem 4.6.1, we obtain (2.5.2).

Consider now the case $q>0$. Fix an algebraic closure $\bar{K}_{0}$ of $K_{0}$, and let $\mathbb{k}_{i}$ be the algebraic closure of $\mathbb{Q}\left(X_{1}, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{q}\right)$ in $\bar{K}_{0}$. Thus $A_{0}$ is contained in $\mathbb{k}_{i}\left[X_{i}\right]$. Define

$$
L_{i}:=\mathbb{k}_{i}\left(X_{i}, w^{(1)}, \ldots, w^{(D)}\right),
$$

where $w^{(1)}, \ldots, w^{(D)}$ are the conjugates of $w$ over $K_{0}$.
First assume that $x \in \mathbb{k}_{i}$ for $i=1, \ldots, q$. Then, by Lemma 9.2.1, $x$ and $y$ belong to the algebraic number field $\overline{\mathbb{Q}} \cap K$. We are now going to apply Theorem 4.6.1.

Let $\mathbf{u}=\left(u_{1}, \ldots, u_{q}\right) \in \mathbb{Z}^{q}$, and put $|\mathbf{u}|=\max \left(\left|u_{1}\right|, \ldots,\left|u_{q}\right|\right)$. As in Section 7.4, we extend the ring homomorphism from a subring of $K_{0}$ to $\mathbb{Q}$,

$$
\varphi_{\mathbf{u}}: \alpha \mapsto \alpha(\mathbf{u}):\left\{g_{1} / g_{2}: g_{1}, g_{2} \in A_{0}, g_{2}(\mathbf{u}) \neq 0\right\} \rightarrow \mathbb{Q},
$$

defined by the substitution $X_{1} \mapsto u_{1}, \ldots, X_{q} \mapsto u_{q}$, to a ring homomorphism from $B$ to $\overline{\mathbb{Q}}$ for which we need to impose some restriction on $\mathbf{u}$. Denote by $\Delta_{\mathcal{F}}$ the discriminant of $\mathcal{F}$, and let $\mathcal{T}=\Delta_{\mathcal{F}} \mathcal{F}_{D} g$. Obviously $\mathcal{T} \in A_{0}$. Since $\Delta_{\mathcal{F}}$ is a polynomial of degree $2 D-2$ with integer coefficients in $\mathcal{F}_{1}, \ldots, \mathcal{F}_{D}$, it follows that $\operatorname{deg} \mathcal{T} \leq(2 d)^{\exp O(r)}$.

Lemma 7.4.4 implies that

$$
\mathcal{S}:=\left\{\mathbf{u} \in \mathbb{Z}^{q}:|\mathbf{u}| \leq N, \mathcal{T}(\mathbf{u}) \neq 0\right\}
$$

is non-empty, provided that $N=(2 d)^{\exp O(r)}$ and the constant implied by the $O$-symbol is sufficiently large. Take $\mathbf{u} \in \mathcal{S}$ and consider the polynomial $\mathcal{F}_{\mathbf{u}}(X):=X^{D}+\mathcal{F}_{1}(\mathbf{u}) X^{D-1}+\cdots+\mathcal{F}_{D}(\mathbf{u})$. It has distinct zeros, say $w_{1}(\mathbf{u}), \ldots, w_{D}(\mathbf{u})$, which are all different from 0 . Then, for $j=1, \ldots, D$,

$$
X_{1} \mapsto u_{1}, \ldots, X_{q} \mapsto u_{q}, \mathbf{u} \mapsto w_{j}(\mathbf{u})
$$

defines, as was claimed in Section 7.4, a ring homomorphism $\varphi_{\mathbf{u}, j}$ from $B$ to $\overline{\mathbb{Q}}$. The image of $\alpha \in B$ under $\varphi_{\mathbf{u}, j}$ is again denoted by $\alpha_{j}(\mathbf{u})$. It is clear that $\varphi_{\mathbf{u}, j}$ is the identity on $B \cap \mathbb{Q}$. Consequently, if $\alpha \in B \cap \overline{\mathbb{Q}}$, then $\varphi_{\mathbf{u}, j}$ has the same minimal polynomial as $\alpha$ and hence it is conjugate to $\alpha$.

Consider the algebraic number field $K_{\mathbf{u}, j}:=\mathbb{Q}\left(w_{j}(\mathbf{u})\right)$ and denote by $D_{K_{\mathbf{u}, j}}$ its discriminant for $j=1, \ldots, D$. By Lemma 7.4 .5 we have

$$
\left[K_{\mathbf{u}, j}: \mathbb{Q}\right] \leq D, \quad\left|D_{K_{\mathbf{u}, j}}\right| \leq D^{2 D-1}\left(d_{3}^{q} e^{h_{3}} \max (1,|\mathbf{u}|)^{d_{3}}\right)^{2 D-2},
$$

where

$$
d_{3}=\max \left(d, \operatorname{deg} \mathcal{F}_{1}, \ldots, \operatorname{deg} \mathcal{F}_{D}\right), h_{3}=\max \left(d, h\left(\mathcal{F}_{1}\right), \ldots, h\left(\mathcal{F}_{D}\right)\right) .
$$

We have $d_{3} \leq(2 d)^{\exp O(r)}$, $h_{3} \leq(2 d)^{\exp O(r)} h$ which, together with $D \leq d^{r}$, gives

$$
\left|D_{K_{\mathbf{u}, j}}\right| \leq \exp \left((2 d)^{\exp O(r)} h\right) .
$$

Fix now any of $j=1, \ldots, D$. In $K_{\mathbf{u}, j}$ denote by $S$ the set consisting of the infinite places and of the finite places corresponding to the prime ideal divisors of $g(\mathbf{u})$. Then $\varphi_{\mathbf{u}, j}$ maps $B$ to the ring of $S$-integers of $K_{\mathbf{u}, j}$. To apply Theorem 4.6.1 we still need to bound $P_{S}, Q_{S}$ and $s$.

It is easy to see that for any $\mathbf{u} \in \mathbb{Z}^{q}$

$$
\log |g(\mathbf{u})| \leq q \log \operatorname{deg} g+h(g)+\operatorname{deg} g \log \max (1,|\mathbf{u}|)
$$

which together with (9.4.1) and with the choice of $N$ gives

$$
\begin{aligned}
|g(\mathbf{u})| & \leq(2 d)^{q \exp O(r)} \cdot \exp \left((2 d)^{\exp O(r)} \cdot h\right) \cdot(2 d)^{(2 d)^{\exp O(r)}} \\
& \leq \exp \left((2 d)^{\exp O(r)} \cdot h\right) .
\end{aligned}
$$

Thus we get

$$
\begin{align*}
& Q_{S} \leq|g(\mathbf{u})|^{D} \leq \exp \left((2 d)^{\exp O(r)} h\right)  \tag{9.4.2}\\
& P_{S} \leq \exp \left((2 d)^{\exp O(r)} h\right) \tag{9.4.3}
\end{align*}
$$

and

$$
\begin{equation*}
s \leq(2 d)^{\exp O(r)} h \tag{9.4.4}
\end{equation*}
$$

Combining Theorem 4.6.1 with the bounds in (9.4.2) to (9.4.4) we obtain
again (2.5.3).
Finally, we deal with the case $x \notin \mathbb{k}_{i}$ for some $i$. Fix such an $i$. Let $S$ denote the set of valuations of $L_{i}$ over $\mathbb{k}_{i}$ such that $v\left(X_{i}\right)<0, v(g)>0$. Let $v$ be any valuation of $L_{i}$ over $\mathbb{k}_{i}$ with $v \notin S$. Then $v\left(X_{i}\right) \geq 0$. We recall that $w$ is integral over $\mathbb{k}_{i}\left[X_{i}\right]$. This implies that $v(w) \geq 0$. We also have that $v(g) \leq 0$, whence $v\left(g^{-1}\right) \geq 0$. Consequently, the ring $B=A_{0}\left[w, g^{-1}\right]$ is a subring of the ring of $S$-integers $\mathcal{O}_{S}$ of $L_{i}$.

Since $x, y \in A$ and $A \subseteq B$, it follows that $x, y \in \mathcal{O}_{S} \backslash \mathbb{k}_{i}$. Now in view of Theorem 5.3.1, (i), we get

$$
\begin{equation*}
m H_{L_{i}}(x), n H_{L_{i}}(y) \leq 6\left(|S|+2 g_{L_{i} / \mathbb{k}_{i}}-2\right) . \tag{9.4.5}
\end{equation*}
$$

Putting $K_{i}=\mathbb{k}_{i}\left(X_{i}, w\right), \Delta_{i}=\left[L_{i}: \mathbb{k}\left(X_{i}\right)\right]$ and using the fact that $x, y \in K_{i}$ and $\left[K_{i}: \mathbb{k}_{i}\left(X_{i}\right)\right] \leq D$ we infer that

$$
H_{L_{i}}(x)=\left[L_{i}: K_{i}\right] H_{K_{i}}(x) \geq\left[L_{i}: K_{i}\right] \geq \Delta_{i} /\left[K_{i}: \mathbb{k}_{i}\left(X_{i}\right)\right] \geq \Delta_{i} / D
$$

and similarly for $H_{L_{i}}(y)$. Together with 9.4.5) this gives

$$
\begin{equation*}
\max (m, n) \leq \frac{6 D}{\Delta_{i}}\left(|S|+2 g_{L_{i} / \mathbb{k}_{i}}-2\right) . \tag{9.4.6}
\end{equation*}
$$

We are now going to estimate from above $|S|$ and $g_{L_{i} / \mathbb{k}_{i}}$. Every valuation of $\mathbb{k}_{i}\left(X_{i}\right)$ can be extended to at most $\Delta_{i}$ valuations of $L_{i}$. Thus $L_{i}$ has at most $\Delta_{i}$ valuations with $v\left(X_{i}\right)<0$ and at most $\Delta_{i} \operatorname{deg}_{X_{i}} g$ valuations with $v(g)>0$. Using also $\operatorname{deg} g \leq(2 d)^{\exp O(r)}$ from Proposition 7.2.7, we deduce that

$$
\begin{equation*}
|S| \leq \Delta_{i}+\Delta_{i} \operatorname{deg}_{X_{i}} g \leq \Delta_{i}(1+\operatorname{deg} g) \leq \Delta_{i}(2 d)^{\exp O(r)} . \tag{9.4.7}
\end{equation*}
$$

Further, $L_{i}$ being the splitting field of $\mathcal{F}$ over $\mathbb{k}_{i}\left(X_{i}\right)$, by Lemma 5.1.1 and Proposition 7.2.5 we obtain

$$
\begin{equation*}
g_{L_{i} / \mathbb{k}_{i}} \leq \Delta_{i} D \max _{j} \operatorname{deg}_{X_{i}} \mathcal{F}_{j} \leq \Delta_{i} D \max _{j} \operatorname{deg} \mathcal{F}_{j} \leq \Delta_{i} D(2 d)^{\exp O(r)} \tag{9.4.8}
\end{equation*}
$$

Thus it follows from $D \leq d^{r}$, (9.4.6), (9.4.7) and (9.4.8) that

$$
\max (m, n) \leq(2 d)^{\exp O(r)}
$$

which completes the proof of (2.5.3) and hence that of Theorem 2.5.1.

## Chapter 10

## Proofs of the results from Sections 2.6-2.8; reduction to unit equations

In Section 10.1 we prove our central results Theorem 2.6.1 and Corollary 2.6.2 on decomposable form equations, stated in Section 2.6, together with the corollaries stated in that section. Their proofs are based on Theorem 2.2.1 on unit equations. In fact, we apply Győry's method to reduce the decomposable form equation under consideration to unit equations in two unknowns, which was originally developed in an effective form over number fields e.g. in Győry (1976,1981a), Győry and Papp (1978), and, in an ineffective form, over arbitrary finitely generated domains in Győry (1982). We make this fully effective and quantitative by employing some of the degree-height estimates from Chapter 8 .

In Section 10.2 we prove the results for norm form equations stated in Section 2.7, and in Section 10.3 the results for discriminant form equations and discriminant equations stated in Section 2.8. These are all consequences of Theorem 2.6.1.

### 10.1 Proofs of the central results on decomposable form equations

Keeping the notation of Sections 2.6 and 8.1, we assume that $A, \delta, \mathcal{L}=$ $\left(\ell_{1}, \ldots, \ell_{n}\right)$ satisfy the conditions of Theorem 2.6.1. Denote as before by $\mathcal{G}(\mathcal{L})$
the triangular graph of $\mathcal{L}$, defined by (2.6.4), and let $\mathcal{L}_{1}, \ldots, \mathcal{L}_{k}$ be the vertex systems of the connected components of $\mathcal{G}(\mathcal{L})$, and $\left[\mathcal{L}_{j}\right]$ the $\bar{K}$-vector space generated by $\mathcal{L}_{j}$, for $j=1, \ldots, k$.

Proof of Theorem 2.6.1. Take $\mathrm{x} \in A^{m}$ such that

$$
\begin{equation*}
F(\mathbf{x})=\ell_{1}(\mathbf{x}) \cdots \ell_{n}(\mathbf{x})=\delta, \text { there is } \ell \in\left[\mathcal{L}_{1}\right] \cap \cdots \cap\left[\mathcal{L}_{k}\right] \text { with } \ell(\mathbf{x}) \neq 0 . \tag{2.6.7}
\end{equation*}
$$

We have to show that the $\operatorname{coset} \mathbf{x}+\mathcal{Z}_{A, F}$ is represented by $\widetilde{\mathbf{x}} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]^{m}$ with (2.6.8). The proof is divided into a couple of steps.
Step 1. Construction of certain scalar multiples $\ell_{i}^{\prime}$ of $\ell_{i}$, for $i=1, \ldots, n$, and a finitely generated domain $A^{\prime} \supset A$, such that $\ell_{1}^{\prime}(\mathbf{x}), \ldots, \ell_{n}^{\prime}(\mathrm{x})$ are units of $A^{\prime}$.
Write

$$
\ell_{i}=\alpha_{i, 1} X_{1}+\cdots+\alpha_{i, m} X_{m} \text { for } i=1, \ldots, n
$$

Put

$$
\mathcal{R}:=2 m n \cdot \nu^{m n} d, \quad \mathcal{R}^{\prime}:=2 m n \cdot \nu^{\nu m n} d .
$$

Let $G$ be the extension of $K$ generated by the $\alpha_{i, j}(i=1, \ldots, n, j=1, \ldots, m)$. We may assume that also $\delta \in G$, since otherwise (2.6.7) cannot hold. By Corollary 8.3.4, there is $\theta \in G$, such that $G=K(\theta), \theta$ has monic minimal polynomial $F_{\theta} \in A[X]$ over $K$, and

$$
\begin{equation*}
\theta \stackrel{\mathrm{int}}{\prec}\left(\mathcal{R}^{\exp O(r)}, \mathcal{R}^{\exp O(r)} h\right) . \tag{10.1.1}
\end{equation*}
$$

Further, letting $\mathcal{E}:=[G: K]$, we have

$$
\begin{align*}
& \alpha_{i, j}=\sum_{t=0}^{\mathcal{E}-1} a_{i, j, t} \theta^{t}, \quad \delta=\sum_{t=0}^{\mathcal{E}-1} b_{t} \theta^{t}, \\
& \text { with } b_{i, j, t}, b_{t} \in K, \quad b_{t}, b_{i, j, t} \prec\left(\mathcal{R}^{\prime \exp O(r)}, \mathcal{R}^{\prime \exp O(r)} h\right) \tag{10.1.2}
\end{align*}
$$

for $i=1, \ldots, n, j=1, \ldots, m, t=0, \ldots, \mathcal{E}-1$. We clear the denominators of the $a_{i, j, t}, b_{t}$. According to the definitions, we have for all $i, j, t$ that

$$
a_{i, j, t}=\frac{g_{i, j, t}^{\prime}\left(z_{1}, \ldots, z_{r}\right)}{g_{i, j, t}^{\prime \prime}\left(z_{1}, \ldots, z_{r}\right)}, \quad b_{t}=\frac{g_{t}^{\prime}\left(z_{1}, \ldots, z_{r}\right)}{g_{t}^{\prime \prime}\left(z_{1}, \ldots, z_{r}\right)}
$$

where $g_{t}^{\prime}, g_{i, j, t}^{\prime}, g_{t}^{\prime \prime}, g_{i, j, t}^{\prime \prime}$ are polynomials in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ of total degree at most $\mathcal{R}^{\prime \exp O(r)}$ and logarithmic height at most $\mathcal{R}^{\prime \exp O(r)} h$. Now define

$$
\begin{aligned}
\gamma_{0} & :=\prod_{t=0}^{\mathcal{E}-1} g_{t}^{\prime \prime}\left(z_{1}, \ldots, z_{r}\right), \\
\gamma_{i} & :=\prod_{t=0}^{\mathcal{E}-1} \prod_{j=1}^{m} g_{i, j, t}^{\prime \prime}\left(z_{1}, \ldots, z_{r}\right) \text { for } i=1, \ldots, n .
\end{aligned}
$$

Then by Lemma 4.1.7 and $\mathcal{E} \leq \nu^{m n}$,

$$
\begin{equation*}
\gamma_{0}, \gamma_{1}, \ldots, \gamma_{n} \stackrel{\text { int }}{\prec}\left(\mathcal{R}^{\prime \exp O(r)}, \mathcal{R}^{\prime \exp O(r)} h\right) . \tag{10.1.3}
\end{equation*}
$$

Define the quantities

$$
\left\{\begin{array}{l}
a_{i, j, t}^{\prime}:=\gamma_{0} \gamma_{i} a_{i, j, t},  \tag{10.1.4}\\
\alpha_{i, j}^{\prime}:=\sum_{t=0}^{\mathcal{E}-1} a_{i, j, t}, t^{t}=\gamma_{0} \gamma_{i} \alpha_{i, j}, \ell_{i}^{\prime}:=\sum_{j=1}^{m} \alpha_{i, j}^{\prime} X_{j}=\gamma_{0} \gamma_{i} \ell_{i}
\end{array}\right.
$$

for $i=1, \ldots, n, j=1, \ldots, m, t=0, \ldots, \mathcal{E}-1$, and

$$
\left\{\begin{align*}
b_{t}^{\prime} & :=\gamma_{0}^{n} \gamma_{1} \cdots \gamma_{n} b_{t} \text { for } t=0, \ldots, \mathcal{E}-1  \tag{10.1.5}\\
\delta^{\prime} & :=\sum_{t=0}^{\mathcal{E}-1} b_{t}^{\prime} \theta^{t}=\gamma_{0}^{n} \gamma_{1} \cdots \gamma_{n} \delta
\end{align*}\right.
$$

With this construction we have

$$
\begin{equation*}
a_{i, j, t}^{\prime}, b_{t}^{\prime} \in A \text { for all } i, j, t \tag{10.1.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\ell_{1}^{\prime}(\mathbf{x}) \cdots \ell_{n}^{\prime}(\mathbf{x})=\delta^{\prime} \tag{10.1.7}
\end{equation*}
$$

Further, by Lemma 4.1.7,

$$
\begin{equation*}
a_{i, j, t}^{\prime}, b_{t}^{\prime} \text { int }\left(\mathcal{R}^{\prime \exp O(r)}, \mathcal{R}^{\prime \exp O(r)} h\right) . \tag{10.1.8}
\end{equation*}
$$

Now let $A^{\prime}:=A\left[\theta, \delta^{\prime-1}\right]=\mathbb{Z}\left[z_{1}, \ldots, z_{r}, \theta, \delta^{\prime-1}\right]$. Then from (10.1.6) it is
clear that $\ell_{1}^{\prime}(\mathrm{x}), \ldots, \ell_{n}^{\prime}(\mathrm{x}) \in A^{\prime}$ and subsequently by 10.1.7,

$$
\begin{equation*}
\ell_{i}^{\prime}(\mathbf{x}) \in A^{\prime *} \text { for } i=1, \ldots, n . \tag{10.1.9}
\end{equation*}
$$

To apply Theorem 2.2.1, we need an ideal representation for $A^{\prime}$. By (10.1.1), the generator $\theta$ of $G$ has monic minimal polynomial over $A$,

$$
F_{\theta}=X^{\mathcal{E}}+p_{1}\left(z_{1}, \ldots, z_{r}\right) X^{\mathcal{E}-1}+\cdots+p_{\mathcal{E}}\left(z_{1}, \ldots, z_{r}\right),
$$

where $p_{1}, \ldots, p_{\mathcal{E}}$ are polynomials in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ of total degree at most $\mathcal{R}^{\exp O(r)}$ and logarithmic height at most $\mathcal{R}^{\exp O(r)} h$. Let

$$
f_{M+1}:=X_{r+1}^{\mathcal{E}}+p_{1} X_{r+1}^{\mathcal{E}-1}+\cdots+p_{\mathcal{E}} .
$$

Then using $\mathcal{E} \leq \nu^{m n}$ we get

$$
\begin{equation*}
\operatorname{deg} f_{M+1} \leq \mathcal{R}^{\exp O(r)}, \quad h\left(f_{M+1}\right) \leq \mathcal{R}^{\exp O(r)} h \tag{10.1.10}
\end{equation*}
$$

Further,

$$
A[\theta] \cong \mathbb{Z}\left[X_{1}, \ldots, X_{r}, X_{r+1}\right] / \mathcal{I}^{\prime}, \text { with } \mathcal{I}^{\prime}:=\left(f_{1}, \ldots, f_{M}, f_{M+1}\right)
$$

The quantity $\delta^{\prime}$ corresponds to the residue class modulo $\mathcal{I}^{\prime}$ of $\sum_{t=0}^{\mathcal{E}-1} \widetilde{b_{t}} X_{r+1}^{t}$, where $\widetilde{b_{t}} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ is a representative for $b_{t}$, of total degree at most $\mathcal{R}^{\prime \exp O(r)}$, and logarithmic height at most $\mathcal{R}^{\prime \exp O(r)} h$. By Lemma 9.1 .2 we have

$$
\begin{equation*}
A^{\prime} \cong \mathbb{Z}\left[X_{1}, \ldots, X_{r}, X_{r+1}, X_{r+2}\right] /\left(f_{1}, \ldots, f_{M}, f_{M+1}, f_{M+2}\right) \tag{10.1.11}
\end{equation*}
$$

where

$$
f_{M+2}:=X_{r+2}\left(\sum_{t=0}^{\mathcal{E}-1} \widetilde{b}_{t} X_{r+1}^{t}\right)-1 .
$$

Using again $\mathcal{E} \leq \nu^{m n}$ we infer that $f_{M+2}$ has total degree at most $\mathcal{R}^{\prime \exp O(r)}$ and logarithmic height at most $\mathcal{R}^{\prime \exp O(r)} h$. Combined with (10.1.10) and our assumptions $\operatorname{deg} f_{i} \leq d, h\left(f_{i}\right) \leq h$ this gives

$$
\begin{equation*}
\operatorname{deg} f_{i} \leq \mathcal{R}^{\prime \exp O(r)}, \quad h\left(f_{i}\right) \leq \mathcal{R}^{\prime \exp O(r)} h \text { for } i=1, \ldots, M+2 \tag{10.1.12}
\end{equation*}
$$

Step 2. Let $\ell_{i}, \ell_{j}$ with $i \neq j$ be connected by an edge in $\mathcal{G}(\mathcal{L})$. Then

$$
\begin{equation*}
\frac{\ell_{i}(\mathbf{x})}{\ell_{j}(\mathbf{x})} \prec\left(\exp \left(\mathcal{R}^{\prime \exp O(r)} h\right), \exp \left(\mathcal{R}^{\prime \exp O(r)} h\right)\right) \tag{10.1.13}
\end{equation*}
$$

To prove this, we first assume that $\ell_{i}, \ell_{j}$ are linearly dependent over $\bar{K}$. Then $\ell_{i}(\mathrm{x}) / \ell_{j}(\mathrm{x})$ is equal to the quotient of a coefficient of $\ell_{i}$ and a coefficient of $\ell_{j}$. So by Corollary 8.3.3 and 8.3.2), we have

$$
\frac{\ell_{i}(\mathbf{x})}{\ell_{j}(\mathbf{x})} \prec\left(\left(2 \nu^{2} d\right)^{\exp O(r)},\left(2 \nu^{2} d\right)^{\exp O(r)} h\right),
$$

which is much stronger than what we want to prove.
Next assume that $\ell_{i}, \ell_{j}$ are linearly independent over $\bar{K}$. Here we have to apply our Theorem 2.2.1 on unit equations. There is $q \neq i, j$ such that $\ell_{i}, \ell_{j}, \ell_{q}$ are linearly dependent over $\bar{K}$. Then clearly, $\ell_{i}^{\prime}, \ell_{j}^{\prime}, \ell_{q}^{\prime}$ are also linearly dependent over $\bar{K}$. In fact, we have

$$
\lambda_{i} \ell_{i}^{\prime}+\lambda_{j} \ell_{j}^{\prime}+\lambda_{q} \ell_{q}^{\prime}=0
$$

where $\lambda_{i}, \lambda_{j}, \lambda_{q}$ are certain $2 \times 2$-determinants of the coefficients of $\ell_{i}^{\prime}, \ell_{j}^{\prime}, \ell_{k}^{\prime}$. By (10.1.4) we have

$$
\lambda_{i}=\widetilde{\lambda_{i}}\left(z_{1}, \ldots, z_{r}, \theta, \delta^{\prime-1}\right),
$$

where $\widetilde{\lambda}_{i} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}, X_{r+1}, X_{r+2}\right]$ is a polynomial of total degree at most $\mathcal{R}^{\prime \exp O(r)}$ and logarithmic height at most $\mathcal{R}^{\prime \exp O(r)} h$. We have something similar for $\lambda_{j}$ and $\lambda_{q}$. Clearly,

$$
\lambda_{i} \cdot \frac{\ell_{i}^{\prime}(\mathbf{x})}{\ell_{j}^{\prime}(\mathbf{x})}+\lambda_{q} \cdot \frac{\ell_{q}^{\prime}(\mathbf{x})}{\ell_{j}^{\prime}(\mathbf{x})}=-\lambda_{j},
$$

while $\ell_{i}^{\prime}(\mathbf{x}) / \ell_{j}^{\prime}(\mathrm{x}), \ell_{q}^{\prime}(\mathrm{x}) / \ell_{j}^{\prime}(\mathrm{x}) \in A^{\prime *}$ by 10.1.9). Now invoking 10.1.11), (10.1.12) and applying Theorem 2.2.1 we obtain

$$
\frac{\ell_{i}^{\prime}(\mathbf{x})}{\ell_{j}^{\prime}(\mathbf{x})}=g\left(z_{1}, \ldots, z_{r}, \beta, \delta^{\prime-1}\right)
$$

where $g$ is a polynomial in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}, X_{r+1}, X_{r+2}\right]$ of total degree and logarithmic height at most $\exp \left(\mathcal{R}^{\prime \exp O(r)} h\right)$. An application of 8.3.2) and

Corollary 8.3.3 yields

$$
\frac{\ell_{i}^{\prime}(\mathbf{x})}{\ell_{j}^{\prime}(\mathbf{x})} \prec\left(\exp \left(\mathcal{R}^{\prime \exp O(r)} h\right), \exp \left(\mathcal{R}^{\prime \exp O(r)} h\right)\right)
$$

Finally, using $\ell_{i}(\mathbf{x}) / \ell_{j}(\mathbf{x})=\left(\gamma_{j} / \gamma_{i}\right)\left(\ell_{i}^{\prime}(\mathbf{x}) / \ell_{j}^{\prime}(\mathbf{x})\right)$, estimate 10.1.3) and again 8.3.2) and Corollary 8.3.3, we arrive at 10.1.13).

Step 3. Let $\ell_{i}, \ell_{j}$ belong to the same connected component of $\mathcal{G}(\mathcal{L})$. Then we have again (10.1.13).
There is a sequence $\ell_{i_{0}}, \ell_{i_{1}}, \ldots, \ell_{i_{s}}$ with $i_{0}=i, i_{s}=j$ of length $s \leq n$ of which any two consecutive linear forms are connected by an edge in $\mathcal{G}(\mathcal{L})$. Now apply Step 2 and Corollary 8.3.3 to

$$
\frac{\ell_{j}(\mathbf{x})}{\ell_{i}(\mathbf{x})}=\prod_{t=0}^{s-1} \frac{\ell_{i_{t+1}}(\mathbf{x})}{\ell_{i_{t}}(\mathbf{x})}
$$

to finish Step 3.
Step 4. Let $\ell_{i}, \ell_{j}$ be any two distinct linear forms from $\mathcal{L}$ with $i \neq j$. Then we have again (10.1.13).

This is clear if $\mathcal{G}(\mathcal{L})$ is connected, so assume that its number $k$ of connected components is $>1$. By assumption, there is $\ell \in\left[\mathcal{L}_{1}\right] \cap \cdots \cap\left[\mathcal{L}_{k}\right]$ such that $\ell(\mathbf{x}) \neq 0$. We can partition $\{1, \ldots, n\}$ into $I_{1} \cup \cdots \cup I_{k}$ such that

$$
\mathcal{L}_{t}=\left(\ell_{s}: s \in I_{t}\right) \text { for } t=1, \ldots, k .
$$

Notice that $\left[\mathcal{L}_{1}\right] \cap \cdots \cap\left[\mathcal{L}_{k}\right]$ consists of the linear forms $\ell$ of the shape

$$
\begin{equation*}
\sum_{s \in I_{1}} c_{s} \ell_{s}=\cdots=\sum_{s \in I_{k}} c_{s} \ell_{s} \tag{10.1.14}
\end{equation*}
$$

for certain $c_{1}, \ldots, c_{n} \in \bar{K}$. We recall that in general, if $\mathcal{A}$ is an $a \times b$-matrix of rank $t$, say, with elements from a field $L$, then the solution space of vectors $\mathrm{x} \in L^{b}$ with $\mathcal{A} \mathrm{x}=0$ has a basis consisting of vectors whose coordinates are $t \times t$-subdeterminants of $\mathcal{A}$. In particular, the linear subspace of $\bar{K}^{n}$, consisting of the vectors $\mathbf{c}=\left(c_{1}, \ldots, c_{n}\right)$ satisfying (10.1.14), has a basis, consisting of vectors whose coordinates are determinants of order at most $m$, with elements from the coefficients of $\pm \ell_{i}(i=1, \ldots, n)$. So by Corollary 8.3.3, these
coordinates have degree-height estimates

$$
\prec\left(\left(2 m \nu^{m^{2}} d\right)^{\exp O(r)},\left(2 m \nu^{m^{2}} d\right)^{\exp O(r)} h\right) .
$$

This basis contains a vector $\mathbf{c}=\left(c_{1}, \ldots, c_{n}\right)$ such that the linear form $\ell$ given by (10.1.14) does not vanish at $\mathbf{x}$. Now suppose for instance that $\ell_{i} \in \mathcal{L}_{1}$, $\ell_{j} \in \mathcal{L}_{2}$. Then

$$
\frac{\ell(\mathbf{x})}{\ell_{i}(\mathbf{x})}=\sum_{s \in I_{1}} c_{s} \frac{\ell_{s}(\mathbf{x})}{\ell_{i}(\mathbf{x})} .
$$

By Corollary 8.3.3 and what we established in Step 3, we get

$$
\frac{\ell(\mathbf{x})}{\ell_{i}(\mathbf{x})} \prec\left(\exp \left(\mathcal{R}^{\prime \exp O(r)} h\right), \exp \left(\mathcal{R}^{\prime \exp O(r)} h\right)\right)
$$

We get a similar estimate for $\ell(\mathbf{x}) / \ell_{j}(\mathbf{x})$. Another application of Corollary 8.3.3, in combination with 8.3.2), completes Step 4.

Step 5. For $i=1, \ldots, n$ we have

$$
\begin{equation*}
\ell_{i}(\mathbf{x}) \prec\left(\exp \left(\mathcal{R}^{\prime \exp O(r)} h\right), \exp \left(\mathcal{R}^{\prime \exp O(r)} h\right)\right) \tag{10.1.15}
\end{equation*}
$$

To prove this, observe that

$$
\ell_{i}(\mathbf{x})^{n}=\delta \prod_{j=1}^{n} \frac{\ell_{i}(\mathbf{x})}{\ell_{j}(\mathbf{x})}
$$

Now apply Step 4 and Proposition 8.3.2.
Step 6. Completion of the proof.
Write $\beta_{i}:=\ell_{i}(\mathbf{x})$ for $i=1, \ldots, n$. Note that $\beta_{i} \in G$, ${\operatorname{so~} \operatorname{deg}_{K} \beta_{i} \leq \nu^{m n} \text { for }}$ $i=1, \ldots, n$. By Corollary 8.3.5, there is $\mathrm{x}^{\prime}=\left(x_{1}^{\prime}, \ldots, x_{m}^{\prime}\right) \in A^{m}$ such that

$$
\ell_{i}\left(\mathbf{x}^{\prime}\right)=\beta_{i} \text { for } i=1, \ldots, n
$$

and

$$
x_{i}^{\prime} \stackrel{\text { int }}{\gtrless}\left(\exp \left(\mathcal{R}^{\prime \exp O(r)} h\right), \exp \left(\mathcal{R}^{\prime \exp O(r)} h\right)\right) \text { for } i=1, \ldots, m,
$$

which means precisely that $\mathbf{x}^{\prime}$ has a representative $\widetilde{\mathbf{x}^{\prime}} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]^{m}$ with
$s\left(\tilde{\mathbf{x}^{\prime}}\right) \leq \exp \left(\mathcal{R}^{\prime \exp O(r)} h\right)$, which is 2.6.8). Further, $\mathbf{x}^{\prime}-\mathbf{x} \in \mathcal{Z}_{A, F}$, so in fact, $\mathrm{x}^{\prime}$ represents the $\operatorname{coset} \mathrm{x}+\mathcal{Z}_{A, F}$. This completes the proof of Theorem 2.6.1.

Proof of Corollary 2.6.2. We prove only the effective part of the statement of Corollary 2.6.2.

Let $\alpha$ be either $\delta$ or one of the coefficients of $\ell_{1}, \ldots, \ell_{n}$. Then an effective representation for $\alpha$ is given, and from this, we can compute effective representations for $\alpha^{2}, \ldots, \alpha^{[G: K]}$. There is a divisor $\nu$ of $[G: K]$ such that $1, \alpha, \ldots, \alpha^{\nu}$ are linearly dependent over $K$. Using the effective representations for the $\alpha^{i}$, we can determine the smallest such $\nu$, a $K$-linear relation between $1, \alpha, \ldots, \alpha^{\nu}$, the monic minimal polynomial of $\alpha$ over $K$ and finally, a degree-height estimate for $\alpha$. Now Theorem 2.6.1 gives an effectively computable number $C$ such that every $\mathcal{Z}_{A, F}$-coset of solutions of 2.6.7) is represented by some $\widetilde{\mathbf{x}} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]^{m}$ with $s(\widetilde{\mathbf{x}}) \leq C$.

There are only finitely many such tuples $\widetilde{\mathbf{x}}$ which can be determined effectively, say $\widetilde{\mathbf{x}}_{1}, \ldots, \widetilde{\mathbf{x}}_{s}$. These represent tuples $\mathbf{x}_{1}, \ldots, \mathbf{x}_{s} \in A^{m}$. Now using the effective representations of $\delta$ and the coefficients of $\ell_{1}, \ldots, \ell_{n}$, one can check for each $\mathbf{x}_{i}$ whether $F\left(\mathbf{x}_{i}\right)=\delta$.

Using the effective representations for $G$ and the coefficients of $\ell_{1}, \ldots, \ell_{n}$, one can compute the triangular graph $\mathcal{G}(\mathcal{L})$ and thus, the vertex systems $\mathcal{L}_{1}, \ldots, \mathcal{L}_{k}$ of its connected components. The intersection $\mathcal{V}:=\left[\mathcal{L}_{1}\right] \cap \cdots \cap\left[\mathcal{L}_{k}\right]$ was defined as a $\bar{K}$-vector space, but since the $\mathcal{L}_{i}$ consist of linear forms with coefficients from $G$, the space $\mathcal{V}$ is defined by a system of linear equations with coefficients from $G$. By solving this system using standard linear algebra, one can decide whether $\mathcal{V}$ is non-zero and if so, compute a basis $\mathcal{B}$ of $\mathcal{V}$, consisting of linear forms from $G$. If for one of the $\mathbf{x}_{i}$ mentioned above there is $\ell \in \mathcal{V}$ with $\ell\left(\mathbf{x}_{i}\right) \neq 0$, then there is such $\ell \in \mathcal{B}$ and one can find it by computing $\ell\left(\mathbf{x}_{i}\right)$ for all $\ell \in \mathcal{B}$. Thus, for each $\mathbf{x}_{i}$ it can be checked whether it satisfies (2.6.7).

Finally, using the effective representations, one can decide for each pair $\mathbf{x}_{i}, \mathbf{x}_{j}$ with $i, j=1, \ldots, s$ whether $\mathbf{x}_{i}, \mathbf{x}_{j}$ lie in the same $\mathcal{Z}_{A, F^{-} \text {-coset, i.e., }}$ $\ell_{t}\left(\mathbf{x}_{i}\right)=\ell_{t}\left(\mathbf{x}_{j}\right)$ for $t=1, \ldots, n$. This shows that one can compute a finite set, consisting of one representative for each $\mathcal{Z}_{A, F}$-coset of solutions of (2.6.7).

Proof of Corollary 2.6.3. By dividing $F$ and $\delta$ by one of the coefficients of
$F$, we can transform 2.3.1 into an equation

$$
F^{\prime}(x, y)=\ell_{1}(x, y) \cdots \ell_{n}(x, y)=\delta^{\prime}
$$

where $\delta^{\prime}$ and the coefficients of $F^{\prime}$ lie in $K$ and are all represented by pairs in $\mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]^{2}$ of total degree at most $d$ and logarithmic height at most $h$, and where each $\ell_{i}$ is either of the form $Y$, or of the form $X-\alpha Y$, with $\alpha \in \bar{K}$. Since $F$ is divisible by at least three pairwise non-proportional linear forms, the system $\mathcal{L}=\left(\ell_{1}, \ldots, \ell_{n}\right)$ is triangularly connected. Further, the module of $(x, y) \in A^{2}$ with $\ell_{i}(x, y)=0$ for $i=1, \ldots, n$ is equal to $\{\mathbf{0}\}$. Lastly, each of the above $\alpha$ has degree at most $n$ over $K$, and by Proposition 8.2.3 it is represented by a tuple of degree at most $(n d)^{\exp O(r)}$ and logarithmic height at most $(n d)^{\exp O(r)}$. Now an application of Theorem 2.6.1 immediately gives the bound (2.6.9), and then using Proposition 2.1.1 one can determine the solutions.

Proof of Corollary 2.6.4 Recall that every solution $\mathbf{x}=\left(x_{1}, x_{2}, x_{3}\right) \in A^{3}$ of the double Pell equation (2.6.10) is a solution of the decomposable form equation (2.6.12), with the decomposable form $F$ given by (2.6.14) and $\delta$ given by (2.6.13). It is easily shown that the linear factors of the decomposable form $F$ in (2.6.12) form a triangularly connected system, and moreover, $\mathcal{Z}_{A, F}=\{\mathbf{0}\}$. Further, by Proposition 8.3.2 and Corollary 8.3.3, both $\delta$ and the coefficients of the linear factors of $F$ are represented by tuples of degree at most $(2 d)^{\exp O(r)}$ and logarithmic height at most $(2 d)^{\exp O(r)} h$. An application of Theorem 2.6.1 directly gives the bound (2.6.15), and the solutions can be found by means of Proposition 2.1.1.

### 10.2 Proofs of the results for norm form equations

Proof of Theorem 2.7.1. Denote by $\mathcal{L}$ the set of conjugates of the linear form $\ell:=\alpha_{1} X_{1}+\cdots+\alpha_{m} X_{m}$ with respect to $K^{\prime} / K$. The coefficients of the linear forms in $\mathcal{L}$ all have degree at most $n=\left[K^{\prime}: K\right]$ over $K$. Further, $\alpha_{1}, \ldots, \alpha_{m}$ and their conjugates over $K$ are all represented by tuples of degree at most $d$ and height at most $h$. The rank of $\mathcal{L}$ is equal to $m$, since $\alpha_{1}, \ldots, \alpha_{m}$ are assumed to be linearly independent over $K$. So with $F:=N_{K^{\prime} / K}\left(\alpha_{1} X_{1}+\right.$ $\cdots+\alpha_{m} X_{m}$ ), the module $\mathcal{Z}_{A, F}$ is $\{0\}$. Letting $\mathcal{L}_{1}, \ldots, \mathcal{L}_{k}$ denote the vertex systems of the connected components of $\mathcal{G}(\mathcal{L})$, we verify below that $X_{m} \in$
$\left[\mathcal{L}_{1}\right] \cap \cdots \cap\left[\mathcal{L}_{k}\right]$. Once this has been done, Theorem 2.7 .1 follows directly from Theorem 2.6.1, taking $\nu=n$.

First observe that any two distinct conjugates of $\ell$ are linearly independent, since $K^{\prime}=K\left(\alpha_{1}, \ldots, \alpha_{m}\right)$. Partition the linear forms in $\mathcal{L}$ into subsets such that $\ell^{\prime}, \ell^{\prime \prime}$ belong to the same subset if the coefficients of $X_{1}, \ldots, X_{m-1}$ in $\ell^{\prime}, \ell^{\prime \prime}$ coincide. Then we get a partition $\mathcal{L}_{1}^{\prime}, \ldots, \mathcal{L}_{k^{\prime}}^{\prime}$ of $\mathcal{L}$ with $k^{\prime}$ denoting the degree of $K^{\prime \prime}:=K\left(\alpha_{1}, \ldots, \alpha_{m-1}\right)$ over $K$. Further, in view of the condition that $\alpha_{m}$ be of degree $\geq 3$ over $K^{\prime \prime}$, each of the sets $\mathcal{L}_{1}^{\prime}, \ldots, \mathcal{L}_{k^{\prime}}^{\prime}$ has cardinality at least 3 . As is easily seen, any three linear forms from the same set $\mathcal{L}_{i}^{\prime}$ are linearly dependent, hence any two linear forms from the same set $\mathcal{L}_{i}^{\prime}$ are connected by an edge in $\mathcal{G}(\mathcal{L})$. That is, each set $\mathcal{L}_{i}^{\prime}$ is contained in the vertex system of one of the connected components of $\mathcal{G}(\mathcal{L})$. Next, the difference of any two linear forms from the same set $\mathcal{L}_{i}^{\prime}$ is proportional to $X_{m}$. This shows

$$
X_{m} \in\left[\mathcal{L}_{1}^{\prime}\right] \cap \cdots \cap\left[\mathcal{L}_{k^{\prime}}^{\prime}\right] \subseteq\left[\mathcal{L}_{1}\right] \cap \cdots \cap\left[\mathcal{L}_{k}\right] .
$$

As mentioned above, this completes the proof of Theorem 2.7.1.

Proof of Corollary 2.7.2. By assumption, an irreducible monic polynomial $P \in K[X]$ is given such that $K^{\prime} \cong K[X] /(P)$. Denote by $G$ the splitting field of $P$ and put $\mathcal{E}:=[G: K]$. Notice that $G$ is the normal closure of $K^{\prime} / K$. By Corollaries 6.2.5 and 6.2.6 we can compute $\theta$ such that $G=K(\theta)$, together with the monic minimal polynomial of $\theta$ over $K$, and express $\alpha_{1}, \ldots, \alpha_{m}$ and their powers as $K$-linear combinations of $1, \theta, \ldots, \theta^{\mathcal{E}-1}$. With these expressions we can compute which $K$-linear combinations of powers of $\alpha_{i}$ are 0 and thus, compute the monic minimal polynomial of $\alpha_{i}$ over $K$ for $i=1, \ldots, m$. These monic minimal polynomials have all their roots in $G$ and we can compute these using Theorem 6.2.3. In this way, we can compute the conjugates of $\alpha_{1}, \ldots, \alpha_{m}$ over $K$. Consequently, the conjugates of the linear form $\ell$ are effectively given. Now Corollary 2.6.2 applies to equation (2.7.1) and the assertion follows.

Proof of Corollary 2.7.3. Let $\mathbf{x}=\left(x_{1}, \ldots, x_{m}\right) \in A^{m}$ be a solution of (2.7.1), and denote by $m^{\prime}$ the greatest integer with $x_{m^{\prime}} \neq 0$. If $m^{\prime} \geq 2$, Corollary 2.7.2 applies with $m^{\prime}$ instead of $m$, while for $m^{\prime}=1$ we get $N\left(\alpha_{1}\right) x_{1}^{n}=\delta$, whence, by Theorems 6.2.3 and 6.3.2, we can check that $x_{1} \in K$ and $x_{1} \in A$.

### 10.3 Proofs of the results for discriminant form equations and discriminant equations

Let $\Omega$ be a finite étale $K$-algebra with $[\Omega: K]=n$. Recall that $\Omega$ is given in the form $K[X] /(P)$, where $P \in K[X]$ is monic and separable of degree $n$. Thus $\Omega=K[\theta]$, with $P(\theta)=0$. Let $G$ be the splitting field of $P$ and $\theta^{(i)}$ $(i=1, \ldots, n)$ the zeros of $P$ in $G$. We can express $\alpha \in \Omega$ as

$$
\alpha=\sum_{j=0}^{n-1} a_{j} \theta^{j} \text { with } a_{j} \in K \text { for } j=0, \ldots, n-1
$$

Thus, the images of $\alpha$ under the $n K$-homomorphisms of $\Omega$ are given by

$$
\alpha^{(i)}=\sum_{j=0}^{n-1} a_{j}\left(\theta^{(i)}\right)^{j} \quad(i=1, \ldots, n) .
$$

By Vandermonde's identity we have $\left.\operatorname{det}\left(\theta^{(i)}\right)^{j-1}\right)_{i, j=1, \ldots, n} \neq 0$. This implies that

$$
\begin{align*}
\alpha^{(1)}=\cdots=\alpha^{(n)} & \Longleftrightarrow a_{1}=\cdots=a_{n-1}=0 \\
& \Longleftrightarrow \alpha \in K . \tag{10.3.1}
\end{align*}
$$

In what follows, let $\omega_{1}, \ldots, \omega_{m} \in \Omega$.
Proof of Theorem 2.8.1 Let $\ell^{(i)}:=\omega_{1}^{(i)} X_{1}+\cdots+\omega_{m}^{(i)} X_{m}$ for $i=1, \ldots, n$, and $\ell_{i, j}:=\ell^{(i)}-\ell^{(j)}$ for $i, j=1, \ldots, n, i \neq j$. We want to apply Theorem 2.6.1 to equation (2.8.3) in the form

$$
\begin{equation*}
F(\mathbf{x}):=\prod_{\substack{i, j=1 \\ i \neq j}}^{n} \ell_{i, j}(\mathbf{x})=(-1)^{n(n-1) / 2} \delta \quad \text { in } \quad \mathbf{x} \in A^{m} \tag{10.3.2}
\end{equation*}
$$

Let $\mathcal{L}$ denote the system of the linear forms $\ell_{i, j}$. We first show that $\mathcal{L}$ is triangularly connected. Indeed, using $\ell_{i, i^{\prime}}+\ell_{i^{\prime}, i^{\prime \prime}}+\ell_{i^{\prime \prime}, i}=0$ for any three distinct $i, i^{\prime}, i^{\prime \prime} \in\{1, \ldots, n\}$, one infers that if $\{i, j\},\left\{i^{\prime}, j^{\prime}\right\}$ are any two distinct subsets of $\{1, \ldots, n\}$, then $\ell_{i, j}, \ell_{i^{\prime}, j^{\prime}}$ are connected by an edge in $\mathcal{G}(\mathcal{L})$ if $\{i, j\} \cap\left\{i^{\prime}, j^{\prime}\right\} \neq \emptyset$. If $\{i, j\} \cap\left\{i^{\prime}, j^{\prime}\right\}=\emptyset$, then there are edges from $\ell_{i, j}$ to $\ell_{i^{\prime}, j}$ and from $\ell_{i^{\prime}, j}$ to $\ell_{i^{\prime}, j^{\prime}}$, implying that $\ell_{i, j}$ and $\ell_{i^{\prime}, j^{\prime}}$ belong to the same connected component of $\mathcal{G}(\mathcal{L})$.

Next, using (10.3.1) it follows that

$$
\begin{aligned}
\mathrm{x} \in \mathcal{Z}_{A, F} & \Longleftrightarrow \ell_{i, j}(\mathbf{x})=0 \text { for all } i, j \\
& \Longleftrightarrow \ell^{(1)}(\mathbf{x})=\cdots=\ell^{(n)}(\mathbf{x}) \Longleftrightarrow \ell(\mathbf{x}) \in K \Longleftrightarrow \mathbf{x} \in \mathcal{Z}_{A, D} .
\end{aligned}
$$

Notice that there are $n(n-1)$ linear forms $\ell_{i, j}$ and that their coefficients $\omega_{t}^{(i)}-\omega_{t}^{(j)}$ have degree at most $n^{2}$ over $K$. Now Theorem 2.8.1 is proved by applying Theorem 2.6.1 with $k=1$, with $n(n-1)$ instead of $n$ and with $n^{2}$ instead of $\nu$ to equation (10.3.2).

Proof of Corollary 2.8.2. Recall that $\Omega$ is given in the form $K[X] /(P)$, with $P$ an effectively given, monic separable polynomial in $K[X]$. Further, a set of $A$-module generators $\omega_{1}, \ldots, \omega_{m} \in \Omega$ of $\mathcal{M}$ is effectively given. Similarly as in the proof of Corollary 2.7.2, we can compute an effective representation for the splitting field $G$ of $P$, as well as effective representations for $\omega_{j}^{(i)}$, for $i=1, \ldots, n, j=1, \ldots, m$. So the coefficients of the linear forms $\ell_{i, j}$ defined in the proof of Theorem 2.8.1 are effectively given. So by Corollary 2.6.2, we can compute a finite set of solutions $\mathrm{x} \in A^{n}$ of (2.8.2), consisting of one representative from each $\mathcal{Z}_{A, D}$-coset. But this means precisely that we can compute a finite set of solutions $\xi \in \mathcal{M}$ of equation (2.8.2], consisting of one element from each $\mathcal{M} \cap K$-coset.

Proof of Corollary 2.8.3. By Corollary 2.8.2, we can effectively compute a full system of representatives, say $\left\{\xi_{1}, \ldots, \xi_{s}\right\}$, for the $\mathcal{O} \cap K$-cosets of solutions of

$$
\begin{equation*}
D_{\Omega / K}(\xi)=\delta \text { in } \xi \in \mathcal{O} \tag{10.3.3}
\end{equation*}
$$

Further, by Corollary 6.3.9, we can compute a full system of representatives, say $\left\{\eta_{1}, \ldots, \eta_{t}\right\}$, for the cosets of $\mathcal{O} \cap K$ modulo $A$. Then the finite set $\left\{\xi_{i}+\right.$ $\left.\eta_{j}: i=1, \ldots, s, j=1, \ldots, t\right\}$ is an effectively computable full system of representatives for the $A$-cosets of solutions of (10.3.3).

In the proof of Theorem 2.8.4 we shall need the following.
Lemma 10.3.1. For every integral domain $A$ of characteristic 0 which is finitely generated over $\mathbb{Z}$ and every two monic polynomials $f, f^{\prime} \in A[X]$, all effectively given, we can determine effectively whether $f, f^{\prime}$ are strongly $A$-equivalent.

Proof. It suffices to consider the case when $f, f^{\prime}$ have equal degrees. Write $f(X)=X^{n}+a_{1} X^{n-1}+\cdots, f^{\prime}(X)=X^{n}+b_{1} X^{n-1}+\cdots$. We have to check whether there is an $a \in A$ such that $f^{\prime}(X)=f(X+a)$. Comparing the coefficients of $X^{n-1}$ we see that for such $a$ we must have $n a=b_{1}-a_{1}$. Using Theorem 6.3.2 we can check whether $a \in A$ and then whether indeed $f^{\prime}(X)=f(X+a)$.

Proof of Theorem 2.8.4 Let $A, G, \delta, n$ be effectively given and satisfy the conditions of Theorem 2.8.4. Denote by $A_{K}$ the integral closure of $A$ in $K$ and by $A_{G}$ the integral closure of $A$ in $G$. Let $f$ be a polynomial with

$$
\begin{align*}
D(f)=\delta, & f \text { is monic, } f \in A[X] \\
& \operatorname{deg} f=n, f \text { has all its zeros in } G . \tag{2.8.6}
\end{align*}
$$

Write

$$
f=\left(X-\xi_{1}\right) \cdots\left(X-\xi_{n}\right)
$$

Then

$$
\begin{align*}
& \xi_{1}, \ldots, \xi_{n} \in A_{G}, \quad \xi_{1}+\cdots+\xi_{n} \in A  \tag{10.3.4}\\
& \prod_{1 \leq i<j \leq n}\left(\xi_{i}-\xi_{j}\right)^{2}=\delta \tag{10.3.5}
\end{align*}
$$

For arbitrary $\xi_{1}, \ldots, \xi_{n}$ with 10.3.4, 10.3.5), we write $\xi:=\left(\xi_{1}, \ldots, \xi_{n}\right)$, $f_{\xi}:=\left(X-\xi_{1}\right) \cdots\left(X-\xi_{n}\right)$. It is important to notice that conversely, if $\xi$ satisfies (10.3.4), 10.3.5) then $f_{\xi}$ satisfies all conditions in (2.8.6), except that it need not belong to $A[X]$.

Two tuples $\xi^{\prime}=\left(\xi_{1}^{\prime}, \ldots, \xi_{n}^{\prime}\right), \xi^{\prime \prime}=\left(\xi_{1}^{\prime \prime}, \ldots, \xi_{n}^{\prime \prime}\right)$ with 10.3.4), (10.3.5) are said to lie in the same $A$-coset if

$$
\xi_{1}^{\prime}-\xi_{1}^{\prime \prime}=\cdots=\xi_{n}^{\prime}-\xi_{n}^{\prime \prime} \in A .
$$

Notice that in this case, the corresponding polynomials $f_{\xi^{\prime}}, f_{\xi^{\prime \prime}}$ are strongly $A$-equivalent. We show that there are only finitely many $A$-cosets of tuples $\xi$ with (10.3.4), (10.3.5), and determine a full system of representatives, and subsequently select those tuples $\xi$ from this system for which $f_{\xi} \in A[X]$. Then every polynomial $f$ with (2.8.6) is strongly $A$-equivalent to one of these $f_{\xi}$.

By Theorem 6.3.6, $A_{G}$ is finitely generated as an $A$-module, and we can effectively determine a system of $A$-module generators for $A_{G}$, say $\left\{\omega_{1}, \ldots, \omega_{m}\right\}$.

Thus, we can express $\xi_{1}, \ldots, \xi_{n}$ with (10.3.4) as

$$
\begin{align*}
& \xi_{i}=\ell_{i}(\mathbf{x}):=x_{i, 1} \omega_{1}+\cdots+x_{i, m} \omega_{m}(i=1, \ldots, n-1),  \tag{10.3.6}\\
& \xi_{n}=\ell_{n}(\mathbf{x}):=x_{0}-\ell_{1}(\mathbf{x})-\cdots-\ell_{n-1}(\mathbf{x})
\end{align*}
$$

where

$$
\mathbf{x}=\left(x_{1,1}, \ldots, x_{1, m}, \ldots, x_{n-1,1}, \ldots, x_{n-1, m}, x_{0}\right) \in A^{m(n-1)+1}
$$

and then 10.3 .5 ) translates into the decomposable form equation

$$
\begin{equation*}
F(\mathbf{x}):=\prod_{\substack{1 \leq i, j \leq n \\ i \neq j}}\left(\ell_{i}(\mathbf{x})-\ell_{j}(\mathbf{x})\right)=(-1)^{n(n-1) / 2} \delta \text { in } \mathbf{x} \in A^{m(n-1)+1} \tag{10.3.7}
\end{equation*}
$$

Completely similarly as in the proof of Theorem 2.8.1, one shows that the system of linear forms

$$
\mathcal{L}:=\left(\ell_{i}-\ell_{j}: 1 \leq i, j \leq n, i \neq j\right)
$$

is triangularly connected. Further, we have

$$
\mathbf{x} \in \mathcal{Z}_{A, F} \Longleftrightarrow \ell_{1}(\mathbf{x})=\cdots=\ell_{n}(\mathbf{x}) .
$$

By Corollary 2.6.2, equation (10.3.7) has only finitely many $\mathcal{Z}_{A, F}$-cosets of solutions, and a full system of representatives of these can be determined effectively. Notice that if $\mathbf{x}=\left(\ldots, x_{0}\right) \in \mathcal{Z}_{A, F}$, then

$$
\ell_{1}(\mathbf{x})=\cdots=\ell_{n}(\mathbf{x})=\frac{1}{n} x_{0} \in \frac{1}{n} A \cap A_{G}=\frac{1}{n} A \cap A_{K} .
$$

Translating this back to (10.3.4), (10.3.5), we see that the tuples $\xi$ with (10.3.4), (10.3.5) lie in only finitely many $\left(\frac{1}{n} A \cap A_{K}\right)$-cosets. Moreover, a full system of representatives for these cosets can be determined effectively. Let $\mathcal{C}$ be such a full system of representatives. By Corollary 6.3.7, there are only finitely many cosets of $\frac{1}{n} A \cap A_{K}$ modulo $A$, and a full system of representatives of these can be determined effectively. Let $\mathcal{C}^{\prime}$ be such a system. From $\mathcal{C}$ and $\mathcal{C}^{\prime}$ we compute

$$
\mathcal{C}^{\prime \prime}:=\left\{\xi+a^{*}: \xi \in \mathcal{C}, a \in \mathcal{C}^{\prime}\right\} \text { where } a^{*}:=\underbrace{(a, \ldots, a)}_{n \text { times }}
$$

which is a full system of representatives for the $A$-cosets of tuples $\xi$ with (10.3.4), (10.3.5). From the tuples $\xi \in \mathcal{C}^{\prime \prime}$ we effectively select those for which $f_{\xi} \in A[X]$; namely, using the effective representations of the coefficients of $f_{\xi}$ we can first decide whether $f_{\xi} \in K[X]$, and then subsequently whether $f_{\xi} \in A[X]$ by means of Theorem 6.3.2. Further, using Lemma 10.3.1, from the $f_{\xi}$ with the remaining $\xi$ we select a maximal set of pairwise not strongly $A$ equivalent polynomials. What is left is a finite, full system of representatives for the strong $A$-equivalence classes of polynomials $f$ with (2.8.6).

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## Glossary of frequently used notation

## General notation

| $\|\mathcal{A}\|$ | cardinality of a finite set $\mathcal{A}$ |
| :---: | :---: |
| $\log ^{*} x$ | $\max (1, \log x), \log ^{*} 0:=1$. |
| $\ll, \gg$ | Vinogradov symbols; $A(x) \ll B(x)$ or $B(x) \gg$ $A(x)$ means that there is a constant $c>0$ such that $\|A(x)\| \leq c B(x)$ for all $x$ in the specified domain. The constant $c$ may depend on certain specified parameters independent of $x$ |
| $<_{a, b, \ldots}$ | the positive constants implied by $<_{a, b, \ldots}$ depends only on $a, b, \ldots$ and are effectively computable |
| $O(\cdot)$ | $c \times$ the expression between the parentheses, where $c$ is an effectively computable positive absolute constant. The $c$ may be different at each occurrence of $O(\cdot)$ |
| $\mathbb{Z}, \mathbb{Z}_{>0}, \mathbb{Z}_{\geq 0}$ | integers, positive integers, non-negative integers |
| $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ | rational numbers, real numbers, complex numbers |
| gcd | greatest common divisor |
| $D(f)$ | discriminant of a polynomial $f(X)$ |
| $\bar{K}$ | algebraic closure of a field $K$ |


| A | integral domain (i.e., commutative ring with 1 and without divisors of 0 ) |
| :---: | :---: |
| $A^{*}$ | unit group (multiplicative group of invertible elements) of $A$ |
| $A_{G}$ | integral closure of $A$ in an extension $G$ of the quotient field of $A$ |
| $A\left[X_{1}, \ldots, X_{n}\right]$ | ring of polynomials in $n$ variables with coefficients in $A$ |
| $A\left[\alpha_{1}, \ldots, \alpha_{n}\right]$ | $\begin{aligned} & \left\{f\left(\alpha_{1}, \ldots, \alpha_{n}\right): f \in A\left[X_{1}, \ldots, X_{r}\right]\right\}, A \text {-algebra } \\ & \text { generated by } \alpha_{1}, \ldots, \alpha_{n} \end{aligned}$ |
| $\xi+\mathcal{M}$ | $\{\xi+\eta: \eta \in \mathcal{M}\}, \mathcal{M}$-coset, where $\mathcal{M}$ is an $A$ module and $\xi$ belongs to an $A$-module containing $\mathcal{M}$ |
| $\mathcal{M}^{\prime} / \mathcal{M}$ | quotient $A$-module of two $A$-modules $\mathcal{M}^{\prime}, \mathcal{M}$, where $\mathcal{M}^{\prime} \supseteq \mathcal{M} ; \mathcal{M}^{\prime} / \mathcal{M}$ consists of the $\mathcal{M}$ cosets $\xi+\mathcal{M}$ with $\xi \in \mathcal{M}^{\prime}$, and is endowed with addition $\left(\xi_{1}+\mathcal{M}\right)+\left(\xi_{2}+\mathcal{M}\right):=\left(\xi_{1}+\xi_{2}\right)+\mathcal{M}$ and scalar multiplication $a \cdot(\xi+\mathcal{M}):=a \xi+\mathcal{M}$, for $\xi_{1}, \xi_{2}, \xi \in \mathcal{M}^{\prime}$ and $a \in A$ |
| $H(Q), L(Q)$ | maximum of the absolute values resp. the sum of the absolute values of the coefficients of $Q \in$ $\mathbb{Z}\left[X_{1}, \ldots, X_{n}\right]$ |
| $\operatorname{deg} Q, h(Q)$ $s(Q)$ | the total degree of $Q \in \mathbb{Z}\left[X_{1}, \ldots, X_{n}\right]$, resp. the logarithmic height $\log H(Q)$ of $Q$ $\max (1, \operatorname{deg} Q, h(Q))$, the size of $Q$ |

## Finite étale algebras over fields

$$
\begin{aligned}
& \Omega / K \\
& \\
& {[\Omega: K]} \\
& x \mapsto x^{(i)} \\
& D_{\Omega / K}(\alpha) \\
& A_{\Omega}
\end{aligned}
$$

finite étale algebra over a field $K$, i.e., a direct product $L_{1} \times \ldots \times L_{q}$ of finite separable field extensions of $K$
$\operatorname{dim}_{K} \Omega$
non-trivial $K$-algebra homomorphisms $\Omega \rightarrow \bar{K}$ discriminant of $\alpha \in \Omega$ over $K$
integral closure of an integral domain $A$ with quotient field $K$ in a finite étale $K$-algebra $\Omega$
$\mathcal{O}$ $A$-order of $\Omega$, i.e., a subring of $A_{\Omega}$ containing $A$ and generating $\Omega$ as a $K$-vector space

## Algebraic number fields

| $\operatorname{ord}_{p}(a)$ | exponent of a prime number $p$ in the unique prime factorization of $a \in \mathbb{Q}$, and $\operatorname{ord}_{p}(0)=\infty$ |
| :---: | :---: |
| $\|a\|_{p}$ | $p^{-\operatorname{ord}_{p}(a)}, p$-adic absolute value of $a \in \mathbb{Q}$ |
| $\|a\|_{\infty}$ | $\max (a,-a)$, ordinary absolute value of $a \in \mathbb{Q}$ |
| $\mathbb{Q}_{p}$ | $p$-adic completion of $\mathbb{Q}, \mathbb{Q}_{\infty}=\mathbb{R}$ |
| $\mathcal{M}_{\mathbb{Q}}$ | $\{\infty\} \cup\{$ primes $\}$, set of places of $\mathbb{Q}$ |
| $\mathcal{O}_{K}, D_{K}, h_{K}, R_{K}$ | ring of integers, discriminant, class number, regulator of a number field $K$ |
| $\mathfrak{p}, \mathfrak{a}$ | non-zero prime ideal, fractional ideal of $\mathcal{O}_{K}$ |
| $\begin{aligned} & {[\alpha]=\alpha \mathcal{O}_{K}} \\ & \operatorname{ord}_{\mathfrak{p}}(\mathfrak{a}) \end{aligned}$ | fractional ideal generated by $\alpha$ exponent of $\mathfrak{p}$ in the unique prime ideal factorization of $\mathfrak{a}$ |
| $\operatorname{ord}_{\mathfrak{p}}(\alpha)$ | exponent of $\mathfrak{p}$ in the unique prime ideal factorization of $(\alpha)$ for $\alpha \in K$, with $\operatorname{ord}_{\mathfrak{p}}(0):=\infty$. |
| $N_{K}(\mathfrak{a})$ | absolute norm of a fractional ideal $\mathfrak{a}$ of $\mathcal{O}_{K}$ (written as $N(\mathfrak{a})$ if it is clear which is the underlying number field) |
| $\mathcal{M}_{K}$ | set of places of a number field $K$ |
| $\|\cdot\|_{v}\left(v \in \mathcal{M}_{K}\right)$ | normalized absolute values of $K$, satisfying the product formula, with $\|\alpha\|_{v}:=N_{K}(\mathfrak{p})^{-\operatorname{ord}_{\mathfrak{p}}(\alpha)}$ if $\alpha \in K$ and $\mathfrak{p}$ is the prime ideal of $\mathcal{O}_{K}$ corresponding to the finite place $v$ |
| $K_{v}$ | completion of $K$ at $v$ |

$S_{\infty}$
$S$
$\mathcal{O}_{S}$
$\mathcal{O}_{S}^{*}$
$N_{S}(\alpha)$
$R_{S}$
$P_{S}, Q_{S}$
$|\mathbf{x}|_{v}\left(v \in \mathcal{M}_{K}\right)$
$H^{\text {hom }}(\mathbf{x})$
$H(\mathbf{x})$
$H(\alpha)$
$h^{\text {hom }}(\mathbf{x}), h(\mathbf{x}), h(\alpha)$
$h(P)$

## Function fields

k
$\mathbb{k}((z))$
$g_{K / \mathbf{k}}$
$\mathcal{M}_{K}$
$v(\mathbf{x})\left(v \in \mathcal{M}_{K}\right)$
$H_{K}^{\mathrm{hom}}(\mathbf{x})$
$H_{K}(x)$
$S$
set of infinite (archimedean) places
finite set of places of $K$, containing $S_{\infty}$
$\left\{\alpha \in K:|\alpha|_{v} \leq 1\right.$ for $\left.v \in \mathcal{M}_{K} \backslash S\right\}$, ring of $S$-integers, written as $\mathbb{Z}_{S}$ if $K=\mathbb{Q}$
$\left\{\alpha \in K:|\alpha|_{v}=1\right.$ for $\left.v \in \mathcal{M}_{K} \backslash S\right\}$, group of $S$-units, written as $\mathbb{Z}_{S}^{*}$ if $K=\mathbb{Q}$
$\prod_{v \in S}|\alpha|_{v}, S$-norm of $\alpha \in K$
$S$-regulator
$\max \left\{N_{K}\left(\mathfrak{p}_{1}\right), \ldots, N_{K}\left(\mathfrak{p}_{t}\right)\right\}, \prod_{i=1}^{t} N_{K}\left(\mathfrak{p}_{i}\right)$, where $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{t}$ are the prime ideals of $\mathcal{O}_{K}$ corresponding to the finite places of $S$
$\max _{i}\left|x_{i}\right|_{v}, v$-adic norm of $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right) \in K^{n}$
$\left(\prod_{v \in \mathcal{M}_{K}}|\mathbf{x}|_{v}\right)^{1 /[K: \mathbb{Q}]}, \quad$ absolute $\quad$ homogeneous height of $\mathbf{x} \in K^{n}$
$\left(\prod_{v \in \mathcal{M}_{K}} \max \left(1,|\mathbf{x}|_{v}\right)\right)^{1 /[K: \mathbb{Q}]}$, absolute height of $\mathbf{x} \in K^{n}$
$\left(\prod_{v \in \mathcal{M}_{K}} \max \left(1,|\alpha|_{v}\right)\right)^{1 /[K: \mathbb{Q}]}$, absolute height of $\alpha \in K$
$\log H^{\text {hom }}(\mathbf{x}), \log H(\mathbf{x}), \log H(\alpha)$, absolute logarithmic heights ( $\mathrm{x} \in K^{n}, \alpha \in K$ )
$h\left(\mathrm{x}_{P}\right), \mathbf{x}_{P}$ vector consisting of the non-zero coefficients of a polynomial $P \in K\left[X_{1}, \ldots, X_{n}\right]$
field of constants (always algebraically closed) field of Laurent series in $z$
genus of function field $K$ with constant field $\mathbb{k}$ ( $K / \mathbb{k}$ is always assumed to be of transcendence degree 1)
set of (normalized discrete) valuations of $K$, trivial on $\mathbb{k}$
$\min _{i} v\left(x_{i}\right), v$-adic norm of $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right) \in K^{n}$
$-\sum_{v \in \mathcal{M}_{K}} v(\mathbf{x})$, homogeneous height of $\mathbf{x} \in K^{n}$
$\sum_{v \in \mathcal{M}_{K}} \max (0,-v(x))$, height of $x \in K$
a finite subset of $\mathcal{M}_{K}$

| $\mathcal{O}_{S}$ | $\left\{\alpha \in K: v(\alpha) \geq 0\right.$ for $\left.v \in \mathcal{M}_{K} \backslash S\right\}$, ring of |
| :--- | :--- |
| $\mathcal{O}_{S}^{*}$ | $S$-integers |
|  | $\left\{\alpha \in K: v(\alpha)=0\right.$ for $\left.v \in \mathcal{M}_{K} \backslash S\right\}$, group of |
|  | $S$-units |

## Finitely generated domains

| $A=\mathbb{Z}\left[z_{1}, \ldots, z_{r}\right]$ | $\left\{f\left(z_{1}, \ldots, z_{r}\right): f \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]\right\}$, finitely generated integral domain over $\mathbb{Z}$ with quotient field $K=\mathbb{Q}\left(z_{1}, \ldots, z_{r}\right)$ |
| :---: | :---: |
| $A \simeq \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right] / \mathcal{I}$ | $\begin{aligned} & \mathcal{I}:=\left\{f \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]: f\left(z_{1}, \ldots, z_{r}\right)=0\right\}, \\ & \text { finitely generated ideal in } \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right] \end{aligned}$ |
| $\mathcal{I}=\left(f_{1}, \ldots, f_{M}\right)$ | ideal representation for $A$ |
| $\tilde{\alpha} \in \mathbb{Z}\left[X_{1}, \ldots, X_{r}\right]$ | representative for $\alpha \in A$ if $\alpha=\tilde{\alpha}\left(z_{1}, \ldots, z_{r}\right)$ |
| $A$ effectively given | if an ideal representation $\left(f_{1}, \ldots, f_{M}\right)$ for $A$ is given |
| $\alpha \in A$ effectively given (computable) | if a representative for $\alpha$ is given (can be computed) |
| $\left\{z_{1}=X_{1}, \ldots, z_{q}=X_{q}\right\}$ | transcendence basis for $K=\mathbb{Q}\left(z_{1}, \ldots, z_{r}\right)$ over $\mathbb{Q}$ |
| $A_{0}=\mathbb{Z}\left[X_{1}, \ldots, X_{q}\right]$ | subring of $A$ with unique factorization |
| $\operatorname{deg} \alpha, h(\alpha)$ for $\alpha \in A_{0}$ $K_{0}=\mathbb{Q}\left(X_{1}, \ldots, X_{q}\right)$ | the total degree and logarithmic height of $\alpha$ quotient field of $A_{0}$ |
| $K=K_{0}(w)$ | where $w \in A$, integral over $A_{0}$ with degree $D$ over $K_{0}$ |
| $\overline{\operatorname{deg}} \alpha(\alpha \in K)$ | $m a x\left(\operatorname{deg} P_{\alpha, 0}, \ldots, \operatorname{deg} P_{\alpha, D-1}, \operatorname{deg} Q_{\alpha}\right), \quad$ where |
|  | $P_{\alpha, 0}, \ldots, P_{\alpha, D-1}, Q_{\alpha} \in A_{0}$ are relatively prime, and $\alpha=Q_{\alpha}^{-1} \sum_{j=0}^{D-1} P_{\alpha, j} \omega^{j}$ |
| $\bar{h}(\alpha)(\alpha \in K)$ | $\max \left(h\left(P_{\alpha, 0}\right), \ldots, h\left(P_{\alpha, D-1}\right), h\left(Q_{\alpha}\right)\right)$ |


[^0]:    ${ }^{1}$ With our definitions of strong $A$-equivalence and $A$-equivalence (sec), which is considered below, we follow Győry (1982) and Evertse and Győry (2017b).

[^1]:    ${ }^{1}$ If $\mathcal{M}_{1}, \mathcal{M}_{2}$ are modules over a ring $R$ with $\mathcal{M}_{1} \subset \mathcal{M}_{2}$, then by an $\mathcal{M}_{1}$-coset in $\mathcal{M}_{2}$ we mean a set of the shape $a+\mathcal{M}_{1}=\left\{a+x: x \in \mathcal{M}_{1}\right\}$ with $a \in \mathcal{M}_{2}$.

